Understanding
Fox and Glynn's "Computing Poisson probabilities"

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Abstract

This article gives a consistent presentation of Computing Poisson Probabilities by Fox and Glynn [3] and its proofs (with a few additions from my hand). It makes the FINDER algorithm, sketched only in [3], explicit.

1 Introduction

This text reproduces a part of the article Computing Poisson Probabilities by Fox and Glynn [3] with additions from my own hand. Some readers may find this important article difficult to follow because a part of the proofs are to be found in another article [4] using a different notation and the FINDER algorithm is not given explicitly. Additionally, in a few places I could not follow the argumentation upon first reading. The present report brings this algorithm and all proofs together in a common notation, makes some proof steps more explicit and corrects a few typos. I wrote this text first for my personal understanding and publish it now as a technical report in the hope that also others may find it helpful. In most of the text, I just copy from the two mentioned articles, with my additions here and there.

Newer literature confirms that the approach of Fox and Glynn is feasible. For example, [7] compares uniformization with other, particularly stiffness-tolerant methods and concludes that for non-stiff or mildly stiff Markov chains, uniformization using Fox and Glynn to calculate the Poisson probabilities is best. [6] generalizes the approach of Fox and Glynn to some other distributions. (However, the general formulation does not allow to calculate the lower and upper truncation points beforehand.) [8] combines uniformisation with *adaptive* uniformisation, a method that takes into account that some states with high exit rates are not reachable quickly, to speed up transient analysis of CTMCs. The same article also contains an appendix showing an improved error bound for [3]. Unfortunately, that appendix seems to confuse $\Phi(k)$ and $\overline{\Phi}(k) = 1 - \Phi(k)$ as a consequence of a typo in [3]. (see Propositions 2 and 3 in [3] below).

2 Finding truncation points

The article [3] exposes clearly an algorithm to find weights. However, the correct way to find truncation points is only sketched, and one has to find out for oneself which bounds apply exactly in which case. My understanding is shown as Algorithm 1.

Fox and Glynn distinguish three cases: $\lambda < 25$, $25 \leq \lambda < 400$ and $400 \leq \lambda$. We actually test the mode, $m = \lfloor \lambda \rfloor$, because testing an integer against a constant is in most cases simpler (lines 1.3, 1.16, 1.30, 1.39).

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Algorithm 1 FINDER algorithm proposal

function FINDER(λ, ε)
  $m := \lfloor \lambda \rfloor$
  if $m < 25$ **then**
    if $-\lambda < \ln \tau$ **then** Error: Underflow near $w(0)$.
    $L := 0$
  else
    $rhs_b := \frac{\varepsilon}{2}\sqrt{2\pi} / \left[\left(1 + \frac{1}{\lambda}\right) \exp \frac{1}{8\lambda}\right]$
    $k := 4$
    loop
      $L := m - \left\lceil k\sqrt{\lambda} + \frac{1}{2} \right\rceil$
      if $L \leq 0$ **then** $L := 0$; **quit loop**
      if $\exp\left(-k^2/2\right)/k \leq rhs_b$ **then quit loop**
      $k := k + 1$
    end loop
  end if
  if $m < 400$ **then**
    $m_{max} := 400$; $r_{2\lambda} := \sqrt{2 \cdot 400}$; $rhs_a := \frac{\varepsilon}{2}\sqrt{2\pi} / \left[\left(1 + \frac{1}{400}\right)\sqrt{2} \exp \frac{1}{16}\right]$; $max_R := 400 + \left\lceil \frac{400+1}{2} \right\rceil$
  else
    $m_{max} := m$; $r_{2\lambda} := \sqrt{2\lambda}$; $rhs_a := \frac{\varepsilon}{2}\sqrt{2\pi} / \left[\left(1 + \frac{1}{\lambda}\right)\sqrt{2} \exp \frac{1}{16}\right]$; $max_R := m + \left\lceil \frac{\lambda+1}{2} \right\rceil$
  end if
  $k := 4$
  loop
    $R := m_{max} + \left\lceil k \cdot r_{2\lambda} + \frac{1}{2} \right\rceil$
    if $R > max_R$ **then** Error: Cannot bound the right tail.
    $d(k, \lambda) := 1 / \left\langle 1 - \exp\left(-\frac{266}{401}\left[k \cdot r_{2\lambda} + \frac{3}{2}\right]\right)\right\rangle$
    {or simplify this by setting $d(k, \lambda) := 1$.}
    if $d(k, \lambda) \exp\left(-k^2/2\right)/k \leq rhs_a$ **then quit loop**
    $k := k + 1$
  end loop
  $w(m) := 10^{-10}\Omega/(R - L + 1)$
  if $m \geq 25$ **then**
    $lc_m := -1 - \frac{1}{12 \cdot 25} - \ln\sqrt{2\pi} - \frac{1}{2}\ln m$
    $i := m - L$
    if $i \leq L$ **then** {This test is equivalent to $2i \leq m$, which is equivalent to $i \leq \frac{\lambda}{2}$.}
      $llb_{p_\lambda(L)} := -i(i+1)\left(\frac{2i+1}{6\lambda} + \frac{1}{2}\right)\frac{1}{\lambda} + lc_m$
    else
      $llb_{p_\lambda(L)} := \max\left\{ i \ln\left(1 - \frac{i}{m+1}\right) + lc_m, -\lambda \right\}$
    end if
    if $llb_{p_\lambda(L)} < \ln\tau - \ln w(m)$ **then** Error: Underflow near $w(L)$.
    if $m \geq 400$ **then**
      $llb_{p_\lambda(R)} := -\frac{(R-m+1)^2}{2\lambda} + lc_m$
      if $llb_{p_\lambda(R)} < \ln\tau - \ln w(m)$ **then** Error: Underflow near $w(R)$.
    end if
  end if
  return $L, R, w(m)$
end function

