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Note

Filter models with polymorphic types*

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Abstract

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Using ideas and results from Barendrecht et al. (1983) and Coppo et al. (1984) on intersection types, a comparable theory is developed for (second order) polymorphic types. The set of filters constructed with polymorphic type forms, with inclusion, a continuous lattice which yields a model of what we call $\beta\eta$-expansion (i.e. the value of a term increases under $\beta\eta$-reduction), but not of β-conversion. Combining intersection with polymorphic types does give filter λ-models, but the two standard ways of interpreting λ-terms do not coincide.

Introduction

With the intersection types, introduced in [4], a kind of polymorphism can be achieved: if a λ-term M has type $\sigma \wedge \tau$, then M has type σ and τ, too. In [2] the λ-model $\mathcal{F}$ is constructed from filters of intersection types. The interpretation of a term is defined directly — in the style of [11] — as the filter of types derivable for that term. In [3] the approach is more semantical in the sense that functions F and G (for application and abstraction, see [1, 5.4]) are defined and used to interpret

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λ-terms. It is proved that also in this way the interpretation of a term is the filter of types derivable for that term.

In the same way as (second-order) universal quantification over propositions can be seen as a generalisation of conjunction, the usual form of polymorphism, see [7, 14, 15], can be viewed as a generalisation of the intersection, by taking $\forall$ instead of $\wedge$. Indeed, if M has type $\forall\alpha.\sigma$ this implies that M has type $\sigma[\alpha := \tau]$ for all types τ.

In this note we extend the notion of filter λ-models to the setting of polymorphic types. The filter domain constructed this way (Sections 2 and 3) does not give rise to a λ-model because it is not algebraic, but we obtain an *expansion λ-model* since the value of a term increases under reduction. A first example — to the authors' knowledge — of an expansion model has been described in [13] for combinators and weak reduction as *Fool's model.*

In this approach our addition of intersection types in Sections 4 and 5, results quite natural. With a construction similar to the one of the domain of polymorphic filters, a model of β-conversion is obtained. Nevertheless it is worthwhile to note that in this context the two different approaches to term interpretation in filter λ-models, outlined above for the intersection, do not coincide.

Finally we would like to remark that familiarity with [2] and [3] may be beneficial in understanding this note.

1. Models of reduction, expansion and conversion

Let $\mathcal{M}$ be a structure in which λ-terms can be interpreted; $\mathcal{M}$ is supposed to contain an ordering $\leqslant$. We intuitively take it that

- $\mathcal{M}$ is a model of *β-expansion* iff $M \twoheadrightarrow_\beta N \Rightarrow \mathcal{M} \models M \leqslant N$
- $\mathcal{M}$ is a model of *β-reduction* iff $M \twoheadrightarrow_\beta N \Rightarrow \mathcal{M} \models M \geqslant N$
- $\mathcal{M}$ is a model of *β-conversion* iff $M =_\beta N \Rightarrow \mathcal{M} \models M = N$.

A model of $\beta\eta$-expansion (resp. reduction or conversion) is defined analogously. We choose the names expansion and reduction because term models of expansion (with objects $[M]^{\mathrm{e}} = \{N \in \Lambda \mid N \twoheadrightarrow_\beta M\}$) and of reduction (with objects $[M]^{\mathrm{r}} = \{N \in \Lambda \mid M \twoheadrightarrow_\beta N\}$) are natural examples. The ordering here is inclusion.

Let D be a cpo with ordering $\leqslant$ and $[D \to D]$ its set of Scott continuous functions. Assume that there are continuous functions $F: D \to [D \to D]$ and $G: [D \to D] \to D$. We often write $x \cdot y$ for $F(x)(y)$. Now λ-terms can be interpreted in the standard way, i.e. for a valuation function ρ from variables to D put

$$[\![x]\!]_\rho = \rho(x)$$

$$[\![MN]\!]_\rho = F([\![M]\!]_\rho)([\![N]\!]\rho)$$

$$[\![\lambda x.M]\!]_\rho = G(\lambda x \in D.[\![M]\!]_{\rho(x:=d)}).$$

Lemma 1.1. *A structure $\mathcal{M} = \langle D, F, G\rangle$, as described above is a model of*

(i) *β-expansion if $F \circ G \leq \mathrm{id}$ and of $\beta\eta$-expansion if moreover $G \circ F \leq \mathrm{id}$;*

(ii) *β-reduction if $F \circ G \geq \mathrm{id}$ and of $\beta\eta$-reduction if moreover $G \circ F \geq \mathrm{id}$;*

(iii) *β-conversion if $F \circ G = \mathrm{id}$ and of $\beta\eta$-conversion if moreover $G \circ F = \mathrm{id}$.*

In [8, 1.15(iii)], $\beta\eta$-expansion models of the form $\langle D, F, G\rangle$ are considered. It is mentioned that for a term M, the set $\{[\![N]\!]_\rho \mid M \twoheadrightarrow_{\beta\eta} N\}$ is directed (due to the Church–Rosser property) and that it consequently has a supremum. This might be interesting in case M is not strongly normalizing.

Remark 1.2. We still consider $\mathcal{M} = \langle D, F, G\rangle$.

(i) If $F \circ G = \mathrm{id}$, $\mathcal{M}$ is usually called *reflexive*; if moreover $G \circ F \geq \mathrm{id}$, then $\mathcal{M}$ is called *additive*; *coadditive* if $G \circ F \leq \mathrm{id}$ and *extensional* if $G \circ F = \mathrm{id}$. Examples of additive models are P_ω (due to Scott and Plotkin) and D_A (Engeler), see e.g. [1]. The intersection filter model $\mathcal{F}$ from [2] (see also [3]) is coadditive and the well-known D_∞-models due to Scott, see e.g. [1] again, are extensional.

(ii) If $\mathcal{M}$ is reflexive, then

$$\mathcal{M} \models 1 \leq I \iff G \circ F \leq \mathrm{id} \quad (\iff \mathcal{M} \text{ is coadditive})$$

$$\mathcal{M} \models 1 \geq I \iff G \circ F \geq \mathrm{id} \quad (\iff \mathcal{M} \text{ is additive})$$

$$\mathcal{M} \models 1 = I \iff G \circ F = \mathrm{id} \quad (\iff \mathcal{M} \text{ is extensional})$$

where $1 \equiv \lambda xy.xy$ and $I \equiv \lambda x.x$. The relation between these terms in the model gives information about the values of terms whose Böhm trees are η-expansions of each other, see [1]. One may further notice that for $f \in [D \to D]$ the element $G(f)$ is the minimal representation of f if $\mathcal{M}$ is coadditive and it is the maximal one if $\mathcal{M}$ is additive.

2. Expansion models of filters with polymorphic types

Definition 2.1. The set of types $T_\forall$ is as in the second order polymorphic λ-calculus, except that a type constant ω is added. Var is some infinite set of type variables. $T_\forall$ is defined as the smallest set satisfying

$$\{\omega\} \cup \mathrm{Var} \subseteq T_\forall$$

$$\sigma, \tau \in T_\forall \Rightarrow (\sigma \to \tau) \in T_\forall$$

$$\alpha \in \mathrm{Var},\ \sigma \in T_\forall \Rightarrow (\forall \alpha.\sigma) \in T_\forall.$$

We adopt the following notational convention: $\alpha, \beta, \gamma, \ldots$ denote type variables and $\sigma, \tau, \mu, \upsilon, \phi, \psi, \ldots$ denote types in general (i.e. are meta type variables). Outermost parentheses are omitted and $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_n$ is written for

$$\sigma_1 \to (\sigma_2 \to \cdots \to (\sigma_{n-1} \to \sigma_n) \cdots)$$

and if $\bar{\alpha} = \alpha_1, \ldots, \alpha_m$ $(m \geq 0)$ then $\forall \bar{\alpha}.\sigma$ means

$$\forall \alpha_1.(\forall \alpha_2.(\cdots(\forall \alpha_m.\sigma)\cdots)).$$

$\mathrm{FV}(\sigma)$ is the set of free variables in σ; σ is closed if $\mathrm{FV}(\sigma) = \emptyset$. The sign $\equiv$ denotes syntactical equality. Types which differ only in the names of their bound variables are identified. $\sigma[\alpha := \tau]$ denotes the substitution of τ for free occurrences of α in σ, where we assume that no variable free in τ becomes bound in $\sigma[\alpha := \tau]$ (in order to avoid such a clash of variables, renaming may be necessary). $\sigma[\bar{\alpha} := \bar{\tau}]$ will be used as a shorthand for $\sigma[\alpha_1 := \tau_1]\cdots[\alpha_n := \tau_n]$. If $\alpha \notin \mathrm{FV}(\mu)$, one has

$$\sigma[\alpha := \tau][\beta := \mu] \equiv \sigma[\beta := \mu][\alpha := \tau[\beta := \mu]].$$

