Unipotent group actions on affine varieties

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Abstract. Algebraic actions of unipotent groups $U$ acting on affine $k$-varieties $X$ (for $k$ an algebraically closed field of characteristic 0) for which the algebraic quotient $X//U$ has small dimension are considered.

In case $X$ is factorial, $O(X)^* = k^*$, and $X//U$ is one-dimensional, it is shown that $O(X)^U = k[f]$, and if some point in $X$ has trivial isotropy, then $X$ is $U$ equivariantly isomorphic to $U \times A^1(k)$. The main results are given distinct geometric and algebraic proofs. Links to the Abhyankar-Sathaye conjecture and a new equivalent formulation of the Sathaye conjecture are made.

1. Preliminaries and Introduction

Throughout, $k$ will denote a field of characteristic zero, $k[n]$ the polynomial ring in $n$ variables over $k$, and $U$ a unipotent algebraic group over $k$. Our interest is in algebraic actions of such $U$ on affine $k$-varieties $X$ (equivalently, on their coordinate rings $O(X)$). An algebraic action of the one dimensional unipotent group $G_a = (k, +)$ is conveniently described through the action of a locally nilpotent derivation $D$ of $O(X)$. Specifically, for $u \in G_a = k$, we have the automorphism $u^*$ acting on $O(X)$ and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation $D$ of $O(X)$. Specifically, for $u \in G_a = k$, we have the automorphism $u^*$ acting on $O(X)$ and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation $D$ of $O(X)$. Specifically, for $u \in G_a = k$, we have the automorphism $u^*$ acting on $O(X)$ and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation $D$ of $O(X)$.

If the action is faithful, there is a canonical isomorphism of $\text{Lie}(G_a^n)$ with $kD_1 + \ldots + kD_n$. In this case, the $D_i$ commute.

The situation is similar for a general unipotent group action $U \times X \rightarrow X$. Because the action is algebraic, each $f \in O(X)$ is contained in a finite dimensional $U$ stable subspace $V_f$ on which $U$ acts by linear transformations. Since $U$ is unipotent, for each $u \in U$, $u^* - id$ is nilpotent on $V_f$, so that

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\[ \ln(u)(g) = \sum_{j=1}^{\infty} \frac{(u^* - id)^j g}{j} \] is a finite sum for all \( g \in V_f \). One checks that \( D_u \equiv \ln(u) \) is a (locally nilpotent) derivation of \( \mathcal{O}(X) \) and \( u^* = \exp(D_u) \).

If the action is faithful, i.e. \( U \to Aut(X) \) is injective, there is a canonical isomorphism of \( \text{Lie}(U) \) with \( \{ D_u \mid u \in U \} \). In fact, \( \text{Lie}(U) = kD_1 + \ldots + kD_m \) (\( m = \dim(U) \)) for some locally nilpotent derivations \( D_i \). In general the \( D_i \) do not commute. In fact, all of them commute if and only if \( U = \mathcal{G}_m \).

Two useful facts about unipotent group actions on quasiaffine varieties \( V \) can be immediately derived from these observations:

1. Because each \( u \in U \) acts via a locally nilpotent derivation of \( \mathcal{O}(V) \), the ring of invariants \( \mathcal{O}(V)^U \) is the intersection of the kernels of locally nilpotent derivations.
2. Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, i.e. \( \mathcal{O}(V)^U \) is factorially closed. In particular if \( \mathcal{O}(V) \) is a UFD so is \( \mathcal{O}(V)^U \).

We will use the fact that \( U \) is a special group in the sense of Serre. This means that a \( U \) action which is locally trivial for the étale topology is locally trivial for the Zariski topology. If \( G \) is a group acting on a variety \( X \), we denote by \( X//G \) the algebraic quotient \( X//G := \text{Spec} \mathcal{O}(X)^G \) and by \( X/G \) the geometric quotient (when it exists). By a free action we mean an action for which the isotropy subgroup of each element consists only of the identity. (A free action is faithful.)

