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CONSTRUCTIVE THEORY OF BANACH ALGEBRAS

THIERRY COQUAND AND BAS SPITTERS

ABSTRACT. We present a way to organize a constructive development of the theory of Banach algebras, inspired by works of Cohen, de Bruijn and Bishop [Coh61, dBvdM67, BB85, CS91]. We illustrate this by giving elementary proofs of Wiener’s result on the inverse of Fourier series and Wiener’s Tauberian Theorem, in a sequel to this paper we show how this can be used in a localic, or point-free, description of the spectrum of a Banach algebra.

INTRODUCTION

The applications of the theory of Banach algebras to “concrete” theorems in analysis, such as Wiener’s theorem on the inverse of Fourier series, or Wiener’s Tauberian Theorem [Wie32], constitute a striking example of the power of abstract methods in mathematics. The abstract argument is short and easy to grasp, when compared to Wiener’s explicit constructions. Furthermore, it is highly non constructive and uses Zorn’s Lemma. As such, it is a perfect illustration of Hilbert’s defense of the use of the law of excluded-middle against Brouwer [Hil26]. A natural question is whether the abstract argument does not contain, in some implicit way, an actual construction. This question has been analysed and answered by P. Cohen [Coh61]. Closely related are later works by de Bruijn and van der Meiden [dBvdM67], and by Bishop and Bridges [BB85]. Here we present a slightly different analysis of the non-constructive argument, thus providing an elementary treatment. Furthermore it allows us to reformulate some key results of Bishop and Bridges [BB85] in a localic, or point-free, setting. This will be done in a sequel to the present paper.

This paper is organised as follows. The first section explains informally the main idea of the paper. The next sections, until Section 7, are presented in elementary Bishop style mathematics, in a detailed way so that they can hopefully be readily implemented in type theory using the framework presented in [OS08]. This presentation follows ideas presented in [BS99].

Notation. We use the following conventions, unless otherwise indicated:

A, B for Banach algebras.

a, b, c, \dots for elements of a Banach algebra.

φ, ψ for functionals.

Γ_R denotes the circle with radius R , Γ is the circle with radius 1.

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1. ANALYSIS OF THE ABSTRACT REASONING

Let A be a commutative Banach algebra with unit and let $\text{MFn}(A)$ be the compact space of non-zero multiplicative functionals $A \rightarrow \mathbb{C}$. Let us assume that an element f of A satisfies $\varphi(f) \neq 0$ for all φ in $\text{MFn}(A)$; and we want to show that f is then invertible in A . To obtain a contradiction, we assume that f is *not* invertible. Then the ideal $\langle f \rangle$ is proper and hence, using Zorn's Lemma, included in a maximal ideal \mathfrak{m} . This maximal ideal is necessarily closed. Using the Gelfand-Mazur theorem, the algebra A/\mathfrak{m} is isomorphic to \mathbb{C} and the corresponding multiplicative functional $\varphi_{\mathfrak{m}} : A \rightarrow A/\mathfrak{m}$ is such that $\varphi_{\mathfrak{m}}(f) = 0$. This contradicts the hypothesis on f . For example, the application of this argument to $A = l^1(\mathbb{Z})$ gives Wiener's theorem on inverse of Fourier series.

Cohen [Coh61] observes that, instead of working with a maximal ideal, one can work just as well with the closure I of ideal generated by f . Moreover, an ideal contains 1 if, and only if, its closure contains 1. This follows directly from the fact that x is invertible if $|1 - x| < 1$. Finally, the proof of the Gelfand-Mazur Theorem actually gives a more general result: if A is a non trivial Banach algebra, the spectrum of any element u in A is non empty. A combination of these remarks eliminates the use of Zorn's Lemma.