Definition 2.2. (i) Let $\leq$ be a type inclusion relation on $T_\forall$ that satisfies the following conditions.

$$\sigma \leq \omega$$

$$\omega \leq \omega \to \omega$$

$$\sigma \leq \tau \;\&\; \tau \leq \mu \;\Rightarrow\; \sigma \leq \mu$$

$$\sigma \leq \sigma' \;\&\; \tau \leq \tau' \;\Rightarrow\; \sigma' \to \tau \leq \sigma \to \tau'$$

$$\sigma \leq \tau \;\Rightarrow\; \forall \alpha.\sigma \leq \forall \alpha.\tau$$

$$\forall \alpha.\sigma \leq \sigma[\alpha := \tau]$$

$$\sigma \leq \forall \alpha.\sigma \qquad \text{if } \alpha \notin \mathrm{FV}(\sigma)$$

$$\forall \alpha.(\sigma \to \tau) \leq \sigma \to (\forall \alpha.\tau) \qquad \text{if } \alpha \notin \mathrm{FV}(\sigma)$$

We write $\sigma = \tau$ iff $\sigma \leq \tau \;\&\; \tau \leq \sigma$.

(ii) $\mathcal{T}_\forall = \langle T_\forall, \leq \rangle$ is called a *polymorphic type structure* or simply a $\forall$-*type structure*. Note that $\leq$ is a parameter in such a structure.

(iii) For types σ with $\mathrm{FV}(\sigma) \subseteq \{\alpha_1, \ldots, \alpha_n\}$ we introduce a closure $\underline{\sigma}$ of σ as $\forall \alpha_1 \ldots \alpha_n.\sigma$.

Remark 2.3. (i) In a $\forall$-type structure the following relations between types can be easily derived.

$$\sigma = \sigma \qquad \text{(so } \leq \text{ is a preorder relation on } T_\forall)$$

$$\sigma \to \omega = \omega$$

$$\forall \alpha.\alpha \leq \sigma$$

$$\forall \alpha\beta.\sigma = \forall \beta\alpha.\sigma$$

$$\forall \alpha.(\sigma \to \tau) = \sigma \to (\forall \alpha.\tau) \qquad \text{if } \alpha \notin \mathrm{FV}(\sigma)$$

$$\forall \alpha.(\sigma \to \tau) \leq (\forall \alpha.\sigma) \to (\forall \alpha.\tau)$$

$$\forall \bar{\alpha}.\sigma \leq \forall \beta.\sigma[\bar{\alpha} := \bar{\tau}] \qquad \text{if } \bar{\beta} \notin \mathrm{FV}(\forall \bar{\alpha}.\sigma).$$

These last two results are used in [14] to define a type inclusion relation. In fact they are equivalent with the last three conditions of Definition 2.2(i).

(ii) By (i), closures are equal modulo =, so we speak of the closure $\underline{\sigma}$ of σ.

Definition 2.4. (i) $d \subseteq T_\forall$ is a ∀-*filter* in $\mathcal{T}_\forall$ iff

$$\omega \in d;$$

$$\sigma \in d \;\Rightarrow\; \forall \alpha.\sigma \in d;$$

$$\sigma \in d \;\&\; \sigma \leqslant \tau \;\Rightarrow\; \tau \in d.$$

(ii) $\mathcal{F}_\forall = \{d \subseteq T_\forall \mid d \text{ is a } \forall\text{-filter in } \mathcal{T}_\forall\}$.

(iii) For $A \subseteq T_\forall$ we take $\uparrow A = \bigcap \{d \in \mathcal{F}_\forall \mid A \subseteq d\}$ as the ∀-*filter generated by* A; for $\sigma \in T_\forall$ we write $\uparrow\sigma$ for $\uparrow\{\sigma\}$.

(iv) For $d, e \in \mathcal{F}_\forall$, application is defined as

$$d \cdot e = \{\tau \in T_\forall \mid \exists \sigma \in e.\ \sigma \to \tau \in d\}.$$

A number of properties of ∀-filters can now be established.

Lemma 2.5. *In* $\langle \mathcal{F}_\forall, \subseteq \rangle$ *the following holds (for definitions see* [1, 1.2]).

(i) $\uparrow\omega$ *and* $T_\forall = \uparrow\forall\alpha.\alpha$ *are the least and greatest elements.*

(ii) *For* $X \subseteq \mathcal{F}_\forall$, $\sup X$ *exists and equals* $\bigcup X$. *Hence* $\langle \mathcal{F}_\forall, \subseteq \rangle$ *is a* complete lattice.

(iii) $d = \sup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$. *Hence* $\langle \mathcal{F}_\forall, \subseteq \rangle$ *is a* continuous lattice.

(iv) ∀-*filters of the form* $\uparrow\sigma$ *are compact.*

(v) *For* $A \subseteq T_\forall$ *with* $A \neq \emptyset$ *one has* $\sigma \in \uparrow A \Leftrightarrow \exists \tau \in A\ \ \exists \bar{\alpha}.\ \sigma \geqslant \forall \bar{\alpha}.\tau$. (*if* $A = \emptyset$ *then* $\uparrow A = \uparrow\omega$).

(vi) $\uparrow\sigma = \{\tau \in T_\forall \mid \tau \geqslant \underline{\sigma}\}$.

Proof. (i) and (ii) Easy.

(iii) ($\subseteq$) $\tau \in d \Rightarrow \uparrow\tau \subseteq d \Rightarrow \uparrow\tau \subseteq \bigcup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\} = \sup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$, by (ii). ($\supseteq$) $\tau \in \sup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\} \Rightarrow \exists\ \uparrow\sigma \subseteq d.\ \tau \in \uparrow\sigma \Rightarrow \tau \in d.$

(iv) Let $\uparrow\sigma \subseteq \sup X$, for some directed set $X \subseteq \mathcal{F}_\forall$; then $\sigma \in \uparrow\sigma \subseteq \bigcup X$ and thus $\sigma \in d$ for a ∀-filter $d \in X$. So $\uparrow\sigma \subseteq d$ and hence $\uparrow\sigma$ is compact.

(v) ($\Rightarrow$) By using the last result of Remark 2.3(i) it is easy to prove that $\{\sigma \mid \exists \tau \in A \exists \bar{\alpha}.\sigma \geqslant \forall \bar{\alpha}.\tau\}$ is a ∀-filter that contains A, so it contains $\uparrow A$. ($\Leftarrow$) Obvious by the ∀-filter definition.

(vi) By (v). □

Note that from (iii) and (iv) we cannot conclude that $\langle \mathcal{F}_\forall, \subseteq \rangle$ is an *algebraic* lattice since we did not prove that the set $\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$ is directed (and thus that all compact elements are of the form $\uparrow\sigma$). Remark 3.6 elaborates on this point.

Lemma 2.6. (i) $d, e \in \mathcal{F}_\forall \Rightarrow d \cdot e \in \mathcal{F}_\forall$.

(ii) *Application is continuous.*

(iii) $\tau \in d \cdot \uparrow\sigma \Leftrightarrow \underline{\sigma} \to \tau \in d$, for $d \in \mathcal{F}_\forall$.

Proof. (i) We check the condition $\tau \in d \cdot e \Rightarrow \forall\alpha.\tau \in d \cdot e$ from Definition 2.4(i); the others are trivial. So let $\sigma \in e$ be such that $\sigma \to \tau \in d$. Then $(\forall\alpha.\sigma) \to (\forall\alpha.\tau) \geqslant \forall\alpha.(\sigma \to \tau) \in d$, by Remark 2.3(i). Because $\forall\alpha.\sigma \in e$ we can conclude that $\forall\alpha.\tau \in d \cdot e$.

