Unipotent group actions on affine varieties

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Abstract. Algebraic actions of unipotent groups $U$ actions on affine $k$-varieties $X$ (over an algebraically closed field of characteristic zero) for which the algebraic quotient $X//U$ has small dimension are considered. In case $X$ is factorial, $\mathcal{O}(X)^* = k^*$, and $X//U$ is one-dimensional, it is shown that $\mathcal{O}(X)^U = k[f]$, and if some point in $X$ has trivial isotropy, then $X$ is $U$ equivariantly isomorphic to $U \times \mathbb{A}^1(k)$. The main results are given distinct geometric and algebraic proofs. Links to the Abhyankar-Sathaye conjecture and a new equivalent formulation of the Sathaye conjecture are made.

1. Preliminaries and Introduction

Throughout, $k$ will denote a field of characteristic zero, $k^n$ the polynomial ring in $n$ variables over $k$, and $U$ a unipotent algebraic group over $k$. Our interest is in algebraic actions of such $U$ on affine $k$-varieties $X$ (equivalently on their coordinate rings $\mathcal{O}(X)$). An algebraic action of the one dimensional unipotent group $G_a = (k, +)$ is conveniently described through the action of a locally nilpotent derivation $D$ of $\mathcal{O}(X)$. Specifically, for $u \in G_a = k$, we have the automorphism $u^*$ acting on $\mathcal{O}(X)$ and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation $D$ of $\mathcal{O}(X)$. Specifically, for $u \in G_a = k$, we have the automorphism $u^*$ acting on $\mathcal{O}(X)$ and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation $D$ of $\mathcal{O}(X)$ such that $u^* = \exp(uD)$. (One can obtain $D$ by taking $D(f) = \frac{u^*f-f|_{u=0}}{u}$.) Similarly, if $G_a^n$ acts on $X$, then we have for each component $G_a$-action a locally nilpotent derivation $D_i$, and for each element $u = (u_1, \ldots, u_n) \in G_a^n$, we have the derivation $D := u_1D_1 + \ldots + u_nD_n$. If the action is faithful, there is a canonical isomorphism of Lie($G_a^n$) with $kD_1 + \ldots + kD_n$. In this case, the $D_i$ commute.

The situation is similar for a general unipotent group action $U \times X \rightarrow X$. Because the action is algebraic, each $f \in \mathcal{O}(X)$ is contained in a finite dimensional $U$ stable subspace $V_f$ on which $U$ acts by linear transformations. Since $U$ is unipotent, for each $u \in U$, $u^* - id$ is nilpotent on $V_f$, so that

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\[ \ln(u)(g) = \sum_{j=1}^{\infty} \frac{(u^* - id)^j g}{j} \] is a finite sum for all \( g \in V_f \). One checks that \( D_u \equiv \ln(u) \) is a (locally nilpotent) derivation of \( O(X) \) and \( u^* = \exp(D_u) \).

If the action is faithful, i.e. \( U \to Aut(X) \) is injective, there is a canonical isomorphism of \( \text{Lie}(U) \) with \( \{ D_u \mid u \in U \} \). In fact, \( \text{Lie}(U) = kD_1 + \ldots + kD_m \) (\( m = \dim(U) \)) for some locally nilpotent derivations \( D_i \). In general the \( D_i \) do not commute. In fact, all of them commute if and only if \( U = G_m^a \).

Two useful facts about unipotent group actions on quasiaffine varieties \( V \) can be immediately derived from these observations:

1. Because each \( u \in U \) acts via a locally nilpotent derivation of \( O(V) \), the ring of invariants \( O(V)^U \) is the intersection of the kernels of locally nilpotent derivations.

2. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, i.e. \( O(V)^U \) is factorially closed. In particular if \( O(V) \) is a UFD so is \( O(V)^U \).

We will use the fact that \( U \) is a special group in the sense of Serre. This means that a \( U \) action which is locally trivial for the étale topology is locally trivial for the Zariski topology. If \( G \) is a group acting on a variety \( X \), we denote by \( X//G \) the algebraic quotient \( X//G := \text{Spec} O(X)^G \) and by \( X/G \) the geometric quotient (when it exists). By a free action we mean an action for which the isotropy subgroup of each element consists only of the identity. (A free action is faithful.)

