Unipotent group actions on affine varieties

H.Derksen*, A.van den Essen, D. R.Finston, and S.Maubach**

Abstract. Algebraic actions of unipotent groups U actions on affine $k-$varieties X (k an algebraically closed field of characteristic 0) for which the algebraic quotient $X//U$ has small dimension are considered. In case X is factorial, $O(X)^* = k^*$, and $X//U$ is one-dimensional, it is shown that $O(X)^U{=}k[f]$, and if some point in X has trivial isotropy, then X is U equivariantly isomorphic to $U \times A^1(k)$. The main results are given distinct geometric and algebraic proofs. Links to the Abhyankar-Sathaye conjecture and a new equivalent formulation of the Sathaye conjecture are made.

1. Preliminaries and Introduction

Throughout, k will denote a field of characteristic zero, $k^{[n]}$ the polynomial ring in n variables over k, and U a unipotent algebraic group over k. Our interest is in algebraic actions of such U on affine $k-$varieties X (equivalently on their coordinate rings $\mathcal{O}(X)$). An algebraic action of the one dimensional unipotent group $\mathcal{G}_a = (k, +)$ is conveniently described through the action of a locally nilpotent derivation D of $\mathcal{O}(X)$. Specifically, for $u \in \mathcal{G}_a = k$, we have the automorphism u^* acting on $\mathcal{O}(X)$ and it is well-known (see for example [**1**] page 16-17) that there exists a unique locally nilpotent derivation $D : \mathcal{O}(X) \longrightarrow \mathcal{O}(X)$ such that $u^* = \exp(uD)$. (One can obtain D by taking $D(f) = \frac{u^*f-f}{u}|_{u=0}$.) Similarly, if $\mathcal{G}_a^n$ acts on X, then we have for each component $\mathcal{G}_a$-action a locally nilpotent derivation D_i, and for each element $u = (u_1, \ldots, u_n) \in \mathcal{G}_a^n$ we have the derivation $D := u_1D_1 + \ldots + u_nD_n$. If the action is faithful, there is a canonical isomorphism of $\mathrm{Lie}(\mathcal{G}_a^n)$ with $kD_1 + \ldots + kD_n$. In this case, the D_i commute.

The situation is similar for a general unipotent group action $U \times X \longrightarrow X$. Because the action is algebraic, each $f \in \mathcal{O}(X)$ is contained in a finite dimensional U stable subspace V_f on which U acts by linear transformations. Since U is unipotent, for each $u \in U$, $u^* - id$ is nilpotent on V_f , so that

*Funded by NSF CAREER grant, DMS 0349019, Invariant Theory, Algorithms and Applications.

**Funded by Veni-grant of council for the physical sciences, Netherlands Organisation for scientific research (NWO).

$\ln(u)(g) = \sum_{j=1}^{\infty} \frac{(u^*-id)^j g}{j}$ is a finite sum for all $g \in V_f$. One checks that $D_u \equiv \ln(u)$ is a (locally nilpotent) derivation of $\mathcal{O}(X)$ and $u^* = \exp(D_u)$. If the action is faithful, i.e. $U \to Aut(X)$ is injective, there is a canonical isomorphism of Lie(U) with $\{D_u \mid u \in U\}$. In fact, Lie$(U) = kD_1 + \ldots + kD_m$ ($m = \dim(U)$) for some locally nilpotent derivations D_i. In general the D_i do not commute. In fact, all of them commute if and only if $U = \mathcal{G}_a^m$.

Two useful facts about unipotent group actions on quasiaffine varieties V can be immediately derived from these observations:

(1) Because each $u \in U$ acts via a locally nilpotent derivation of $O(V)$, the ring of invariants $O(V)^U$ is the intersection of the kernels of locally nilpotent derivations.
(2) Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, i.e. $O(V)^U$ is factorially closed. In particular if $O(V)$ is a UFD so is $O(V)^U$.