(ii) As usual.

(iii) $(\Rightarrow)\, \tau \in d \cdot {\uparrow}\sigma \Rightarrow \exists\mu \in {\uparrow}\sigma$ such that $\mu \to \tau \in d$; by Lemma 2.5(vi)$(\Rightarrow)\, \underline{\sigma} \to \tau \in d$. $(\Leftarrow)$ Obvious, since $\underline{\sigma} \in {\uparrow}\sigma$. □

Definition 2.7. For the interpretation of λ-terms, functions $F: \mathscr{F}_\forall \to [\mathscr{F}_\forall \to \mathscr{F}_\forall]$ and $G: [\mathscr{F}_\forall \to \mathscr{F}_\forall] \to \mathscr{F}_\forall$ are defined by $F(d) = \lambda e.d \cdot e$ and $G(f) = {\uparrow}\{\underline{\sigma} \to \tau \mid \tau \in f({\uparrow}\sigma)\}$.

It is easy to see that F and G are continuous functions. The fact that in the definition of G the closure of σ is taken comes from Lemma 2.6(iii). We can immediately notice that $G \circ F \leqslant \mathrm{id}$ since, by Lemma 2.6(iii) one has $\tau \in d \cdot {\uparrow}\sigma \Rightarrow \underline{\sigma} \to \tau \in d$. Hence $(G \circ F)(d) = {\uparrow}\{\underline{\sigma} \to \tau \mid \tau \in d \cdot {\uparrow}\sigma\} \subseteq d$.

The *balance* property to be defined next, can be understood as some kind of reversal of the clause $\sigma \leqslant \sigma'$ & $\tau \leqslant \tau' \Rightarrow \sigma' \to \tau \leqslant \sigma \to \tau'$, from Definition 2.2(i). Balanced polymorphic type structures are of importance due to the subsequent result (Theorem 2.9).

Definition 2.8. We call $\mathscr{T}_\forall$ a *balanced* polymorphic type structure iff for all $\sigma, \tau, \upsilon \in T_\forall$ and all closed $\mu \in T_\forall$ one has

$$\sigma \to \tau \geqslant \mu \to \upsilon \;\&\; \tau \neq \omega \;\Rightarrow\; \underline{\sigma} \leqslant \mu \;\&\; \tau \geqslant \underline{\upsilon}.$$

Theorem 2.9. (i) *$\mathscr{T}_\forall$ is a balanced $\forall$-type structure* $\Leftrightarrow F \circ G \leqslant \mathrm{id}$.

(ii) *$\mathscr{T}_\forall$ is a balanced $\forall$-type structure $\Rightarrow \langle \mathscr{F}_\forall, F, G \rangle$ is a $\beta\eta$-expansion model.*

Proof. (i)$(\Rightarrow)$ Suppose $\tau \in (F \circ G)(f)(d) = G(f) \cdot d$; if $\tau = \omega$ it is obvious that $\tau \in f(d)$, so assume $\tau \neq \omega$. Find a $\sigma \in d$ with $\sigma \to \tau \in G(f) = {\uparrow}\{\underline{\mu} \to \upsilon \mid \upsilon \in f({\uparrow}\mu)\}$. Then $\sigma \to \tau \geqslant \forall\bar{\alpha}.(\underline{\mu} \to \upsilon)$ for some μ, υ with $\upsilon \in f({\uparrow}\mu)$ by Lemma 2.5(v) and so $\sigma \to \tau \geqslant \underline{\mu} \to \forall\bar{\alpha}.\upsilon$. Hence by the balance property we have $\underline{\sigma} \leqslant \underline{\mu}$ and $\tau \geqslant \underline{\forall\bar{\alpha}.\upsilon} = \underline{\upsilon}$. But then $\underline{\mu} \in d$ and thus $\upsilon \in f({\uparrow}\mu) \subseteq f(d)$. So $\tau \in f(d)$, too. $(\Leftarrow)$ Suppose $\mu \to \upsilon \leqslant \sigma \to \tau$ and $\tau \neq \omega$, for μ closed. The (continuous) step function $f_{{\uparrow}\mu{\uparrow}\upsilon}$ is defined as usual by $f_{{\uparrow}\mu{\uparrow}\upsilon}(d) = \textbf{if } {\uparrow}\mu \subseteq d \textbf{ then } {\uparrow}\upsilon \textbf{ else } {\uparrow}\omega$. Since ${\uparrow}\upsilon = f_{{\uparrow}\mu{\uparrow}\upsilon}({\uparrow}\mu)$, one has $\mu \to \upsilon \in G(f_{{\uparrow}\mu{\uparrow}\upsilon})$, because μ is closed. Now $\sigma \to \tau \geqslant \mu \to \upsilon$ implies that $\sigma \to \tau \in {\uparrow}(\mu \to \upsilon) \subseteq G(f_{{\uparrow}\mu{\uparrow}\upsilon})$; so $\tau \in G(f_{{\uparrow}\mu{\uparrow}\upsilon}) \cdot {\uparrow}\sigma \subseteq f_{{\uparrow}\mu{\uparrow}\upsilon}({\uparrow}\sigma)$, since $F \circ G \leqslant \mathrm{id}$, and thus $\tau \in \textbf{if } {\uparrow}\mu \subseteq {\uparrow}\sigma \textbf{ then } {\uparrow}\upsilon \textbf{ else } {\uparrow}\omega$. Because $\tau \neq \omega$ we must have $\tau \in {\uparrow}\upsilon$ and ${\uparrow}\mu \subseteq {\uparrow}\sigma$. This gives $\tau \geqslant \underline{\upsilon}$ and $\mu \geqslant \underline{\sigma}$.

(ii) Obvious from (i) and the fact that $G \circ F \leqslant \mathrm{id}$. □

In a $\beta\eta$-expansion model the value of a term increases under $\beta\eta$-reduction. In order to find a set of terms for which the interpretation remains invariant under β-conversion, the following terminology is useful.

Definition 2.10. (i) Let D be a complete lattice. A function $f \in [D \to D]$ is *distributive* if for all nonempty (and not just for the directed) sets $X \subseteq D$ one has $f(\sup X) = \sup f(X)$.

(ii) A λ-term M is *simple* if all bound variables occur at most once.

Lemma 2.11. (i) *The functions F and G from Definition 2.7 are distributive.*

(ii) *If the variable x does not occur in M or only once, then* $\lambda d.\ [\![M]\!]_{\rho(x:=d)}$ *is distributive.*

(iii) *If* $f \in [\mathscr{F}_\forall \to \mathscr{F}_\forall]$ *is distributive, then* $(F \circ G)(f) \geqslant f$.

(iv) *If M is simple and* $M \twoheadrightarrow_\beta N$ *then N is simple.*

Proof. Easy. □

Proposition 2.12. *Let* $\mathscr{T}_\forall$ *be a balanced polymorphic type structure and M be a simple term. Then* $M \twoheadrightarrow_\beta N \Rightarrow [\![M]\!]_\rho = [\![N]\!]_\rho$ *in* $\mathscr{F}_\forall$.

Hence $\mathscr{F}_\forall$ is a model of the linear λ-calculus, see [10]. Our set of types $T_\forall$ however contains the constant ω which is not the case there.

As mentioned in Definition 2.2(ii), $\forall$-type structures $\langle T_\forall, \leqslant\rangle$ contain the relation $\leqslant$ as a parameter. We now consider a particular choice, $\leqslant_0$, which is the smallest relation which satisfies the conditions from Definition 2.2(i) and which yields a balanced $\forall$-type structure.

In the following lemma some useful properties of the relation $\leqslant_0$ are presented. Proof by induction on the definition can be used because $\leqslant_0$ is minimal.

Lemma 2.13. (i) *Substitution in* $\langle T_\forall, \leqslant_0\rangle$ *is monotonic, i.e.*

$$\sigma \leqslant_0 \tau \ \Rightarrow\ \sigma[\beta := \mu] \leqslant_0 \tau[\beta := \mu];$$

moreover the length of the proof remains the same under substitution.