The paper is organized as follows: Section 2 contains some examples which illustrate the main results and clarify their hypotheses. The main results are proved in Section 3 from a geometric perspective, and Section 4 gives them an algebraic interpretation. (The algebraic and geometric viewpoint both have their merits: the geometric viewpoint lends itself to possible generalizations, while the algebraic proofs are constructive and can be more easily used in algorithms.) In section 5 we elaborate on some implications of the main results for the Sathaye conjecture, and on the motivation for studying this problem.

### 2. Examples

The following examples are valuable in various parts of the text.

**Example 1.** Let \( X = k^3 \), and \( U := \{ u_{a,b,c} \mid a, b, c \in k \} \) where

\[
\begin{pmatrix}
1 & a & b \\
0 & 1 & c \\
0 & 0 & 1
\end{pmatrix}
\]

a unipotent group acting by \( u_{a,b,c}(x, y, z) = (x + a, y + az + b, z + c) \) (which indeed is an algebraic action). For each \( (a, b, c) \in k^3 \) we thus have an automorphism, and its associated derivation on \( k[X, Y, Z] \) is \( D_{a,b,c} = a\partial_X + (aZ + b - \frac{ac}{2})\partial_Y + c\partial_Z \). Set \( D_1 = \partial_Y, D_2 := \partial_X + Z\partial_Y, D_3 = \partial_Z \). As a Lie algebra \( \text{Lie}(U) \) is generated by \( D_1, D_2, D_3 \). One checks that \( D_1 \) commutes
with $D_2, D_3$, but $[D_2, D_3] = D_1$. However, restricted to $k[X,Y,Z]^D_1 = k[X,Z]$, $D_2$ and $D_3$ do commute, as they coincide with the derivations $\partial_X$ and $\partial_Z$. Furthermore, as a $k$ vector space $\text{Lie}(U)$ has basis $\partial_X, \partial_Y, \partial_Z$.

**Example 2.** Let $\mathcal{O}(X) = A = k[X,Y,Z]$, and $D_1 = Z\partial_X, D_2 = \partial_Y$. These locally nilpotent derivations generate a $U = (G_a)^2$-action on $k^3$ given by $(a,b) \cdot (x,y,z) \mapsto (x + az, y + b, z)$. Now $k[Z] = A^{D_1, D_2} = \mathcal{O}(X/U)$. $D_1, D_2$ are linearly independent over $k[Z]$. When calculating modulo $Z - \alpha$ where $\alpha \in k$, we notice that $D_1 \mod (Z - \alpha), D_2 \mod (Z - \alpha)$ are linearly independent over $A/(Z - \alpha)$ except when $\alpha = 0$. However, defining $\mathcal{M} := (\text{Lie}(U) \otimes k(Z)) \cap \text{DER}(A) = (k[Z]D_1 + k[Z]D_2) \cap \text{DER}(A)$ we see that $\mathcal{M} = k[Z]\partial_X + k[Z]\partial_Y$. The derivations $\partial_X, \partial_Y$ are linearly independent modulo each $Z - \alpha$. And for each $\alpha \in k$, we have $A/(Z - \alpha) \cong k^2$.

**Example 3.** Let $P := X^2Y + X + Z^2 + T^3$, $\mathcal{X} := \{(x,y,z,t) | P(x,y,z,t) = 0\}$. Let $A := k[x,y,z,t] := k[X,Y,Z,T] / (P) = \mathcal{O}(\mathcal{X})$. The commuting locally nilpotent derivations $2Z\partial_Y - X^2\partial_Z, 3T^2\partial_Y - X^2\partial_T$ on $k[X,Y,Z,T]$ map $P$ to zero, and hence induce derivations $D_1, D_2$ on $A$. They are linearly independent over $A^{D_1, D_2} = k[X]$ and since they commute, induce a $(G_a)^2$-action on $\mathcal{X}$. Modulo $X - \alpha$, $D_1, D_2$ are linearly independent, except when $\alpha = 0$. Now defining $\mathcal{M} := (\text{Lie}(U) \otimes k(X)) \cap \text{DER}(A) = k[X]D_1 + k[X]D_2 = \text{Lie}(U) \otimes k[X]$, we see that $\mathcal{M}$ modulo $X - \alpha$ is a $k$-module of dimension $2$ except when $\alpha = 0$, when it is of dimension $1$. Also, $A/(X - \alpha) \cong k^2$ except when $\alpha = 0$, when it is isomorphic to $R[X]$ where $R = k[Z,T] / (Z^2 + T^3)$.