To simplify, we consider the case where A is generated by one element u : for any a in A and any $r > 0$ there is a polynomial P such that $|a - P(u)| < r$. This covers for instance the example of the disc algebra [Coh61]. In this case the spectrum $\text{MFn}(A)$ can be identified with the spectrum, $\sigma(u)$, of u : the set of complex λ such that $\lambda - u$ is not invertible. Any such element λ defines a multiplicative functional $\varphi(P(u)) := P(\lambda)$ on polynomials in u . If $|P(\lambda)| > |P(u)|$, then $P(\lambda) - P(u)$ is invertible and hence $\lambda - u$ is invertible. Since this is not the case, $|P(\lambda)| \leq |P(u)|$, so φ can be extended to a multiplicative functional $\varphi(g) := g(\lambda)$. Conversely, for φ in $\text{MFn}(A)$, $\varphi(\varphi(u) - u) = \varphi(u) - \varphi(u) = 0$ and hence $\varphi(u) - u$ cannot be invertible, i.e. $\varphi(u) \in \sigma(u)$. The hypothesis on f becomes $f(\lambda) \neq 0$ for all λ in the spectrum of u . The spectrum of u is empty in A/I . Indeed, the closure of the ideal generated by $\lambda - u$ contains $f(\lambda) - f = f(\lambda)$ modulo I which is invertible and hence $\lambda - u$ is always invertible modulo I . Hence 1 belongs to I and hence f is invertible. The maximal ideal of the abstract argument has been replaced by the "big enough" ideal I . Essentially, this is the method followed by de Bruijn and van der Meiden [dBvdM67], who advocated a point-free approach to the description of the spectrum of a Banach algebra.

In this way we eliminate the use of Zorn's Lemma, but the argument is still non-constructive. Cohen shows that we can follow the proof of Gelfand-Mazur and produce an actual computation of the inverse of f . Bishop and Bridges' work is similar [BB85]. Coquand and Stolzenberg [CS91] show that in most cases, we can obtain a relatively short and explicit formula for the inverse.

In this paper we suggest a slightly different constructive argument. It is enough to have a constructive proof of the following result.

Theorem 1. *If we have u in A such that for all λ the element $\lambda - u$ is invertible and furthermore the inverse is uniformly bounded, then $1 = 0$ in A .*

Notice that, constructively, we have to state explicitly that the inverse is uniformly bounded. At first the conclusion seems purely negative, since it says that a ring is trivial, and not informative. But if we apply the Theorem to a quotient

algebra A/I the conclusion shows that 1 is in I . Choosing I to be the closure of $\langle f \rangle$ actually builds an inverse of f , provided we work constructively. Such use of “trivial rings” occurs often in constructive algebra [Ric88, LQ08]. In the applications to Wiener’s Theorems on Fourier series and on the Tauberian Theorem, we believe that our treatment is simpler than the one in [BB85]. Moreover, this theorem has a rather direct proof, for instance, it does not rely on Cauchy’s formula like the treatment in [Coh61, BB85, CS91].

However, one should notice that this constructive reading does not yet have the “elegance” of the classical proofs, since we have to deal explicitly with a bound of the inverse. In the sequel to this paper, we explain how to “hide” explicit mentions to this bound, by using a localic, or pointfree, presentation of the spectrum. In this way we obtain a rather faithful constructive explanation of the classical arguments.

2. INTEGRATION AND DIFFERENTIATION WITH VALUES IN A BANACH SPACE

We consider a vector space E over the real numbers with an upper real valued semi-norm: if a is in E then $|a|$ is an *upper real* (open nonempty upper set of rationals) such that $|ra| = |r||a|$ and $|a + b| \leq |a| + |b|$. This is an important difference with Bishop’s treatment, for whom the norm is a Cauchy real¹.

The equality in E is such that $|a| = 0$ iff $a = 0$ in E . If $f : [a, b] \rightarrow E$ is a (uniformly) continuous map, we say that f is *differentiable* iff there exists a continuous $f' : \mathbb{R} \rightarrow E$ such that for all $\varepsilon > 0$ there exists η such that $|f(y) - f(x) - (y - x)f'(x)| \leq \varepsilon|y - x|$ if $|y - x| \leq \eta$.