(ii) $\sigma \to \tau =_0 \omega \ \Leftrightarrow\ \tau =_0 \omega.$

(iii) $\forall \bar{\alpha}\,.\,(\mu \to \upsilon) \leqslant_0 \xi \ \&\ \xi \neq_0 \omega \ \Rightarrow\ \xi \equiv \forall \bar{\beta}\,.\,(\sigma \to \tau).$

(iv) $\forall \bar{\alpha}\,.\,(\mu \to \upsilon) \leqslant_0 \forall \bar{\beta}\,.\,(\sigma \to \tau) \ \&\ \tau \neq_0 \omega$

$$\Rightarrow\ \sigma \leqslant_0 \mu[\bar{\gamma} := \bar{\xi}] \ \&\ \forall \bar{\delta}.\upsilon \leqslant_0 \tau,$$

for some type variables $\bar{\gamma}$, $\bar{\delta}$ *and types* $\bar{\xi}$.

Proof. (i) and (iii) By induction on the definition of $\leqslant_0$.

(ii) Similar to [2, 2.4(i)].

(iv) By induction on the definition of $\leqslant_0$. □

Theorem 2.14. $\langle \mathscr{F}^0_\forall, F, G\rangle$ *is a* $\beta\eta$*-expansion model.*

Proof. By Lemma 2.13(iv), $\langle T_\forall, \leqslant_0\rangle$ is balanced, hence Theorem 2.9(ii) applies. □

3. Type assignment

Definition 3.1. (i) A *statement* is an expression of the form $M{:}\sigma$ where M (the subject) is a λ-term and σ (the predicate) is an element from $T_\forall$. A *basis* is a set of statements with only variables as subjects. A basis is called *closed* if all its predicates are closed; to emphasize that, we often write $\underline{B}$ in such cases.

(ii) The *type assignment* induced by the $\forall$-type structure $\langle T_\forall, \leqslant_0\rangle$ is defined by the following natural deduction rules.

$$(\to \mathrm{I})\ \frac{\begin{matrix}[x{:}\sigma]\\ \vdots \\ M{:}\tau\end{matrix}}{\lambda x.M{:}\ \sigma\to\tau} \qquad (\to \mathrm{E})\ \frac{M{:}\ \sigma\to\tau \qquad N{:}\sigma}{MN{:}\tau}$$

$$(\forall \mathrm{I})\ \frac{M{:}\sigma}{M{:}\ \forall\alpha.\sigma} \qquad (\leqslant_0)\ \frac{M{:}\sigma \qquad \sigma\leqslant_0\tau}{M{:}\tau} \qquad (\omega)\ \frac{}{M{:}\omega}$$

The rules $(\to\mathrm{I})$ and $(\forall\mathrm{I})$ are restricted in the following sense. In $(\to\mathrm{I})$ the term variable x may not be free in assumptions on which $M{:}\tau$ depends other than $x{:}\sigma$ and in $(\forall\mathrm{I})$ the type variable α may not be free in any statement assumption on which $M{:}\sigma$ depends.

(iii) $B\vdash M{:}\sigma$ means that the statement $M{:}\sigma$ is derivable from the basis B in this type assignment system. $\mathscr{D}: B\vdash M{:}\sigma$ expresses that $\mathscr{D}$ is a deduction that establishes $B\vdash M{:}\sigma$.

The next two lemmas are devoted to technical properties of the deduction relation that has just been defined. The approach is a combination of [2, 2.7, 2.8) and [6, A2]. We write $B\backslash x=\{y{:}\sigma\in B \mid y\not\equiv x\}$. A derivation $\mathscr{D}: B\vdash M{:}\sigma$ uses only a finite part of the basis B; this part is sometimes called the *relevant part of B with respect to* $\mathscr{D}$.

Lemma 3.2. (i) $B\backslash x\cup\{x{:}\sigma\}\vdash x{:}\tau \Rightarrow \tau\geqslant_0\sigma$.

(ii) $B\vdash M{:}\sigma \Rightarrow B[\alpha:=\tau]\vdash M{:}\sigma[\alpha:=\tau]$.

(iii) $B\vdash MN{:}\tau \Rightarrow \exists\sigma\in T_\forall.\ B\vdash M{:}\ \sigma\to\tau\ \&\ B\vdash N{:}\sigma$.

(iv) *If for all* $\sigma,\tau\in T_\forall.\ [B\backslash x\cup\{x{:}\sigma\}\vdash M{:}\tau \Rightarrow B\backslash x\cup\{x{:}\sigma\}\vdash N{:}\tau]$, *then for all* $\mu\in T_\forall.\ [B\vdash\lambda x.M{:}\mu \Rightarrow B\vdash \lambda x.N{:}\mu]$.

(v) $\mathscr{D}: B\vdash\lambda x.M{:}\xi\ \&\ \xi\neq_0\omega \Rightarrow \xi$ *is of the form* $\forall\vec{\beta}.\ (\sigma\to\tau)$. *with* $\vec{\beta}$ *not occurring in the relevant part of B w.r.t.* $\mathscr{D}$.

Proof. By induction on the length of the derivations.

(i) The only rules that can be applied are $(\leqslant_0)$, (ω) and $(\forall\mathrm{I})$; for $(\forall\mathrm{I})$ the type variable involved cannot occur in the basis and so especially not in σ.

(ii) Use Lemma 2.13(i) for the rule $(\leqslant_0)$; for $(\forall\mathrm{I})$ a change of bound variables might be needed.

(iii) For $(\forall\mathrm{I})$ use $\forall\alpha.(\sigma\to\tau)\leqslant_0(\forall\alpha.\sigma)\to(\forall\alpha.\tau)$, from Remark 2.3(i).

(iv) In case the last step was $(\rightarrow\mathrm{I})$, the result follows from the assumption.

(v) In case of $(\leqslant_0)$ use Lemma 2.13(iii) and possibly a change of bound variables (which can be done because the relevant part of the basis with respect to the deduction is finite). $\square$

Lemma 3.3. (i) *If* $\mathscr{D}: B\backslash x \cup \{x{:}\mu\} \vdash M{:}\upsilon$ *and* $\forall\bar{\alpha}.(\mu\rightarrow\upsilon) \leqslant_0 \forall\bar{\beta}.(\sigma\rightarrow\tau)$ *with* $\bar{\alpha}$ *not free in the relevant part of* $B\backslash x$ *w.r.t.* $\mathscr{D}$, *then* $B\backslash x \cup \{x{:}\sigma\} \vdash M{:}\tau$.

(ii) $B\backslash x \cup \{x{:}\sigma\} \vdash M{:}\tau \Leftrightarrow B\backslash x \vdash \lambda x.M{:}\sigma\rightarrow\tau$.

(iii) $B \vdash \lambda x.Mx{:}\xi \Rightarrow B \vdash M{:}\xi$, *if* $x \notin \mathrm{FV}(M)$.

(iv) $B \vdash \lambda x.M{:}\xi \Rightarrow B \vdash \lambda y.M[x:=y]{:}\,\xi$, *if* $y \notin \mathrm{FV}(M)$.

Proof. (i) By induction on the definition of $\leqslant_0$. In the transitivity case one needs Lemma 2.13(iii) and possibly a change of bound variables. The condition that the $\bar{\alpha}$ do not occur in the relevant part of $B\backslash x$ w.r.t. $\mathscr{D}$ is needed in the cases $\forall\alpha.(\mu\rightarrow\upsilon) \leqslant_0 \mu\rightarrow(\forall\alpha.\upsilon)$, if $\alpha\notin\mathrm{FV}(\mu)$, and $\forall\alpha.(\mu\rightarrow\upsilon) \leqslant_0 \mu[\alpha:=\xi]\rightarrow\upsilon[\alpha:=\xi]$; in the latter case one also needs Lemma 3.2(ii).