[3] do not make completely clear in what cases the underflow test in line 1.4 is to be executed; I supposed that it only applies to $\lambda < 25$, as the later underflow tests in lines 1.30–1.43 are not executed then. I propose to calculate all underflow tests (lines 1.4, 1.38, 1.41) using natural logarithms; this allows to check for underflow without actually calculating numbers that are (almost) zero. If $\lambda \geq 25$, we calculate a left bound L such that the left tail $T_\lambda(L-1)$ is at most $\frac{\varepsilon}{2}$. Corollary 2 in [3] gives us an upper bound to the left tail: $T_\lambda(L-1) = T_\lambda(\lfloor m - k\sqrt{\lambda} - \frac{3}{2} \rfloor) \leq b_\lambda \frac{\exp(-k^2/2)}{k\sqrt{2\pi}}$. Note that $b_\lambda/\sqrt{2\pi} = \left(1 + \frac{1}{\lambda}\right) \exp\left(\frac{1}{8\lambda}\right)/\sqrt{2\pi}$ does not depend on k; therefore, we calculate this part of the bound outside the loop in line 1.7. When we have found the left bound, we could improve the efficiency of finding the right bound slightly by allowing $\varepsilon -$ (the maximal error in $T_\lambda(L-1)$) as error on the right tail.

The right bound is calculated in lines 1.16–1.28 using corollary 1 in [3]. That corollary gives an upper bound on the right tail $Q_\lambda(R+1)$; however, that error bound only holds if R is not too large. For small λ, we calculate the bound as if $\lambda = 400$, because that will give us some extra room for R. The funny constant $\frac{266}{401}$ in line 1.25 is explained in our proof of Proposition 2 in [3] below; why $d(k, \lambda) := 1$ is a valid simplification is explained near the definition of $d(k, \lambda)$ on page 7.

Finally, in lines 1.30–1.43, we check whether underflow may occur during the calculation of weights. This part uses propositions 5 and 6 in [3] to find lower bounds on $p_\lambda(L)$ and $p_\lambda(R)$. (Fox and Glynn actually propose to use their corollaries 3 and 4, based on the loop variables k, but I do not see how that would simplify the calculation.)

3 Rationale: Why it works

In the following sections, I copy most text from [3], starting from their section 4, together with a few proofs from [4], and include my own remarks (in blue) in between. Corrections of what I consider wrong are in red. I silently adapted the notation of [4] to that of [3].

4 Bounding Poisson Tails

Let

$$p_\lambda(i) = \frac{\exp(-\lambda)\lambda^i}{i!}$$ (the probability that exactly i events happen in a time interval when λ are expected)

$$Q_\lambda(i) = \sum_{j=\lceil i \rceil}^{\infty} p_\lambda(j)$$ (the probability that at least $\lceil i \rceil$ events happen in a time interval when λ are expected)

$$T_\lambda(i) = \sum_{j=0}^{\lfloor i \rfloor} p_\lambda(j)$$ (the probability that at most $\lfloor i \rfloor$ events happen in a time interval when λ are expected)

$$\varphi(x) = \frac{\exp(x^2/2)}{\sqrt{2\pi}}$$ (the probability density of the normal distribution)

$$\overline{\Phi}(x) = \int_x^\infty \varphi(t)\, dt$$ (the *complementary* cumulative distribution of the normal distribution)

$$a_\lambda = \left(1 + \frac{1}{\lambda}\right) \sqrt{2} \exp\left(\tfrac{1}{16}\right)$$

$$b_\lambda = \left(1 + \frac{1}{\lambda}\right) \exp\left(\frac{1}{8\lambda}\right)$$ (two helper variables which will be used below)

Note that $Q_\lambda(i) + T_\lambda(i) = 1 + p_\lambda(i)$ if $i \in \mathbb{N}$, so $Q_\lambda(i)$ is *not* the complementary cumulative distribution function. It does hold that $Q_\lambda(i+1) + T_\lambda(i) = 1$.