**Example 4.** The $U = G_a \times G_a$ action on $\mathbb{A}^2(k)$ given by

$$U \times \mathbb{A}^2 \ni ((s,t),(x,y)) \mapsto (x,y + t + sx) \in \mathbb{A}^2$$

is faithful and fixed point free. However every point in $\mathbb{A}^2$ has a non-trivial isotropy subgroup. If $x \neq 0$, then $((s,-sx),(x,y)) \mapsto (x,y)$ and $((s,0),(0,y)) \mapsto (0,y)$.

### 3. Main Results

The following simple lemma is useful in a number of places.

**Lemma 1.** Let $U$ be a unipotent algebraic group acting algebraically on a factorial quasiaffine variety $X$ of dimension $n$ satisfying $\mathcal{O}(X)^* = k^*$. If the action is not transitive and some point $x \in X$ has orbit of dimension $n - 1$, then $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$

**Proof.** Since $n - 1$ is the maximum orbit dimension there is a Zariski open subset $V$ of $X$ for which the geometric quotient $V/U$ exists as a variety. Then the transcendence degree of the quotient field $K$ of $\mathcal{O}(V/U)$ is equal to $1$. Since $K = qf(\mathcal{O}(X)^U)$ and

$$\mathcal{O}(X)^U = \mathcal{O}(X) \cap K,$$
is a ufd, $\mathcal{O}(X)^U$ is finitely generated over $k$. From $(\mathcal{O}(X)^U)^* = k^*$, we conclude that $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$.

### 3.1. Unipotent actions having zero-dimensional quotient.

**Theorem 1.** Let $U$ be an $n$-dimensional unipotent group acting faithfully on an affine $n$-dimensional variety $X$ satisfying $\mathcal{O}(X)^* = k^*$. If either

a) Some $x \in X$ has trivial isotropy subgroup or

b) $n = 2$, $X$ is factorial, and $U$ acts without fixed points, then the action is transitive. In particular $X \cong k^n$.

**Proof.** In case a) there is an open affine subset $V$ of $X$ on which $U$ acts without fixed points. Since $U$ has the same dimension as $V$, $V//U$ is zero-dimensional, hence $\mathcal{O}(V//U)$ is a field. This field contains $k$, and its units are contained in $\mathcal{O}(X)^* = k^*$, hence $\mathcal{O}(V//U) = k$. It follows that there exists an open set $V'$ of $X$ for which $V'/U \cong \text{Spec } k$. Thus $V' \cong U$ as a variety, and therefore $V' \cong k^n$. If $v \in V'$, then $Uv = V'$. Since $U$ is unipotent, all orbits are closed, hence $V'$ is closed in $X$. Since it is of dimension $n$, and $X$ is irreducible of dimension $n$, we have that $V' = X$.

In case b) $X$ is necessarily smooth since it is smooth in codimension 1 and every orbit is infinite. If $X$ has a two dimensional (i.e. dense) orbit then the conclusion follows as in a). So we assume for each $x \in X$ that the orbit $Ux$ is one dimensional, given as $\exp(uD)x$, and therefore isomorphic to $\mathbb{A}^1(k)$ by the discussion in the introduction. From Lemma 1 we conclude that $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$.