(ii) ($\Rightarrow$) By $(\rightarrow\mathrm{I})$. ($\Leftarrow$) We may assume $\tau \neq_0 \omega$. Find the statement $\lambda x.M{:}\ \mu\rightarrow\upsilon$ in the derivation $\mathscr{D}$ of $B\backslash x \vdash \lambda x.M{:}\ \sigma\rightarrow\tau$ on which $\sigma\rightarrow\tau$ depends and which is conclusion of $(\rightarrow\mathrm{I})$. Then $\sigma\rightarrow\tau \geqslant_0 \forall\bar{\alpha}.(\mu\rightarrow\upsilon)$ with $\bar{\alpha}$ not in the relevant part of B w.r.t. $\mathscr{D}$, because the only rules we can use to derive $\lambda x.M{:}\ \sigma\rightarrow\tau$ from $\lambda x.M{:}\ \mu\rightarrow\upsilon$, are $(\forall\mathrm{I})$ and $(\leqslant_0)$. Since $B\backslash x\cup\{x{:}\mu\} \vdash M{:}\upsilon$, then $B\backslash x \cup\{x{:}\sigma\}\vdash M{:}\tau$ by (i).

(iii) Easy using (ii), Lemma 3.2(iii), Lemma 3.2(v) and the fact that $x\notin\mathrm{FV}(M)$.

(iv) First one proves $B\backslash x\cup\{x{:}\sigma\}\vdash M{:}\tau \Rightarrow B\backslash x\cup\{y{:}\sigma\}\vdash M[x:=y]{:}\tau$, by induction on the length of the derivation. Using this in case of $(\rightarrow\mathrm{I})$, the result follows easily. $\square$

Since a λ-term is interpreted as a set of types in the expansion λ-model $\langle\mathscr{F}^0_\forall, F, G\rangle$ one might find a more syntactical characterization of the value of such a term.

Definition 3.4. (i) Let ρ be a valuation in $\mathscr{F}^0_\forall$; $\underline{B}_\rho = \{x{:}\underline{\sigma} \mid \sigma\in\rho(x)\}$.

(ii) A deduction $\mathscr{D}$ is said to be *closed* if for all cancelled premises $x{:}\sigma$ that occur in $\mathscr{D}$, σ is closed. $B \Vdash M{:}\tau$ denotes that there is a closed deduction $\mathscr{D}$ with $\mathscr{D}: B\vdash M{:}\tau$.

Whenever the statement $\lambda x.M{:}\ \sigma\rightarrow\tau$ occurs as conclusion of $(\rightarrow\mathrm{I})$ in a closed deduction it is required that σ is closed, since it occurs in a cancelled premise. Hence $\nVdash \lambda x.x{:}\forall\alpha.(\alpha\rightarrow\alpha)$ but we do have $\Vdash \lambda x.x{:}\ (\forall\alpha.\alpha)\rightarrow(\forall\alpha.\alpha)$. In the proof of the next proposition the use of closed deductions will be related to the closure of σ in the definition of G from 2.7.

Proposition 3.5. *In* $\langle\mathscr{F}^0_\forall, F, G\rangle$ *one has*

$$[\![M]\!]_\rho = \{\sigma\in T_\forall \mid \underline{B}_\rho \Vdash M{:}\sigma\}.$$

Proof. First of all, we have to notice that $\{\sigma \in T_\forall \mid \underline{B}_\rho \Vdash M:\sigma\}$ is a $\forall$-filter (for this reason the basis $\underline{B}_\rho$ needs to be closed).

($\subseteq$) By induction on the structure of M; the only interesting case is $M \equiv \lambda x.P$. $\upsilon \in [\![P]\!]_{\rho(x:=\uparrow\mu)} \Rightarrow \underline{B}_{\rho(x:=\uparrow\mu)} \Vdash P:\upsilon$, by the induction hypothesis on P

$\Rightarrow \underline{B}_\rho \backslash x \cup \{x:\underline{\phi} \mid \phi \in \uparrow\mu\} \Vdash P:\upsilon$

$\Rightarrow \underline{B}_\rho \backslash x \cup \{x:\underline{\phi}_1, \ldots, x:\underline{\phi}_n\} \Vdash P:\upsilon$, for some $\phi_1 \geqslant_0 \mu, \ldots, \phi_n \geqslant_0 \underline{\mu}$

$\Rightarrow \underline{B}_\rho \backslash x \cup \{x:\underline{\mu}\} \Vdash P:\upsilon$, by some closed deduction obtained from the previous one by adding the rule ($\leqslant_0$) on top when an assumption of the form $x:\underline{\phi}_i$ is needed.

$\Rightarrow \underline{B}_\rho \backslash x \Vdash \lambda x.P:\underline{\mu} \to \upsilon$, since $\underline{\mu}$ is closed

$\Rightarrow \underline{B}_\rho \Vdash \lambda x.P:\underline{\mu} \to \upsilon$.

Hence $\{\underline{\mu} \to \upsilon \mid \upsilon \in [\![P]\!]_{\rho(x:=\uparrow\mu)}\} \subseteq \{\sigma \mid \underline{B}_\rho \Vdash \lambda x.P:\sigma\}$. So we are done.

($\supseteq$) By induction on closed deductions; the only interesting case is when the last applied rule is ($\to$I). Suppose $\underline{B}_\rho \Vdash \lambda x.P: \mu \to \upsilon$ is obtained by a closed deduction $\mathscr{D}$ with ($\to$I) as last step; then one has $\mathrm{FV}(\mu) = \emptyset$ and $\mathscr{D}': \underline{B}_\rho \cup \{x:\mu\} \Vdash P:\upsilon$ where $\mathscr{D}'$ followed by ($\to$I) is $\mathscr{D}$. Thus $\mathscr{D}'$ is a closed deduction of $\underline{B}_{\rho(x:=\uparrow\mu)} \Vdash P:\upsilon$ and so $\upsilon \in [\![P]\!]_{\rho(x:=\uparrow\mu)}$ by the induction hypothesis applied to $\mathscr{D}'$. Hence $\mu \to \upsilon \in [\![\lambda x.P]\!]_\rho$ because μ is closed. □

Remark 3.6. Given the connection between the interpretation of a λ-term and the types derivable for that term, one can understand why we could not prove that $\mathscr{F}^0_\forall$ is a model of β-conversion. Since if it were so, type assignment would be closed under β-equality, i.e. $\underline{B} \vdash M:\sigma \;\&\; M =_\beta N \Rightarrow \underline{B} \vdash N:\sigma$ (or with $\Vdash$). We take a closer look.

The implication $\vdash(\lambda x.M)N:\tau \Rightarrow \vdash M[x:=N]:\tau$, known as the *subject reduction theorem*, holds for most type assignment systems. The reverse implication, which involves the expansion, is more complicated. In general, there are two different problems involved.

(1) x does not occur in M and N is an untypeable term. This problem does not arise in the present situation, since we can always assign ω to N.

(2) x occurs more than once in M and in the deduction of $\vdash M[x:=N]:\tau$ different types are assigned to N.

For the sake of simplicity, suppose x occurs twice in M and types σ_1 and σ_2 are assigned to N. In general formulation, the difficulty is solved when one can find a type σ that can be assigned to N with $\sigma \leqslant \sigma_1$ and $\sigma \leqslant \sigma_2$.

An important observation is that this is essentially the same problem as that of the directedness of the set $\{\uparrow\sigma \mid \underline{\sigma} \in d\}$. (Given $\underline{\sigma}_1, \underline{\sigma}_2 \in d$, find a $\underline{\sigma} \in d$ with $\uparrow\underline{\sigma}_1, \uparrow\underline{\sigma}_2 \subseteq \uparrow\sigma$, i.e. $\underline{\sigma} \leqslant \underline{\sigma}_1$ and $\underline{\sigma} \leqslant \underline{\sigma}_2$.) And if the set $\{\uparrow\sigma \mid \underline{\sigma} \in d\}$ is directed, then we have that $\mathscr{F}^0_\forall$ is algebraic and also that $F \circ G = \mathrm{id}$.