Proposition 1 (ii) in [4]. Assume $\lambda > 0$. If $n > \lambda - 1$ and $l \geq 1$, then

$$Q_\lambda(n) \leq \left(1 - \left(\frac{\lambda}{n+1}\right)^l\right)^{-1} \cdot \sum_{k=\lceil n \rceil}^{\lceil n \rceil + l - 1} p_\lambda(k)$$

Proof. It is evident that

$$\begin{aligned} Q_\lambda(n) &= \sum_{k=\lceil n \rceil}^{\lceil n \rceil + l - 1} p_\lambda(k) + \lambda^l \sum_{k=\lceil n \rceil}^{\infty} p_\lambda(k) \times \frac{k!}{(k+l)!} \\ &\leq \sum_{k=\lceil n \rceil}^{\lceil n \rceil + l - 1} p_\lambda(k) + \lambda^l \sum_{k=\lceil n \rceil}^{\infty} p_\lambda(k) \times (n+1)^{-l} = \sum_{k=\lceil n \rceil}^{\lceil n \rceil + l - 1} p_\lambda(k) + \left(\frac{\lambda}{n+1}\right)^l Q_\lambda(n) \end{aligned}$$

Solving for $Q_\lambda(n)$ (which requires $n + 1 > \lambda$ and $l > 0$) yields Proposition 1 (ii). □

Let $m = \lfloor \lambda \rfloor$.

Proposition 2 in [4]. Let $\lambda, i \geq 1$. Then,

1. $$p_\lambda(m - i) \leq \frac{1}{\sqrt{2\pi m}} \exp\left(-\frac{i(i-1)}{2\lambda}\right),$$

2. $$p_\lambda(m + i) \leq \frac{1}{\sqrt{2\pi m}} \exp\left(-\frac{i(i-1)}{2\lambda} + \frac{(i-1)i(2i-1)}{12\lambda^2}\right)$$

Proof. For $1 \leq i \leq m$, we use the fact that $m \leq \lambda$ to obtain

$$\begin{aligned} p_\lambda(m-i) &= p_\lambda(m) \cdot \left(\frac{m}{\lambda}\right) \cdot \left(\frac{m-1}{\lambda}\right) \cdots \left(\frac{m-i+1}{\lambda}\right) \\ &\leq p_\lambda(m) \cdot (1) \cdot \left(1 - \frac{1}{\lambda}\right) \cdots \left(1 - \frac{i-1}{\lambda}\right) = p_\lambda(m) \exp\left(\sum_{k=0}^{i-1} \ln\left(1 - \frac{k}{\lambda}\right)\right). \end{aligned}$$

By Lemma 1 (i) in [4][1], $\ln(1 - k/\lambda) \leq -k/\lambda$, so

$$p_\lambda(m-i) \leq p_\lambda(m) \exp\left(-\sum_{k=0}^{i-1} \frac{k}{\lambda}\right) = p_\lambda(m) \exp\left(-\frac{i(i-1)}{2\lambda}\right),$$

by a standard summation formula. To bound $p_\lambda(m)$, we use a Stirling formula-type inequality (see [2, page 54]):

$$m! > \sqrt{2\pi m} \cdot m^m \exp(-m) \exp\left(\frac{1}{12m+1}\right)$$

which yields ($b = \lambda - m$):

$$\begin{aligned} p_\lambda(m) = \frac{\exp(-\lambda)\lambda^m}{m!} &\leq \frac{\exp(-\lambda)\lambda^m}{\sqrt{2\pi m} \cdot m^m \exp(-m)} = \frac{1}{\sqrt{2\pi m}} \exp(-b) \left(1 + \frac{b}{m}\right)^m \\ &\leq \frac{1}{\sqrt{2\pi m}} \exp(-b) \exp(b) = \frac{1}{\sqrt{2\pi m}}; \qquad (1) \end{aligned}$$

[1] For $y > -1$, we have $\ln(1+y) \leq y$. For the proof, let $g(y) = y - \ln(1+y)$; it is easy to see that $g(0) = 0$ is a global minimum using $g'(y)$.

the last inequality is obtained by exponentiating both sides of $\ln(1+b/m) \le b/m$ (see Lemma 1 (i) in [4]). This proves Proposition 2 (1) for $i \le m$; for $i > m$, the inequality is trite.

As for $p_\lambda(m+i)$, use $m \ge \lambda - 1$ to obtain

$$p_\lambda(m+i) = p_\lambda(m) \cdot \frac{\lambda^i}{(m+1)(m+2)\cdots(m+i)} \le p_\lambda(m) \cdot \frac{\lambda^i}{\lambda(\lambda+1)\cdots(\lambda+i-1)}$$
$$= p_\lambda(m) \exp\left(-\sum_{k=0}^{i-1} \ln\left(1+\frac{k}{\lambda}\right)\right) \le p_\lambda(m) \exp\left(-\sum_{k=0}^{i-1} \frac{k}{\lambda} - \frac{k^2}{2\lambda^2}\right),$$

the latter inequality by Lemma 1 (ii) in [4][2]. Using standard summation formulae and inequality (1) gives Proposition 2 (ii). $\square$

Theorem 1 (ii) in [4]. Suppose $\lambda \ge 2$. If $2 \le i \le (\lambda+3)/2$, then,

$$Q_\lambda(m+i) \le \exp\left(\frac{1}{8\lambda}\right)\left(1+\frac{1}{\lambda}\right)\sqrt{2}\left(1-\left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \overline{\Phi}\left(\frac{i-\frac{3}{2}}{\sqrt{2\lambda}}\right).$$

Proof. We first use Proposition 1 (ii) in [4] with $l = m$ to obtain

$$Q_\lambda(m+i) \le \left(1-\left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \cdot \sum_{k=\lceil i\rceil}^{m+\lceil n\rceil-1} p_\lambda(m+k)$$

For $i \le k \le m+i-1$, we have

$$-\frac{k(k-1)}{2\lambda} + \frac{(k-1)k(2k-1)}{12\lambda^2} \le -\frac{k(k-1)}{2\lambda} \cdot \beta,$$

where $\beta = 1 - \frac{m+i}{3\lambda} + \frac{1}{2\lambda}$.