The paper is organized as follows: Section 2 contains some examples which illustrate the main results and clarify their hypotheses. The main results are proved in Section 3 from a geometric perspective, and Section 4 gives them an algebraic interpretation. (The algebraic and geometric viewpoint both have their merits: the geometric viewpoint lends itself to possible generalizations, while the algebraic proofs are constructive and can be more easily used in algorithms.) In section 5 we elaborate on some implications of the main results for the Sathaye conjecture, and on the motivation for studying this problem.

## 2. Examples

The following examples are valuable in various parts of the text.

**Example 1.** Let \( X = k^3 \), and \( U := \{ u_{a,b,c} \mid a, b, c \in k \} \) where

\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]

a unipotent group acting by \( u_{a,b,c}(x, y, z) = (x + a, y + az + b, z + c) \) (which indeed is an algebraic action). For each \( (a, b, c) \in k^3 \) we thus have an automorphism, and its associated derivation on \( k[X, Y, Z] \) is \( D_{a,b,c} = a\partial_X + (aZ + b - \frac{ac}{2})\partial_Y + c\partial_Z \). Set \( D_1 = \partial_Y, D_2 := \partial_X + Z\partial_Y, D_3 = \partial_Z \). As a Lie algebra \( \text{Lie}(U) \) is generated by \( D_1, D_2, D_3 \). One checks that \( D_1 \) commutes
with $D_2, D_3$, but $[D_2, D_3] = D_1$. However, restricted to $k[X, Y, Z]^{D_1} = k[X, Z]$, $D_2$ and $D_3$ do commute, as they coincide with the derivations $\partial_X$ and $\partial_Z$. Furthermore, as a $k$ vector space $\text{Lie}(U)$ has basis $\partial_X, \partial_Y, \partial_Z$.

**Example 2.** Let $\mathcal{O}(X) = A = k[X, Y, Z]$, and $D_1 = Z\partial_X, D_2 = \partial_Y$. These locally nilpotent derivations generate a $U = (\mathfrak{g}_a)^2$-action on $k^3$ given by $(a, b) \cdot (x, y, z) \mapsto (x + az, y + b, z)$. Now $k[Z] = A^{D_1, D_2} = \mathcal{O}(X/U)$. $D_1, D_2$ are linearly independent over $k[Z]$. When calculating modulo $Z - \alpha$, where $\alpha \in k$, we notice that $D_1 \mod (Z - \alpha), D_2 \mod (Z - \alpha)$ are linearly independent over $A/(Z - \alpha)$ except when $\alpha = 0$. However, defining $\mathcal{M} := (\text{Lie}(U) \otimes k(Z)) \cap \text{DER}(A) = (k[Z]D_1 + k[Z]D_2) \cap \text{DER}(A)$ we see that $\mathcal{M} = k[Z]\partial_X + k[Z]\partial_Y$. The derivations $\partial_X, \partial_Y$ are linearly independent modulo each $Z - \alpha$. And for each $\alpha \in k$, we have $A/(Z - \alpha) \cong k[2]$.

**Example 3.** Let $P := X^2Y + X + Z^2 + T^3$, $\mathcal{X} := \{(x, y, z, t) \mid P(x, y, z, t) = 0\}$. Let $A := k[x, y, z, t] := k[X, Y, Z, T]/(P) = \mathcal{O}(\mathcal{X})$. The commuting locally nilpotent derivations $2Z\partial_Y - X^2\partial_Z, 3T^2\partial_Y - X^2\partial_T$ on $k[X, Y, Z, T]$ map $P$ to zero, and hence induce derivations $D_1, D_2$ on $A$. They are linearly independent over $A^{D_1, D_2} = k[X]$ and since they commute, induce a $(\mathfrak{g}_a)^2$-action on $\mathcal{X}$. Modulo $X - \alpha$, $D_1, D_2$ are linearly independent, except when $\alpha = 0$. Now defining $\mathcal{M} := (\text{Lie}(U) \otimes k(X)) \cap \text{DER}(A) = [k[X]D_1 + k[X]D_2 = \text{Lie}(U) \otimes k[X]$, we see that $\mathcal{M}$ modulo $X - \alpha$ is a $k$-module of dimension 2 except when $\alpha = 0$, when it is of dimension 1. Also, $A/(X - \alpha) \cong k[2]$ except when $\alpha = 0$, when it is isomorphic to $R[X]$ where $R = k[Z, T]/(Z^2 + T^3)$.