We will use the fact that U is a special group in the sense of Serre. This means that a U action which is locally trivial for the étale topology is locally trivial for the Zariski topology. If G is a group acting on a variety X, we denote by $X//G$ the algebraic quotient $X//G :=$**Spec** $\mathcal{O}(X)^G$ and by X/G the geometric quotient (when it exists). By a free action we mean an action for which the isotropy subgroup of each element consists only of the identity. (A free action is faithful.)

The paper is organized as follows: Section 2 contains some examples which illustrate the main results and clarify their hypotheses. The main results are proved in Section 3 from a geometric perspective, and Section 4 gives them an algebraic interpretation. (The algebraic and geometric viewpoint both have their merits: the geometric viewpoint lends itself to possible generalizations, while the algebraic proofs are constructive and can be more easily used in algorithms.) In section 5 we elaborate on some implications of the main results for the Sathaye conjecture, and on the motivation for studying this problem.

2. Examples

The following examples are valuable in various parts of the text.

Example 1. *Let $X = k^3$, and $U := \{u_{a,b,c} \mid a, b, c \in k\}$ where*

$$u_{a,b,c} := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

a unipotent group acting by $u_{a,b,c}(x, y, z) = (x + a, y + az + b, z + c)$ (which indeed is an algebraic action). For each $(a, b, c) \in k^3$ we thus have an automorphism, and its associated derivation on $k[X, Y, Z]$ is $D_{a,b,c} = a\partial_X + (aZ + b - \frac{ac}{2})\partial_Y + c\partial_Z$. Set $D_1 = \partial_Y, D_2 := \partial_X + Z\partial_Y, D_3 = \partial_Z$. As a Lie algebra Lie$(U)$ *is generated by D_1, D_2, D_3. One checks that D_1 commutes* *with D_2, D_3, but $[D_2, D_3] = D_1$. However, restricted to $k[X,Y,Z]^{D_1} = k[X,Z]$, D_2 and D_3 do commute, as they coincide with the derivations ∂_X and ∂_Z. Furthermore, as a k vector space $\mathrm{Lie}(U)$ has basis $\partial_X, \partial_Y, \partial_Z$.*

Example 2. *Let $\mathcal{O}(X) = A = k[X,Y,Z]$, and $D_1 = Z\partial_X, D_2 = \partial_Y$. These locally nilpotent derivations generate a $U = (\mathcal{G}_a)^2$-action on k^3 given by $(a,b)\cdot(x,y,z) \longrightarrow (x+az, y+b, z)$. Now $k[Z] = A^{D_1,D_2} = \mathcal{O}(X/U)$. D_1, D_2 are linearly independent over $k[Z]$. When calculating modulo $Z-\alpha$ where $\alpha \in k$, we notice that $D_1 \mod (Z-\alpha), D_2 \mod (Z-\alpha)$ are linearly independent over $A/(Z-\alpha)$ except when $\alpha = 0$. However, defining $\mathcal{M} := (\mathrm{Lie}(U) \otimes k(Z)) \cap DER(A) = (k(Z)D_1 + k(Z)D_2) \cap DER(A)$ we see that $\mathcal{M} = k[Z]\partial_X + k[Z]\partial_Y$. The derivations ∂_X, ∂_Y are linearly independent modulo each $Z-\alpha$. And for each $\alpha \in k$, we have $A/(Z-\alpha) \cong k^{[2]}$.*

Example 3. *Let $P := X^2Y + X + Z^2 + T^3$, $\mathcal{X} := \{(x,y,z,t) \mid P(x,y,z,t) = 0\}$. Let $A := k[x,y,z,t] := k[X,Y,Z,T]/(P) = \mathcal{O}(\mathcal{X})$. The commuting locally nilpotent derivations $2Z\partial_Y - X^2\partial_Z$, $3T^2\partial_Y - X^2\partial_T$ on $k[X,Y,Z,T]$ map P to zero, and hence induce derivations D_1, D_2 on A. They are linearly independent over $A^{D_1,D_2} = k[X]$ and since they commute, induce a $(\mathcal{G}_a)^2$-action on $\mathcal{X}$. Modulo $X-\alpha$, D_1, D_2 are linearly independent, except when $\alpha = 0$. Now defining $\mathcal{M} := (\mathrm{Lie}(U) \otimes k(X)) \cap DER(A) = k[X]D_1 + k[X]D_2 = \mathrm{Lie}(U) \otimes k[X]$, we see that $\mathcal{M}$ modulo $X-\alpha$ is a k-module of dimension 2 except when $\alpha = 0$, when it is of dimension 1. Also, $A/(X-\alpha) \cong k^{[2]}$ except when $\alpha = 0$, when it is isomorphic to $R[X]$ where $R = k[Z,T]/(Z^2+T^3)$.*

