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# A Decomposition Theorem for Domains

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A domain constructor that generalizes the product is defined. It is shown that with this constructor exactly the prime-algebraic coherent Scott-domains can be generated from two-chains and boolean flat domains. © 1996 Academic Press, Inc.

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## 1. INTRODUCTION

This work was motivated by the idea of understanding a database as a subset of a product of flat domains. In relational database theory a database is understood as a relation, i.e., a subset of a product of sets. By adding a least element to every factor, meaning “no information,” we get a product of flat domains. In [BJO91], concepts of relational database theory are generalized to arbitrary Scott-domains. That is, concepts are formulated for arbitrary Scott-domains in such a way that they coincide with the corresponding concepts in relational database theory on products of flat domains.

A natural question to ask is whether there is a nice axiomatic characterization of those Scott-domains that are decomposable into products of flat domains. Jung and Puhmann investigated this question in [JLP92]; also [Puh90], [Haa93]. They found a way to decompose dI-domains, not into products of flat domains, but into hyperlimits of indecomposable domains. A hyperlimit of domains is a subset of their product that is specified by a family of connecting morphisms between the factors. The hyperlimit construction is finer than the product construction, in the sense that products are just hyperlimits with no connecting morphisms.

Unfortunately, the indecomposable dI-domains of the decomposition theorem in [JLP92] need not be flat domains. The reason for this is that the authors restricted themselves to decompositions into hyperlimits over *cycle-free diagrams*. That is the major difference to this work. Here we try to decompose Scott-domains into hyperlimits over *arbitrary diagrams*. We find a nice axiomatic characterization of hyperlimits of flat domains. Namely, the nonempty hyperlimits of flat domains are exactly the prime-algebraic coherent Scott-domains. Even more: Any prime-algebraic coherent Scott-domain is decomposable into a

hyperlimit of two-chains and boolean flat domains. The two-chains and the boolean flat domains are thus the only non-trivial prime-algebraic coherent Scott-domains that are indecomposable with respect to the hyperlimit construction.

A few words about prime-algebraic coherent Scott-domains: They form a class of domains that is well-known in domain theory. They arise for example as a domain-theoretic model of concurrency, namely, as configuration sets of event structures [NPW81, Win87]. In fact, configuration sets of (not necessarily finitary) prime event structures that are determined by their conflict relation can be characterized as prime-algebraic coherent Scott-domains [NPW81]. Girard’s coherence spaces for linear logic are another example of prime-algebraic coherent Scott-domains [Gir87]. (Of course, coherence spaces are just prime event structures with a discrete causal dependency relation.) The category of prime-algebraic coherent Scott-domains and continuous functions has nice closure properties. It is cartesian closed, and closed under lifting and coalesced sum. It is not closed under arbitrary limits, but under limits over diagrams whose connecting morphisms preserve existing suprema.

The hyperlimit construction looks similar to the familiar limit construction. In fact, bilimits (i.e., inverse limits of diagrams of domains whose connecting morphisms are upper adjoints) as well as products of domains can be viewed as special instances of hyperlimits. But while the bilimit construction applied to flat domains does not lead out of the class of flat domains at all and the product construction generates a class of domains which seems to be hard to characterize axiomatically, the hyperlimit construction generates a rich and nice class of domains.

## 2. DEFINITIONS

Most of the definitions are as in [AJ94]. A partially ordered set is called *directed complete* if every nonempty directed subset has a supremum. A partially ordered set is called *bounded complete* if every subset that has an upper bound has a supremum. A partially ordered set is bounded complete if and only if every nonempty subset has an infimum. A *complete lattice* is a bounded complete partially ordered set with a greatest element. If  $A$  denotes a nonempty