**Example 4.** The $U = \mathfrak{g}_a \times \mathfrak{g}_a$ action on $\mathbb{A}^2(k)$ given by

$$U \times \mathbb{A}^2 \ni ((s, t), (x, y)) \mapsto (x, y + t + sx) \in \mathbb{A}^2$$

is faithful and fixed point free. However every point in $\mathbb{A}^2$ has a non-trivial isotropy subgroup. If $x \neq 0$, then $((s, -sx), (x, y)) \mapsto (x, y)$ and $((s, 0), (0, y)) \mapsto (0, y)$.

3. Main Results

The following simple lemma is useful in a number of places.

**Lemma 1.** Let $U$ be a unipotent algebraic group acting algebraically on a factorial quasiaffine variety $X$ of dimension $n$ satisfying $\mathcal{O}(X)^* = k^*$. If the action is not transitive and some point $x \in X$ has orbit of dimension $n - 1$, then $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$.

**Proof.** Since $n - 1$ is the maximum orbit dimension there is a Zariski open subset $V$ of $X$ for which the geometric quotient $V/U$ exists as a variety. Then the transcendence degree of the quotient field $K$ of $\mathcal{O}(V/U)$ is equal to 1. Since $K = qf(\mathcal{O}(X)^U)$ and

$$\mathcal{O}(X)^U = \mathcal{O}(X) \cap K,$$
is a ufd, \( \mathcal{O}(X)^U \) is finitely generated over \( k \). From \( (\mathcal{O}(X)^U)^* = k^* \), we conclude that \( \mathcal{O}(X)^U = k[f] \) for some \( f \in \mathcal{O}(X) \).

### 3.1. Unipotent actions having zero-dimensional quotient.

**Theorem 1.** Let \( U \) be an \( n \)-dimensional unipotent group acting faithfully on an affine \( n \)-dimensional variety \( X \) satisfying \( \mathcal{O}(X)^* = k^* \). If either

1. Some \( x \in X \) has trivial isotropy subgroup or
2. \( n = 2 \), \( X \) is factorial, and \( U \) acts without fixed points,

then the action is transitive. In particular \( X \cong k^n \).

**Proof.** In case a) there is an open affine subset \( V \) of \( X \) on which \( U \) acts without fixed points. Since \( U \) has the same dimension as \( V \), \( V//U \) is zero-dimensional, hence \( \mathcal{O}(V//U) \) is a field. This field contains \( k \), and its units are contained in \( \mathcal{O}(X)^* = k^* \), hence \( \mathcal{O}(V//U) = k \). It follows that there exists an open set \( V' \) of \( X \) for which \( V'/U \cong \text{Spec } k \). Thus \( V' \cong U \) as a variety, and therefore \( V' \cong k^n \). If \( v \in V' \), then \( Uv = V' \). Since \( U \) is unipotent, all orbits are closed, hence \( V' \) is closed in \( X \). Since it is of dimension \( n \), and \( X \) is irreducible of dimension \( n \), we have that \( V' = X \).

In case b) \( X \) is necessarily smooth since it is smooth in codimension 1 and every orbit is infinite. If \( X \) has a two dimensional (i.e. dense) orbit then the conclusion follows as in a). So we assume for each \( x \in X \) that the orbit \( Ux \) is one dimensional, given as \( \exp(uD)x \), and therefore isomorphic to \( \mathbb{A}^1(k) \) by the discussion in the introduction. From Lemma 1 we conclude that \( \mathcal{O}(X)^U = k[f] \) for some \( f \in \mathcal{O}(X) \).