Example 4. *The $U = \mathcal{G}_a \times \mathcal{G}_a$ action on $\mathbb{A}^2(k)$ given by*

$$U \times \mathbb{A}^2 \ni ((s,t),(x,y) \mapsto (x, y+t+sx) \in \mathbb{A}^2$$

is faithful and fixed point free. However every point in $\mathbb{A}^2$ has a non-trivial isotropy subgroup. If $x \neq 0$, then $((s,-sx),(x,y) \mapsto (x,y)$ and $((s,0),(0,y)) \mapsto (0,y)$.

3. Main Results

The following simple lemma is useful in a number of places.

Lemma 1. *Let U be a unipotent algebraic group acting algebraically on a factorial quasiaffine variety X of dimension n satisfying $\mathcal{O}(X)^* = k^*$. If the action is not transitive and some point $x \in X$ has orbit of dimension $n-1$, then $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$*

Proof. Since $n-1$ is the maximum orbit dimension there is a Zariski open subset V of X for which the geometric quotient V/U exists as a variety. Then the transcendence degree of the quotient field K of $\mathcal{O}(V/U)$ is equal to 1. Since $K = qf(\mathcal{O}(X)^U)$ and

$$\mathcal{O}(X)^U = \mathcal{O}(X) \cap K,$$

is a ufd, $\mathcal{O}(X)^U$ is finitely generated over k. From $(\mathcal{O}(X)^U)^* = k^*$, we conclude that $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$ □

3.1. Unipotent actions having zero-dimensional quotient.

Theorem 1. *Let U be an n-dimensional unipotent group acting faithfully on an affine n-dimensional variety X satisfying $\mathcal{O}(X)^* = k^*$. If either*
a) Some $x \in X$ has trivial isotropy subgroup or
b) $n = 2$, X is factorial, and U acts without fixed points,
then the action is transitive. In particular $X \cong k^n$.

Proof. In case a) there is an open affine subset V of X on which U acts without fixed points. Since U has the same dimension as V, $V//U$ is zero-dimensional, hence $\mathcal{O}(V//U)$ is a field. This field contains k, and its units are contained in $\mathcal{O}(X)^* = k^*$, hence $\mathcal{O}(V//U) = k$. It follows that there exists an open set $V'$of X for which $V'/U \cong$**Spec** k. Thus $V' \cong U$ as a variety, and therefore $V' \cong k^n$. If $v \in V'$, then $Uv = V'$. Since U is unipotent, all orbits are closed, hence V' is closed in X. Since it is of dimension n, and X is irreducible of dimension n, we have that $V' = X$.

In case b) X is necessarily smooth since it is smooth in codimension 1 and every orbit is infinite. If X has a two dimensional (i.e. dense) orbit then the conclusion follows as in a). So we assume for each $x \in X$ that the orbit Ux is one dimensional, given as $\exp(uD)x$, and therefore isomorphic to $\mathbb{A}^1(k)$ by the discussion in the introduction. From Lemma 1 we conclude that $\mathcal{O}(X)^U = k[f]$ for some $f \in \mathcal{O}(X)$

Note that factorial closure of $\mathcal{O}(X)^U$ implies that $f - \lambda$ is irreducible for every $\lambda \in k$. The absence of nonconstant units implies that $X \to$**Spec**$(k[f])$ is surjective, and all fibers are U orbits. Smoothness of X implies in addition that this mapping is flat, hence an $\mathbb{A}^1$ bundle over $\mathbb{A}^1$. But any such bundle is trivial, so we conclude again that $X \cong \mathbb{A}^2$. □

Example 4 of the previous section illustrates case *b)*.