bounded complete partially ordered set we will often write  $\perp_A$  for its least element. If  $A$  denotes a complete lattice we will often write  $\top_A$  for its greatest element. An element  $x$  of a directed complete partially ordered set is called *compact* if any nonempty directed set whose supremum is greater than or equal to  $x$  already has an element that is greater than or equal to  $x$ . A directed and bounded complete partially ordered set is called *algebraic* if each of its elements is supremum of the compact elements below it. In a bounded and directed complete partially ordered set the set of all compact elements below some fixed element is directed, because a supremum of a finite set of compact elements is again compact. A *Scott-domain* is a bounded complete, directed complete and algebraic partially ordered set. Note that, according to this definition, a Scott-domain can be empty and can have an uncountable basis. A continuous function  $f: A \rightarrow B$  between Scott-domains  $A$  and  $B$  is a monotone function that preserves directed suprema. A continuous function between nonempty Scott-domains is called *strict* if it preserves the least element. We say that a continuous function  $f: A \rightarrow B$  *preserves suprema* (or is *sup-preserving*) if it preserves all suprema existing in  $A$ . Note that a sup-preserving function between nonempty Scott-domains is strict. A *Scott-closed* subset of a Scott-domain is a lower set that is closed under directed suprema.

An *adjunction* between partially ordered sets  $P$  and  $Q$  is a pair of monotone maps  $(l: P \rightarrow Q, u: Q \rightarrow P)$ , such that  $l(x) \leq y$  if and only if  $x \leq u(y)$  for all  $x \in P, y \in Q$ . Then  $l$  will be called the *lower adjoint* of  $u$ , and  $u$  the *upper adjoint* of  $l$ . A pair of monotone maps  $(l, u)$  is an adjunction iff  $l \circ u \leq id_P$  and  $u \circ l \geq id_Q$  in the pointwise ordering. This is the case iff  $l(x)$  is the smallest element of  $u^{-1}(\uparrow x)$  for all  $x \in P$ , or, dually, iff  $u(y)$  is the greatest element of  $l^{-1}(\downarrow y)$  for all  $y \in Q$ . An (*internal*) *projection* of a partially ordered set  $P$  is a monotone map  $p: P \rightarrow P$ , such that  $p \circ p = p$  and  $p \leq id_P$  in the pointwise ordering. If  $(l: P \rightarrow Q, u: Q \rightarrow P)$  is an adjunction between partially ordered sets, then  $l \circ u$  is an internal projection of  $Q$ . If  $M$  is a subset of a Scott-domain that is closed under existing suprema then there is a projection onto  $M$ , which is given by  $p(x) = \bigvee (\downarrow x \cap M)$ .

An element  $x$  of a Scott-domain  $A$  is called *completely prime* if  $\bigvee M \geq x$  implies that there is an  $m \in M$  such that  $m \geq x$ , for any bounded subset  $M$  of  $A$ . Completely prime elements are compact, but the opposite is not necessarily true. An example of a completely prime element that is not compact is the least element of a nonempty Scott-domain. We will write  $\text{Prime}(A)$  for the set of all completely prime elements of a Scott-domain  $A$ . A Scott-domain is called *prime-algebraic* if each of its elements is supremum of the completely prime elements below it. In this work coherence is not the topological property as defined in [AJ94], but it is the completeness property as in [NPW81]. Namely, a Scott-domain  $A$  is called *coherent* if any subset  $M \subseteq A$  that has the property that  $\{x, y\}$  is bounded for all  $x, y \in M$ , is

itself bounded. Note that a coherent Scott-domain is non-empty. A *flat domain* is an unordered set  $X$ , which is enriched by a least element. We will denote the resulting domain by  $X_\perp$ . By the *two-chain* we mean the flat domain  $\{\top\}_\perp$ . By the *boolean flat domain* we mean the domain  $\{\mathbf{t}, \mathbf{f}\}_\perp$ . Throughout this text we will denote it by  $\mathbb{B}$ .

### 3. A DECOMPOSITION THEOREM

**DEFINITION 3.1.** Let  $(I, K)$  be a directed graph, with  $I$  being a set of vertices and  $K$  a set of arrows. Let  $A = (A_i)_{i \in I}$  be a family of Scott-domains, and let  $f = (f_k)_{k \in K}$  be a family of continuous maps such that  $f_k: A_i \rightarrow A_j$ , if  $k: i \rightarrow j$ . We call  $(A, f)$  a *diagram* of Scott-domains. For each  $i \in I$  and  $k \in K$  we will call  $A_i$  an *object* and  $f_k$  a *connecting morphism* of the diagram and  $\{A_i | i \in I\}$  will be called *object set* of the diagram. The set

$$\left\{ (x_i)_{i \in I} \in \prod_{i \in I} A_i \mid \forall k: i \rightarrow j \in K. f_k(x_i) \leq x_j \right\}$$

in the componentwise ordering shall be called the *hyperlimit* of the diagram and we will write  $\text{hyp}(A, f)$  for it.

