Weak coupling limits in a stochastic model of heat conduction

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Abstract

We study the Brownian momentum process, a model of heat conduction, weakly coupled to heat baths. In two different settings of weak coupling to the heat baths, we study the non-equilibrium steady state and its proximity to the local equilibrium measure in terms of the strength of coupling. For three and four site systems, we obtain the two-point correlation function and show it is generically not multilinear.

Keywords: weak coupling limit, local equilibrium, Brownian momentum process, inclusion process, duality.

1 Introduction

In the study of non-equilibrium systems, exactly solvable models can serve as test-cases with which general statements about non-equilibrium, such as in [3], [11] can be tested. Recently, in [6], [7], [8], we studied the Brownian momentum process (BMP) and showed that this models is exactly solvable via duality with a particle system, the symmetric inclusion process. In this paper, we look at the close-to-equilibrium states of the BMP. First, we consider a close-to-equilibrium scenario where the temperature of the right heat bath is close to the temperature of the left heat bath, and show that the distance between the local equilibrium measure and the true non-equilibrium steady state is of order at most the square of the temperature difference, in agreement with the theory of Mc Lennan ensembles, see [11]. Next, we consider a situation where the linear chain is coupled weakly to heat baths to left and right ends (with fixed and different temperatures), and study which equilibrium measure is selected in the limit where the coupling strength λ tends to zero, as well as how far the true non-equilibrium steady state is from the local equilibrium measure for small coupling strengths. The temperature profile can be computed for all values of λ and is only linear in the chain including the extra sites associated to the heat baths for $\lambda = 1$, and linear if these sites are not included for all values of $\lambda > 0$. Finally, we explicitly compute the two-point correlation for all $\lambda > 0$ for a three and four sites system and show that the multilinear ansatz of the two-point function introduced in [6], see also [3], [4] fails for a system of four sites, except when $\lambda = 1$.

2 The model

The Brownian momentum process on a linear chain $\{1, \ldots, N\}$ coupled at the left and right end to a heat bath is a Markov process $\{x(t) : t \geq 0\}$ on the state space $\Omega_N = \mathbb{R}^{\{1,\ldots,N\}}$. The configuration $x(t) = x_i(t) : i \in \{1, \ldots N\}$ is interpreted as momenta associated to the sites $i \in \{1, \ldots, N\}$. The process is defined via its generator working on the core of smooth functions $f : \Omega_N \to \mathbb{R}$ which is given by

$$L = \lambda B_1 + \lambda B_N + \sum_{i,j}^{N} p(i,j) L_{i,j} \tag{1}$$

with

$$L_{i,j} = (x_i \partial_j - x_j \partial_i)^2$$

and where ∂_j is shorthand for $\frac{\partial}{\partial x_j}$. The underlying random walk transition rate $p(i,j)$ is chosen to be symmetric and nearest neighbor, i.e., $p_{i,i+1} = p_{i+1,i} = 1, i \in \{1, \ldots, N-1\}$, $p(i,j) = 0$ otherwise. Since $L_{i,j} = L_{j,i}$ the symmetry of $p(i,j)$ is no loss of generality.

The boundary operators B_1, B_N model the contact with the heat baths, and are chosen to be Ornstein-Uhlenbeck generators corresponding to the temperatures of the left and right heat bath, i.e.,

$$B_1 = T_L \partial_1^2 - x_1 \partial_1$$

$$B_N = T_R \partial_N^2 - x_N \partial_N$$

Finally, $\lambda > 0$ measures the strength of the coupling to the heat baths. The process with generator (1) is abbreviated as BMP_λ.

If $T_L = T_R = T$, then, for all $\lambda > 0$, the unique stationary measure of the process $\{x(t) : t \geq 0\}$ is the product of Gaussian measures with mean zero and variance T. If $T_L \neq T_R$ there exists a unique stationary measure; the so-called non-equilibrium steady state denoted by $\mu^\lambda_{T_L,T_R}$. The existence and uniqueness of the measure $\mu^\lambda_{T_L,T_R}$ follows from duality (see next section).

We will look at two different close-to-equilibrium scenarios:

1. $\lambda = 1$, $T_R = T_L + \epsilon$ and $\epsilon \to 0$,
2. $T_L \neq T_R$, and $\lambda \to 0$.

In both cases we look at the behavior of the measure $\mu^\lambda_{T_L,T_R}$, in case two, as $\lambda \to 0$, and in case one as $\epsilon \to 0$. Since for $\lambda = 0$, the system has infinitely many equilibrium measures, in the second case it is of interest to find out which of these measure is selected in the limit $\lambda \to 0$. Both in the first and second case, we want to understand how close the true non-equilibrium steady state is to the local equilibrium measure.

3 Duality

The BMP_λ can be analyzed via duality. The dual process is an interacting particle system, the so-called symmetric inclusion process [8], where particles are jumping on the lattice $\{0, 1, \ldots, N, N+1\}$ and interacting by "inclusion" (i.e., particles at site i can attract particles at site j). The "extra sites" $0, N+1$ -associated to the heat baths- are absorbing. I.e., a dual particle configuration is a map

$$\xi : \{0, \ldots, N+1\} \to \mathbb{N}$$

specifying at each site the number of particles present at that site. The space of dual particle configurations is denoted by Ω^d_N For $\xi \in \Omega^d_N$, $\xi^{i,j}$ denotes the configuration obtained from ξ by removing a particle from i and putting it at j.