3.2. Unipotent actions having one-dimensional quotient.

The following theorem is the main result of this paper.

Theorem 2 (**Main theorem**)**.** *Let U be a unipotent algebraic group of dimension n, acting on X, a factorial variety of dimension $n+1$ satisfying $\mathcal{O}(X)^* = k^*$.*
(1) If at least one $x \in X$ has trivial stabilizer then $\mathcal{O}(X)^U = \mathcal{O}(X/\!\!/U) = k[f]$. Furthermore, $f^{-1}(\lambda) \cong k^n$ for all but finitely many $\lambda \in k$.
(2) If U acts freely, then X is U-isomorphic to $U \times k$. In particular, $X \simeq k^{n+1}$ and f is a coordinate.

An important example to keep in mind is example 1, as this satisfies (1) but not (2). (There $U = \mathcal{G}_a^2$.)

Proof of theorem 2.

CLAIM 1: $\mathcal{O}(X)^U = k[f]$.

Proof of claim 1: This follows from lemma 1.

CLAIM 2: $f : X \to k$ is surjective and has fibers isomorphic to U. The fibers are the U-orbits.

Proof of claim 2: The fibers $f^{-1}(\lambda)$ are the zero loci of the irreducible $f - \lambda$, and are invariant under U. Since U acts freely on each fiber and orbits of unipotent group actions are closed, we see that the f fibers are exactly the U orbits in X. Thus f is a U-fibration (and, as the underlying variety of U is k^n, an $\mathbb{A}^n$-fibration).

CLAIM 3: X is smooth.

Proof of claim 3: The set X_{sing} is U-stable, hence it is a union of U-orbits. The U-orbits are the zero sets $f - \lambda$, hence of codimension 1. So X_{sing} is of codimension 1 or empty. But X is factorial, so in particular normal, which implies that the set of singular points of X, denoted by X_{sing}, is of codimension at least 2. This means that X_{sing} can only be empty.

CLAIM 4: f is smooth.

Proof of claim 4: All fibers of f are isomorphic to U, hence to k^n, by claim 2. Thus the fibers of f are geometrically regular of dimension n. Since X is smooth, f is flat, and proposition 10.2 of [**2**] yields that f is smooth.

CLAIM 5: $X \times_f X$ is smooth.

Proof of claim 5: $X \times_f X$ is smooth since it is a base extension of the smooth X by the smooth morphism f.

CLAIM 6: $g : U \times X \to X \times_f X$ given by $(u, x) \mapsto (x, ux)$ is an isomorphism.

Proof of claim 6: The map g restricted to $U \times f^{-1}(\lambda)$ is a bijection onto $\{(x, y) \mid f(x) = f(y) = \lambda\}$. Taking the union over $\lambda \in k$, we get that g is a bijection. Since both $U \times X$ and $X \times_f X$ are smooth and g is a bijection, Zariski's Main Theorem implies that g is an open immersion if it is birational. If so then g must be an isomorphism since it is bijective.

From Rosenlicht's cross section theorem [**6**], X has a U stable open subset $\tilde{X}$ on which the U action has a geometric quotient $\tilde{X}/U$ and a U equivariant isomorphism $\tilde{X} \cong U \times \tilde{X}/U$. Restricting g to $U \times \tilde{X} \to \tilde{X} \times_f \tilde{X}$ is clearly an isomorphism, so that g is birational.