By Proposition 2 (2) in [4], it follows that

$$Q_\lambda(m+i) \le \left(1-\left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \cdot \frac{1}{\sqrt{2\pi m}} \sum_{k=\lceil i\rceil}^{m+\lceil i\rceil-1} \exp\left(\frac{-k(k-1)}{2\lambda}\cdot\beta\right).$$

As in the bound for $T_\lambda(m-i)$, the latter sum is dominated by (if $i \ge 2$)

$$\int_{i-1}^{\infty} \exp\left(\frac{-u(u-1)\beta}{2\lambda}\right) du \qquad \text{(substitute } t = \left(u-\tfrac{1}{2}\right)\sqrt{\beta/\lambda}\text{)}$$
$$= \int_{(i-\frac{3}{2})\sqrt{\beta/\lambda}}^{\infty} \exp\left(\frac{-(t\sqrt{\lambda/\beta}+\frac{1}{2})(t\sqrt{\lambda/\beta}-\frac{1}{2})\beta}{2\lambda}\right)\sqrt{\lambda/\beta}\, dt$$
$$= \sqrt{\frac{\lambda}{\beta}} \int_{(i-\frac{3}{2})\sqrt{\beta/\lambda}}^{\infty} \exp\left(-\frac{t^2}{2} + \frac{(\frac{1}{2})^2\beta}{2\lambda}\right) dt = \sqrt{\frac{\lambda}{\beta}} \exp\left(\frac{\beta}{2^2\cdot 2\lambda}\right)\sqrt{2\pi}\cdot\overline{\Phi}\left(\left(i-\tfrac{3}{2}\right)\sqrt{\frac{\beta}{\lambda}}\right)$$

Now, $\beta \le 1$ since $m+i \ge i \ge 2$; furthermore, since $i \le (\lambda+3)/2$, it follows that $\beta \ge 1 - \frac{\lambda+(\lambda+3)/2}{3\lambda} + \frac{1}{2\lambda} = \frac{1}{2}$. Hence, $1/\sqrt{\beta} \le \sqrt{2}$, $\exp(\frac{\beta}{8\lambda}) \le \exp(\frac{1}{8\lambda})$, and

$$\overline{\Phi}\left(\left(i-\tfrac{3}{2}\right)\sqrt{\frac{\beta}{\lambda}}\right) \le \overline{\Phi}\left(\left(i-\tfrac{3}{2}\right)\cdot\frac{1}{\sqrt{2\lambda}}\right)$$

[2] For $y > 0$, $-\ln(1+y) \le -y + y^2/2$. For the proof, let $g(y) = \ln(1+y) - y + y^2/2$; it is easy to see that $g(0) = 0$ is a global minimum using $g'(y)$.

Combining these inequalities and the previously obtained $\sqrt{\lambda/m} \le 1 + \frac{1}{\lambda}$, we get Theorem 1 (ii):

$$Q_\lambda(m+i) \le \left(1 - \left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \cdot \frac{1}{\sqrt{2\pi m}} \sqrt{\frac{\lambda}{\beta}} \exp\left(\frac{\beta}{2^2 \cdot 2\lambda}\right) \sqrt{2\pi} \cdot \overline{\Phi}\left(\left(i - \tfrac{3}{2}\right)\sqrt{\frac{\beta}{\lambda}}\right)$$
$$\le \left(1 - \left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \cdot \sqrt{\frac{\lambda}{m}} \sqrt{2} \exp\left(\frac{1}{8\lambda}\right) \overline{\Phi}\left(\left(i - \tfrac{3}{2}\right)\sqrt{\frac{1}{2\lambda}}\right)$$
$$\le \left(1 - \left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \cdot \left(1 + \frac{1}{\lambda}\right) \sqrt{2} \exp\left(\frac{1}{8\lambda}\right) \overline{\Phi}\left(\frac{i - \frac{3}{2}}{\sqrt{2\lambda}}\right)$$

□

Proposition 2 in [3] (= Proposition 3 in [4]). Suppose $\lambda \ge 2$ and $2 \le i \le (\lambda+3)/2$. Then

$$Q_\lambda(m+i) \le \frac{a_\lambda}{1 - \exp\left(-\frac{2}{9} i\right)} \overline{\Phi}\left(\frac{i - \frac{3}{2}}{\sqrt{2\lambda}}\right)$$

Proof. [3] give the proposition without proof. We copy it from [4].