Surely, products of Scott-domains are special hyperlimits, the family of connecting morphisms being empty. The hyperlimit construction looks similar to the familiar limit construction for domains. In general, the hyperlimit of a given diagram of domains has more elements than the limit of the same diagram. The hyperlimit “leaves more room in the product” than the limit. In O-categories a categorical definition of the hyperlimit can be given (see [Haa93]). In fact, the hyperlimit construction is a special case of a more general concept developed in the theory of 2-categories, namely, lax limits. Here are some examples:

- (a) Take the index graph  $(\mathbb{N}, \{(k_n: (n+1) \rightarrow n) \mid n \in \mathbb{N}\})$ . For each  $n \in \mathbb{N}$  let  $A_n$  be the two-chain and  $f_{k_n}$  be the identity map. It is easy to check that the hyperlimit of  $(A, f)$  is an  $\omega$ -chain with an additional top element.
- (b) Take the index graph  $(\{0, 1\}, \{a: 0 \rightarrow 1\})$ , the objects  $B_0 = B_1 = \mathbb{B}$  and the function  $g_a$  that maps  $\mathbf{t}$  to itself and the other two elements to the least element. Figure 1 shows the hyperlimit of this diagram.

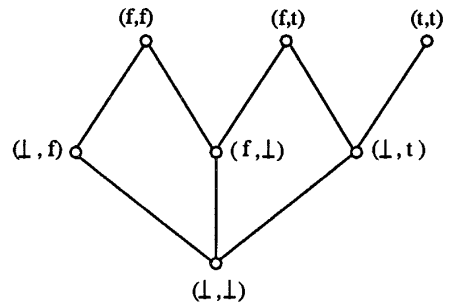
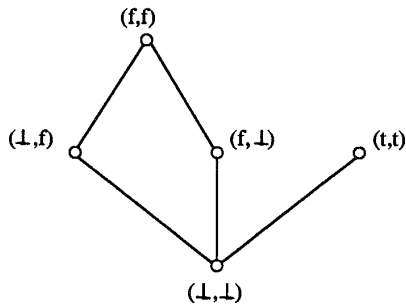


FIG. 1.  $\text{hyp}(B, g)$ .

FIG. 2.  $\text{hyp}(C, h)$ .

(c) Let us now add a second connecting morphism to the diagram  $(B, g)$ , namely the same function in the other direction. So the index graph will now be  $(\{0, 1\}, \{a: 0 \rightarrow 1, b: 1 \rightarrow 0\})$ , the objects  $C_0$  and  $C_1$  will both be  $\mathbb{B}$  and  $h_a$  and  $h_b$  will both be the function that maps  $t$  to itself and the other two elements to the least elements. Figure 2 shows the hyperlimit of this diagram.

Note that in all of the examples the projections onto the factors have lower adjoints, and thus copies of the factors can be found in the hyperlimit-object as images of internal projections. We will see that this is so in general.

It is noteworthy that not only the product, but also the continuous function space  $[A \rightarrow B]$  of two algebraic domains can be viewed as a special case of a hyperlimit: Take  $(K(A), \leq_{K(A)})$  as index graph, where  $K(A)$  denotes the set of compact elements of  $A$  and  $\leq_{K(A)}$  is interpreted as a directed graph in the standard way. Let every compact element indicate a copy of  $B$  and every  $(k, h) \in \leq_{K(A)}$  indicate the identity on  $B$ . Then the hyperlimit of this diagram is isomorphic to  $[A \rightarrow B]$ . Using this observation it can be proved that the category of hyperlimits of flat domains and continuous functions is cartesian closed. In the light of the characterization theorem, which we are going to establish, this is not so interesting, since the the category of prime-algebraic coherent Scott-domains and continuous functions is well-known to be Cartesian closed. But it can also be shown that the category of hyperlimits of *cycle-free* diagrams of flat domains and continuous functions is cartesian closed. This category is strictly included in the category of prime-algebraic coherent Scott-domains and continuous functions. On the other hand, it strictly includes the smallest cartesian closed full subcategory of the category of Scott-domains and continuous functions [Haa93].