Lemma 1. *If $|f'(x)| \leq M$ for all x in $[a, b]$ then $|f(b) - f(a)| \leq M(b - a)$. In particular, if $f'(x) = 0$ for all x in $[a, b]$ then $f(b) = f(a)$.*

Proof. For any $\varepsilon > 0$ we show $|f(b) - f(a)| \leq (M + \varepsilon)(b - a)$, by cutting $[a, b]$ in subintervals $[a_i, a_{i+1}]$ such that $|f(a_{i+1}) - f(a_i) - (a_{i+1} - a_i)f'(a_i)| \leq \varepsilon(a_{i+1} - a_i)$. \square

Let E be a *Banach space*, i.e. a normed vector space which is complete: any Cauchy approximation converges. If $g : [a, b] \rightarrow E$ is uniformly continuous we define $\int_a^b g(x)dx$. If we write $G(y) = \int_a^y g(x)dx$ then $G : [a, b] \rightarrow E$ is differentiable and $G'(y) = g(y)$.

Corollary 1. *If $|g(x)| \leq M$ for all x in $[a, b]$, then $|\int_a^b g(x)dx| \leq M(b - a)$.*

3. EXPONENTIAL FUNCTION

A (commutative) *Banach algebra* A is a Banach space with a ring structure, multiplication ab being such that $|ab| \leq |a||b|$. In this section, we assume also that A has a unit element 1.

Lemma 2. *If $|1 - x| < 1$, then x is invertible with inverse $\sum(1 - x)^n$.*

¹The fact that the norm is not a Cauchy real constructively arises if one wants to quotient a Banach space by a closed subspace which may not be located. (This will indeed occur in the applications.)

Another motivation is the interpretation of these results in Banach algebra bundles which we will discuss in Section 8. In a Banach algebra bundle the function $x \mapsto |a_x|$ is *upper semi continuous*, but generally not continuous; see [KR94].

Bishop and Bridges [BB85, p.462] state: ‘It would be interesting, and probably nontrivial, to extend the theory to cover such algebras [where the norm is not a Cauchy real].’

Proof. We write $y = 1 - x$. We have $x = 1 - y$ and $\sum y^n$ is the inverse of $1 - y$ since $|y| < 1$. \square

We define the exponential

$$e^a = \sum \frac{a^n}{n!}$$

of an element a of A . The map $x \mapsto e^{ax}$ defines a function $e_a : \mathbb{R} \rightarrow A$ such that e_a is differentiable, $e'_a(x) = ae_a(x)$ and $e_a(0) = 1$. Furthermore, we have $e_a(x + y) = e_a(x)e_a(y)$.

Lemma 3. *Let $f : \mathbb{R} \rightarrow A$ be a continuous function such that $f(0) = 1$ and $f(x + y) = f(x)f(y)$. Then f is differentiable and $f = e_a$ with $a = f'(0)$.*

Proof. We have $1/t \int_0^t f(x)dx \rightarrow 1$ if $t \rightarrow 0$. So we can find $t > 0$ such that $|1 - 1/t \int_0^t f(z)dz| < 1$. The element $v = 1/t \int_0^t f(z)dz$ is then invertible by Lemma 2. On the other hand we have, using $f(x + y) = f(x)f(y)$

$$tf(x)v = \int_0^t f(x + z)dz = \int_x^{x+t} f(z)dz$$

and hence

$$tf(x) = v^{-1} \int_x^{x+t} f(z)dz = v^{-1} \left(\int_0^{t+x} f(z)dz - \int_0^x f(z)dz \right).$$

It follows that f is differentiable and

$$tf'(x) = v^{-1}(f(x + t) - f(x)).$$

Since $f(x + y) = f(x)f(y)$, we have $f'(x + y) = f(x)f'(y)$. In particular, if we write $a = f'(0)$ we have $f'(x) = af(x)$. If we write $g(x) = f(x)e_a(-x)$ we have $g(0) = 1$ and $g'(x) = af(x)e_a(-x) - af(x)e_a(-x) = 0$. Hence by Lemma 1 we have $g(x) = 1$ for all x and so $f(x) = e_a(x)$. \square