Recall that $a_\lambda = \left(1 + \frac{1}{\lambda}\right)\sqrt{2}\exp\left(\frac{1}{16}\right)$. For $\lambda \ge 2$, $\exp(\frac{1}{8\lambda}) \le \exp(\frac{1}{16})$, so it remains only to show that

$$\left(1 - \left(\frac{\lambda}{m+i+1}\right)^m\right)^{-1} \le \left(1 - \exp\left(-\tfrac{2}{9} i\right)\right)^{-1}. \tag{2}$$

Since $m+1 \ge \lambda$, it follows that $\lambda/(m+i+1) \le \lambda/(\lambda+i) = 1 - (i/(\lambda+i))$. Now,

$$1 - \frac{i}{\lambda+i} \le \exp\left(-\frac{i}{\lambda+i}\right)$$

(exponentiate both sides of Lemma 1 (i) in [4]), so

$$\left(\frac{\lambda}{m+i+1}\right)^m \le \exp\left(\frac{-im}{\lambda+i}\right). \tag{3}$$

The function $f(x) = (x-1)/(x+1)$ is non-decreasing on $[0, \infty)$ so $f(x) \ge f(2) = \frac{1}{3}$ for $x \ge 2$. Thus, for $\lambda \ge 2$, $(\lambda-1)/(\lambda+1) \ge \frac{1}{3}$, proving that $m \ge \frac{1}{3}(\lambda+1)$. Hence, $m(\lambda+i)^{-1} \ge m(\lambda + \frac{\lambda+3}{2})^{-1} \ge \frac{1}{3}(\lambda+1)/(\frac{3}{2}(\lambda+1)) = \frac{2}{9}$. (However, we are going to use this proposition only for $\lambda \ge 400$; therefore, we can improve the bound in this equation to $m(\lambda+i)^{-1} \ge \frac{266}{401}$, using $m \ge f(400)(\lambda+1) = \frac{399}{401}(\lambda+1)$.) Relation (3) then yields

$$\left(\frac{\lambda}{m+i+1}\right)^m \le \text{ (if } \lambda \ge 400) \ \exp\left(-\tfrac{266}{401} i\right) \le \exp\left(-\tfrac{2}{9} i\right),$$

from which (2) follows immediately. □

Proposition 3 in [3] (= Theorem 1 (i) in [4]). Suppose $\lambda \ge 2$ and $i \ge 2$. Then

$$T_\lambda(m-i) \le b_\lambda \overline{\Phi}\left(\frac{i - \frac{3}{2}}{\sqrt{\lambda}}\right).$$

Proof. We also copy this proof from [4].

By Proposition 2 (1) of [4],

$$T_\lambda(m-i) = \sum_{k=\lceil i\rceil}^{m} p_\lambda(m-k) \le \frac{1}{\sqrt{2\pi m}} \cdot \sum_{k=\lceil i\rceil}^{m} \exp\left(-\frac{k(k-1)}{2\lambda}\right).$$

Since $g(x) = -x(x-1)/(2\lambda)$ is non-increasing on $[\frac{1}{2}, \infty)$,

$$\exp\left(-\frac{x(x-1)}{2\lambda}\right) \le \int_{x-1}^{x} \exp\left(-\frac{u(u-1)}{2\lambda}\right)\, du$$
$$= \int_{x-1}^{x} \exp\left(\frac{-(u-\frac{1}{2})^2 + (\frac{1}{2})^2}{2\lambda}\right) dt = \exp\left(\frac{(\frac{1}{2})^2}{2\lambda}\right) \int_{x-1}^{x} \exp\left(-\frac{(u-\frac{1}{2})^2}{2\lambda}\right) du$$

for $x \ge \frac{3}{2}$. Thus, if $i \ge 2$,

$$T_\lambda(m-i) \le \frac{1}{\sqrt{2\pi m}} \exp\left(\frac{1}{8\lambda}\right) \int_{i-1}^{\infty} \exp\left(-\frac{(u-\frac{1}{2})^2}{2\lambda}\right) du \qquad \left(\text{substitute } t = \frac{u-\frac{1}{2}}{\sqrt{\lambda}}\right)$$
$$= \frac{1}{\sqrt{2\pi m}} \exp\left(\frac{1}{8\lambda}\right) \int_{(i-\frac{3}{2})/\sqrt{\lambda}}^{\infty} \exp\left(-\frac{t^2}{2}\right) \sqrt{\lambda}\, dt$$
$$= \frac{1}{\sqrt{2\pi m}} \exp\left(\frac{1}{8\lambda}\right) \sqrt{\lambda} \cdot \sqrt{2\pi} \cdot \overline{\Phi}\left(\frac{i-\frac{3}{2}}{\sqrt{\lambda}}\right) = \exp\left(\frac{1}{8\lambda}\right) \sqrt{\frac{\lambda}{m}} \cdot \overline{\Phi}\left(\frac{i-\frac{3}{2}}{\sqrt{\lambda}}\right). \quad (4)$$

By Lemma 1 (iii) in [4][3], $\sqrt{\lambda/m} = \sqrt{1+(b/m)} \le 1 + b/(2m)$. For $\lambda \ge 2$, $\lambda \le \lfloor\lambda\rfloor + 1 \le \lfloor\lambda\rfloor + 2 - \frac{2}{\lambda} = \lfloor\lambda\rfloor + (2/\lambda)\cdot(\lambda-1) \le \lfloor\lambda\rfloor + (2/\lambda)\cdot\lfloor\lambda\rfloor$ and therefore $b = \lambda - \lfloor\lambda\rfloor \le 2\lfloor\lambda\rfloor/\lambda = 2m/\lambda$, so that $b/(2m) \le 1/\lambda$; substituting into (4) yields Theorem 1 (i). □