The lifted coalesced sum  $(\bigoplus_{i \in I} A_i)_\perp$  of a family of domains with least element can also be viewed as a hyperlimit of the summands plus an additional flat domain: As object family take  $(A_i)_{i \in I}$  plus the additional object  $I_\perp$ . For each  $i \in I$  take a connecting morphism from  $A_i$  to  $I_\perp$ , that maps  $\perp_{A_i}$  to  $\perp_{I_\perp}$  and all other elements of  $A_i$ , to  $i$ . Then the hyperlimit of this diagram is isomorphic to  $(\bigoplus_{i \in I} A_i)_\perp$ .

A bilimit (i.e., an inverse limit of a diagram of domains, whose connecting morphisms are upper adjoints) can be viewed as a hyperlimit of a diagram with the same objects. The connecting morphisms of the hyperlimit diagram are the connecting morphisms of the limit diagram plus all the lower adjoints of those. Detailed proofs of the remarks in the preceding paragraph are given in [Haa93].

We will now show that the class of Scott-domains is closed under the hyperlimit-construction.

**PROPOSITION 3.1.** *Let  $(A, f)$  be a diagram of Scott-domains. Then  $\text{hyp}(A, f)$  as a subset of  $\prod_{i \in I} A_i$  is closed under directed suprema and infima of nonempty sets. Moreover,  $\text{hyp}(A, f)$  is algebraic. Thus it is again a Scott-domain.*

*Proof.* Let  $(x_d)_{d \in D}$  be a directed family in  $\text{hyp}(A, f)$  and let  $f_k: A_i \rightarrow A_j$  be a connecting morphism. Then

$$f_k \left( \bigvee_{d \in D} x_{di} \right) = \bigvee_{d \in D} f_k(x_{di}) \leq \bigvee_{d \in D} x_{dj},$$

where the equality holds by continuity of  $f_k$  and the inequality because  $x_d \in \text{hyp}(A, f)$  for all  $d \in D$ . Let  $L$  be a nonempty set,  $(x_l)_{l \in L}$  a family of elements of  $\text{hyp}(A, f)$  and  $f_k: A_i \rightarrow A_j$  a connecting morphism. Then, by monotonicity of  $f_k$ ,

$$f_k \left( \bigwedge_{l \in L} x_{li} \right) \leq f_k(x_{l_0i}) \leq x_{l_0j} \quad \text{for all } l_0 \in L.$$

Therefore,  $f_k(\bigwedge_{l \in L} x_{li}) \leq \bigwedge_{l \in L} x_{lj}$ . In order to show algebraicity we need the following lemma. ■

**LEMMA 3.1.** *Let  $(A, f)$  be a diagram of Scott-domains.*

(i) *Let  $p_i: \text{hyp}(A, f) \rightarrow A_i$  denote the projection into the  $i$ -th component. Then there is a Scott-closed subset  $\tilde{A}_i$  of  $A_i$ , which contains the image of  $p_i$ , such that the corestriction of  $p_i$  to  $\tilde{A}_i$  has a lower adjoint. Moreover, the diagram restricts to  $\tilde{A}$ . That is, if  $k: i \rightarrow j \in K$  then  $f_k$  maps  $\tilde{A}_i$  into  $\tilde{A}_j$ . If  $\tilde{f}_k$  denotes the restriction to  $\tilde{A}_i$  and corestriction to  $\tilde{A}_j$  of  $f_k$ , then  $\text{hyp}(A, f) = \text{hyp}(\tilde{A}, \tilde{f})$ .*

(ii) *Let  $\tilde{p}_i: \text{hyp}(A, f) \rightarrow \tilde{A}_i$  denote the corestriction of  $p_i$  and  $l_i: \tilde{A}_i \rightarrow \text{hyp}(A, f)$  its lower adjoint. Then the identity on  $\text{hyp}(A, f)$  is pointwise supremum of  $\{l_i \circ \tilde{p}_i \mid i \in I\}$ .*

*Proof.* (i) Define  $\tilde{A}_i := \{x \in A_i \mid p_i^{-1}(\uparrow x) \neq \emptyset\}$ . It is easy to see that  $\tilde{A}_i$  is a lower set and contains the image of  $p_i$ . We have already proved that  $\text{hyp}(A, f)$  as a subset of  $\prod_{i \in I} A_i$  is closed under nonempty infima. That means that  $p_i$  preserves nonempty infima. Moreover, the corestriction  $\tilde{p}_i$  of  $p_i$  to  $\tilde{A}_i$  has the property that the preimage of any principal filter is nonempty. Therefore  $\tilde{p}_i^{-1}(\uparrow x)$  has a smallest element for all  $x \in \tilde{A}_i$ . This means that  $\tilde{p}_i$  has a lower adjoint.

