Supersymmetric QCD from noncommutative geometry

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ABSTRACT

We derive supersymmetric quantum chromodynamics from a noncommutative spin manifold. We extend the model of Chamseddine and Connes that leads to the Einstein–Yang–Mills action and apply the spectral action principle to derive the Lagrangian of supersymmetric QCD, including soft supersymmetry breaking (negative sign) mass terms for the squarks. We find that these results are in good agreement with the physics literature.

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Physics and geometry share a fruitful history. The most prominent example of that is Einstein’s General Relativity, described in the language of Riemannian geometry. Noncommutative geometry [1] provides, amongst others, a generalization of the latter. Over the past years it has emerged as a successful tool for deriving models in high-energy physics from geometrical principles. Its main appeal is that it unifies gauge theories with General Relativity. The prime result [2] (following [3,4], see also [5]) by Chamseddine, Connes and Marcolli, who continued on the path set out by Connes and Lott [6], is a geometrical derivation of the full Standard Model, coupled to gravity and automatically including the famous Higgs mechanism. On top of that, in the noncommutative description of the Standard Model there naturally appear relations between Standard Model parameters, allowing for experimentally testable predictions [2, §5].

Ever since the early days of the field, there was an interest in the possible compatibility of noncommutative geometry and supersymmetry: can noncommutative geometry describe supersymmetric theories? This has so far led to some tentative results [7–10]. In this Letter we take a rather different approach. In a way fully compatible with the paradigm of noncommutative geometry we derive the action of the supersymmetric extension of QCD, the part of the Standard Model that describes quarks and gluons. This Letter can be considered as the more conceptual counterpart of [14], in which focus lies primarily on the calculations and technical details.

1. The Einstein–Yang–Mills system from a noncommutative manifold

The best way to introduce the key ideas is to derive SU(N)-Yang–Mills theory, coupled to fermions in the adjoint representation, from a noncommutative manifold [4]. In order to understand what the latter means, it is important to make the step of first algebraizing ordinary Riemannian (spin) geometry. Given a compact Riemannian spin manifold M, say of dimension 4, (local) coordinate functions xμ are continuous maps from M to R. This leads us to consider the algebra of continuous functions C(M) instead of M itself. Then, given the spin structure on M we have a Hilbert space of square-integrable spinors, denoted by L2(S). There is an action of C(M) on it by pointwise multiplication. Also, we have a Dirac operator ⧸ := iγμ(∂μ + ωμ) on L2(S), where ωμ is the spin connection accounting for M not being flat. The triple (C(M), L2(S), ⧸) is an example of a so-called spectral triple, the basic device in noncommutative geometry [1]. We will not dwell on the mathematical properties that it satisfies, but work in the explicit examples of interest.

The chirality operator γ5 defines a grading operator on L2(S) (since γ52 = 1) and charge conjugation gives rise to an operator J1: L2(S) → L2(S) which satisfies J1L2(S) = −1 (note the Euclidean signature).

Up to now, nothing noncommutative has happened: the algebra of functions on M is still commutative. We would like to change
corresponding noncommutative algebra is other words, the coordinates on we consider cut-off parameter \( \Lambda \) would be missing quartic interactions), but the following not traceclass) and incomplete from the physics point of view (we \( [4,11] \) does work: action Here, we restrict to unitary operators of the form \( U \) on the Hilbert space \( \mathcal{L}^2(S) \otimes M_N(\mathbb{C}) \). The expression \( JUJ^\dagger \) is right multiplication with \( U^\dagger \). In this case, the above transformation leaves \( \gamma \) and \( J \) invariant, but changes \( D \) to \( D + JA^J J^\dagger =: DA \), i.e. left and right multiplication with \( A = u[D, U^\dagger] \equiv \gamma^\mu A_\mu \). Here \( A_\mu \) is a pure gauge, \( u(N) \)-valued function on \( M \). One derives that the so-called fluctuated Dirac operator (the covariant derivative) is