We reparameterize the bounds in the above propositions with the subsitutions $(i-\frac{3}{2})/\sqrt{2} = k\sqrt{\lambda}$ and $i - \frac{3}{2} = k\sqrt{\lambda}$ respectively. This gives:

$$Q_\lambda(\lceil m + k\sqrt{2\lambda} + \tfrac{3}{2}\rceil) = Q_\lambda(m + k\sqrt{2\lambda} + \tfrac{3}{2}) \le a_\lambda d(k,\lambda)\overline{\Phi}(k)$$
$$T_\lambda(\lfloor m - k\sqrt{\lambda} - \tfrac{3}{2}\rfloor) = T_\lambda(m - k\sqrt{\lambda} - \tfrac{3}{2}) \le b_\lambda \overline{\Phi}(k)$$

where

$$d(k,\lambda) = \frac{1}{1 - \exp\left(-\frac{2}{9}\left[k + \sqrt{2\lambda} + \frac{3}{2}\right]\right)}$$

and $T_\lambda(j) = 0$ for $j < 0$. For $\lambda \ge 25$ and $k \ge 3$, we get $d(k,\lambda) \le 1.007$. For $k \ge 4$ and the improved definition of $d(k,\lambda) = 1/\left\langle 1 - \exp\left(-\frac{266}{401}\left[k + \sqrt{2\lambda} + \frac{3}{2}\right]\right)\right\rangle$, valid for $\lambda \ge 400$, we can even say that $d(k,\lambda) < 1 + 10^{-33}$. Note that this number in current floating-point arithmetic will be rounded to 1. Therefore, we can simplify line 1.25 to $d(k,\lambda) := 1$ in a practical implementation.

[3] now continues with an approximation of $\overline{\Phi}(k)$, as that function is difficult to compute; however, in the standard library math.h of C (99), there is a function erfc() that one could use to calculate $\overline{\Phi}(k) = \mathsf{erfc}(k/\sqrt{2})/2$. If we do so in the Finder algorithm above, we have to replace the test in line 1.12 by $\mathsf{erfc}(k/\sqrt{2}) < \varepsilon/b_\lambda$, and the test in line 1.26 by $d(k,\lambda)\mathsf{erfc}(k/\sqrt{2}) < \varepsilon/a_\lambda$.

Proposition 4 in [3] (= 26.2.12 in [1, page 932]). If $x > 0$, then $\overline{\Phi}(x) \lesssim \varphi(x)/x$ with error less than $\varphi(x)/x^3$, i. e. $\overline{\Phi}(x) \le \left(1 + x^{-2}\right)\varphi(x)/x$.

If we follow this proposition strictly, we will have to multiply the subsequent bounds with a factor $1 + x^{-2}$. However, in the intended range of use, for $4 \le x \le 7$, the expression $\phi(x)/x$ already overestimates $\overline{\Phi}(x)$ by 2–6 %. Apply proposition 4 to Glynn's reparameterized bound to get

[3] For $y \ge 0$, $\sqrt{1+y} \le 1 + y/2$. For the proof, let $g(y) = 1 + y/2 - \sqrt{1+y}$; it is easy to see that $g(0) = 0$ is a global minimum using $g'(y)$.

Corollary 1 in [3]. If $\lambda \geq 2$ and $\frac{1}{2\sqrt{2\lambda}} \leq k \leq \frac{\sqrt{\lambda}}{2\sqrt{2}}$, then

$$Q_\lambda(\lceil m + k\sqrt{2\lambda} + \tfrac{3}{2} \rceil) \leq a_\lambda d(k, \lambda) \frac{\exp(-k^2/2)}{k\sqrt{2\pi}}$$

Proof. Let's just look at the bounds. The approximation holds if $2 \leq i = k\sqrt{2\lambda} + \frac{3}{2} \leq \frac{\lambda+3}{2}$, i. e., $\frac{1}{2} \leq k\sqrt{2\lambda} \leq \frac{\lambda}{2}$. □

Corollary 2 in [3]. If $\lambda \geq 2$ and $k \geq 1/(2\sqrt{\lambda})$, then

$$T_\lambda(\lfloor m - k\sqrt{\lambda} - \tfrac{3}{2} \rfloor) \leq b_\lambda \frac{\exp(-k^2/2)}{k\sqrt{2\pi}}$$

Proof. The approximation holds if $2 \leq i = k\sqrt{\lambda} + \frac{3}{2}$, i. e., $\frac{1}{2} \leq k\sqrt{\lambda}$. □

Corollary 1 does not contradict the fact that, for large enough truncation points, the mass in the right Poisson tail is an order of magnitude greater than the mass in the corresponding normal tail. In corollary 1, the truncation point is at most $\lceil m + \lambda/2 + \frac{3}{2} \rceil$.