The generator of the dual process then reads

$$\begin{aligned} & L_d\phi(\xi) = 2\lambda\xi_1[\phi(\xi^{1,0}) - \phi(\xi)] + \\ + & \sum_{i,j=1}^{N-1} p(i,j) \left(2\xi_j(2\xi_i+1)[\phi(\xi^{j,i}) - \phi(\xi)] + 2\xi_i(2\xi_j+1)[\phi(\xi^{i,j}) - \phi(\xi)]\right) \\ & +2\lambda\xi_N[\phi(\xi^{N,N+1}) - \phi(\xi)] \end{aligned} \tag{2}$$

In words, this means particles at site i jump to j at rate $2p(i,j)(2\xi_j + 1)$. At the boundary site 1 (resp. N) particles can jump at rate 2λ to the site 0 (resp. $N+1$) where they are absorbed. Absorbed particles do not interact with non-absorbed ones. The dual process is abbreviated as SIP_λ. The duality functions for duality between BMP_λ and SIP_λ are independent of λ and given by

$$D(\xi, x) = T_L^{\xi_0} T_R^{\xi_{N+1}} \prod_{i=1}^{N} \frac{x_i^{2\xi_i}}{(2\xi_i - 1)!!}$$

for $\xi \in \Omega_N^d$ a dual particle configuration, and $x \in \Omega_N$.

The duality relation then reads

$$LD(\xi, x) = L_d D(\xi, x) \tag{3}$$

where L works on x and L_d on ξ. By passing to the semigroup, from (3) we obtain the duality relation

$$\mathbb{E}_x D(\xi, x(t)) = \mathbb{E}_\xi^d D(\xi(t), x) \tag{4}$$

where $\mathbb{E}_x$ is expectation in BMP_λ starting from $x \in \Omega_N$, and $\mathbb{E}_\xi^d$ is expectation in SIP_λ starting from $\xi \in \Omega_N^d$.

For $\xi \in \Omega_N^d$ we denote $|\xi| = \sum_{i=0}^{N+1} \xi_i$ the total number of particles in ξ. Since eventually all particles in a particle configuration $\xi \in \Omega_N^d$ will be absorbed, we have a unique stationary distribution μ_{T_L,T_R}^λ with

$$\int D(\xi, x)\mu_{T_L,T_R}^\lambda(dx) = \sum_{k,l:k+l=|\xi|} T_L^k T_R^l \mathbb{P}_\xi^d \left(\xi(t=\infty) = k\delta_0 + l\delta_{N+1}\right) \tag{5}$$

where $\xi(t = \infty)$ denotes the final configuration when all particles are absorbed and $k\delta_0 + l\delta_{N+1}$ the configuration with k particles at 0 and l particles at $N + 1$.

4 Temperature profile

The local temperature at site $i \in \{1, \ldots, N\}$ is defined as

$$T_i = \int x_i^2 \mu_{T_L,T_R}^\lambda(dx)$$

and by definition $T_0 = T_L, T_{N+1} = T_R$. We say that the temperature profile is linear in the lattice interval $[K, L]$ if there exist $a, b \in \mathbb{R}$ with $T_i = ai + b$, for all $i \in [K, L]$. For the computation of the temperature profile we only need a single dual walker, which performs a continuous-time random walk with rates $2p(i, j)$ and absorption at rate 2λ from the sites $1, N$.

Indeed, using (5) we have

$$T_i = T_L \mathbb{P}_{\delta_i}^d \left(\xi(\infty) = \delta_0\right) + T_R \left(1 - \mathbb{P}_{\delta_i}^d \left(\xi(\infty) = \delta_0\right)\right) \tag{6}$$

From this expression, one obtains the following equations for the temperature profile:

$$\begin{aligned}
\sum_{i=1}^{N} p(i,1)T_i &= T_1 - \lambda(T_L - T_1) \\
\sum_{i=1}^{N} p(i,k)T_i &= T_k \\
\sum_{i=1}^{N} p(i,N)T_i &= T_N - \lambda(T_R - T_N)
\end{aligned} \tag{7}$$

The second equation expresses that the temperature profile is a harmonic function of the transition probabilities, whereas the first and third equation are boundary conditions. In the case $\lambda = 1$ and p corresponding to the simple nearest neighbor random walk, the equation for $T_i, i = 0, \dots, N$ is the discrete Laplace equation, which gives a linear temperature profile in $[0, N+1]$.

Remark 4.1. *In this paper we restrict to the symmetric nearest neighbor walk kernel $p(i,j)$. The equations (7) hold for general symmetric $p(i,j)$. However, in the cases where it is not translation-invariant and/or not nearest neighbor, the temperature profile will not be linear.*

We have the following theorem that follows immediately from the equations (7).

Theorem 4.1. *For all $\lambda > 0$, the temperature profile is linear in $[1, N]$ and is given by*

$$T_i = ai + b \tag{8}$$

$i = 1, \dots, N$ with

$$a = \frac{\lambda(T_R - T_L)}{\lambda(N-1)+2}$$

$$b = \frac{T_L + T_R + \lambda(NT_L - T_R)}{\lambda(N-1)+2}$$

We can now look at different limiting cases:

1. In the case $\lambda = 1$ we recover the result from [6]:

   $$T = T_L + \frac{T_R - T_L}{N+1} i$$

   In this case (only) the temperature profile is linear in $[0, N+1]$.

2. In the limit $\lambda \to 0$ we obtain for all $i \in \{1, \dots, N\}$

   $$\lim_{\lambda \to 0} T_i^{(\lambda)} = \frac{T_L + T_R}{2}$$

3. In the limit $\lambda \to \infty$ we obtain $T_1 = T_L, T_N = T_R$ and the profile is linear in $[1, N]$, similar to a system with $\lambda = 1$ and $N - 2$ sites.