Note that factorial closure of $\mathcal{O}(X)^U$ implies that $f - \lambda$ is irreducible for every $\lambda \in k$. The absence of nonconstant units implies that $X \to \text{Spec}(k[f])$ is surjective, and all fibers are $U$ orbits. Smoothness of $X$ implies in addition that this mapping is flat, hence an $\mathbb{A}^1$ bundle over $\mathbb{A}^1$. But any such bundle is trivial, so we conclude again that $X \cong \mathbb{A}^2$. $\square$

Example 4 of the previous section illustrates case b).

### 3.2. Unipotent actions having one-dimensional quotient.

The following theorem is the main result of this paper.

**Theorem 2 (Main theorem).** Let $U$ be a unipotent algebraic group of dimension $n$, acting on $X$, a factorial variety of dimension $n + 1$ satisfying $\mathcal{O}(X)^* = k^*$.

1. If at least one $x \in X$ has trivial stabilizer then $\mathcal{O}(X)^U = \mathcal{O}(X/U) = k[f]$. Furthermore, $f^{-1}(\lambda) \cong k^n$ for all but finitely many $\lambda \in k$.

2. If $U$ acts freely, then $X$ is $U$-isomorphic to $U \times k$. In particular, $X \cong k^{n+1}$ and $f$ is a coordinate.

An important example to keep in mind is example [1] as this satisfies (1) but not (2). (There $U = \mathcal{G}_a^2$.)
**Proof of theorem**

**Claim 1:** \( \mathcal{O}(X)^U = k[f] \).

*Proof of claim 1:* This follows from lemma 1.

**Claim 2:** \( f : X \to k \) is surjective and has fibers isomorphic to \( U \). The fibers are the \( U \)-orbits.

*Proof of claim 2:* The fibers \( f^{-1}(\lambda) \) are the zero loci of the irreducible \( f - \lambda \), and are invariant under \( U \). Since \( U \) acts freely on each fiber and orbits of unipotent group actions are closed, we see that the \( f \) fibers are exactly the \( U \) orbits in \( X \). Thus \( f \) is a \( U \)-fibration (and, as the underlying variety of \( U \) is \( k^n \), an \( \mathbb{A}^n \)-fibration).

**Claim 3:** \( X \) is smooth.

*Proof of claim 3:* The set \( X_{\text{sing}} \) is \( U \)-stable, hence it is a union of \( U \)-orbits. The \( U \)-orbits are the zero sets \( f - \lambda \), hence of codimension 1. So \( X_{\text{sing}} \) is of codimension 1 or empty. But \( X \) is factorial, so in particular normal, which implies that the set of singular points of \( X \), denoted by \( X_{\text{sing}} \), is of codimension at least 2. This means that \( X_{\text{sing}} \) can only be empty.

**Claim 4:** \( f \) is smooth.

*Proof of claim 4:* All fibers of \( f \) are isomorphic to \( U \), hence to \( k^n \), by claim 2. Thus the fibers of \( f \) are geometrically regular of dimension \( n \). Since \( X \) is smooth, \( f \) is flat, and proposition 10.2 of [2] yields that \( f \) is smooth.

**Claim 5:** \( X \times_f X \) is smooth.

*Proof of claim 5:* \( X \times_f X \) is smooth since it is a base extension of the smooth \( X \) by the smooth morphism \( f \).

**Claim 6:** \( g : U \times X \to X \times_f X \) given by \( (u, x) \mapsto (x, ux) \) is an isomorphism.

*Proof of claim 6:* The map \( g \) restricted to \( U \times f^{-1}(\lambda) \) is a bijection onto \( \{(x, y) \mid f(x) = f(y) = \lambda \} \). Taking the union over \( \lambda \in k \), we get that \( g \) is a bijection. Since both \( U \times X \) and \( X \times_f X \) are smooth and \( g \) is a bijection, Zariski’s Main Theorem implies that \( g \) is an open immersion if it is birational. If so then \( g \) must be an isomorphism since it is bijective.