Let  $D$  be a directed subset of  $\tilde{A}_i$  and let  $l_i$  denote the lower adjoint of  $\tilde{p}_i$ . By monotonicity of  $l_i$ , then  $l_i(D)$  is directed and

$$p_i\left(\bigvee l_i(D)\right) = \bigvee p_i \circ l_i(D) = \bigvee \tilde{p}_i \circ l_i(D) \geq \bigvee D.$$

The first equality holds by continuity of  $p_i$  and the inequality because  $(l_i, \tilde{p}_i)$  is an adjunction. Let  $f_k: A_i \rightarrow A_j$  be a connecting morphism and  $x \in \tilde{A}_i$ . Then there is a  $y \in \text{hyp}(A, f)$  such that  $p_i(y) \geq x$ . Then  $f_k(x) \leq f_k(p_i(y)) \leq p_j(y)$ , where the first inequality holds by monotonicity of  $f_k$  and the second one because  $y \in \text{hyp}(A, f)$ . Thus,  $f_k(x) \in \text{hyp}(A, f)$ . It is clear that  $\text{hyp}(A, f) = \text{hyp}(\tilde{A}, \tilde{f})$ .

(ii) Because  $(l_i, \tilde{p}_i)$  is an adjunction, we have  $l_i \circ \tilde{p}_i \leq id$  for all  $i \in I$ . Together with the bounded completeness of  $\text{hyp}(A, f)$  this guarantees the existence of the pointwise supremum of  $\{l_i \circ \tilde{p}_i \mid i \in I\}$ . On the other hand, we have that

$$\begin{aligned} \tilde{p}_i\left(\bigvee_{j \in I} (l_j \circ \tilde{p}_j)(x)\right) &\geq \bigvee_{j \in I} \tilde{p}_i((l_j \circ \tilde{p}_j)(x)) \\ &\geq \tilde{p}_i((l_i \circ \tilde{p}_i)(x)) \\ &= \tilde{p}_i(x) \end{aligned}$$

for all  $i \in I$  and  $x \in \text{hyp}(A, f)$ . The first of those inequalities holds by monotonicity of  $\tilde{p}_i$  and the equality holds, because  $(l_i, \tilde{p}_i)$  is an adjunction. Because the inequality holds for all  $i \in I$  and  $x \in \text{hyp}(A, f)$ , we have that  $\bigvee_{i \in I} l_i \circ \tilde{p}_i \geq id$ . ■

Now we can finish the proof of Proposition 3.1.

*Proof of Proposition 3.1 (continued).* We want to prove the algebraicity of  $\text{hyp}(A, f)$ : For each  $i \in I$  consider the map  $\tilde{p}_i: \text{hyp}(A, f) \rightarrow \tilde{A}_i$  and its lower adjoint  $l_i$  from Lemma 3.1. Note that, as a Scott-closed subset of a Scott-domain,  $\tilde{A}_i$  is again a Scott-domain. For each  $x \in \text{hyp}(A, f)$  we have

$$\begin{aligned} x &= \bigvee_{i \in I} l_i \circ \tilde{p}_i(x) \\ &= \bigvee_{i \in I} l_i\left(\bigvee K_i\right) \\ &= \bigvee_{i \in I} \bigvee (l_i(K_i)) \\ &= \bigvee \left(\bigcup_{i \in I} l_i(K_i)\right), \end{aligned}$$

where  $K_i$  denotes the set of compact elements below  $\tilde{p}_i(x)$ . The first of those equations holds by Lemma 3.1(ii), the second one by algebraicity of  $\tilde{A}_i$  and the third one, because a lower adjoint preserves suprema. As a lower adjoint of a continuous map,  $l_i$  preserves compact elements. Therefore  $\bigcup_{i \in I} l_i(K_i)$  is a set of compact elements. ■

Are coherence and prime-algebraicity preserved under the hyperlimit construction? This is not so in general. We have to put a condition on the diagram, namely that its connecting morphism preserve suprema.