\[
D_A = i\gamma^\mu [\partial_\mu + \omega_\mu \gamma] \otimes 1 + 1 \otimes A_\mu, \tag{1}
\]

where \( \delta_\mu := -i \mathrm{ad}^* A_\mu \) (\( \mathrm{ad}^* A_\mu \) means taking the commutator with \( A_\mu \)) is skew-Hermitian due to the self-adjointness of \( A_\mu \) and is thus \( su(N) \)-valued, but not necessarily pure gauge. The gauge group thus is \( C(M, U(N)) \), consisting of the unitary elements in \( C(M, M_N(\mathbb{C})) \). Note that the non-abelian nature of this group is a direct consequence of the noncommutativity in \( C(M, M_N(\mathbb{C})) \).

It is then natural to seek for gauge invariant functionals of the gauge fields and the simplest formula one could come up with is the trace of \( D_A^2 \). This is, however, ill-defined (because \( D_A^2 \) is not traceclass) and incomplete from the physics point of view (we would be missing quartic interactions), but the following spectral action [4,11] does work:

\[
S_{\delta}[A] := \text{Tr} \left( D_A \right). \tag{2}
\]

Here \( A \) is a cut-off scale and \( f \) some positive, even function that should in the end be fixed by comparison with known results. The cut-off parameter \( \Lambda \) can be used to obtain an asymptotic expansion for the spectral action; the terms with a positive power of \( \Lambda \) as a coefficient are then the physically relevant ones. The fermionic action is given by

\[
S_f[A, \psi] := \langle \psi, D_A \psi \rangle, \tag{3}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product on the Hilbert space \( \mathcal{H} \). It turns out ([4], see also [5, Sect. 11.4]) that the spectral action has an asymptotic expansion (as \( \Lambda \to \infty \)) of the form:

<table>
<thead>
<tr>
<th>Bosons</th>
<th>Continuous</th>
<th>Finite</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( N^2 - 1 )</td>
<td>( N^2 - 1 )</td>
</tr>
<tr>
<td>Fermions</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1

The number of degrees of freedom for both fields appearing in the Einstein–Yang–Mills system.

We would like to obtain a realization of supersymmetry for the Einstein–Yang–Mills system in the framework of noncommutative geometry. This can be seen as a first step towards supersymmetric QCD, since it essentially describes gluons and gluinos. The possibility of such a supersymmetry was suggested in [4]. A necessary condition for supersymmetry is that we have an equal number of bosonic and fermionic degrees of freedom. This is not the case yet. Indeed, both in the spinorial as in the finite part the fermionic degrees of freedom exceed those of the bosons: a spinor \( \psi(x) \) has eight real (four complex) degrees of freedom whereas the continuous part of the gauge potential has only four: \( A_\mu, \mu = 1, \ldots, 4 \), and on the finite part we automatically got a reduction for the bosons from \( 2N^2 \) to \( N^2 - 1 \) real degrees of freedom. The degrees of freedom are summarized in Table 1. In this Euclidean setup \( J^* = -1 \), so no Majorana fermions exist [12] and we have to use Weyl spinors instead. In contrast to [2] we cannot restrict to Weyl fermions by altering the inner product, for \( J\gamma = -\gamma J \) here. Instead we employ a scheme due to Van Nieuwenhuizen and Waldron [13]; we both relax the reality condition on the action and Wick rotate the spinors \( \psi, \bar{\psi} \) and gamma-matrices, as appearing in a Minkowskian setup, to the Euclidean case, resulting in spinors \( \psi, \bar{\psi} \) of opposite chirality. The path integral is insensitive to such a rotation since the system still contains two fermionic variables (\( \psi \) instead of \( \bar{\psi} \) and \( \bar{\psi} \)). To summarize, the solution is thus to take as the fermionic part of the action

\[
S_f[A, \psi, \bar{\psi}] := \langle \chi, D_A \psi \rangle; \quad \psi \in \mathcal{H}^+, \ \chi \in \mathcal{H}^-.
\]