Note that factorial closure of \( \mathcal{O}(X)^U \) implies that \( f - \lambda \) is irreducible for every \( \lambda \in k \). The absence of nonconstant units implies that \( X \to \text{Spec}(k[f]) \) is surjective, and all fibers are \( U \) orbits. Smoothness of \( X \) implies in addition that this mapping is flat, hence an \( \mathbb{A}^1 \) bundle over \( \mathbb{A}^1 \). But any such bundle is trivial, so we conclude again that \( X \cong \mathbb{A}^2 \).

Example 4 of the previous section illustrates case b).

### 3.2. Unipotent actions having one-dimensional quotient.

The following theorem is the main result of this paper.

**Theorem 2 (Main theorem).** Let \( U \) be a unipotent algebraic group of dimension \( n \), acting on \( X \), a factorial variety of dimension \( n + 1 \) satisfying \( \mathcal{O}(X)^* = k^* \).

1. If at least one \( x \in X \) has trivial stabilizer then \( \mathcal{O}(X)^U = \mathcal{O}(X//U) = k[f] \). Furthermore, \( f^{-1}(\lambda) \cong k^n \) for all but finitely many \( \lambda \in k \).
2. If \( U \) acts freely, then \( X \) is \( U \)-isomorphic to \( U \times k \). In particular, \( X \cong k^{n+1} \) and \( f \) is a coordinate.

An important example to keep in mind is example 1, as this satisfies (1) but not (2). (There \( U = G_a^2 \).)
Proof of theorem 2.

Claim 1: \( \mathcal{O}(X)^U = k[f] \).

Proof of claim 1: This follows from lemma 1.

Claim 2: \( f : X \to k \) is surjective and has fibers isomorphic to \( U \). The fibers are the \( U \)-orbits.

Proof of claim 2: The fibers \( f^{-1}(\lambda) \) are the zero loci of the irreducible \( f - \lambda \), and are invariant under \( U \). Since \( U \) acts freely on each fiber and orbits of unipotent group actions are closed, we see that the \( f \) fibers are exactly the \( U \) orbits in \( X \). Thus \( f \) is a \( U \)-fibration (and, as the underlying variety of \( U \) is \( k^n \), an \( \mathbb{A}^n \)-fibration).

Claim 3: \( X \) is smooth.

Proof of claim 3: The set \( X_{\text{sing}} \) is \( U \)-stable, hence it is a union of \( U \)-orbits. The \( U \)-orbits are the zero sets \( f - \lambda \), hence of codimension 1. So \( X_{\text{sing}} \) is of codimension 1 or empty. But \( X \) is factorial, so in particular normal, which implies that the set of singular points of \( X \), denoted by \( X_{\text{sing}} \), is of codimension at least 2. This means that \( X_{\text{sing}} \) can only be empty.

Claim 4: \( f \) is smooth.

Proof of claim 4: All fibers of \( f \) are isomorphic to \( U \), hence to \( k^n \), by claim 2. Thus the fibers of \( f \) are geometrically regular of dimension \( n \). Since \( X \) is smooth, \( f \) is flat, and proposition 10.2 of [2] yields that \( f \) is smooth.

Claim 5: \( X \times_f X \) is smooth.

Proof of claim 5: \( X \times_f X \) is smooth since it is a base extension of the smooth \( X \) by the smooth morphism \( f \).

Claim 6: \( g : U \times X \to X \times_f X \) given by \( (u, x) \mapsto (x, ux) \) is an isomorphism.

Proof of claim 6: The map \( g \) restricted to \( U \times f^{-1}(\lambda) \) is a bijection onto \( \{(x, y) \mid f(x) = f(y) = \lambda \} \). Taking the union over \( \lambda \in k \), we get that \( g \) is a bijection. Since both \( U \times X \) and \( X \times_f X \) are smooth and \( g \) is a bijection, Zariski’s Main Theorem implies that \( g \) is an open immersion if it is birational. If so then \( g \) must be an isomorphism since it is bijective.