Now we are ready to prove the theorem. Using def. 0.10 p.16 of [**5**], and the fact (4) that f is smooth, together with (6), yields that $f : X \to \mathbb{A}^1$ is an étale principal U-bundle and therefore a Zariski locally trivial principal U bundle as U is special. Such bundles are classified by the cohomology group $H^1(U, k)$, which is trivial because U is unipotent. Thus the bundle $f : X \to k$ is trivial, which means that $X \cong U \times k$. □

Remark 1. (1) *To obtain $\tilde{X}$ explicitly and avoid the use of Rosenlicht's theorem, recall that the action of U is generated by a finite set of $\mathcal{G}_a$ actions each one given as the exponential of some locally nilpotent derivation D_i of $\mathcal{O}(X)$, indeed $D_i \in \mathfrak{u}$, the Lie algebra of*

U. As such there is an open subset X_i of X on which D_i has a slice, and the corresponding $\mathcal{G}_a$ acts by translation. Then $\tilde{X} := \cap_{i=1}^{s} X_i$.

(2) *One can avoid the use of the étale topology by applying a "Seshadri cover" [**7**]. One constructs a variety Z finite over X, necessarily affine, to which the U action extends so that*
   (a) *$k(Z)/k(X)$ is Galois. Denote the Galois group by Γ.*
   (b) *The Γ and U actions commute on Z.*
   (c) *The U action on Z is Zariski locally trivial and, because the action on X is proper by claim 6,*
   (d) *$Y \equiv Z/U$ exists as a separated scheme of dimension 1, hence a curve, and affine because of the existence of nonconstant globally defined regular functions, namely $\mathcal{O}(Z)^U$.*
   (e) *$\mathcal{O}(X)^U \cong \mathcal{O}(Y)^\Gamma$ and $X//U \cong X/U \cong Y/\Gamma$ shows that $X \to X/U$ is Zariski locally trivial.*

4. Algebraic Version

4.1. Unipotent actions having zero dimensional kernel. Let X be a quasiaffine variety, and U an algebraic group acting on X. We write $A := \mathcal{O}(X)$ and denote by $\mathfrak{u}$ the Lie algebra of U. In this section, we will make the following assumptions:

(P) *a)* X and U are of dimension n.
b) There is a point $x \in X$ such that $\mathrm{stab}(x) = \{e\}$.
c) $\mathcal{O}(X)^* = k^*$

Definition 1. *Assume (P). We say that $D_1, \ldots, D_n$ is a triangular basis of $\mathfrak{u}$ (with respect to the action on X) if*

(1) $\mathfrak{u} = kD_1 \oplus kD_2 \oplus \ldots \oplus kD_n$ and
(2) With subalgebras A_i of A given by $A_1 := A$, $A_i := A^{D_1} \cap \ldots \cap A^{D_{i-1}}$, the restriction of D_i to A_i commutes with the restrictions of $D_{i+1}, \ldots, D_n$.

For a triangular basis, it is clear that $D_j(A_j) \subseteq A_j$ for each j.

If U is unipotent then the existence of a triangular basis is a consequence of the Lie-Kolchin theorem. Indeed, the Lie algebra $\mathfrak{u}$ of U is isomorphic to a Lie subalgebra of the full Lie algebra of upper triangular matrices over k. In particular $\mathfrak{u}$ has a basis $D_1, \ldots, D_n$ satisfying $[D_i, D_j] \in span\{D_1, \ldots D_{\min\{i,j\}-1}\}$.. By definition of the A_i this basis is triangular with respect to the action and D_1 is in the center of $\mathfrak{u}$.

Proposition 1. *Assume (P) and U unipotent. Then $A \cong k[s_1, \ldots, s_n] = k^{[n]}$ where $D_i(s_i) = 1$, and $D_i(s_j) = 0$ if $j > i$.*

Proof. We proceed by induction $n = \dim \mathfrak{u}$. If $n = 1$, then we have one nonzero LND on a dimension one k-algebra domain A satisfying $A^* = k^*$. It is well-known that this means that $A \cong k[x]$ and the derivation is simply ∂_x. Suppose the theorem is proved for $n-1$. Let $D_1, D_2, \ldots D_n$ be a triangular basis for $\mathfrak{u}$. Restricting to A^{D_1} and noting that D_1 is in the center of $\mathfrak{u}$, we have an action of the Lie algebra $\mathfrak{u}/kD_1$ which has the triangular basis $k\overline{D_2} + \ldots + k\overline{D_n}$ ($\overline{D_i}$ denotes residue class modulo kD_1). By construction $\overline{D}_i(a) := D_i(a)$ is well defined, and by induction we find $s_2, \ldots, s_n \in A^{D_1}$ satisfying $D_i(s_i) = 1, D_i(s_j) = 0$ if $j > i \geq 2$. $D_i(s_j) = \delta_{ij}$.