4. PATH INTEGRATION

Let E be a Banach space over of the complex numbers. We say that $f : \mathbb{C} \rightarrow E$ is *differentiable* iff there exists a (uniformly) continuous function $f' : \mathbb{C} \rightarrow E$ such that for all $\varepsilon > 0$ there exists $\eta > 0$ such that

$$|f(z') - f(z) - (z' - z)f'(z)| \leq \varepsilon|z' - z|$$

whenever $|z' - z| \leq \eta$.

If $\gamma : [0, 1] \rightarrow E$ is a differentiable function and $f : \mathbb{C} \rightarrow E$ is a continuous function, we define $\int_\gamma f(z)dz$ to be $\int_0^1 f(\gamma(t))\gamma'(t)dt$. We say that γ is a *loop* iff $\gamma(0) = \gamma(1)$.

Lemma 4. *If $f : \mathbb{C} \rightarrow E$ is differentiable and $\gamma : [0, 1] \rightarrow \mathbb{C}$ is a loop then $\int_\gamma f(z)dz = 0$*

Proof. We consider $g : [0, 1] \rightarrow E$ defined by

$$g(s) = \int_0^1 f(s\gamma(t))s\gamma'(t)dt.$$

Then $g(1) = \int f$ and $g(0) = 0$. It is direct to see that g is differentiable and that

$$g'(s) = \int_0^1 (f(s\gamma(t)) + s\gamma(t)f'(s\gamma(t)))\gamma'(t)dt = \int_0^1 h'(t)dt = h(1) - h(0) = 0$$

where $h(t) = \gamma(t)f(s\gamma(t))$. By Lemma 1, we have $g(1) = g(0) = 0$, hence the result. \square

5. INVERSE FUNCTION

Let A be a Banach algebra over the complex numbers with a unit element.

Lemma 5. *If a is invertible, with inverse b such that $|b| \leq M$, then $a - u$ is invertible if $|u| \leq c/M$ and $c < 1$. Its inverse is bounded by $M/(1 - c)$.*

Proof. The element $a - u$ is invertible iff the element $1 - ub$ is. Since $|u| < 1/M$ and $|b| < M$, $|ub| \leq c$ and hence $1 - ub$ is invertible by Lemma 2. The inverse is bounded by $(1 - c)^{-1}$. Hence the inverse of $a - u$ is bounded by $M/(1 - c)$. \square

It follows from this lemma that the set of invertible elements A^\times of A is an open subset of A .

Theorem 2. *Let u be in A . If for all z in \mathbb{C} the element $z - u$ is invertible, with inverse $f(z)$, and f is uniformly bounded, then $1 = 0$ in A .*

Notice that classically, the hypothesis that f is bounded is not needed².

Proof. We have $f(z') - f(z) = (z - z')f(z)f(z')$ and hence f is Lipschitz continuous and differentiable with $f'(z) = -f(z)^2$. We consider $\gamma : [0, 1] \rightarrow \mathbb{C}$ to be the circle loop $\Gamma_R(t) = Re^{2\pi it}$ for some $R > |u|$. By Lemma 4 we have

$$\int_{\Gamma_R} f(z)dz = 0.$$

On the other hand, for R big enough $f(z)$ is equal to $\sum u^n/z^{n+1}$ over Γ_R and so $\int_{\Gamma_R} f(z)dz$ is equal to $\int_{\Gamma_R} dz/z = 2\pi i$. So, we have $1 = 0$ in A . \square

Before giving the applications we remark that if I is an ideal of a Banach algebra A , we can define a new Banach algebra A/I by taking the new norm to be: $|a|_I < r$ iff there exists b in I such that $|a - b| < r$. Then $a = 0$ in A/I iff a belongs to the closure of I . It follows from Lemma 2 that $1 = 0$ in A/I iff 1 belongs to I .