From Rosenlicht’s cross section theorem [6], \( X \) has a \( U \) stable open subset \( \tilde{X} \) on which the \( U \) action has a geometric quotient \( \tilde{X}/U \) and an \( U \) equivariant isomorphism \( \tilde{X} \cong U \times \tilde{X}/U \). Restricting \( g \) to \( U \times \tilde{X} \to \tilde{X} \times_f \tilde{X} \) is clearly an isomorphism, so that \( g \) is birational.

Now we are ready to prove the theorem. Using def. 0.10 p.16 of [5], and the fact (4) that \( f \) is smooth, together with (6), yields that \( f : X \to \mathbb{A}^1 \) is an étale principal \( U \)-bundle and therefore a Zariski locally trivial principal \( U \) bundle as \( U \) is special. Such bundles are classified by the cohomology group \( H^1(U, k) \), which is trivial because \( U \) is unipotent. Thus the bundle \( f : X \to k \) is trivial, which means that \( X \cong U \times k \). \( \square \)

Remark 1. (1) To obtain \( \tilde{X} \) explicitly and avoid the use of Rosenlicht’s theorem, recall that the action of \( U \) is generated by a finite set of \( G_a \) actions each one given as the exponential of some locally nilpotent derivation \( D_i \) of \( \mathcal{O}(X) \), indeed \( D_i \in \mathfrak{u} \), the Lie algebra of
U. As such there is an open subset $X_i$ of $X$ on which $D_i$ has a slice, and the corresponding $G_a$ acts by translation. Then $\bar{X} := \cap_{i=1}^s X_i$.

(2) One can avoid the use of the étale topology by applying a "Seshadri cover" \[7\]. One constructs a variety $Z$ finite over $X$, necessarily affine, to which the $U$ action extends so that

(a) $k(Z)/k(X)$ is Galois. Denote the Galois group by $\Gamma$.

(b) The $\Gamma$ and $U$ actions commute on $Z$.

(c) The $U$ action on $Z$ is Zariski locally trivial and, because the action on $X$ is proper by claim 6,

(d) $Y \equiv Z/U$ exists as a separated scheme of dimension 1, hence a curve, and affine because of the existence of nonconstant globally defined regular functions, namely $O(Z)$.

(e) $O(X)^U = O(Y)^\Gamma$ and $X//U \equiv Y/\Gamma$ shows that $X \to X/U$ is Zariski locally trivial.

4. Algebraic Version

4.1. Unipotent actions having zero dimensional kernel. Let $X$ be a quasiaffine variety, and $U$ an algebraic group acting on $X$. We write $A := O(X)$ and denote by $u$ the Lie algebra of $U$. In this section, we will make the following assumptions:

(P) $a)$ $X$ and $U$ are of dimension $n$.

$b)$ There is a point $x \in X$ such that stab$(x) = \{e\}$.

$c)$ $O(X)^* = k^*$

Definition 1. Assume (P). We say that $D_1, \ldots, D_n$ is a triangular basis of $u$ (with respect to the action on $X$) if

(1) $u = kD_1 \oplus kD_2 \oplus \ldots \oplus kD_n$ and

(2) With subalgebras $A_i$ of $A$ given by $A_1 := A$, $A_i := A^{D_1} \cap \ldots \cap A^{D_i-1}$, the restriction of $D_i$ to $A_i$ commutes with the restrictions of $D_{i+1}, \ldots, D_n$.

For a triangular basis, it is clear that $D_j(A_j) \subseteq A_j$ for each $j$.

If $U$ is unipotent then the existence of a triangular basis is a consequence of the Lie-Kolchin theorem. Indeed, the Lie algebra $u$ of $U$ is isomorphic to a Lie subalgebra of the full Lie algebra of upper triangular matrices over $k$. In particular $u$ has a basis $D_1, \ldots, D_n$ satisfying $[D_i, D_j] \in \text{span}\{D_1, \ldots, D_{\min\{i,j\}}\}$. By definition of the $A_i$ this basis is triangular with respect to the action and $D_1$ is in the center of $u$.