**PROPOSITION 3.2.** *Let  $(A, f)$  be a diagram of Scott-domains and functions that preserve suprema. Then*

- (i)  $\text{hyp}(A, f)$  is closed under suprema existing in  $\prod_{i \in I} A_i$ .
- (ii) If all objects of  $(A, f)$  are prime-algebraic then  $\text{hyp}(A, f)$  is prime-algebraic.
- (iii) If all objects of  $(A, f)$  are coherent, then  $\text{hyp}(A, f)$  is coherent.

*Proof.* (i) Closedness under existing suprema is proved like closedness under directed suprema in Proposition 3.1.

(ii) In order to prove prime-algebraicity one imitates the proof of algebraicity in Proposition 3.1, using the fact that a lower adjoint of a sup-preserving map preserves completely prime elements.

(iii) Let  $M$  be a subset of  $\text{hyp}(A, f)$  such that each two-element subset of  $M$  has an upper bound in  $\text{hyp}(A, f)$ . Because  $\prod_{i \in I} A_i$ , as a product of coherent Scott-domains, is again a coherent Scott-domain,  $M$  has a supremum in  $\prod_{i \in I} A_i$ . Because  $\text{hyp}(A, f)$  is closed under suprema existing in  $\prod_{i \in I} A_i$ , this is an element of  $\text{hyp}(A, f)$ . ■

Neither the category of prime-algebraic Scott-domains and continuous functions nor the category of coherent Scott-domains and continuous functions is closed under the hyperlimit construction [Haa93]. However, the last proposition says that the category of prime-algebraic Scott-domains and sup-preserving functions is closed under this construction. Domain categories whose morphisms are sup-preserving functions arise as special categorical models of linear logic. In the coherence space model [Gir87] and the similar event structure model [Zha91] the morphisms are stable in addition to preserving suprema. However, in [Hut95], it is shown how to interpret the linear logic connectives in the category of prime-algebraic Scott-domains and sup-preserving functions.

From Proposition 3.2, we get the following corollary for hyperlimits of flat domains.

**COROLLARY 3.1.** *The hyperlimit of a diagram of flat domains and strict connecting morphisms is a prime-algebraic coherent Scott-domain.*

*Proof.* Surely, a flat domain is a prime-algebraic coherent Scott-domain. A strict continuous function between flat domains preserves suprema. Hence the statement follows from Propositions 3.1 and 3.2. ■

Dropping the strictness condition in Corollary 3.1 does not cause much harm. It can be shown that the empty domain is the only domain that is isomorphic to a hyperlimit of flat domains, but not a prime-algebraic coherent Scott-domain.

We will next give an internal characterization of hyperlimits of flat domains in the spirit of the well-known characterization of SFP-domains as those domains that have a sequence of deflations that approximates the identity. To this end we need the following definition.

**DEFINITION 3.2.** Given a Scott-domain  $B$  and a set  $\{q_i \mid i \in I\}$  of internal projections of  $B$ . If a tuple  $(x_i)_{i \in I} \in \prod_{i \in I} \text{im}(q_i)$  satisfies

$$q_i(x_j) \leq x_i \quad \text{for all } i, j \in I,$$

we call it *weakly commuting* (w.r.t. the given set of projections).

**LEMMA 3.2.** (a) *Let  $B$  be a nonempty Scott-domain. If  $B$  is isomorphic to a hyperlimit of flat domains, it has a set of internal projections  $\{q_i \mid i \in I\}$  onto flat domains that has the following properties:*

- (i) *The identity is pointwise supremum of  $\{q_i \mid i \in I\}$ .*
- (ii)  *$q_i$  preserves suprema for all  $i \in I$ .*
- (iii) *Any weakly commuting tuple is bounded.*

(b) *Conversely, if  $B$  is a Scott-domain and  $\{q_i \mid i \in I\}$  a set of internal projections that has the above three properties, then  $B$  is isomorphic to a hyperlimit of a diagram whose object set is  $\{\text{im}(q_i) \mid i \in I\}$  and all of whose connecting morphisms are strict continuous functions.*