5 Bounding Poisson probabilities

We bound the Poisson probabilities $p_\lambda(i)$ from below to guarantee that, properly scaled, they do not underflow for $L_\lambda \leq i \leq R_\lambda$. By the monotonicity of $p_\lambda(i)$ to the left and to the right of $m = \lfloor \lambda \rfloor$, it suffices to check only $p_\lambda(L_\lambda)$ and $p_\lambda(R_\lambda)$. The programs Finder and Weighter use corollaries 3 and 4 below only for $\lambda \geq 25$. For $0 < \lambda < 25$, we set $L_\lambda = 0$ and $R_\lambda = R_{400}$. The latter is justified since the mass in the right tail increases with λ. Weighter checks that $R_{400} \leq 600$; for ε corresponding to $k = 7$, $R_{400} = 600$. It then assures that properly-scaled probabilities do not underflow, resetting R_λ if necessary. The error bound is then $\frac{1}{2}\varepsilon + 10^{10}(R_{400} - R_\lambda)\tau/\Omega \leq \frac{1}{2}\varepsilon + 6 \cdot 10^{12}\tau/\Omega$. The second term is negligible when $\varepsilon \gg 6 \cdot 10^{12}\tau/\Omega$, which holds for $\varepsilon = 10^{-10}$ and the computers considered in section 3.

Let

$$c_m = \frac{1}{\sqrt{2\pi m}} \exp\left(m - \lambda - \frac{1}{12m}\right).$$

According to [2, page 54], the following bound supplements Stirling's formula:

$$m! < \sqrt{2\pi m} \cdot m^m \exp(-m) \exp\left(\frac{1}{12m}\right).$$

It readily follows that

$$p_\lambda(m) = \frac{\exp(-\lambda)\lambda^m}{m!} \geq \frac{\exp(-\lambda)\lambda^m}{\sqrt{2\pi m} \cdot m^m \exp(-m) \exp\left(\frac{1}{12m}\right)} = \left(\frac{\lambda}{m}\right)^m \frac{1}{\sqrt{2\pi m}} \exp\left(-\lambda + m - \frac{1}{12m}\right) \geq c_m.$$

We prove

Proposition 5 in [3]. For $i > 0$,

$$p_\lambda(m+i) \geq p_\lambda(m) \exp\left(-\frac{i(i+1)}{2\lambda}\right) \geq c_m \exp\left(-\frac{(i+1)^2}{2\lambda}\right).$$

Proof. We use the following known fact: $\ln(1+x) \leq x$ for $x \geq 0$. Right of mode:

$$p_\lambda(m+i) = p_\lambda(m) \exp\left(-\sum_{k=1}^{i} \ln\left(\frac{m+k}{\lambda}\right)\right) \geq p_\lambda(m) \exp\left(-\sum_{k=1}^{i} \ln\left(1+\frac{k}{\lambda}\right)\right)$$
$$\geq p_\lambda(m) \exp\left(-\sum_{k=1}^{i} \frac{k}{\lambda}\right) = p_\lambda(m) \exp\left(-\frac{i(i+1)}{2\lambda}\right).$$

□

Corollary 3 in [3]. Let $\hat{k} = k\sqrt{2} + \frac{3}{2\sqrt{\lambda}}$. Then for $k > 0$ and $\lambda \geq 1$,

$$p_\lambda(\lfloor m + k\sqrt{2\lambda} + \tfrac{3}{2}\rfloor) = p_\lambda(\lfloor m + \hat{k}\sqrt{\lambda}\rfloor) \geq c_m \exp\left(-\frac{(\hat{k}+1)^2}{2}\right).$$

Proof. Simple consequence of Proposition 5 in [3]. We require $\lambda \geq 1$ to show that $(\frac{i}{\sqrt{\lambda}}+1)^2/2 \geq (i+1)^2/(2\lambda)$. □

Proposition 6 in [3].

1. For $0 < i \leq \lambda/2$,

$$p_\lambda(m-i) \geq p_\lambda(m) \exp\left(-\frac{i(i+1)}{2\lambda} - \frac{i(i+1)(2i+1)}{6\lambda^2}\right) \geq c_m \exp\left(-\frac{(i+1)^2}{2\lambda} - \frac{(i+1)^3}{3\lambda^2}\right).$$

2. For $0 < i \leq m$,

$$p_\lambda(m-i) \geq c_m \left[1 - \frac{i}{m+1}\right]^i.$$

Proof. We use the following known fact: $\ln(1-x) \leq -x - x^2$ for $0 \leq x \leq \frac{1}{2}$. Left of mode:

$$p_\lambda(m-i) = p_\lambda(m) \exp\left(\sum_{k=1}^{i} \log\left(\frac{m-k+1}{\lambda}\right)\right) \geq p_\lambda(m) \exp\left(\sum_{k=1}^{i} \log\left(1-\frac{k}{\lambda}\right)\right)$$
$$\geq p_\lambda(m) \exp\left(-\sum_{k=1}^{i}\left(\frac{k}{\lambda}+\frac{k^2}{\lambda^2}\right)\right) = p_\lambda(m) \exp\left(-\frac{i(i+1)}{2\lambda} - \frac{i(i+1)(2i+1)}{6\lambda^2}\right)$$

$$p_\lambda(m-i) \geq p_\lambda(m)\left(\frac{m-i+1}{\lambda}\right)^i \geq c_m \left(\frac{m-i+1}{m+1}\right)^i = c_m\left[1-\frac{i}{m+1}\right]^i$$

□

Corollary 4 in [3]. Let $\tilde{k} = k + \frac{3}{2\sqrt{\lambda}}$.