Next we consider a preslice $p \in A$ such that $D_1(p) = q, D_1(q) = 0$, i.e. $q = q(s_2, \ldots, s_n)$. We pick p in such a way that q is of lowest possible lexicographic degree w.r.t $s_2 >> s_3 >> \ldots >> s_n$. Now $D_1(D_2(p)) = D_2D_1(p) = D_2(q)$. Restricted to $k[s_2, \ldots, s_n]$, $D_2 = \partial_{s_2}$, so $D_2(q)$ is of lower s_2-degree than q. Unless $D_2(q) = 0$, we get a contradiction with the degree requirements of q, as $D_2(p)$ would be a "better" preslice having a lower degree derivative. Thus, $q \in k[s_3, \ldots, s_n]$. Using the same argument for D_3, D_4 etc. we get that $q \in k^*$. Hence, p is in fact a slice. □

4.2. Unipotent actions having one-dimensional quotient. With the same notations as in the previous section, we also denote the ring of U invariants in A by A^U and **Spec** A^U by $X//U$. Note that $A^U = \{a \in A \mid D(a) = 0 \text{ for all } D \in \mathfrak{u}\}$. If U is unipotent and $D_1, \ldots, D_n$ is a triangular basis of $\mathfrak{u}$, we again write $A_1 := A$, $A_{i+1} = A_i \cap A^{D_i}$, noting that $A^U = A_n$. In this section we consider the conditions :

(Q1) *U is a unipotent algebraic group of dimension n acting on an affine variety X of dimension $n+1$ with $A^* = k^*$.*

and:

(Q) *$A^U = k[f]$ for some irreducible $f \in A \backslash k$.*

Remark 2. *According to Lemma 1, condition (Q1) along with the assumption that X is factorial and the existence of a point $x \in X$ with* $\mathrm{stab}(x) = \{e\}$, *implies that (Q) holds.*

Notation 1. *Assuming (Q), let $\alpha \in k$. Set $\overline{A} := A/(f - \alpha)$ and write $\overline{a}$ for the residue class of a in $\overline{A}$ and $\overline{D}$ for the derivation induced by $D \in \mathfrak{u}$ on $\overline{A}$.*

Our goal is to prove the following constructively:

Theorem 3. *Assume (Q1) and (Q). Let $D_1, \ldots, D_n$ be a triangular basis of $\mathfrak{u}$.*

(1) *For $\alpha \in k$,*
   (a) *If $\overline{D_1}, \ldots, \overline{D_n}$ are independent over $A/(f - \alpha)$, then*
   $$A/(f - \alpha) \cong k^{[n]}.$$
   (b) *There are only finitely many α for which $\overline{D_1}, \ldots, \overline{D_n}$ are dependent over $A/(f - \alpha)$.*

(2) *In the case that $\overline{D_1}, \ldots, \overline{D_n}$ are independent over $A/(f - \alpha)$ for each $\alpha \in k$, then there are $s_1, \ldots, s_n \in A$ with $A = k[s_1, \ldots, s_n, f]$, hence A is isomorphic to a polynomial ring in $n+1$ variables (and f is a coordinate).*

Definition 2. *Assume (Q1) and (Q), and a triangular basis $D_1, \ldots, D_n$ of $\mathfrak{u}$. Define*

$$\mathcal{P}_i := \{p \in A \mid D_i(p) \in k[f],\ D_j(p) = 0 \text{ if } j < i\}$$

and

$$\mathcal{J}_i := D_i(\mathcal{P}_i) \subseteq k[f].$$

Thus $\mathcal{P}_i$ is the set of "preslices" of D_i that are compatible with the triangular basis $D_1, \ldots, D_n$.