4. In the limit $N \to \infty$, such that $i/N \to r \in [0, 1]$ fixed,

$$\lim_{N\to\infty, \frac{i}{N}\to r} T_i = T_L + r(T_R - T_L)$$

   This means that the macroscopic profile is linear and does not depend on λ.

Remark 4.2. *The expectation of the heat current in the steady state in the system is $J = T_{i+1} - T_i$. Heat conductivity κ is defined via the equation $J = \kappa \Delta T$. From Theorem 4.1 it follows that $\kappa = \frac{\lambda}{\lambda(N-1)+2}$ which is independent of the temperature (i.e. the system obeys the Fourier's law for all values of $\lambda > 0$).*

5 The stationary measure for $\epsilon \to 0$

We consider the first weak coupling setting, i.e, $\lambda = 1$, $T_R = T_L + \epsilon$. We will prove that up to corrections of order ϵ^2, the stationary measure is given by a product of Gaussian measures corresponding to the temperature profile, i.e., the local equilibrium measure.

Let us denote this local equilibrium measure

$$\nu_{T_L,T_R} = \otimes_{i=1}^{N} G_{T_i}(x_i)dx_i$$

with T_i given by (8),

$$G_T(x) = \frac{1}{\sqrt{2\pi T}} \exp(-x^2/2T)$$

and $\mu_{T_L,T_L+\epsilon}$ the true non-equilibrium steady state (with $\lambda = 1$). Then we have the following result.

Theorem 5.1. *The true equilibrium measure and the local equilibrium measure are at most order ϵ^2 apart, i.e., there exists $\epsilon_0 > 0$ such that for all $\xi \in \Omega_N^d$ there exists a constant $C = C(\xi) < \infty$ such that for all $0 \le \epsilon \le \epsilon_0$ we have*

$$\left| \int D(\xi, x)\mu_{T_L,T_L+\epsilon}(dx) - \int D(\xi, x)\nu_{T_L,T_L+\epsilon}(dx) \right| \le C(\xi)\epsilon^2 \tag{9}$$

Proof. For the local equilibrium measure we have

$$\int D(\xi, x)\nu_{T_L,T_L+\epsilon}(dx) = \prod_{i=1}^{N} T_i^{\xi_i} \tag{10}$$

expanding this up to order ϵ we find,

$$\prod_i T_i^{\xi_i} = \prod_i \left(T_L + \frac{\epsilon i}{N+1}\right)^{\xi_i} = T_L^{|\xi|}\left(1 + \frac{\epsilon}{T_L(N+1)}\sum_i i\xi_i\right) + O(\epsilon^2)$$

Start now from (5) and expand up to order ϵ:

$$\begin{aligned} & \int D(\xi,x)\mu_{T_L T_L+\epsilon}(dx) \\ = \; & T_L^{|\xi|}\left(1 + \frac{\epsilon}{T_L}\sum_{k,l:k+l=|\xi|} l\mathbb{P}_\xi^d(\xi(\infty) = k\delta_0 + l\delta_{N+1})\right) + O(\epsilon^2) \quad (11)\end{aligned}$$

Upon identification of (10) and (11) we see that we have to prove

$$\begin{aligned}\sum_{k,l:k+l=|\xi|} l\mathbb{P}_\xi^d(\xi(\infty) = k\delta_0 + l\delta_{N+1}) &= \mathbb{E}_\xi(\xi_\infty(N+1)) \\ &= \frac{1}{(N+1)}\sum_{i=0}^{N+1} i\xi_i =: \psi(\xi) \quad (12)\end{aligned}$$

The function $\phi(\xi) := \mathbb{E}_\xi(\xi_\infty(N+1))$ is the harmonic function for the dual process, i.e.,

$$L_d\phi = 0$$

which satisfies the boundary conditions

$$\phi\left(k\delta_0 + \sum_{i=1}^N \xi_i\delta_i + l\delta_{N+1}\right) = \phi\left(\sum_{i=1}^N \xi_i\delta_i\right) + l \qquad (13)$$

Therefore, it suffices to show that

$$\frac{1}{(N+1)}\sum_{i=0}^{N+1} i\xi_i =: \psi(\xi)$$

both satisfies

$$L_d\psi = 0$$

and the boundary conditions (13). That ψ satisfies the boundary conditions is immediately clear. The fact that ψ is harmonic follows from explicit computation:

$$\begin{aligned} L_d\psi(\xi) &= 2\xi_1[\psi(\xi^{1,0}) - \psi(\xi)] \\ &+ \sum_{i=1}^{N-1} \big(2\xi_{i+1}(2\xi_i+1)[\psi(\xi^{i+1,i}) - \psi(\xi)] + 2\xi_i(2\xi_{i+1}+1)[\psi(\xi^{i,i+1}) - \psi(\xi)]\big) \\ &+ 2\xi_N[\psi(\xi^{N,N+1}) - \psi(\xi)] \\ &= \frac{1}{N+1}\Big(2\xi_1[-1] + \\ &+ \sum_{i=1}^{N-1} (2\xi_{i+1}(2\xi_i+1)[-1] + 2\xi_i(2\xi_{i+1}+1)[+1]) \\ &+ 2\xi_N[+1]\Big) \\ &= \frac{1}{N+1}\left(2\xi_1[-1] + +2\sum_{i=1}^{N-1}(\xi_i - \xi_{i+1}) + 2\xi_N[+1]\right) \end{aligned}$$

and since $\sum_{i=1}^{N-1}(\xi_i - \xi_{i+1}) = \xi_1 - \xi_N$ we indeed have

$$L_d\psi(\xi) = 0$$

□

6 The case $\lambda \to 0$

Next, we consider the second weak coupling setting, i.e., we fix $T_L \neq T_R$ and study the behavior of the measure $\mu^{\lambda}_{T_L,T_R}$ as a function of λ.