From Rosenlicht’s cross section theorem [6], \( X \) has a \( U \) stable open subset \( \tilde{X} \) on which the \( U \) action has a geometric quotient \( \tilde{X}/U \) and a \( U \) equivariant isomorphism \( \tilde{X} \cong U \times \tilde{X}/U \). Restricting \( g \) to \( U \times \tilde{X} \to \tilde{X} \times_f \tilde{X} \) is clearly an isomorphism, so that \( g \) is birational.

Now we are ready to prove the theorem. Using def. 0.10 p.16 of [5], and the fact (4) that \( f \) is smooth, together with (6), yields that \( f : X \to \mathbb{A}^1 \) is an étale principal \( U \)-bundle and therefore a Zariski locally trivial principal \( U \) bundle as \( U \) is special. Such bundles are classified by the cohomology group \( H^1(U, k) \), which is trivial because \( U \) is unipotent. Thus the bundle \( f : X \to k \) is trivial, which means that \( X \cong U \times k \). \( \square \)

**Remark 1.** (1) To obtain \( \tilde{X} \) explicitly and avoid the use of Rosenlicht’s theorem, recall that the action of \( U \) is generated by a finite set of \( G_a \) actions each one given as the exponential of some locally nilpotent derivation \( D_i \) of \( \mathcal{O}(X) \), indeed \( D_i \in u \), the Lie algebra of
U. As such there is an open subset $X_i$ of $X$ on which $D_i$ has a slice, and the corresponding $G_a$ acts by translation. Then $\tilde{X} := \cap_{i=1}^s X_i$.

(2) One can avoid the use of the étale topology by applying a "Seshadri cover" [7]. One constructs a variety $Z$ finite over $X$, necessarily affine, to which the $U$ action extends so that
(a) $k(Z)/k(X)$ is Galois. Denote the Galois group by $\Gamma$.
(b) The $\Gamma$ and $U$ actions commute on $Z$.
(c) The $U$ action on $Z$ is Zariski locally trivial and, because the action on $X$ is proper by claim 6,
(d) $Y \equiv Z/U$ exists as a separated scheme of dimension 1, hence a curve, and affine because of the existence of nonconstant globally defined regular functions, namely $O(Z)$.
(e) $O(X)^U \cong O(Y)^\Gamma$ and $X//U \cong Y/\Gamma$ shows that $X \to X/U$ is Zariski locally trivial.

4. Algebraic Version

4.1. Unipotent actions having zero dimensional kernel. Let $X$ be a quasiaffine variety, and $U$ an algebraic group acting on $X$. We write $A := O(X)$ and denote by $u$ the Lie algebra of $U$. In this section, we will make the following assumptions:

(P) a) $X$ and $U$ are of dimension $n$.

b) There is a point $x \in X$ such that $\text{stab}(x) = \{e\}$.

c) $O(X)^* = k^*$

Definition 1. Assume (P). We say that $D_1, \ldots, D_n$ is a triangular basis of $u$ (with respect to the action on $X$) if

(1) $u = kD_1 \oplus kD_2 \oplus \ldots \oplus kD_n$ and

(2) With subalgebras $A_i$ of $A$ given by $A_1 := A$, $A_i := A^{D_1} \cap \ldots \cap A^{D_{i-1}}$, the restriction of $D_i$ to $A_i$ commutes with the restrictions of $D_{i+1}, \ldots, D_n$.

For a triangular basis, it is clear that $D_j(A_j) \subseteq A_j$ for each $j$.

If $U$ is unipotent then the existence of a triangular basis is a consequence of the Lie-Kolchin theorem. Indeed, the Lie algebra $u$ of $U$ is isomorphic to a Lie subalgebra of the full Lie algebra of upper triangular matrices over $k$. In particular $u$ has a basis $D_1, \ldots, D_n$ satisfying $[D_i, D_j] \in \text{span}\{D_1, \ldots, D_{\min\{i,j\}}\}$. By definition of the $A_i$ this basis is triangular with respect to the action and $D_1$ is in the center of $u$.