which is the Euclidean counterpart of the action for \( \psi \) and \( \bar{\psi} \) in Minkowskian space.
For the finite part we obtain the reduction from \( \mathfrak{su}(N) \) to \( \mathfrak{su}(N) \) by first using the fact that any complex matrix can be written as the sum of a Hermitian and an anti-Hermitian matrix, i.e. it is the complexification of \( \mathfrak{su}(N) \), followed by splitting a fermion into a trace and a traceless part: \( \psi = \text{Tr} \psi + (A + J A \gamma^3) \) are parametrized by an \( SU(3) \) gauge potential \( A_\mu(x) \) and a \( C^3 \)-valued function \( \tilde{q}(x) \): \( D_\tilde{q}(\psi_q, \psi_{\tilde{g}}, \psi_{\tilde{q}}) = (\psi_q \tilde{g}, \psi_{\tilde{q}}) + (\psi_{\tilde{g}} \tilde{q}, \psi_q) + (\chi_{\tilde{g}} \tilde{q}, \psi_q) \). We will identify \( \tilde{q} \) and \( \tilde{q} \) as the squark and anti-squark, respectively. As before, \( \tilde{g} \) will be the gluon, and \( \psi_{\tilde{g}} \in L^2(S) \otimes M_3(\mathbb{C}) \) the gluino. This nomenclature is justified by the fact that the quarks and squarks are in the same representation of the gauge group, as are the gluons and gluinos.

The corresponding spectral action is determined to be (see [14] for the technical details): \( S_b[\tilde{q}, A] = S_b^f[A] + \int_M \left[ -\frac{f_2}{\pi^2} A^2(\tilde{q}(x))^2 + \frac{f_0}{4\pi^2} \times (8|\tilde{q}(x)|^2 + 6|D_\mu \tilde{q}(x)|^2 - 3R(\tilde{q}(x))^2) \right] + O(A^{-2}) \) where \( S_b^f[A] \) is the action of the Einstein–Yang–Mills system (cf. (4)). The fermionic action becomes \( S_f[A, \psi_q, \psi_{\tilde{g}}, \chi_q, \psi_{\tilde{q}}] \equiv (\psi_q, \chi_q, \psi_{\tilde{g}}, \psi_{\tilde{q}}), D_A(\psi_q, \psi_{\tilde{g}}, \psi_{\tilde{q}}) = (\psi_q, (\psi_q, \chi_q, \psi_{\tilde{g}}, \psi_{\tilde{q}}) + (\psi_{\tilde{g}}, (\psi_q, \chi_q, \psi_{\tilde{g}}, \psi_{\tilde{q}}) + (\chi_q \tilde{g}, \psi_q) + (\chi_q \tilde{g}, \psi_q) \psi_{\tilde{q}} \tilde{g} \right) \), in which we recognize additional quark-squark-gluino and gluino-gluino-gluino interactions.

Upon switching to flat Minkowski space these results are seen to be in excellent agreement with the literature on the Minimally Supersymmetric Standard Model (e.g. [15,16]); all interactions are present and their form is precisely the same. Not for all terms, however, do the coefficients match exactly. Some of these coefficients depend on the dimension of the representations in the finite Hilbert space and thus differ from the value one would obtain from a description of the full MSSM. Addressing this question is part of future research.

One observation that we cannot refrain from making is that the sum \( S_{\tilde{g}} + S_f \) is in fact not supersymmetric; there appear (negative sign) squark mass terms as allowed in soft supersymmetry breaking (see e.g. [16]). We consider the presence of these terms as a merit of the above model, leaving the question open whether a soft SUSY-breaking mechanism, responsible for these terms, can be found within noncommutative geometry. Possibly, one of the noncommutative manifolds that appear in the classification of [17] will describe the supersymmetric theory with a spontaneous supersymmetry breaking mechanism.

References