It is not clear how to solve this difficulty for polymorphic type structures. E.g. both $(\forall\alpha.\alpha) \to (\forall\alpha.\alpha)$ and $(\forall\alpha.(\alpha \to \alpha)) \to (\forall\alpha.(\alpha \to \alpha))$ are types for $\lambda x.xx$, but there seems to be no type for $\lambda x.xx$ below both of them. If x occurs in a term at most once, this difficulty does not arise; in fact for linear λ-terms one obtains

λ-models. (Proposition 2.12 can also be proved using Proposition 3.5.). The easiest way to overcome this difficulty in case of multiple variable occurrences is to add intersection types. Then we simply have $\sigma_1 \wedge \sigma_2 \leqslant \sigma_1$ and $\sigma_1 \wedge \sigma_2 \leqslant \sigma_2$. By a construction similar to the one of Section 2, a model of β-conversion is obtained.

4. Conversion models of filters with polymorphic and intersection types

Definition 4.1. (i) The set of types $T_{\forall\wedge}$ is formed as $T_\forall$, except that the following formation rule, $\sigma, \tau \in T_{\forall\wedge} \Rightarrow (\sigma \wedge \tau) \in T_{\forall\wedge}$, is added.

(ii) A type inclusion relation $\leqslant$ on $T_{\forall\wedge}$ is defined as the ordering satisfying, besides the conditions of Definition 2.2(i), also following:

$$\sigma \wedge \tau \leqslant \sigma$$

$$\sigma \wedge \tau \leqslant \tau$$

$$(\sigma \to \tau) \wedge (\sigma \to \mu) \leqslant \sigma \to (\tau \wedge \mu)$$

$$\sigma \leqslant \sigma \wedge \sigma$$

$$\sigma \leqslant \sigma' \;\&\; \tau \leqslant \tau' \;\Rightarrow\; \sigma \wedge \tau \leqslant \sigma' \wedge \tau'.$$

Again we write $\sigma = \tau$ iff $\sigma \leqslant \tau \;\&\; \tau \leqslant \sigma$.

(iii) $\mathcal{T}_{\forall\wedge} = \langle T_{\forall\wedge}, \leqslant \rangle$ is called a *polymorphic intersection type structure* or simply a $\forall\wedge$*-type structure.*

Note that $\forall$ and $\wedge$ are interchangeable in the following way: $(\forall\alpha.\sigma) \wedge (\forall\beta.\tau) = \forall\gamma\delta.(\sigma[\alpha := \gamma] \wedge \tau[\beta := \delta])$, with γ and δ fresh.

Definition 4.2. (i) A $\forall\wedge$*-filter* d in $\mathcal{T}_{\forall\wedge}$ is defined by adding to the clauses of Definition 2.4(i) the following: $\sigma, \tau \in d \Rightarrow \sigma \wedge \tau \in d$.

(ii) $\mathcal{F}_{\forall\wedge} = \{d \subseteq T_{\forall\wedge} \mid d \text{ is a } \forall\wedge\text{-filter in } \mathcal{T}_{\forall\wedge}\}$.

$\uparrow A$ and $\uparrow\sigma$ are used as before (but are $\forall\wedge$-filters in this context).

Lemma 4.3. *In* $\langle \mathcal{F}_{\forall\wedge}, \subseteq \rangle$ *the following holds.*

(i) *For* $X \subseteq \mathcal{F}_{\forall\wedge}$, sup X *exists and equals* $\uparrow\bigcup X$. *If* X *is directed then* $\sup X = \bigcup X$.

(ii) *For a* $\forall\wedge$*-filter* d, *the set* $\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$ *is directed and has supremum* d.

(iii) $\{\uparrow\sigma \mid \sigma \in T_{\forall\wedge}\}$ *is the set of compact elements. Hence* $\langle \mathcal{F}_{\forall\wedge}, \subseteq \rangle$ *is an algebraic complete lattice.*

(iv) *For* $A \subseteq T_{\forall\wedge}$ *with* $A \neq \emptyset$ *one has*:

$$\sigma \in \uparrow A \;\Leftrightarrow\; \exists \tau_1 \ldots \tau_n \in A\; \exists \vec{\alpha}.\; \sigma \geqslant \forall\vec{\alpha}.(\tau_1 \wedge \cdots \wedge \tau_n).$$

(v) $\uparrow\sigma = \{\tau \in T_{\forall\wedge} \mid \tau \geqslant \sigma\}$.

(vi) $\sup\{\uparrow\sigma_1, \uparrow\sigma_2\} = \uparrow(\sigma_1 \wedge \sigma_2)$.

Proof. (i) Obvious.

(ii) Note that $\uparrow\sigma_1, \uparrow\sigma_2 \subseteq \uparrow(\sigma_1 \wedge \sigma_2)$. The rest is as in Lemma 2.5(iii).

(iii) We only show that compact elements are of the form $\uparrow\sigma$. Let d be a compact. By (ii) one has $d = \sup\{\uparrow\sigma \mid \uparrow\sigma \subseteq d\}$. Hence $d \subseteq \uparrow\sigma$ for a $\uparrow\sigma \subseteq d$. Thus $d = \uparrow\sigma$.

(iv) As for Lemma 2.5(v) using the fact that $\forall$ and $\wedge$ are interchangeable.

(v) By (iv).

(vi) First $\uparrow(\sigma_1 \wedge \sigma_2)$ is a majorant: $\uparrow\sigma_1, \uparrow\sigma_2 \subseteq \uparrow(\sigma_1 \wedge \sigma_2)$, moreover it is the smallest one, since $\uparrow\sigma_1, \uparrow\sigma_2 \subseteq d \Rightarrow \sigma_1, \sigma_2 \in d \Rightarrow \sigma_1 \wedge \sigma_2 \in d \Rightarrow \uparrow(\sigma_1 \wedge \sigma_2) \in d$. □

The application in $\mathscr{F}_{\forall\wedge}$ and the functions F and G from Definition 2.4(iv) and Definition 2.7 are taken over. It is easy to see that they are well defined and continuous also in this context and that $G \circ F \leq \mathrm{id}$ in $\mathscr{F}_{\forall\wedge}$. Moreover it is easy to prove that the following holds in any $\forall\wedge$-type structure.

Lemma 4.4. *$F \circ G \geq \mathrm{id}$ in $[\mathscr{F}_{\forall\wedge} \to \mathscr{F}_{\forall\wedge}]$.*

Proof. Let $\tau \in f(d) = \bigcup \{f(\uparrow\sigma) \mid \sigma \in d\}$, by Lemma 4.3(i) and (ii). Then $\tau \in f(\uparrow\sigma)$ for a $\sigma \in d$ and thus $\underline{\sigma} \to \tau \in G(f)$. Hence $\tau \in G(f) \cdot \uparrow\sigma \subseteq G(f) \cdot d = (G \circ F)(f)(d)$. □

The definition of the balance property has to be adapted, though. In its new form it is almost the same as the condition (C3) of [3, definition 2.12]. In fact Theorem 4.6 is very similar to Lemma 2.13(iii) there.

Definition 4.5. We call $\mathscr{T}_{\forall\wedge}$ a *balanced* $\forall\wedge$-type structure iff for closed μ_i $(1 \leq i \leq n)$:

$$\sigma \to \tau \geq (\mu_1 \to \upsilon_1) \wedge \cdots \wedge (\mu_n \to \upsilon_n) \;\&\; \tau \neq \omega \;\Rightarrow$$

$$\underline{\sigma} \leq \textstyle\bigwedge_I \mu_i \text{ and } \tau \geq \bigwedge_I \underline{\upsilon}_i \text{ for some nonempty } I \subseteq \{1, \ldots, n\}.$$

Theorem 4.6. (i) *$\mathscr{T}_{\forall\wedge}$ is a balanced type structure $\Leftrightarrow$ $F \circ G \leq \mathrm{id}$.*

(ii) *$\mathscr{T}_{\forall\wedge}$ is a balanced type structure $\Rightarrow$ $\langle \mathscr{F}_{\forall\wedge}, F, G\rangle$ is a coadditive reflexive λ-model.*

Proof. (i)($\Rightarrow$) Suppose $\tau \in (F \circ G)(f)(d) = G(f) \cdot d$; if $\tau = \omega$ it is obvious that $\tau \in f(d)$. so assume $\tau \neq \omega$. Find a $\sigma \in d$ with $\sigma \to \tau \in G(f) = \uparrow\{\underline{\mu} \to \upsilon \mid \upsilon \in f(\uparrow\mu)\}$. Then $\sigma \to \tau \geq \forall\bar{\alpha}.\,(\bigwedge_{i \leq n} \underline{\mu}_i \to \upsilon_i)$ for some μ_i, υ_i with $\upsilon_i \in f(\uparrow\mu_i)$ by Lemma 4.3(iv) and so $\sigma \to \tau \geq \bigwedge_{i \leq n} \underline{\mu}_i \to \forall\bar{\alpha}.\upsilon_i$. Hence by the balance property we have $\underline{\sigma} \leq \bigwedge_I \underline{\mu}_i$ and $\tau \geq \bigwedge_{i \leq n} \underline{\upsilon}_i$, for some nonempty $I \subseteq \{1, \ldots, n\}$. But then for $i \in I$, $\underline{\mu}_i \in d$ and thus $\upsilon_i \in f(\uparrow\mu_i) \subseteq f(d)$. Thus $\tau \in f(d)$. too.