In this case, the local equilibrium measure is the product of Gaussian measures corresponding to the temperature profile (8), i.e., we have to compare $\mu^{\lambda}_{T_L,T_R}$ with $\nu^{\lambda}_{T_L,T_R}$ where

$$\nu^{\lambda}_{T_L,T_R} = \otimes_{i=1}^{N} G_{T_i^{\lambda}}(x_i)(dx_i)$$

where T_i^{λ} is given by (8). Denote

$$\phi(\xi) = \int D(\xi,x)\ \mu^{\lambda}_{T_L,T_R}(dx) \tag{14}$$

then ϕ is the harmonic function of the dual generator satisfying the boundary conditions

$$\phi(\xi^* = \xi + k\delta_0 + l\delta_{N+1}) = \psi(\xi).\psi(k\delta_0 + l\delta_{N+1}) = T_L^k T_R^l \psi(\xi)$$

On the other hand if we put

$$\psi(\xi) := \int D(\xi,x)\ \nu^{\lambda}_{T_L,T_R}(dx) = T_L^k T_R^l \prod_i (T_i^{(\lambda)})^{\xi_i} \tag{15}$$

then we see immediately that ψ satisfies the boundary conditions.

We will now first prove

Lemma 6.1. *There exists $\lambda_0 > 0$ such that for all $\xi \in \Omega_N^d$ there exists $A(\xi) > 0$ such that for all $0 < \lambda \leq \lambda_0$ we have*

$$|\left(L_d\psi\right)(\xi)| \leq \lambda^2 A(\xi)$$

In particular, since there is only a finite number of dual particle configurations with total number of particles equal to K, we have, for all $0 < \lambda \leq \lambda_0$

$$\sup_{\xi:|\xi|=K} |\left(L_d\psi\right)(\xi)| \leq C(K)\lambda^2$$

for some $C(K) > 0$

Proof. Compute

$$\begin{aligned} L_d\psi(\xi) &= 2\psi(\xi)\left(\lambda\xi_1\left(\frac{T_L}{T_1}-1\right)+\lambda\xi_N\left(\frac{T_R}{T_N}-1\right)\right) \\ &+ 2\psi(\xi)\left(\sum_{i=1}^{N-1}\left(\xi_{i+1}(2\xi_i+1)\left(\frac{T_i}{T_{i+1}}-1\right)+\xi_i(2\xi_{i+1}+1)\left(\frac{T_{i+1}}{T_i}-1\right)\right)\right) \end{aligned}$$

Put $T_R - T_N = T_1 - T_L =: \gamma$

$$\begin{aligned} L_d\psi(\xi) &= 2\psi(\xi)\left(\lambda\xi_1\left(\frac{-\gamma}{T_1}\right)+\lambda\xi_N\left(\frac{\gamma}{T_N}\right)\right) \\ &+ 2\psi(\xi)\left(\sum_{i=1}^{N-1}\left(2\xi_{i+1}\xi_i\frac{(T_i-T_{i+1})^2}{T_iT_{i+1}}+(T_i-T_{i+1})\left(\frac{\xi_{i+1}}{T_{i+1}}-\frac{\xi_i}{T_i}\right)\right)\right) \end{aligned} \quad (16)$$

Remember from Theorem 4.1 that $T_i = \lambda ai + b$, hence $T_i - T_{i+1} = -\lambda\alpha$, with

$$\lambda\alpha = \frac{\lambda(T_R - T_L)}{\lambda(N-1)+2}$$

$$b = \frac{T_L + T_R + \lambda(NT_L - T_R)}{\lambda(N-1)+2}$$

We find

$$\gamma = \frac{T_R - T_L}{\lambda(N-1)+2} = \alpha$$

and hence, from (16)

$$L_d\psi(\xi) = 2\psi(\xi)\left(\lambda\xi_1[\frac{-\alpha}{T_1}]+\lambda\xi_N[\frac{\alpha}{T_N}]+\sum_{i=1}^{N-1}\left(2\lambda^2\alpha^2\frac{\xi_i\xi_{i+1}}{T_iT_{i+1}}-\lambda\alpha\left(\frac{\xi_{i+1}}{T_{i+1}}-\frac{\xi_i}{T_i}\right)\right)\right)$$

We then see that the first order terms form a vanishing telescopic sum:

$$\sum_{i=1}^{N-1}\left(\frac{\xi_i}{T_i}-\frac{\xi_{i+1}}{T_{i+1}}\right)=\frac{\xi_1}{T_1}-\frac{\xi_N}{T_N}$$

and therefore;

$$L_d\psi(\xi)=4\lambda^2a^2\psi(\xi)\sum_{i=1}^{N-1}\left(\frac{\xi_{i+1}\xi_i}{T_iT_{i+1}}\right)$$

□

Given this result, we will prove that the measures $\nu^\lambda_{T_L,T_R}$ and $\mu^\lambda_{T_L,T_R}$ are at most order $O(\lambda\log(1/\lambda))$ apart as $\lambda\to 0$.

Theorem 6.1. *Let ϕ,ψ be the functions defined in* (14) *and* (15)*, then we have the following. There exists $\lambda_0>0$, such that for all $\xi\in\Omega_N^d$ there is $C(\xi)>0$, such that for all $0<\lambda\le\lambda_0$*

$$|\phi(\xi)-\psi(\xi)|\le C(\xi)\lambda\log\frac{1}{\lambda} \tag{17}$$

as a consequence,

$$\lim_{\lambda\to 0}\mu^\lambda_{T_L,T_R}=\otimes_{i=1}^N G_{\frac{T_L+T_R}{2}}(x_i)\,dx_i$$

i.e., in the limit $\lambda\to 0$, the equibrium measure corresponding to temperature $(T_L+T_R)/2$ is selected.