We deduce the following method from the previous theorem: To prove that an element g of A is invertible, it is enough to find u in A such that for all z in \mathbb{C} the element $z - u$ is invertible of bounded inverse in $A/\langle g \rangle$.

We emphasize that classically the inverse is always uniformly bounded. Constructively, there is a meta-theorem, the fan rule, that allows us to always find a bound in concrete case; see e.g. [TvD88]. In the examples below we make an effort to be explicit about the bound.

6. APPLICATION 1: WIENER'S THEOREM ON FOURIER SERIES

We present Wiener's theorem on Fourier series; see e.g. [Loo53]. The Banach algebra $B = l^1(\mathbb{Z})$ is the completion of the algebra of sequences in $\mathbb{C}^{\mathbb{Z}}$ of finite support with the convolution product $(a * b)_n = \sum a_i b_{n-i}$ and the norm $|a| = \sum |a_n|$. This algebra has unit δ_0 . We write u for δ_1 . Then $u^{-1} = \delta_{-1}$ and for every a , $a = \sum a_n u^n$ and B is simply an algebra of infinite series under formal multiplication.

²Classically, this theorem implies the Gelfand-Mazur Theorem: a Banach algebra A which is also a field is isomorphic to \mathbb{C} . Indeed, from Theorem 2, for each u in A , there exists λ such that $u - \lambda$ is not invertible and this implies $u = \lambda$.

If λ is in Γ , i.e. if $|\lambda| = 1$, then we define $a(\lambda) = \sum a_n \lambda^n$. We have $|a(\lambda)| \leq |a|$.

Theorem 3. *If f is an element of B such that $|f(\lambda)| \geq \varepsilon$ for all λ in Γ , then f is invertible.*

Proof. We let A be the quotient $B/\langle f \rangle$. We show that $\lambda - u$ is invertible for all λ in \mathbb{C} and furthermore that the inverse is uniformly bounded. We can then apply Theorem 2.

If $|\lambda| < 1$, then $|\lambda u^{-1}| = |\lambda| \leq r$ for some $r < 1$. Hence, $\sum_{n \geq 1} (\lambda/u)^n$ exists and equals $(1 - \lambda u^{-1})^{-1} = u(u - \lambda)^{-1}$. The inverse $(u - \lambda)^{-1}$ is bounded by $|u^{-1}|(1 - r)^{-1} = (1 - r)^{-1}$.

Similarly, if $|\lambda| > 1$, then $|u/\lambda| = |1/\lambda| \leq r$ for some $r < 1$. Hence, $\sum_{n \geq 1} (u/\lambda)^n$ exists and equals $(1 - (u/\lambda))^{-1} = u^{-1}(\lambda - u)^{-1}$. The inverse $(\lambda - u)^{-1}$ is bounded by $|u|(1 - r)^{-1} = (1 - r)^{-1}$.

Let g be an element of finite support such that $|f - g| < \varepsilon'$. Then

$$|(g(\lambda) - g) - (f(\lambda) - f)| = |(g - f)(\lambda) + (f - g)| \leq 2\varepsilon'.$$

Since $f(\lambda) - f$ is equal to $f(\lambda) \bmod f$, $g(\lambda) - g$ is invertible mod f . By Lemma 5 we have a uniform bound M on the inverse of $g(\lambda) - g$ in Γ . Since $g(\lambda) - g$ is a polynomial in u , $\lambda - u$ divides it, say $(\lambda - u)h = g(\lambda) - g$. So, $\lambda - u$ is invertible for all λ in Γ and the inverse is bounded by $M|h|$. We assume that $M|h| \geq 1$. By Lemma 5, $\lambda - u$ is invertible for all λ with $||\lambda| - 1| \leq 1/(2M|h|)$ and its inverse is bounded by $|Mh|/(1 - \frac{1}{2}) = 2|Mh|$.