1. For $0 < \tilde{k} \leq \sqrt{\lambda}/2$ and $\lambda \geq 1$,
$$p_\lambda(\lceil m - k\sqrt{\lambda} - \tfrac{3}{2}\rceil) = p_\lambda(\lceil m - \tilde{k}\sqrt{\lambda}\rceil) \geq c_m \exp\left(-\frac{(\tilde{k}+1)^2}{2} - \frac{(\tilde{k}+1)^3}{3\sqrt{\lambda}}\right).$$

2. For $\tilde{k} \leq m/\sqrt{m+1}$,
$$p_\lambda(\lceil m - k\sqrt{\lambda} - \tfrac{3}{2}\rceil) \geq p_\lambda(\lceil m - \tilde{k}\sqrt{m+1}\rceil) \geq c_m \left(1 - \frac{\tilde{k}}{\sqrt{m+1}}\right)^{\tilde{k}\sqrt{m+1}}$$

3. For $\tilde{k} \leq m/\sqrt{m+1}$,
$$p_\lambda(\lceil m - k\sqrt{\lambda} - \tfrac{3}{2}\rceil) \geq p_\lambda(0) = \exp(-\lambda).$$

We suggest using 1 when applicable; the bound is then at least $c_m \exp(-\frac{11}{15}(\tilde{k}+1)^2)$ for $\lambda \geq 25$, as

$$\begin{aligned}
-\frac{(\tilde{k}+1)^2}{2} - \frac{(\tilde{k}+1)^3}{3\sqrt{\lambda}} &\geq -\tfrac{22}{30}(\tilde{k}+1)^2 && \text{iff}\\
\frac{(\tilde{k}+1)^3}{3\sqrt{\lambda}} &\leq \tfrac{7}{30}(\tilde{k}+1)^2 && \text{iff}\\
\frac{(\tilde{k}+1)}{\sqrt{\lambda}} &\leq \tfrac{1}{2} + \tfrac{1}{5} = \tfrac{7}{10} && \text{if (using } \tfrac{\sqrt{\lambda}}{5} \geq 1)\\
\tilde{k} + 1 &\leq \frac{\sqrt{\lambda}}{2} + 1 && \text{iff}\\
\tilde{k} &\leq \frac{\sqrt{\lambda}}{2}.
\end{aligned}$$

If only 2 and 3 apply, compute both bounds and use the maximum. Since for m large

$$\left(1 - \frac{\tilde{k}}{\sqrt{m+1}}\right)^{\tilde{k}\sqrt{m+1}} \sim \exp(-\tilde{k}^2), \tag{5}$$

computing the left side of (5) is numerically stable. For example, with $m = 63$ and $\tilde{k} = 7$, 1 does not apply and

$$\begin{aligned}
\left(1 - \tfrac{7}{8}\right)^{56} &\doteq 2.6 \times 10^{-51} && \text{[see Cor. 4 2]}\\
\exp(-49) &\doteq 5.2 \times 10^{-22} && \text{[see (5)]}\\
\exp(-63) &\doteq 4.4 \times 10^{-28} && \text{[see Cor. 4 3]}
\end{aligned}$$

Convergence in (5) is glacial.

For $\lambda \geq 25$, we get
$$c_m \geq \frac{1}{\exp\left(\frac{1}{12 \cdot 25}\right) \cdot e\sqrt{2\pi m}} \geq 0.14627/\sqrt{m}.$$

6 Conclusion

I now leave the texts of Fox and Glynn again to add my own retrospective remarks.

I corrected the implementation of Poisson probability estimation in MRMC [5] according to the above propositions. There were two outstanding differences that became apparent from the MRMC test suite: First and foremost, all warning messages that there were an underflow (line 1.38 in the algorithm) disappeared. The reason for this is the misprint of the bound $\tilde{k} \leq m/\sqrt{m+1}$ as $\tilde{k} \leq \sqrt{m+1}/m$ in Corollary 4 (2 and 3) in [3]. While the correct bound holds whenever the calculated left truncation point is ≥ 0 (and therefore, we did not include a specific test in line 1.36 of our algorithm), the misprinted bound is almost always violated if $\tilde{k} > \sqrt{\lambda}/2 \approx \sqrt{m+1}/2$, i. e. if Corollary 4 (1) in [3] does not apply. Second, the right truncation point for small λ is much larger, as we use the correct $m_{max} = 400$ in line 1.23. In a few cases, this leads to steady-state detection. Further, as a small change, often the truncation points differ by one from the older version; this may be the cause that in four or five tests of the suite, the final probability estimate differed by $\varepsilon/10$ from the earlier result.

I hope that this improvement of MRMC can be incorporated in the next version after 1.5. In the meantime, you may download the MRMC changes from the svn repository with the URL https://svn-i2.informatik.rwth-aachen.de/repos/mrmc/branches/nijmegen-small-improvements.

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