*Proof.* (a) Let  $(A, f)$  be a diagram of flat domains, and  $\phi: B \rightarrow \text{hyp}(A, f)$  an isomorphism. Because  $B$  is nonempty, the image of a projection from  $\text{hyp}(A, f)$  to a factor will be nonempty. A nonempty Scott-closed subset of a flat domain is again a flat domain. By Lemma 3.1(i), we can therefore assume that every projection  $p_i: \text{hyp}(A, f) \rightarrow A_i$  has a lower adjoint  $l_i$ . First we claim that  $\{l_i \circ p_i \mid i \in I\}$  has properties (i), (ii), and (iii):

We have already proved property (i) in Lemma 3.1(ii). Considering that a lower adjoint preserves suprema, property (ii) follows from Proposition 3.2(i). To prove property (iii) we take a weakly commuting tuple  $(x_i)_{i \in I} \in \prod_{i \in I} \text{im}(l_i \circ p_i)$ . We claim that  $(p_i(x_i))_{i \in I} \in \prod_{i \in I} A_i$  is an element of  $\text{hyp}(A, f)$  and an upper bound of  $\{x_i \mid i \in I\}$ : Note first that from  $l_i \circ p_i(x_j) \leq x_i$  it follows that  $p_i(x_j) \leq p_i(x_i)$  for all  $i, j \in I$ . But this means that  $(p_i(x_i))_{i \in I}$

is componentwise greater than  $x_j$  for all  $j \in I$ . It remains to show that it is an element of  $\text{hyp}(A, f)$ : Let  $f_k: A_i \rightarrow A_j$  be a connecting morphism. Then  $f_k(p_i(x_i)) \leq p_j(x_i) \leq p_j(x_j)$ . The first of these inequalities holds, because  $x_i$  is an element of  $\text{hyp}(A, f)$ .

Using that  $\phi$  is an isomorphism, it can now be shown that  $\{\phi^{-1} \circ l_i \circ p_i \circ \phi \mid i \in I\}$  is also a set of internal projections onto flat domains that has properties (i), (ii), and (iii).

(b) We will define a diagram  $(A, f)$ . Let the underlying directed graph be the complete directed graph over  $I$ . For all  $i \in I$  let  $A_i$  be the image of  $q_i$  and for all  $(i, j) \in I \times I$  let  $f_{ij}: \text{im}(q_i) \rightarrow \text{im}(q_j)$  be the restriction and corestriction of  $q_j$ . We define the candidates for the isomorphisms as follows:

$$\begin{aligned} \phi: B &\rightarrow \text{hyp}(A, f), & x &\mapsto (q_i(x))_{i \in I}, \\ \psi: \text{hyp}(A, f) &\rightarrow B, & (x_i)_{i \in I} &\mapsto \bigvee_{i \in I} x_i. \end{aligned}$$

For  $\psi$  to be well-defined we need (iii). Obviously,  $\phi$  and  $\psi$  are monotonic. Let  $(x_i)_{i \in I} \in \text{hyp}(A, f)$  and  $j \in I$ . Then

$$q_j \circ \psi((x_i)_{i \in I}) = q_j \left( \bigvee_{i \in I} x_i \right) \stackrel{\text{(ii)}}{=} \bigvee_{i \in I} q_j(x_i) = q_j(x_j) = x_j.$$

The third of these equalities holds, because  $(x_i)_{i \in I}$  is weakly commuting. The equality holds for all  $j \in I$ , and therefore  $\phi \circ \psi$  is the identity on  $\text{hyp}(A, f)$ . Now let  $x \in B$ . Then

$$\psi \circ \phi(x) = \bigvee_{i \in I} q_i(x) \stackrel{\text{(i)}}{=} x.$$

Thus  $B \simeq \text{hyp}(A, f)$ . ■

**COROLLARY 3.2.** *A nonempty Scott-domain is isomorphic to a hyperlimit of a diagram of flat domains if and only if it has a set of internal projections onto flat domains that has the three properties from Lemma 3.2.*

*Proof.* This is a direct consequence of Lemma 3.2. ■

We will get the final decomposition theorem as a corollary of the following lemma.