Proposition 1. Assume (P) and $U$ unipotent. Then $A \cong k[s_1, \ldots, s_n] = k^{[n]}$ where $D_1(s_i) = 1$, and $D_i(s_j) = 0$ if $j > i$.

Proof. We proceed by induction $n = \dim u$. If $n = 1$, then we have one nonzero LND on a dimension one $k$-algebra domain $A$ satisfying $A^* = k^*$. It is well-known that this means that $A \cong k[x]$ and the derivation is simply $\partial_x$. Suppose the theorem is proved for $n - 1$. Let $D_1, D_2, \ldots, D_n$ be a triangular
Next we consider a preslice $p \in A$ such that $D_1(p) = q, D_1(q) = 0$, i.e. $q = q(s_2, \ldots, s_n)$. We pick $p$ in such a way that $q$ is of lowest possible lexicographic degree w.r.t $s_2 >> s_3 >> \ldots >> s_n$. Now $D_1(D_2(p)) = D_2D_1(p) = D_2(q)$. Restricted to $k[s_2, \ldots, s_n]$, $D_2 = \partial_{s_2}$, so $D_2(q)$ is of lower $s_2$-degree than $q$. Unless $D_2(q) = 0$, we get a contradiction with the degree requirements of $q$, as $D_2(p)$ would be a “better” preslice having a lower degree derivative. Thus, $q \in k[s_3, \ldots, s_n]$. Using the same argument for $D_3, D_4$ etc. we get that $q \in k^*$. Hence, $p$ is in fact a slice. \hfill \Box

4.2. Unipotent actions having one-dimensional quotient. With the same notations as in the previous section, we also denote the ring of $U$ invariants in $A$ by $A^U$ and Spec $A^U$ by $X//U$. Note that $A^U = \{a \in A \mid D(a) = 0 \text{ for all } D \in u\}$. If $U$ is unipotent and $D_1, \ldots, D_n$ is a triangular basis of $u$, we again write $A_1 := A, A_{i+1} = A_i \cap A^{D_1}$, noting that $A^U = A_n$.

In this section we consider the conditions:

(Q1) $U$ is a unipotent algebraic group of dimension $n$ acting on an affine variety $X$ of dimension $n + 1$ with $A^* = k^*$.

and:

(Q) $A^U = k[f]$ for some irreducible $f \in A \setminus k$.

Remark 2. According to Lemma 1, condition (Q1) along with the assumption that $X$ is factorial and the existence of a point $x \in X$ with $\text{stab}(x) = \{e\}$, implies that (Q) holds.

Notation 1. Assuming (Q), let $a \in k$. Set $\overline{A} := A/(a - \alpha)$ and write $\overline{\alpha}$ for the residue class of $a$ in $\overline{A}$ and $\overline{D}$ for the derivation induced by $D \in u$ on $\overline{A}$.

Our goal is to prove the following constructively:

Theorem 3. Assume (Q1) and (Q). Let $D_1, \ldots, D_n$ be a triangular basis of $u$.

(1) For $\alpha \in k$,

(a) If $\overline{D_1}, \ldots, \overline{D_n}$ are independent over $A/(f - \alpha)$, then

$$A/(f - \alpha) \cong k^{[n]}.$$

(b) There are only finitely many $\alpha$ for which $\overline{D_1}, \ldots, \overline{D_n}$ are dependent over $A/(f - \alpha)$.

(2) In the case that $\overline{D_1}, \ldots, \overline{D_n}$ are independent over $A/(f - \alpha)$ for each $\alpha \in k$, then there are $s_1, \ldots, s_n \in A$ with $A = k[s_1, \ldots, s_n, f]$, hence $A$ is isomorphic to a polynomial ring in $n + 1$ variables (and $f$ is a coordinate).
Definition 2. Assume (Q1) and (Q), and a triangular basis $D_1, \ldots, D_n$ of $u$. Define

$$\mathcal{P}_i := \{ p \in A \mid D_i(p) \in k[f], \ D_j(p) = 0 \text{ if } j < i \}$$

and

$$\mathcal{J}_i := D_i(\mathcal{P}_i) \subseteq k[f].$$

Thus $\mathcal{P}_i$ is the set of ”preslices” of $D_i$ that are compatible with the triangular basis $D_1, \ldots, D_n$.