Proposition 1. Assume (P) and $U$ unipotent. Then $A \cong k[s_1, \ldots, s_n] = k^n$ where $D_i(s_i) = 1$, and $D_i(s_j) = 0$ if $j > i$.

Proof. We proceed by induction $n = \dim u$. If $n = 1$, then we have one nonzero LND on a dimension one $k$-algebra domain $A$ satisfying $A^* = k^*$. It is well-known that this means that $A \cong k[x]$ and the derivation is simply $\partial_x$. Suppose the theorem is proved for $n - 1$. Let $D_1, D_2, \ldots, D_n$ be a triangular
basis for \( u \). Restricting to \( A^{D_1} \) and noting that \( D_1 \) is in the center of \( u \), we have an action of the Lie algebra \( u/kD_1 \) which has the triangular basis \( k\overline{D}_2 + \ldots + k\overline{D}_n \) (\( \overline{D}_i \) denotes residue class modulo \( kD_1 \)). By construction \( \overline{D}_i(a) := D_i(a) \) is well defined, and by induction we find \( s_2, \ldots, s_n \in A^{D_1} \) satisfying \( D_i(s_i) = 1 \), \( D_i(s_j) = 0 \) if \( j > i \geq 2 \). 

Next we consider a preslice \( p \in A \) such that \( D_1(p) = q, D_1(q) = 0 \), i.e. \( q = q(s_2, \ldots, s_n) \). We pick \( p \) in such a way that \( q \) is of lowest possible lexicographic degree w.r.t \( s_2 \gg s_3 \gg \ldots \gg s_n \). Now \( D_1(D_2(p)) = D_2D_1(p) = D_2(q) \). Restricted to \( k[s_2, \ldots, s_n] \), \( D_2 = \partial_{s_2} \), so \( D_2(q) \) is of lower \( s_2 \)-degree than \( q \). Unless \( D_2(q) = 0 \), we get a contradiction with the degree requirements of \( q \), as \( D_2(p) \) would be a “better” preslice having a lower degree derivative. Thus, \( q \in k[s_3, \ldots, s_n] \). Using the same argument for \( D_3, D_4 \) etc. we get that \( q \in k^* \). Hence, \( p \) is in fact a slice. \( \square \)

4.2. Unipotent actions having one-dimensional quotient. With the same notations as in the previous section, we also denote the ring of \( U \) invariants in \( A \) by \( A^U \) and \( \text{Spec} \ A^U \) by \( X//U \). Note that \( A^U = \{ a \in A \mid D(a) = 0 \text{ for all } D \in u \} \). If \( U \) is unipotent and \( D_1, \ldots, D_n \) is a triangular basis of \( u \), we again write \( A_1 := A, A_{i+1} = A_i \cap A^{D_1} \), noting that \( A^U = A_n \). In this section we consider the conditions:

\[(Q1)\quad U \text{ is a unipotent algebraic group of dimension } n \text{ acting on an affine variety } X \text{ of dimension } n + 1 \text{ with } A^* = k^*. \]

and:

\[(Q)\quad A^U = k[f] \text{ for some irreducible } f \in A \backslash k.\]

Remark 2. According to Lemma 1, condition \((Q1)\) along with the assumption that \( X \) is factorial and the existence of a point \( x \in X \) with \( \text{stab}(x) = \{e\} \), implies that \((Q)\) holds.

Notation 1. Assuming \((Q)\), let \( \alpha \in k \). Set \( \overline{A} := A/(f - \alpha) \) and write \( \overline{\alpha} \) for the residue class of \( \alpha \) in \( \overline{A} \) and \( \overline{D} \) for the derivation induced by \( D \in u \) on \( \overline{A} \).

Our goal is to prove the following constructively:

Theorem 3. Assume \((Q1)\) and \((Q)\). Let \( D_1, \ldots, D_n \) be a triangular basis of \( u \).

1. For \( \alpha \in k \),
   a. If \( \overline{D}_1, \ldots, \overline{D}_n \) are independent over \( A/(f - \alpha) \), then 
      \[ A/(f - \alpha) \cong k^{[n]} \].
   b. There are only finitely many \( \alpha \) for which \( \overline{D}_1, \ldots, \overline{D}_n \) are dependent over \( A/(f - \alpha) \).