Proof. We start with the following lemma

Lemma 6.2. *For all $\xi\in\Omega_N^d$ a (dual) particle configuration, there exists $c=c(\xi)>0$, $a=a(\xi)>0$ such that for all $\lambda>0$, and for all $t>0$*

$$\left|\int\mathbb{E}_xD(\xi,x_t)\;\nu^\lambda_{T_L,T_R}(dx)-\int D(\xi,x)\;\mu^\lambda_{T_L,T_R}(dx)\right|\le ce^{-\lambda at}$$

Proof. Using duality between BMP_λ and SIP_λ, and (5)

$$\begin{aligned}
&\left|\int\mathbb{E}_xD(\xi,x_t)\;\nu^\lambda_{T_L,T_R}(dx)-\int D(\xi,x)\;\mu^\lambda_{T_L,T_R}(dx)\right|\\
=&\left|\mathbb{E}_\xi\left(\prod_{i=1}^N(T_i^{(\lambda)})^{\xi_i(t)}T_L^{\xi_0(t)}T_R^{\xi_{N+1}(t)}\right)-\mathbb{E}_\xi\left(T_L^{\xi_0(\infty)}T_R^{\xi_{N+1}(\infty)}\right)\right|\\
\le\;& C(\xi)\mathbb{P}^d_\xi(\xi(t)\neq\xi(\infty))\\
\le\;& C(\xi)\mathbb{P}^d_\xi(\text{ there exist particles that are not absorbed at time t})\\
\le\;& C(\xi)e^{-a\lambda t}
\end{aligned}$$

In order to see the last inequality, we remark that for a particle at positions $1, N$, the probability to be absorbed at the next step is of order λ, as the maximal rate to move to the other (non-absorbing) neighbor is at most $2(|\xi| + 1)$. $\square$

Proof of Theorem 6.1: using Lemma 6.1, and duality between SIP_λ and BMP_λ, we have

$$\begin{aligned} \left| \int \mathbb{E}_x D(\xi, x_t) \ \nu^\lambda_{T_L,T_R}(dx) - \int D(\xi, x) \ \nu^\lambda_{T_L,T_R}(dx) \right| &= \left| \int_0^t L_d \psi(\xi_s) ds \right| \\ &\leq C(|\xi|) \lambda^2 t \end{aligned}$$

Combining with Lemma 6.2 we have

$$\int D(\xi, x_t) \ \nu^\lambda_{T_L,T_R}(dx) - \int D(\xi, x) \ \mu^\lambda_{T_L,T_R}(dx) \leq C(\xi) \left(\lambda^2 t + e^{-a\lambda t} \right) \quad (18)$$

Now optimize w.r.t. t by choosing $t = (1/a\lambda) \log(a/\lambda)$ $\square$

7 The two point correlation functions in the limit $\lambda \to 0$

In this section we prove that for the two-point correlation function in the non-equilibrium steady state, the deviation from local equilibrium is of order λ, which strengthens (17) for $\xi = \delta_i + \delta_j$ (i.e., we get rid of the $\log(1/\lambda)$-factor). In the appendix we give explicit expressions for the two-point function of some finite systems, and show in particular that it is not multilinear for $\lambda \neq 1$.

Define for $i, j \in \{1, \ldots, N\}$

$$\mathbf{Y}_{ij} = \int (x_i^2 x_j^2) \mu^{(\lambda)}_{T_L T_R}(dx)$$

and additionally $\mathbf{Y}_{0i} = T_L T_i$, $\mathbf{Y}_{i,N+1} = T_i T_R$.

Denote by $\mathbf{T}$ the matrix with elements $\mathbf{T}_{ij} = T_i T_j$ if $i \neq j$ and $\mathbf{T}_{ij} = 3T_i^2$ if $i = j$ where T_i is the temperature profile of Theorem 4.1

Theorem 7.1. *There exists $C > 0$ such that for all $i, j \in \{1, \ldots, N\}$ we have*

$$|\mathbf{Y}_{ij} - \mathbf{T}_{ij}| \leq C\lambda \quad (19)$$

Proof. From the stationarity of $\mu^{\lambda}_{T_L,T_R}$ we find that $\mathbf{Y}$ satisfies the following system of linear equations for $k,l \in \{1,\ldots,N\}$

$$\begin{aligned} 0 &= (-4Y_{kl} + Y_{k-1l} + Y_{k+1l} + Y_{kl-1} + Y_{kl+1}) \\ &+ 4Y_{kk+1}\delta_{kl} + 4Y_{k-1k}\delta_{kl} - 4Y_{k-1k}\delta_{k,l+1} - 4Y_{kk+1}\delta_{k,l-1} \\ &+ \lambda(T_L T_l - Y_{1l})\delta_{1k} + \lambda(T_L T_k - Y_{1k})\delta_{1l} \\ &+ \lambda(T_R T_l - Y_{Nl})\delta_{Nk} + \lambda(T_R T_k - Y_{Nk})\delta_{Nl} \end{aligned} \tag{20}$$

which has the form

$$\mathbf{M}.\mathbf{Y} = \mathbf{D}$$

By explicit computation we obtain

$$\mathbf{X} := \mathbf{M}.\mathbf{T} - \mathbf{D} = O(\lambda^2) \tag{21}$$

From this we will now derive that

$$\mathbf{Y} = \mathbf{T} + O(\lambda). \tag{22}$$

Put

$$\|\mathbf{Y} - \mathbf{T}\| = \|\mathbf{M^{-1}}\mathbf{M}(\mathbf{Y} - \mathbf{T})\| = \|\mathbf{M^{-1}}\mathbf{X}\|$$

We will show that

$$\|\mathbf{M^{-1}}\mathbf{X}\|^2 \leq \frac{c}{\lambda^2}\|\mathbf{X}\|^2 \tag{23}$$

which combined with (21) gives the desired result (22).