Lemma 2. There exist $p_i \in \mathcal{P}_i \setminus \{0\}, p_i \in A_i$, and $q_i \in k^{[1]} \setminus \{0\}$ such that $\mathcal{J}_i = q_i(f)k[f]$ and $D_i(p_i) = q_i$.

Proof. First note that $\mathcal{J}_i$ is not empty, as theorem $i$ applied to $A(f) := A \otimes k(f)$ gives an $s_i \in A(f)$ which satisfies $D_i(s_i) = 1$, $D_j(s_i) = 0$ if $j < i$. Multiplying $s_i$ by a suitable element of $k[f]$ gives a nonzero element $r(f)s_i$ of $\mathcal{P}_i$, and $D_i(r(f)s_i) = r(f)$. Because $k[f] = \cap \ker(D_i)$, $\mathcal{P}_i$ is a $k[f]$-module, and therefore $\mathcal{J}_i$ is an ideal of $k[f]$. This means that $\mathcal{J}_i$ is a principal ideal, and we take for $q_i$ a generator (and $p_i \in D_i^{-1}(q_i)$). Since $D_j(p_i) = 0$ if $j < i$, we have $p_i \in A_i$. \hfill $\Box$

Corollary 1. The $p_i$, $1 \leq i \leq n$, are algebraically independent over $k$.

Proof. The $s_i$ are certainly algebraically independent, and $p_i \in k[f]s_i$. \hfill $\Box$

Lemma 3. Assume (Q), and take $p_i, q_i$ as in lemma $i$. Then the $D_i$ are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some $i$.

Proof. ($\Rightarrow$): Suppose that $0 \neq D := g_1D_1 + \ldots + g_nD_n$ satisfies $\overline{D} = 0$ where $g_i \in A$, and not all $\overline{\mathcal{P}}_i = 0$. Let $i$ be the highest such that $\overline{g}_i \neq 0$. Then $0 = \overline{D}(p_i) = \overline{g}_i\overline{D}(p_i) = \overline{g}_i\overline{q}_i(f)$. Since $\overline{A}$ is a domain, $q_i(\alpha) = q_i(f) = 0$.

($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $\overline{D}_i$ are linearly dependent over $A/(f - \alpha)$. Consider $\overline{D}_i$ restricted to $\overline{A}_i$. If $j > i$ then $\overline{D}_i(p_j) = \overline{D}_i(p_j) = 0$. Furthermore $\overline{D}_i(p_i) = \overline{q}_i(f) = q(\alpha) = 0$. Hence, $\overline{D}_i$ is zero if restricted to $k[\overline{p}_i, \ldots, \overline{p}_n]$. But since this is of transcendence degree $n$, it follows that $\overline{D}_i = 0$ on $\overline{A}_i$. Reversing the argument of ($\Rightarrow$) yields the linear dependence of the $\overline{D}_i$. \hfill $\Box$

Proof. (of theorem $\mathfrak{3}$) Part 1: If $\overline{D}_1, \ldots, \overline{D}_n$ are independent, then Proposition 1 yields that $\overline{A} \cong k^{[n]}$. Lemma $\mathfrak{3}$ states that for any point $\alpha$ outside the zero set of $q_1q_2 \cdots q_n$ we have $A/(f - \alpha) \cong k^{[n]}$. This zero set is either all of $k$ or finite, yielding part 1.