2. In the case that \( \overline{D}_1, \ldots, \overline{D}_n \) are independent over \( A/(f - \alpha) \) for each \( \alpha \in k \), then there are \( s_1, \ldots, s_n \in A \) with \( A = k[s_1, \ldots, s_n, f] \), hence \( A \) is isomorphic to a polynomial ring in \( n + 1 \) variables (and \( f \) is a coordinate).
Definition 2. Assume (Q1) and (Q), and a triangular basis $D_1, \ldots, D_n$ of $u$. Define

$$\mathcal{P}_i := \{ p \in A \mid D_i(p) \in k[f], \ D_j(p) = 0 \text{ if } j < i \}$$

and

$$\mathcal{J}_i := D_i(\mathcal{P}_i) \subseteq k[f].$$

Thus $\mathcal{P}_i$ is the set of "preslices" of $D_i$ that are compatible with the triangular basis $D_1, \ldots, D_n$.

Lemma 2. There exist $p_i \in \mathcal{P}_i \setminus \{0\}, p_i \in A_i$, and $q_i \in k[1]\setminus\{0\}$ such that $\mathcal{J}_i = q_i(f)k[f]$ and $D_i(p_i) = q_i$.

Proof. First note that $\mathcal{J}_i$ is not empty, as theorem[1] applied to $A(f) := A \otimes k(f)$ gives an $s_i \in A(f)$ which satisfies $D_i(s_i) = 1$, $D_j(s_i) = 0$ if $j < i$. Multiplying $s_i$ by a suitable element of $k[f]$ gives a nonzero element $r(f)s_i$ of $\mathcal{P}_i$, and $D_i(r(f)s_i) = r(f)$. Because $k[f] = \cap \ker(D_i)$, $\mathcal{P}_i$ is a $k[f]$-module, and therefore $\mathcal{J}_i$ is an ideal of $k[f]$. This means that $\mathcal{J}_i$ is a principal ideal, and we take for $q_i$ a generator (and $p_i \in D_i^{-1}(q_i)$). Since $D_j(p_i) = 0$ if $j < i$, we have $p_i \in A_i$.

Corollary 1. The $p_i$, $1 \leq i \leq n$, are algebraically independent over $k$.

Proof. The $s_i$ are certainly algebraically independent, and $p_i \in k[f]s_i$. □

Lemma 3. Assume (Q), and take $p_i, q_i$ as in lemma[2]. Then the $D_i$ are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some $i$.

Proof. ($\Rightarrow$): Suppose that $0 \neq D := g_1D_1 + \ldots + g_nD_n$ satisfies $\overline{D} = 0$ where $g_i \in A$, and not all $\overline{g_i} \neq 0$. Let $i$ be the highest such that $\overline{g_i} \neq 0$. Then $0 = D(p_i) = g_iD(p_i) = \overline{g_i}q_i(f)$. Since $A$ is a domain, $q_i(\alpha) = q_i(f) = 0$.

($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $\overline{D_i}$ are linearly dependent over $A/(f - \alpha)$. Consider $\overline{D_i}$ restricted to $A_i$. If $j > i$ then $\overline{D_i}(p_j) = \overline{D_j}(p_j) = 0$. Furthermore $\overline{D_i}(p_i) = \overline{q_i}(f) = q(\alpha) = 0$. Hence, $\overline{D_i}$ is zero if restricted to $k[\overline{p_i}, \ldots, \overline{p_n}]$. But since this is of transcendence degree $n$, it follows that $\overline{D_i} = 0$ on $A_i$. Reversing the argument of ($\Rightarrow$) yields the linear dependence of the $\overline{D_i}$.

Proof. (of theorem[3]) Part 1: If $\overline{D_1}, \ldots, \overline{D_n}$ are independent, then Proposition 1 yields that $A \cong k[\overline{n}]$. Lemma[3] states that for any point $\alpha$ outside the zero set of $q_1q_2 \cdots q_n$ we have $A/(f - \alpha) \cong k[\overline{n}]$. This zero set is either all of $k$ or finite, yielding part 1.