Write $\alpha := 1/(2M|h|)$. Then either $||\lambda| - 1| \leq \alpha$ or $||\lambda| - 1| \geq \alpha/2$. In the latter case, either $|\lambda| \leq 1 - \alpha/2$ or $|\lambda| \geq 1 + \alpha/2$. We conclude that $\lambda - u$ is invertible for all λ in \mathbb{C} and its inverse is bounded by

$$\sup\{2M|h|, (1 - \frac{\alpha}{2})^{-1}, (1 - (1 + \frac{\alpha}{2})^{-1})^{-1}\}.$$

□

Being constructive, this reasoning can be seen as an algorithm that computes an inverse of f in B .

7. APPLICATION 2: WIENER'S TAUBERIAN THEOREM

Let $C(\mathbb{R})$ be the space of continuous functions of compact support and let $L = L^1(\mathbb{R})$ be the completion of this space for the L^1 norm. If we define $T_x(g)(y) = g(x + y)$, for g in $C(\mathbb{R})$, we have $|T_x(g) - T_x(h)| = |g - h|$ and, by extension, we have a continuous function $x \mapsto T_x(g)$, $\mathbb{R} \rightarrow L^1(\mathbb{R})$ for any g in L such that $|T_x(g)| = |g|$ for all x . For f in $C(\mathbb{R})$ and g in L , we define

$$f * g = \int f(x)T_{-x}(g)dx.$$

It follows that we have $|f * g| \leq |f|_\infty |g|_1$. Hence the product $f * g$ can be defined by extension for f and g both in L . If both f, g are in $C(\mathbb{R})$, so is $f * g$.

Since we have

$$(f * g)(y) = (g * f)(y) = \int f(x)g(y - x)dx$$

for f, g in $C(\mathbb{R})$ this holds also for f, g in L and the algebra L with convolution product is a commutative Banach algebra. (It can be shown that it does not have any unit element.)

If g is a continuous function with compact support we define

$$\hat{g}(p) = \int g(x)e^{-ipx} dx.$$

We then have \hat{g} in $C_0(\mathbb{R})$, that is, \hat{g} is continuous on \mathbb{R} and tends to 0 at infinity. This is first proved for $g \in C^1(\mathbb{R})$ by integration by parts and then for general $g \in C(\mathbb{R})$ by density. Moreover, $|\hat{g}(p)| \leq |g|$ for all p . It follows that we can extend the map $g \mapsto \hat{g}$ and define \hat{g} in $C_0(\mathbb{R})$ for g in L .

We fix an element h of L such that $\hat{h}(p) = 1$ for all p in $[-M, M]$ and for all $\varepsilon > 0$ there exists η such that $|\hat{h}(p)| \leq 1 - \eta$ if $M + \varepsilon \leq |p|$.

Theorem 4. *If $|\hat{f}(p)| > \varepsilon$ for all p in $[-M, M]$, then f divides any element g such that $g * h = g$ in L .*

Proof. We let I be the ideal of all elements $k * h - k$ and B be the Banach algebra L/I . Notice that B has h as unit element. We claim that f is invertible in B .

The map $\alpha : \mathbb{R} \rightarrow B$ defined by $\alpha(x) = T_x(h)$ is continuous and satisfies $\alpha(0) = h$ and $\alpha(x + y) = \alpha(x) * \alpha(y)$. By Lemma 3 there exists an element u in B such that $T_x(h) = e^{ux}$ for all x in \mathbb{R} . In B we have

$$g = g * h = \int g(x)T_{-x}(h)dx = \int g(x)e^{-ux} dx.$$

We show that $u - \lambda$ is invertible mod f and of bounded inverse. We can then apply Theorem 2 and deduce that f is invertible in B .

We will consider three cases: 1. $\lambda = ip$ for $|p| \leq M$, 2. $\lambda = ip$ for $|p| \geq M + \delta$ and $\delta \geq 0$, and 3. $\lambda = r + ip$ and $|r| > \delta > 0$. By continuity and Lemma 5 these cases are sufficient.