Part 2: Lemma $\mathfrak{3}$ tells us directly that for each $1 \leq i \leq n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the $p_i$ can be taken to be actual slices ($s_i = p_i$). Using the fact that $s_i \in A_i$ we obtain that $A = A_1 = A_2[s_1] = A_3[s_2, s_1] = \ldots = A_{n+1}[s_1, \ldots, s_n] = k[s_1, \ldots, s_n, f]$ as claimed. \hfill $\Box$
5. Consequences of the main theorems

This paper is originally motivated by the following result of [4]:

**Theorem 4.** Let \( A = k[x, y, z] \) and \( D_1, D_2 \) two commuting locally nilpotent derivations on \( A \) which are linearly independent over \( A \). Then \( A^{D_1, D_2} = k[f] \) and \( f \) is a coordinate.

Here the notation \( A^{D_1, D_2} \) means \( A^{D_1} \cap A^{D_2} \) the intersections of the kernels of \( D_1 \) and \( D_2 \), which is the set of elements vanishing under \( D_1 \) resp. \( D_2 \). (Note that for the \( G_a \) action associated to \( D \), this notation means \( \mathcal{O}(X/G_a) = \mathcal{O}(X)^{G_a} = \mathcal{O}(X)^D \).) By a coordinate is meant an element \( f \) in \( k^n \) for which there exist \( f_2, \ldots, f_n \) with \( k[f, f_2, \ldots, f_n] = k^n \). Equivalently, \( (f, f_2, \ldots, f_n) : k^n \to k^n \) is an automorphism. The most important ingredient of this theorem is Kaliman’s theorem [3].

In [4] it is conjectured that this result is true also in higher dimensions, i.e. having \( n \) commuting linearly independent locally nilpotent derivations on \( k^{n+1} \) should yield that their common kernel is generated by a coordinate. However, it seems that this conjecture is very hard, on a par with the well-known Sathaye conjecture:

**SC**(\( n \)) **Sathaye-conjecture**: Let \( f \in A := k^n \) such that \( A/(f - \lambda) \cong k^{n-1} \). Then \( f \) is a coordinate.

The Sathaye conjecture is proved for \( n \leq 3 \) by the aforementioned Kaliman’s theorem. Therefore, the original motivation was to find additional requirements in higher dimensions to achieve the result that \( f \) is a coordinate. The results in this paper give one such requirement, namely that \( k^n/(f - \lambda) \cong k^{n-1} \) for all constants \( \lambda \).

Another consequence of the result of this paper is that the Sathaye conjecture is equivalent to

**MSC**(\( n \)) **Modified Sathaye Conjecture**: Let \( A := k^n \), and let \( f \in A \) be such that \( A/(f - \alpha) \cong k^{n-1} \) for all \( \alpha \in k \). Then there exist \( n-1 \) commuting locally nilpotent derivations \( D_1, \ldots, D_{n-1} \) on \( A \) such that \( A^{D_1, \ldots, D_{n-1}} = k[f] \) and the \( D_i \) are linearly independent modulo \( (f - \alpha) \) for each \( \alpha \in k \).

**Proof of equivalence of SC**(\( n \)) **and MSC**(\( n \)). Suppose we have proven the MSC\((n)\). Then for any \( f \) satisfying “\( A/(f - \alpha) \cong k^{n-1} \)” for all \( \alpha \in k \) we can find commuting LNDs \( D_1, \ldots, D_{n-1} \) on \( A \) giving rise to a \( G_a^{n-1} \) action satisfying the hypotheses of Theorem 2. Applying this theorem, we obtain that \( f \) is a coordinate in \( A \). So the SC\((n)\) is true in that case.

Now suppose we have proven the SC\((n)\). Let \( f \) satisfy the requirements of the MSC\((n)\), that is, “\( A/(f - \alpha) \cong k^{n-1} \)” for all \( \alpha \in k \). Since \( f \) satisfies the requirements of SC\((n)\), \( f \) then must be a coordinate. So it has \( n-1 \) so-called mates: \( k[f, f_2, \ldots, f_n] = k^n \). But then the partial derivative with respect to each of these \( n \) polynomials \( f, f_2, \ldots, f_n \) defines a locally nilpotent
derivation. All of them commute, and the intersection of the kernels of the last \( n-1 \) derivations is \( k[f] \); so the MSC holds.

\[ \square \]

References


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