Lemma 2. *There exist $p_i \in \mathcal{P}_i \backslash \{0\}, p_i \in A_i$, and $q_i \in k^{[1]} \backslash \{0\}$ such that $\mathcal{J}_i = q_i(f)k[f]$ and $D_i(p_i) = q_i$.*

Proof. First note that $\mathcal{J}_i$ is not empty, as theorem 1 applied to $A(f) := A \otimes k(f)$ gives an $s_i \in A(f)$ which satisfies $D_i(s_i) = 1$, $D_j(s_i) = 0$ if $j < i$. Multiplying s_i by a suitable element of $k[f]$ gives a nonzero element $r(f)s_i$ of $\mathcal{P}_i$, and $D_i(r(f)s_i) = r(f)$. Because $k[f] = \cap \ker(D_i)$, $\mathcal{P}_i$ is a $k[f]$-module, and therefore $\mathcal{J}_i$ is an ideal of $k[f]$. This means that $\mathcal{J}_i$ is a principal ideal, and we take for q_i a generator (and $p_i \in D_i^{-1}(q_i)$). Since $D_j(p_i) = 0$ if $j < i$, we have $p_i \in A_i$. $\square$

Corollary 1. *The p_i, $1 \leq i \leq n$, are algebraically independent over k.*

Proof. The s_i are certainly algebraically independent, and $p_i \in k[f]s_i$. $\square$

Lemma 3. *Assume (Q), and take p_i, q_i as in lemma 2. Then the D_i are linearly dependent modulo $f - \alpha$ if and only if $q_i(\alpha) = 0$ for some i.*

Proof. ($\Rightarrow$): Suppose that $0 \neq D := g_1 D_1 + \ldots + g_n D_n$ satisfies $\overline{D} = 0$ where $g_i \in A$, and not all $\overline{g}_i = \bar{0}$. Let i be the highest such that $\bar{g}_i \neq \bar{0}$. Then $0 = \overline{D(p_i)} = \bar{g}_i \bar{D} \bar{p}_i = \bar{g}_i \overline{q_i(f)}$. Since $\bar{A}$ is a domain, $q_i(\alpha) = \overline{q_i(f)} = 0$. ($\Leftarrow$): Assume $f - \alpha$ divides $q_i(f)$. We need to show that the $\overline{D_i}$ are linearly dependent over $A/(f-\alpha)$. Consider $\bar{D}_i$ restricted to $\bar{A}_i$. If $j > i$ then $\bar{D}_i(p_j) = \overline{D_i(p_j)} = \overline{0}$. Furthermore $\bar{D}_i(\bar{p}_i) = \bar{q}_i(f) = q(\alpha) = 0$. Hence, $\bar{D}_i$ is zero if restricted to $k[\bar{p}_i, \ldots, \bar{p}_n]$. But since this is of transcendence degree n, it follows that $\bar{D}_i = 0$ on $\bar{A}_i$. Reversing the argument of ($\Rightarrow$) yields the linear dependence of the $\bar{D}_i$. $\square$

Proof. *(of theorem 3)* Part 1: If $\bar{D}_1, \ldots, \bar{D}_n$ are independent, then Proposition 1 yields that $\bar{A} \cong k^{[n]}$. Lemma 3 states that for any point α outside the zero set of $q_1 q_2 \cdots q_n$ we have $A/(f-\alpha) \cong k^{[n]}$. This zero set is either all of k or finite, yielding part 1.
Part 2: Lemma 3 tells us directly that for each $1 \leq i \leq n$ and $\alpha \in k$, we have $q_i(\alpha) \neq 0$. But this means that the $q_i \in k^*$, so the p_i can be taken to be actual slices ($s_i = p_i$). Using the fact that $s_i \in A_i$ we obtain that $A = A_1 = A_2[s_1] = A_3[s_2, s_1] = \ldots = A_{n+1}[s_1, \ldots s_n] = k[s_1, \ldots, s_n, f]$ as claimed. $\square$

5. Consequences of the main theorems

This paper is originally motivated by the following result of [**4**]:

Theorem 4. *Let $A = k[x, y, z]$ and D_1, D_2 two commuting locally nilpotent derivations on A which are linearly independent over A. Then $A^{D_1,D_2} = k[f]$ and f is a coordinate.*

Here the notation A^{D_1,D_2} means $A^{D_1} \cap A^{D_2}$ the intersections of the kernels of D_1 and D_2, which is the set of elements vanishing under D_1 resp. D_2. (Note that for the $\mathcal{G}_a$ action associated to D, this notation means $\mathcal{O}(X/\mathcal{G}_a) = \mathcal{O}(X)^{\mathcal{G}_a} = \mathcal{O}(X)^D$). By a **coordinate** is meant an element f in $k^{[n]}$ for which there exist $f_2, \ldots, f_n$ with $k[f, f_2, \ldots, f_n] = k^{[n]}$. Equivalently, $(f, f_2, \ldots, f_n) : k^{[n]} \longrightarrow k^{[n]}$ is an automorphism. The most important ingredient of this theorem is Kaliman's theorem [**3**].

In [**4**] it is conjectured that this result is true also in higher dimensions, i.e. having n commuting linearly independent locally nilpotent derivations on $k^{[n+1]}$ should yield that their common kernel is generated by a coordinate. However, it seems that this conjecture is very hard, on a par with the well-known Sathaye conjecture:

$SC(n)$ **Sathaye-conjecture:** Let $f \in A := k^{[n]}$ such that $A/(f - \lambda) \cong k^{[n-1]}$. Then f is a coordinate.

The Sathaye conjecture is proved for $n \leq 3$ by the aforementioned Kaliman's theorem. Therefore, the original motivation was to find additional requirements in higher dimensions to achieve the result that f is a coordinate. The results in this paper give one such requirement, namely that $k^{[n]}/(f - \lambda) \cong k^{[n-1]}$ for all constants λ.

Another consequence of the result of this paper is that the Sathaye conjecture is equivalent to

$MSC(n)$ **Modified Sathaye Conjecture:** Let $A := k^{[n]}$, and let $f \in A$ be such that $A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$. Then there exist $n-1$ commuting locally nilpotent derivations $D_1, \ldots, D_{n-1}$ on A such that $A^{D_1,\ldots,D_{n-1}} = k[f]$ and the D_i are linearly independent modulo $(f - \alpha)$ for each $\alpha \in k$.

Proof of equivalence of $SC(n)$ and $MSC(n)$. Suppose we have proven the $MSC(n)$. Then for any f satisfying "$A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$" we can find commuting LNDs $D_1, \ldots, D_{n-1}$ on A giving rise to a $\mathcal{G}_a^{n-1}$action satisfying the hypotheses of Theorem 2. Applying this theorem, we obtain that f is a coordinate in A. So the $SC(n)$ is true in that case.

Now suppose we have proven the $SC(n)$. Let f satisfy the requirements of the $MSC(n)$, that is, "$A/(f - \alpha) \cong k^{[n-1]}$ for all $\alpha \in k$". Since f satisfies the requirements of $SC(n)$, f then must be a coordinate. So it has $n - 1$ so-called mates: $k[f, f_2, \ldots, f_n] = k^{[n]}$. But then the partial derivative with respect to each of these n polynomials $f, f_2, \ldots, f_n$ defines a locally nilpotent derivation. All of them commute, and the intersection of the kernels of the last $n-1$ derivations is $k[f]$; so the MSC holds. □

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Harm Derksen, Department of Mathematics, University of Michigan,, Ann Arbor, MI 48109, USA, hderksen@umich.edu

Arno van den Essen, Radboud University Nijmegen, Toernooiveld 1, The Netherlands, a.vandenessen@math.ru.nl

David R. Finston, Department of Mathematical Sciences, New Mexico State University,, Las Cruces, NM 88003 USA, dfinston@nmsu.edu

Stefan Maubach, Radboud University Nijmegen, Toernooiveld 1, The Netherlands, s.maubach@math.ru.nl