1. We claim that $ip - u$ is invertible modulo f if $|p| \leq M$. Indeed, if we take N such that

$$\left| \int_{-N}^N |f(x)| dx - \int |f(x)| dx \right| \leq \frac{\varepsilon}{2(1 + |h|)},$$

then

$$\begin{aligned} \left| \int f(x)(e^{-ipx} - e^{-ux}) dx - \int_{-N}^N f(x)(e^{-ipx} - e^{-ux}) dx \right| &\leq \\ \left| \int_{\mathbb{R} \setminus [-N, N]} f(x)(e^{-ipx} - e^{-ux}) dx \right| &\leq \\ \left| \int_{\mathbb{R} \setminus [-N, N]} f(x) \right| (1 + |h|) dx &\leq \frac{\varepsilon}{2}. \end{aligned}$$

Since,

$$\int f(x)(e^{-ipx} - e^{-ux}) dx = \hat{f}(p) - f.$$

this element is invertible mod f , by hypothesis its inverse is bounded by $1/\varepsilon$. By Lemma 5, $\int_{-N}^N f(x)(e^{-ipx} - e^{-ux}) dx$ is invertible mod f . Let g be its inverse. Then $|g| \leq 1/\varepsilon(1 - \frac{1}{2}) = 2/\varepsilon$. Since this integral is divisible by $ip - u$, it remains to find an explicit bound for the inverse. We have $e^{-ipx} - e^{-ux} = e^{-ipx}(1 - e^{(ip-u)x})$ and for all r , $r|(1 - e^{rx})$ and $|\frac{1 - e^{rx}}{r}| \leq 1 + e^{|r|x}$, as a simple power series argument

shows. Consequently, the inverse of $ip - u$ is bounded by $\frac{2}{\epsilon} \int_{-N}^N |f(x)|(1 + e^{rx})dx$, where $r > |ip - u|$.

2. We claim that $ip - u$ is invertible with bounded inverse for all p such that $|p| \geq M + \delta$. Indeed, if we take $\eta > 0$ such that $|\hat{h}(p)| \leq 1 - \eta$ and N such that

$$\left| \int_{-N}^N |h(x)|dx - \int |h(x)|dx \right| \leq \eta/2(1 + |h|).$$

then

$$\begin{aligned} \left| \int_{-N}^N h(x)(e^{-ipx} - e^{-ux})dx - \int h(x)(e^{-ipx} - e^{-ux})dx \right| &\leq \\ \left| \int_{\mathbb{R} \setminus [-N, N]} h(x)(e^{-ipx} - e^{-ux})dx \right| &\leq \\ \left| \int_{\mathbb{R} \setminus [-N, N]} h(x) \right| (1 + |h|)dx &\leq \eta/2 \end{aligned}$$

and

$$\left| \int h(x)(e^{-ipx} - e^{-ux})dx \right| = \hat{h}(p) - h$$

the latter term is invertible since h is the unit of B . Hence $\int_{-N}^N h(x)(e^{-ipx} - e^{-ux})dx$ is invertible with inverse bounded by $2/\eta$. As before, this element is divisible by $ip - u$ and its inverse is bounded by $\frac{2}{\eta} \int_{-N}^N |h(x)|(1 + e^{rx})dx$, where $r > |ip - u|$.

We conclude that $ip - u$ is invertible for all p in \mathbb{R} and its inverse is bounded by $d := \frac{2}{\epsilon} \int (|h(x)| + |f(x)|(1 + e^{rx}))dx$, where $r > |ip - u|$. Using Lemma 5 it follows that we can find $\delta > 0$ such that $r + ip - u$ is invertible for all p in \mathbb{R} and r such that $|r| < 1/2d$, the inverse is bounded by $2d$.