To obtain (23) consider

$$\begin{aligned} <\mathbf{M}^{-1}\mathbf{X}, \mathbf{M}^{-1}\mathbf{X}> &= <\mathbf{X}, (\mathbf{M}^{-1})^T\mathbf{M}^{-1}\mathbf{X}> \\ &= <\mathbf{X}, \mathbf{A}^{-1}\mathbf{X}> \end{aligned}$$

with $\mathbf{A} := \mathbf{M}\mathbf{M}^T$ Using the spectral decomposition of $\mathbf{A}$, we get

$$\begin{aligned} <\mathbf{X}, \mathbf{A}^{-1}\mathbf{X}> &= \sum_i \frac{1}{\lambda_i^{(\mathbf{A})}} <\mathbf{X}, \mathbf{e}_i><\mathbf{e}_i, \mathbf{X}> \\ &\leq \frac{1}{\min_i(\lambda_i^{(\mathbf{A})})}||\mathbf{X}||^2 \end{aligned}$$

where λ_i are the eigenvalues of $\mathbf{A}$ with the corresponding eigenvectors $\mathbf{e}_i$. So it suffices now to see that

$$\min_i(\lambda_i^{(\mathbf{A})}) \geq c\lambda^2$$

We have

$$\min_i(\lambda_i^{(\mathbf{A})}) = \inf_{||\mathbf{X}||=1} <\mathbf{X}, \mathbf{A}\mathbf{X}>$$

The matrix $\mathbf{M}$ has the form $\mathbf{M} = \mathbf{K} + \lambda \mathbf{S}$ and hence

$$< \mathbf{X}, \mathbf{AX} > = < (\mathbf{K}^T + \lambda \mathbf{S}^T)\mathbf{X}, (\mathbf{K}^T + \lambda \mathbf{S}^T)\mathbf{X} >$$

Therefore

$$\frac{< \mathbf{X}, \mathbf{AX} >}{\lambda^2} = \frac{\lambda^2 ||\mathbf{S}^T \mathbf{X}||^2 + 2\lambda < \mathbf{S}^T \mathbf{X}, \mathbf{K}^T \mathbf{X} > + ||\mathbf{K}^T \mathbf{X}||^2}{\lambda^2}$$

and so we obtain

$$\liminf_{\lambda \to 0} \frac{\min_i (\lambda_i^{(\mathbf{A})})}{\lambda^2} > 0$$

Indeed, since $\mathbf{M} \equiv \mathbf{K} + \lambda \mathbf{S}$ is not singular, either $\mathbf{K}$ or $\mathbf{S}$ must not be singular, therefore $||\mathbf{S}^T \mathbf{X}||^2$ and $||\mathbf{K}^T \mathbf{X}||^2$ cannot be both zero. □

Remark 7.1. *It follows from the correlation inequalities derived in [8] that $\mathbf{Y}_{ij} \geq \mathbf{T}_{ij}$. Indeed, $\mathbf{T}_{ij}$ would be the correlation function if the dual walkers were walking independently, however, two dual walkers interact by inclusion (attraction), and this leads to a positive covariance.*

8 Acknowledgment

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9 Appendix

Here we derive explicit expressions for the two point correlation function for systems with three and four sites. We start from the equations (20).

Since Y_{kl} is symmetric in k and l it suffices to consider $k \leq l$. the different cases are as follows;

1. $\mathbf{1 < k = l < N}$; $(-2Y_{kk} + 3Y_{kk-1} + 3Y_{kk+1}) = 0$
2. $\mathbf{1 < k = l - 1 < N - 1}$; $(-8Y_{kk+1} + Y_{k-1k+1} + Y_{k+1k+1} + Y_{kk} + Y_{kk+2}) = 0$
3. $\mathbf{1 < k < l + 1 < N + 1}$; $(-4Y_{kl} + Y_{k-1l} + Y_{k+1l} + Y_{kl-1} + Y_{kl+1}) = 0$
4. $\mathbf{k = l = 1}$; $(-2Y_{11} + 3Y_{10} + 3Y_{12}) + \lambda(T_L T_1 - Y_{11}) + \lambda(T_L T_1 - Y_{11}) = 0$
5. $\mathbf{k = l = N}$; $(-2Y_{NN} + 3Y_{NN-1} + 3Y_{NN+1}) + \lambda(T_R T_N - Y_{NN}) + \lambda(T_R T_N - Y_{NN}) = 0$
6. $\mathbf{1 = k = l - 1}$; $(-8Y_{12} + Y_{02} + Y_{22} + Y_{11} + Y_{13}) + \lambda(T_L T_2 - Y_{12}) = 0$
7. $\mathbf{k = l - 1 = N - 1}$; $(-8Y_{N-1N} + Y_{N-2N} + Y_{NN} + Y_{N-1N-1} + Y_{N-1N+1}) + \lambda(T_R T_{N-1} - Y_{NN-1}) = 0$

8. $\mathbf{1 = k < l + 1 < N + 1}$; $(-4Y_{1l} + Y_{0l} + Y_{2l} + Y_{1l-1} + Y_{1l+1}) + \lambda(T_L T_1 - Y_{1l}) = 0$

9. $\mathbf{1 = k < l + 1 = N + 1}$; $(-4Y_{1N} + Y_{0N} + Y_{2N} + Y_{1N-1} + Y_{1N+1}) + \lambda(T_L T_1 - Y_{1N}) + \lambda(T_R T_1 - Y_{1N}) = 0$