($\Leftarrow$) Suppose $(\mu_1 \to \upsilon_1) \wedge \cdots \wedge (\mu_n \to \upsilon_n) \leq \sigma \to \tau$ and $\tau \neq \omega$. for μ_i closed. Take $f = \sup\{f_{\uparrow\mu_i \uparrow\upsilon_i} \mid 1 \leq i \leq n\}$. Then $f(d) = \sup\{\uparrow\upsilon_i \mid 1 \leq i \leq n \;\&\; \mu_i \in d\} = \uparrow(\bigwedge_I \upsilon_i)$, with $I = \{1 \leq i \leq n \mid \mu_i \in d\}$, by Lemma 4.3(vi). Since for $1 \leq i \leq n$ one has $\upsilon_i \in f(\uparrow\mu_i)$ with μ_i closed, it is clear that $\mu_i \to \upsilon_i \in G(f)$. Thus $\bigwedge_{1 \leq i \leq n} (\mu_i \to \upsilon_i) \in G(f)$ and $\uparrow(\bigwedge_{1 \leq i \leq n} (\mu_i \to \upsilon_i)) \subseteq G(f)$. Now define $I = \{1 \leq i \leq n \mid \underline{\sigma} \leq \mu_i\}$; we shall prove

$I \neq \emptyset$ and $\bigwedge_I \upsilon_i \leqslant \tau$. By assumption $\sigma \to \tau \geqslant \bigwedge_{1 \leqslant i \leqslant n} (\mu_i \to \upsilon_i)$, so $\sigma \to \tau \in \uparrow (\bigwedge_{1 \leqslant i \leqslant n} (\mu_i \to \upsilon_i)) \subseteq G(f)$. Hence $\tau \in G(f) \cdot \uparrow\sigma = F(G(f))(\uparrow\sigma) \subseteq f(\uparrow\sigma) =$ **if** $I = \emptyset$ **then** $\uparrow\omega$ **else** $\uparrow(\bigwedge_I \upsilon_i)$. But $\tau \neq \omega$, so $I \neq \emptyset$ and $\tau \in \uparrow(\bigwedge_I \upsilon_i)$, i.e. $\tau \geqslant \bigwedge_I \upsilon_i$.

(ii) Obvious from (i) and Lemma 4.4. □

Definition 4.7. (i) The type inclusion relation $\leqslant_0$ is defined as the smallest relation that satisfies the conditions from Definition 4.1(ii).

(ii) $\mathscr{F}^0_{\forall\wedge}$ is the set of filters in $\langle T_{\forall\wedge}, \leqslant_0 \rangle$.

In the following lemma some useful properties of the relation $\leqslant_0$, similar to those of Lemma 2.13 are presented. We use the following ad hoc notation: $\sigma \wedge^* \tau$ stands for $\forall\bar{\alpha}.\ ((\forall\bar{\beta}.\sigma) \wedge (\forall\bar{\gamma}.\tau))$ for some $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$; the variables occurring in $\bar{\alpha}, \bar{\beta}$ and $\bar{\gamma}$ are called *hidden variables* in $\sigma \wedge^* \tau$.

Lemma 4.8. (i) *Substitution in* $\langle T_{\forall\wedge}, \leqslant_0 \rangle$ *is monotonic; moreover the length of the proof remains the same under substitution.*

(ii) $\sigma \to \tau =_0 \omega \Leftrightarrow \tau =_0 \omega$.

(iii) $\bigwedge^*_I (\mu_i \to \upsilon_i) \leqslant_0 \xi \ \&\ \xi \neq_0 \omega \Rightarrow \xi \equiv \bigwedge^*_J (\sigma_j \to \tau_j)$.

Lemma 4.9. $\langle T_{\forall\wedge}, \leqslant_0 \rangle$ *is a balanced* $\forall\wedge$*-type structure.*

Proof. The lemma follows from the following statement, which can be proved by induction on the definition of $\leqslant_0$.

$$\begin{aligned} \text{If}\quad & (\textstyle\bigwedge^*_I (\mu_i \to \upsilon_i) \wedge^* (\bigwedge^*_{I'} \gamma_{i'}) \wedge^* \omega \wedge^* \cdots \wedge^* \omega \\ & \leqslant_0 (\textstyle\bigwedge^*_J (\sigma_j \to \tau_j) \wedge^* (\bigwedge^*_{J'} \delta_{j'}) \wedge^* \omega \wedge^* \cdots \wedge^* \omega \\ & \text{and } \tau_k \neq_0 \omega, \text{ for all } k \in J \\ \text{then}\quad & \sigma_k \leqslant_0 \textstyle\bigwedge_H \mu_h[\bar{\alpha}_h := \bar{\xi}_h] \text{ and } \bigwedge_H \forall\bar{\beta}_h.\upsilon_h \leqslant_0 \tau_k \\ & \text{for some nonempty } H \subseteq I. \end{aligned}$$ □

Theorem 4.10. $\langle \mathscr{F}^0_{\forall\wedge}, F, G \rangle$ *is a coadditive* λ*-model.*

In the rest of this section we shall compare the domain $\mathscr{F}^0_{\forall\wedge}$ with the intersection domain $\mathscr{F}$ from [2]. It will turn out that these domains are not isomorphic as complete lattices. First, we shall be more precise about isomorphism between lattices.

Definition 4.11 (Sanchis [16, Definition 0.3]). Two complete lattices $\langle A, \leqslant_A \rangle$ and $\langle B, \leqslant_B \rangle$ are *isomorphic* if there are monotonic functions $v: A \to B$ and $w: B \to A$ such that $v \circ w = \mathrm{id}_B$ and $w \circ v = \mathrm{id}_A$.

Lemma 4.12. *Let the complete lattices* $\langle A, \leqslant_A \rangle$ *and* $\langle B, \leqslant_B \rangle$ *be isomorphic via* v, w*; then*

(i) *v and w are continuous;*

(ii) *$x \in A$ is compact $\Leftrightarrow$ $v(x) \in B$ is compact and $x \in B$ is compact $\Leftrightarrow$ $w(x) \in A$ is compact.*

Proof. We only give the proofs for v; by symmetry they hold for w as well.

(i) Let $X \subseteq A$ be directed, then v is continuous if $v(\sup X) =_B \sup v(X)$. Well, $v(x) \leqslant_B v(\sup X)$ for $x \in X$ and so $\sup v(X) \leqslant_B v(\sup X)$. By the same reasoning applied to w and $v(X)$ one has $\sup X \leqslant_A w(\sup v(X))$. Combining these yields $\sup v(X) \leqslant_B v(\sup X) \leqslant_B v(w(\sup v(X))) =_B \sup v(X)$.

(ii) Let $x \in A$ be compact and $X \subseteq B$ be directed, then $v(x) \leqslant_B \sup X \Rightarrow x =_A w(v(x)) \leqslant_A w(\sup X) =_A \sup w(X)$, since w is continuous by (i) $\Rightarrow \exists y \in w(X). x \leqslant_A y$, since x is compact and $w(X)$ is directed $\Rightarrow \exists z \in X. v(x) \leqslant_B z$, take $z = v(y)$.