Part 2: Lemma[3] tells us directly that for each $1 \leq i \leq n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the $p_i$ can be taken to be actual slices ($s_i = p_i$). Using the fact that $s_i \in A_i$ we obtain that $A = A_1 = A_2[s_1] = A_3[s_2, s_1] = \cdots = A_{n+1}[s_1, \ldots, s_n] = k[s_1, \ldots, s_n, f]$ as claimed. □
5. Consequences of the main theorems

This paper is originally motivated by the following result of \cite{4}:

**Theorem 4.** Let $A = k[x, y, z]$ and $D_1, D_2$ two commuting locally nilpotent derivations on $A$ which are linearly independent over $A$. Then $A^{D_1, D_2} = k[f]$ and $f$ is a coordinate.

Here the notation $A^{D_1, D_2}$ means $A^{D_1} \cap A^{D_2}$ the intersections of the kernels of $D_1$ and $D_2$, which is the set of elements vanishing under $D_1$ resp. $D_2$. (Note that for the $G_a$ action associated to $D$, this notation means $O(X/G_a) = O(X)^{G_a} = O(X)^D$. By a coordinate is meant an element $f$ in $k[n]$ for which there exist $f_2, \ldots, f_n$ with $k[f, f_2, \ldots, f_n] = k[n]$. Equivalently, $(f, f_2, \ldots, f_n) : k[n] \rightarrow k[n]$ is an automorphism. The most important ingredient of this theorem is Kaliman’s theorem \cite{3}.

In \cite{4} it is conjectured that this result is true also in higher dimensions, i.e. having $n$ commuting linearly independent locally nilpotent derivations on $k^{[n+1]}$ should yield that their common kernel is generated by a coordinate. However, it seems that this conjecture is very hard, on a par with the well-known Sathaye conjecture:

**SC($n$) Sathaye-conjecture:** Let $f \in A := k[n]$ such that $A/(f - \lambda) \cong k^{[n-1]}$. Then $f$ is a coordinate.

The Sathaye conjecture is proved for $n \leq 3$ by the aforementioned Kaliman’s theorem. Therefore, the original motivation was to find additional requirements in higher dimensions to achieve the result that $f$ is a coordinate. The results in this paper give one such requirement, namely that $k^{[n]}/(f - \lambda) \cong k^{[n-1]}$ for all constants $\lambda$.

Another consequence of the result of this paper is that the Sathaye conjecture is equivalent to

**MSC($n$) Modified Sathaye Conjecture:** Let $A := k[n]$, and let $f \in A$ be such that $A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$. Then there exist $n - 1$ commuting locally nilpotent derivations $D_1, \ldots, D_{n-1}$ on $A$ such that $A^{D_1, \ldots, D_{n-1}} = k[f]$ and the $D_i$ are linearly independent modulo $(f - \alpha)$ for each $\alpha \in k$.

**Proof of equivalence of SC($n$) and MSC($n$).** Suppose we have proven the MSC($n$). Then for any $f$ satisfying “$A/(f - \alpha) \cong k^{[n-1]}$” we can find commuting LNDs $D_1, \ldots, D_{n-1}$ on $A$ giving rise to a $G_{a}^{n-1}$ action satisfying the hypotheses of Theorem 2. Applying this theorem, we obtain that $f$ is a coordinate in $A$. So the SC($n$) is true in that case.

Now suppose we have proven the SC($n$). Let $f$ satisfy the requirements of the MSC($n$), that is, “$A/(f - \alpha) \cong k^{[n-1]}$” for all $\alpha \in k$. Since $f$ satisfies the requirements of SC($n$), $f$ then must be a coordinate. So it has $n - 1$ so-called mates: $k[f, f_2, \ldots, f_n] = k[n]$. But then the partial derivative with respect to each of these $n$ polynomials $f, f_2, \ldots, f_n$ defines a locally nilpotent
derivation. All of them commute, and the intersection of the kernels of the last \( n - 1 \) derivations is \( k[f] \); so the MSC holds. \( \square \)

References


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