10. $\mathbf{1 < k < l + 1 = N + 1}$; $(-4Y_{kN} + Y_{k-1N} + Y_{k+1N} + Y_{kN-1} + Y_{kN+1}) + \lambda(T_R T_k - Y_{kN}) = 0$

9.1 3 Sites System

The equations for the two-point correlation function are of the form $\mathbf{M}.\mathbf{Y} = \mathbf{D}$ where

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{22} \\ Y_{23} \\ Y_{33} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} -\lambda T_L T_3 - \lambda T_R T_1 \\ -3\lambda T_R T_3 \\ -\lambda T_R T_2 \\ -3\lambda T_L T_1 \\ -\lambda T_L T_2 \\ 0 \end{pmatrix}$$

and the matrix $\mathbf{M}$ can be read from the previous equations as;

$$\mathbf{M} = \begin{pmatrix} 0 & 1 & -2(1+\lambda) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & -(1+\lambda) \\ 0 & 0 & 1 & 1 & -(7+\lambda) & 1 \\ -(1+\lambda) & 3 & 0 & 0 & 0 & 0 \\ 1 & -(7+\lambda) & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & -2 & 3 & 0 \end{pmatrix}$$

The explicit solution is via inversion of $\mathbf{M}$. The result for $\mathbf{Y}$ and the correlation functions $\mathbf{C}_{ij} = \mathbf{Y}_{ij} - T_i T_j (1 + 2\delta_{ij})$ then reads as follows:

$$\mathbf{Y}_{11} = \frac{3\left(T_R^2(5+3\lambda) + 2T_L T_R\left(5+9\lambda+2\lambda^2\right) + T_L^2\left(5+23\lambda+24\lambda^2+4\lambda^3\right)\right)}{4(1+\lambda)^2(5+\lambda)}$$

$$\mathbf{Y}_{12} = \frac{T_R^2(5+3\lambda) + 2T_L T_R\left(5+4\lambda+\lambda^2\right) + T_L^2\left(5+13\lambda+2\lambda^2\right)}{4\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{Y}_{13} = \frac{T_L^2\left(5+13\lambda+2\lambda^2\right) + T_R^2\left(5+13\lambda+2\lambda^2\right) + 2T_L T_R\left(5+9\lambda+12\lambda^2+2\lambda^3\right)}{4(1+\lambda)^2(5+\lambda)}$$

$$\mathbf{Y}_{22} = \frac{3\left(2T_L T_R\left(5+4\lambda+\lambda^2\right) + T_L^2\left(5+8\lambda+\lambda^2\right) + T_R^2\left(5+8\lambda+\lambda^2\right)\right)}{4\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{Y}_{23} = \frac{T_L^2(5+3\lambda) + 2T_L T_R\left(5+4\lambda+\lambda^2\right) + T_R^2\left(5+13\lambda+2\lambda^2\right)}{4\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{Y}_{33} = \frac{3\left(T_L^2(5+3\lambda)+2T_LT_R\left(5+9\lambda+2\lambda^2\right)+T_R^2\left(5+23\lambda+24\lambda^2+4\lambda^3\right)\right)}{4(1+\lambda)^2(5+\lambda)}$$

and

$$\mathbf{C}_{11} = \frac{3(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}, \mathbf{C}_{12} = \frac{(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{C}_{13} = \frac{(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}, \mathbf{C}_{22} = \frac{3(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}$$

$$\mathbf{C}_{23} = \frac{(T_L-T_R)^2\lambda}{2\left(5+6\lambda+\lambda^2\right)}, \mathbf{C}_{33} = \frac{3(T_L-T_R)^2\lambda}{2(1+\lambda)^2(5+\lambda)}$$

We see that for all k, l

$$\mathbf{C}_{kl} \propto \lambda(T_L-T_R)^2$$

and also $\mathbf{C}_{kl} \geq 0$.

One might be interested to see if the bi-linear ansatz introduced in [6] for the special case $\lambda = 1$ is also valid here, i.e.

$$\begin{aligned} \mathbf{Y}_{ij} &= a+bi+cj+dij \\ \mathbf{Y}_{ii} &= A+Bi+Di^2 \end{aligned} \qquad (24)$$

with the boundary conditions $\mathbf{Y}_{0i} = T_LT_i$, $\mathbf{Y}_{i,N+1} = T_iT_R$.

However, to check the validity of the ansatz we must calculate the correlation functions for a 4 sites system, since in 3 sites systems we have only 6 correlation functions which are less than the 7 constants of the ansatz.

9.2 4 Sites System

Similar to the calculation for the 3 site system, we have $\mathbf{M}.\mathbf{Y} = \mathbf{D}$ where

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{14} \\ Y_{22} \\ Y_{23} \\ Y_{24} \\ Y_{33} \\ Y_{34} \\ Y_{44} \end{pmatrix} \text{ and } \mathbf{D} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\lambda T_LT_3 \\ -\lambda T_LT_2 \\ -3\lambda T_LT_1 \\ -\lambda T_2T_R \\ -\lambda T_RT_3 \\ -3\lambda T_4T_R \\ -\lambda T_LT_4-\lambda T_1T_R \end{pmatrix}$$

and where the matrix $\mathbf{M}$ is given by

$$\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & -8 & 1 & 1 & 0 & 0 \\
0 & 3 & 0 & 0 & -2 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3 & 0 & -2 & 3 & 0 \\
0 & 1 & -3-\lambda & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -7-\lambda & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1-\lambda & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & -3-\lambda & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -7-\lambda & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & -1-\lambda \\
0 & 0 & 1 & -2-2\lambda & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$$