**LEMMA 3.3.** *Let  $A$  be a prime-algebraic coherent Scott-domain and let  $\mathcal{M}$  be a set of subsets of  $\text{Prime}(A)$  that has the following properties:*

- (i)  $\text{Prime}(A) = \bigcup \mathcal{M}$
- (ii) *For each  $\mathfrak{m} \in \mathcal{M}$  every two-element subset of  $\mathfrak{m}$  has no upper bound in  $A$ .*
- (iii) *For each two-element subset  $\{x, y\}$  of  $\text{Prime}(A)$  that has no upper bound in  $A$  there is an  $\mathfrak{m} \in \mathcal{M}$  such that  $\{x, y\} \subseteq \mathfrak{m}$ .*

Then  $A$  is isomorphic to a hyperlimit of a diagram whose object set is  $\{m \cup \{\perp_A\} \mid m \in \mathcal{M}\}$  and whose connecting morphisms are strict continuous functions.

*Proof.* Let  $m \in \mathcal{M}$ . Because the elements of  $m$  are pairwise unbounded,  $m \cup \{\perp_A\}$  is closed under suprema existing in  $A$ . Hence  $q_m(x) = \bigvee \{y \in m \mid y \leq x\}$  defines a projection onto  $m \cup \{\perp_A\}$ . We will show that  $\{q_m \mid m \in \mathcal{M}\}$  satisfies the three conditions from Lemma 3.2:

From the prime-algebraicity of  $A$  and from  $\bigcup \mathcal{M} = \text{Prime}(A)$  it follows that  $id_A$  is pointwise supremum of  $\{q_m \mid m \in \mathcal{M}\}$ . From the fact that all elements of  $m$  are completely prime it follows easily that  $q_m$  preserves suprema for each  $m \in \mathcal{M}$ . Let  $(x_m)_{m \in \mathcal{M}} \in \prod_{m \in \mathcal{M}} im(q_m)$  be a weakly commuting tuple. We want to show that  $\{x_m \mid m \in \mathcal{M}\}$  is bounded. By the coherence of  $A$ , it suffices to show that  $\{x_n, x_m\}$  is bounded for any  $n, m \in \mathcal{M}$ . Assume  $\{x_n, x_m\}$  were unbounded. Then there would be a  $k \in \mathcal{M}$  that contains  $\{x_n, x_m\}$ . But then  $x_n = q_k(x_n) \leq x_k$  and  $x_m = q_k(x_m) \leq x_k$ , contradicting the assumption that  $\{x_n, x_m\}$  is unbounded.

By Lemma 3.2(b), we can now conclude that  $A$  is isomorphic to a hyperlimit of a diagram whose object set is  $\{m \cup \{\perp_A\} \mid m \in \mathcal{M}\}$  and whose connecting morphisms are strict continuous functions. ■

Now, the final theorem:

**THEOREM 3.1.** *A Scott-domain is coherent and prime-algebraic if and only if it is isomorphic to a hyperlimit of a diagram whose objects are two-chains and boolean flat domains and whose connecting morphisms are strict continuous functions.*

*Proof.* The “if” part follows from Corollary 3.1 and the “only if” part from Lemma 3.3 with

$$\mathcal{M} = \{\{x\} \mid x \in \text{Prime}(A)\} \cup \{\{x, y\} \mid x, y \in \text{Prime}(A) \text{ and } \{x, y\} \text{ is unbounded}\}. \quad \blacksquare$$

We also get a version for prime-algebraic lattices:

**THEOREM 3.2.** *A complete lattice is prime-algebraic if and only if it is isomorphic to a hyperlimit of two-chains and strict continuous functions.*

*Proof.* Let  $(A, f)$  be a diagram of two-chains. It is easy to see that the constant  $\top$ -tuple is an element of  $\text{hyp}(A, f)$ . It is the greatest element of  $\text{hyp}(A, f)$ . Together with Proposition 3.1, this shows that  $\text{hyp}(A, f)$  is a complete

lattice. By Corollary 3.1, it is prime-algebraic. The other direction follows from Lemma 3.3 with  $\mathcal{M}$  being the set of all singleton subsets of  $\text{Prime}(A)$ . ■

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