Thus $v(x) \in B$ is compact. Of course we can show in the same way that $x \in B$ is compact $\Rightarrow w(x) \in A$ is compact. Hence $v(x) \in B$ is compact $\Rightarrow x =_A w(v(x)) \in A$ is compact. □

Theorem 4.13. *$\langle \mathscr{F}^0_{\forall\wedge}, \subseteq \rangle$ and $\langle \mathscr{F}, \subseteq \rangle$ are not isomorphic as complete lattices.*

Proof. Suppose towards a contradiction that there are monotonic $v : \mathscr{F}^0_{\forall\wedge} \to \mathscr{F}$, and $w : \mathscr{F} \to \mathscr{F}^0_{\forall\wedge}$ with $v \circ w = \mathrm{id}$ and $w \circ v = \mathrm{id}$. Let $T_\wedge$ be the set of intersection types as introduced in [2, 3.1]. Then, because for all $e \in \mathscr{F}$, $e \subseteq T_\wedge$, for all $d \in \mathscr{F}^0_{\forall\wedge}$, we must have $d \subseteq w(T_\wedge)$ by surjectivity of w; thus $w(T_\wedge) = T_{\forall\wedge}$. Since $T_{\forall\wedge} = \uparrow\forall\alpha.\alpha$ we have that $T_{\forall\wedge}$ is conpact. We show that $T_\wedge$ is not compact which then, by Lemma 4.12(ii), establishes the result. So suppose that $T_\wedge$ is compact; then $T_\wedge = \uparrow\sigma_0$ for some $\sigma_0 \in T_\wedge$, by [3, 1.8(iv)]. But then, for all $\tau \in T_\wedge$, $\sigma_0 \leqslant' \tau$, where $\leqslant'$ is introduced in [2, 3.3]. Thus certainly, for all $n \in \mathbb{N}$, $\sigma_0 \leqslant' \phi_n$ where $\phi_n \in T_\wedge$ are basic variables. By induction on the definition of $\leqslant'$ one can prove that for all basic types ϕ_n one has

$$\tau \leqslant' \phi_n \text{ or } \tau \leqslant' \phi_n \wedge \mu \Rightarrow \tau \equiv \phi_n \text{ or } \tau \equiv \mu' \wedge \phi_n$$
$$\text{or } \tau \equiv \phi_n \wedge \mu'' \text{ or } \tau \equiv \mu' \wedge \phi_n \wedge \mu''.$$

But then, for all $n \in \mathbb{N}$ for all $\tau \in T_\wedge$, $\tau \leqslant' \phi_n \Rightarrow \phi_n$ occurs in τ. Hence, not for all $n \in \mathbb{N}$, $\sigma_0 \leqslant' \phi_n$, since σ_0 is a type consisting of a finite number of symbols, which yields the contradiction. □

5. Type assignment and ∀∧-filters

Definition 5.1. The *type assignment* induced by the ∀∧-type structure $\langle T_{\forall\wedge}, \leqslant_0 \rangle$ is defined by adding to the rules of Definition 3.1(ii) the following.

$$(\wedge \mathrm{I}) \; \frac{M : \sigma \qquad M : \tau}{M : \sigma \wedge \tau}$$

The properties of type assignment mentioned in Lemmas 3.2 and 3.3 are easily translated to the present context. Proofs are omitted.

Lemma 5.2. (i) $B\backslash x \cup \{x:\sigma_1, \ldots, \sigma_n\} \vdash x:\tau \Rightarrow \tau \geqslant_0 \sigma_1 \wedge \cdots \wedge \sigma_n$.

(ii) $B \vdash M:\sigma \Rightarrow B[\alpha := \tau] \vdash M:\sigma[\alpha := \tau]$.

(iii) $B \vdash MN:\tau \Rightarrow \exists \sigma \in T_{\forall\wedge}.\ B \vdash M:\sigma \to \tau\ \&\ B \vdash N:\sigma$.

(iv) *If for all* $\sigma, \tau \in T_{\forall\wedge}.\ [B\backslash x \cup \{x:\sigma\} \vdash M:\tau \Rightarrow B\backslash x \cup \{x:\sigma\} \vdash N:\tau]$, *then for all* $\mu \in T_{\forall\wedge}.\ [B \vdash \lambda x.M:\mu \Rightarrow B \vdash \lambda x.N:\mu]$.

(v) $\mathscr{D} : B \vdash \lambda x:M:\xi\ \&\ \xi \neq_0 \omega \Rightarrow \xi$ *is of the form* $\bigwedge_J^* (\sigma_j \to \tau_j)$, *with the hidden variables in* $\bigwedge_J^* (\sigma_j \to \tau_j)$ *not occurring in the relevant part of B w.r.t.* $\mathscr{D}$.

Lemma 5.3. (i) *If* $\mathscr{D} : B\backslash x \cup \{x:\mu_i\} \vdash M:\upsilon_i$ *for all* $i \in I$ *and* $\bigwedge_I^*(\mu_i \to \upsilon_i) \leqslant_0 \bigwedge_J^* (\sigma_j \to \tau_j)$ *with all variables hidden in* $\bigwedge_I^* (\mu_i \to \upsilon_i)$ *not free in the relevant part of* $B\backslash x$ *w.r.t.* $\mathscr{D}$, *then* $B\backslash x \cup \{x:\sigma_j\} \vdash M:\tau_j$ *for all* $j \in J$.

(ii) $B\backslash x \cup \{x:\sigma\} \vdash M:\tau \Leftrightarrow B\backslash x \vdash \lambda x.M:\sigma \to \tau$.

(iii) $B \vdash \lambda x.Mx:\xi \Rightarrow B \vdash M:\xi$, *if* $x \notin \mathrm{FV}(M)$.

(iv) $B \vdash \lambda x.M:\xi \Rightarrow B \vdash \lambda y.M[x := y]:\xi$, *if* $y \notin \mathrm{FV}(M)$.

Proposition 5.4 (EQ_β). *Suppose* $M =_\beta N$; *then* $B \vdash M:\tau \Leftrightarrow B \vdash N:\tau$.

Proof. It is sufficient to prove $B \vdash (\lambda x.P)Q:\tau \Leftrightarrow B \vdash P[x := Q]:\tau$.

($\Rightarrow$) Obvious. ($\Leftarrow$) Collect all the statements $Q:\sigma_1, \ldots, Q:\sigma_n$ occurring in $\mathscr{D} : B \vdash P[x := Q]:\tau$. Then $B\backslash x \cup \{x:\sigma_1, \ldots, x:\sigma_n\} \vdash P:\tau$ and thus $B \vdash \lambda x.P: (\sigma_1 \wedge \cdots \wedge \sigma_n) \to \tau$. Also $B \vdash Q: \sigma_1 \wedge \cdots \wedge \sigma_n$ and so $B \vdash (\lambda x.P)Q:\tau$. □

By interpreting λ-terms in $\langle \mathscr{F}^0_{\forall\wedge}, F, G \rangle$ we obtain, as in Proposition 3.5, the following result.

Proposition 5.5. *In* $\langle \mathscr{F}^0_{\forall\wedge}, F, G \rangle$ *one has*

$$[\![M]\!]_\rho = \{\sigma \in T_{\forall\wedge} \mid \underline{B}_\rho \vdash M:\sigma\}.$$

Our filter definition with the clause $\sigma \in d \Rightarrow \forall \alpha.\sigma \in d$, does not faithfully reflect the properties of the ($\forall$I)-rule; hence no one of the filter models can be used to prove the completeness of the type assignment, which, on the contrary, is easily proved (see [12]) using a, by now standard, term model technique [9] and [5].

Note added in proof

Very recently, Gordon Plotkin pointed out to us that the assignment to a term of the set of $\forall\cap$-types derivable for that term does not yield a λ-model. Hence there are no two different interpretations as claimed in the Introduction.

The statement after Proposition 2.12 can be sharpened: the complete lattice $F_\forall$ is a model of the affine (or direct) λ-calculus in which variables may occur at most once in terms. In some sense it is dual to the λI-calculus where variables occur at least once.

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