3. For $\lambda = r + ip$, $\lambda - u$ divides

$$(1) \quad 1 - e^{(\lambda - u)x} = 1 - e^{\lambda x} T_x(h) = (1 - e^{-rx} e^{-ipx} T_x(h))$$

and $|T_x(h)|$ is bounded by $|h|$. For $x = \frac{\log(2|h|)}{r}$, $|e^{-rx} e^{-ipx} T_x(h)| \leq e^{-rx} |h| = \frac{1}{2}$, so the right hand side of equation 1 is invertible with inverse bounded by $2e^{-rx} = 2/(2|h|) = 1/|h|$. Hence $\lambda - u$ has an inverse with bound $1/|h|$ for $|r| > \epsilon$.

We have finished the proof that for each λ , $u - \lambda$ is invertible mod f and of bounded inverse.

It follows that we have k such that $f * k = h \bmod I$. If we assume $g * h = g$ we have $g * I = 0$ and then $f * k * g = g$ so that f divides g . \square

Proposition 1. [Kat04, VI.1.13] *If g in $L^1(\mathbb{R})$ is such that \hat{g} has compact support then there exists h in $L^1(\mathbb{R})$ such that $h * g = g$. In fact, we can use a De la Vallée Poussin's kernel for h .*

We deduce the following version of Wiener's Tauberian Theorem.

Corollary 2. *If f in L is such that \hat{f} never takes the value 0, then every function g such that \hat{g} is of compact support is in the ideal generated by f . Consequently, this ideal is dense in L .*

Proof. Any function in $L^1(\mathbb{R})$ is the limit of functions in $L^1(\mathbb{R})$ having their Fourier transform of compact support [Kat04, VI.1.12]:

Define the Fejér kernel $K_\lambda(x) = \lambda K(\lambda x)$, where $K(x) = \frac{1}{2\pi} \left(\frac{\sin x/2}{x/2}\right)^2$.

By the inversion formula:

$$f(x) = \frac{1}{2\pi} \int \hat{f}(t) e^{itx} dt,$$

$\hat{K}_\lambda(t) = \max(1 - |t|/\lambda, 0)$ and one can derive that $(\widehat{K_\lambda * f})(t) = (1 - |t|/\lambda)\hat{f}(t)$ for $|t| \leq \lambda$ and 0 otherwise. \square

Since $f * g = \int g(x) T_{-x}(f) dx$ another way to state this Corollary is the following one.

Corollary 3. *If f in L is such \hat{f} never takes the value 0 then the vector space generated by the functions $T_x(f)$ is dense in L .*

Constructively, the hypothesis that \hat{f} never takes the value 0 should be read as: \hat{f} is bounded away from 0 on any compact.

8. CONCLUSIONS AND FUTURE WORK

To have a point-free description of the spectrum of a Banach algebra was the goal of the work of de Bruijn and van der Meiden [dBvdM67]. The complexity of this description and the process of finding the proof of the compactness of the spectrum is cited by de Bruijn as one inspiration for his AUTOMATH project [dBvdM67, dB94].

It would be interesting to have an actual implementation of our work, following already existing work formalising basic analysis [OS08].

The work of Krivine [Kri64] contains several examples similar to Wiener's results, but in a 'real' framework. One considers respectively $l^1(\mathbb{N})$ instead of $l^1(\mathbb{Z})$ and $L^1(\mathbb{R}^+)$ instead of $L^1(\mathbb{R})$. It might be interesting to give a constructive interpretation these results using the technique presented in [Coq05, CS05].

Constructive, and choice-free, results on Banach spaces can be interpreted as results on Banach sheaves, or equivalently, Banach bundles [Mul80]. Similarly, constructive results on Banach algebras can be interpreted as results on Banach algebra bundles [KR94].

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THIERRY COQUAND, COMPUTING SCIENCE DEPARTMENT, GÖTEBORG UNIVERSITY

BAS SPITTERS, COMPUTER SCIENCE DEPARTMENT, RADBOD UNIVERSITY NIJMEGEN