The solution for $\mathbf{Y}$ is

$$\begin{aligned}
\mathbf{Y}_{11} &= \frac{6T_R^2(12+\lambda(14+3\lambda))+6T_LT_R(24+\lambda(76+5\lambda(8+\lambda)))+T_L^2(72+3\lambda(172+3\lambda(106+\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{12} &= \frac{2T_R^2(1+\lambda)(12+\lambda(14+3\lambda))+T_LT_R(48+\lambda(152+\lambda(128+\lambda(44+5\lambda))))+2T_L^2(12+\lambda(74+\lambda(121+\lambda(49+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{13} &= \frac{T_L^2(1+\lambda)(24+5\lambda(4+\lambda)(5+\lambda))+T_R^2(24+\lambda(76+\lambda(41+4\lambda)))+2T_LT_R(24+\lambda(76+\lambda(109+\lambda(47+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{14} &= \frac{T_L^2(24+5\lambda(4+\lambda)(5+\lambda))+T_R^2(24+5\lambda(4+\lambda)(5+\lambda))+T_LT_R(48+\lambda(152+\lambda(314+3\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{22} &= \frac{3\big(2T_LT_R(24+\lambda(76+\lambda(73+3\lambda(9+\lambda))))+T_L^2(24+\lambda(124+\lambda(181+7\lambda(10+\lambda))))+T_R^2(24+\lambda(76+\lambda(77+2\lambda(12+\lambda))))\big)}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{23} &= \frac{2T_L^2(12+\lambda(5+2\lambda)(10+\lambda(8+\lambda)))+2T_R^2(12+\lambda(5+2\lambda)(10+\lambda(8+\lambda)))+T_LT_R(2+\lambda)(24+\lambda(64+\lambda(50+7\lambda)))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{24} &= \frac{T_R^2(1+\lambda)(24+5\lambda(4+\lambda)(5+\lambda))+T_L^2(24+\lambda(76+\lambda(41+4\lambda)))+2T_LT_R(24+\lambda(76+\lambda(109+\lambda(47+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{33} &= \frac{3\big(2T_LT_R(24+\lambda(76+\lambda(73+3\lambda(9+\lambda))))+T_R^2(24+\lambda(124+\lambda(181+7\lambda(10+\lambda))))+T_L^2(24+\lambda(76+\lambda(77+2\lambda(12+\lambda))))\big)}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{34} &= \frac{2T_L^2(1+\lambda)(12+\lambda(14+3\lambda))+T_LT_R(48+\lambda(152+\lambda(128+\lambda(44+5\lambda))))+2T_R^2(12+\lambda(74+\lambda(121+\lambda(49+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))} \\
\mathbf{Y}_{44} &= \frac{6T_L^2(12+\lambda(14+3\lambda))+6T_LT_R(24+\lambda(76+5\lambda(8+\lambda)))+3T_R^2(24+\lambda(172+3\lambda(106+\lambda(46+5\lambda))))}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}
\end{aligned}$$

and the corresponding correlation functions are

$$\begin{gathered}
\mathbf{C}_{11} = \frac{3(T_L-T_R)^2\lambda(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}, \mathbf{C}_{12} = \frac{(T_L-T_R)^2\lambda(1+\lambda)(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))} \\
\mathbf{C}_{13} = \frac{2(T_L-T_R)^2\lambda(1+\lambda)(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}, \mathbf{C}_{14} = \frac{2(T_L-T_R)^2\lambda(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))} \\
\mathbf{C}_{22} = \frac{3(T_L-T_R)^2\lambda(2+\lambda(4+\lambda))(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}, \mathbf{C}_{23} = \frac{(T_L-T_R)^2\lambda(24+\lambda(86+\lambda(93+\lambda(27+2\lambda))))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))} \\
\mathbf{C}_{24} = \frac{2(T_L-T_R)^2\lambda(1+\lambda)(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}, \mathbf{C}_{33} = \frac{3(T_L-T_R)^2\lambda(2+\lambda(4+\lambda))(12+\lambda(16+\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))} \\
\mathbf{C}_{34} = \frac{(T_L-T_R)^2\lambda(1+\lambda)(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}, \mathbf{C}_{44} = \frac{3(T_L-T_R)^2\lambda(24+\lambda(50+13\lambda))}{(6+\lambda)(2+3\lambda)^2(8+\lambda(16+5\lambda))}
\end{gathered}$$

We see once more that for all k, l

$$\mathbf{C}_{kl} \propto \lambda(T_L - T_R)^2$$

and $\mathbf{C}_{kl} \geq 0$.

Now we can directly check the validity of the bi-linear ansatz. Direct calculation shows that the diagonal part of the ansatz, i.e., $\mathbf{Y}_{ii} = A + Bi + Di^2$ is valid, but the non-diagonal part $\mathbf{Y}_{ij} = a + bi + cj + dij$ is not.

If we determine the coeficients a, b, c, d by fitting the bilinear ansatz to $\mathbf{Y}_{12}, \mathbf{Y}_{13}, \mathbf{Y}_{23}, \mathbf{Y}_{34}$, then we obtain

$$\mathbf{Y}_{14} - (a + b + 4c + 4d) = \frac{3(T_L - T_R)^2(-1+\lambda)\lambda^2}{(6+\lambda)(2+3\lambda)(8+\lambda(16+5\lambda))}.$$

which shows that the bilinear form can not hold for $\lambda \notin \{0, 1\}$. Remark that also when $\lambda \to \infty$ the deviation from the multilinear form vanishes, which is consistent with the intuition that this limit is the same as having $\lambda = 1$ in a smaller system obtained by removing the sites $1, N$.

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