Ranking Functions for Loops with Disjunctive Exit-Conditions

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Introduction

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Piecewise Ranking Functions

Condition Jumping

Conclusions
**Ranking Function**

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```c
while (i < 15) {
    i++;
}
```

- Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```java
while (i < 15) {
    consumeResource();
    i++;
}
```
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- **Resource Analysis**

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    consumeResource();
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- **Resource Analysis**

```plaintext
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
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Applicable Loops

- The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]
\[ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 \]

- where \( e_i \) are arithmetical expressions
- i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
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**Quadratic Example**

```java
public int m(int a, int b, int c) {
    int count = 0;
    while (a > 0 && c <= b && c > 0) {
        if (c == b) { a--; c = 0; }
        count++;
        c++;
    }
    return count;
}
```

**Test runs**

1\(^{st}\) group: degree 2 NCA on plane
- a=1, b=1, c=1 => count=1
- a=1, b=1, c=2 => count=2
- a=1, b=1, c=3 => count=3
- a=1, b=2, c=2 => count=1
- a=1, b=2, c=3 => count=2
- a=1, b=3, c=3 => count=1

2\(^{nd}\) group: degree 1 NCA on plane
- a=2, b=1, c=1 => count=2
- a=2, b=1, c=2 => count=4
- a=2, b=2, c=2 => count=3

3\(^{rd}\) group: degree 0 NCA on plane
- a=3, b=1, c=1 => count=3

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

\[ RF(a, b, c) = ab - c + 1 \]

Expected degree of polynomial (here: d=2)
Soundness

• The procedure itself is unsound
• Use external prover to verify the inferred ranking functions
• KeY: http://www.key-project.org/
• Ranking function can be expressed in JML as a decreases clause

```java
1  //@ decreases i < 15 ? 15 - i : 0;
2  while (i < 15) {
3     i++;
4  }
```
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Any loop ranking function is piecewise...

```plaintext
1 while (i < 15) {
2     i++;
3 }
```

Its ranking function is actually:

\[
\begin{cases}
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

1 \textbf{while} \ ((i>0 \ \&\& \ i<20) \ || \ i>50) \ \{} \\
2 \hspace{1em} \textbf{if} \ (i>50) \ i--; \\
3 \hspace{1em} \textbf{else} \ i++; \\
4 \ \}\n
It's ranking function is non-trivially piecewise:

$$ \begin{cases} 
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases} $$
Expressing Piecewise Ranking Functions in JML

```java
1  //@ decreases (i>0&&i<20) ? 20-i : (i>50 ? i-50 : 0);
2  while ((i>0 && i<20) || i>50) {
3      if (i>50) i--;
4      else i++;
5  }
```
Applicable Loops

• The extended method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \]

\[ sC := e_1 [\langle, \rangle, \leq, \geq, =, \neq] e_2 \]

• where \( e_i \) are arithmetical expressions

• i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```java
1 while ((i>0 && i<20) || i>50) {
2    if (i>50) i--;
3    else i++;
4 }
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
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   - \( i > 0 \land i < 20 \land i > 50 \)

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1 while ((i>0 && i<20) || i>50) {
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   - $i > 0 \land i < 20 \land i > 50$

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Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2     if (i>50) i--; 
3     else i++; 
4 }

1 Split up the condition into disjunctive parts:
   - i > 0 ∧ i < 20
   - i > 50

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```plaintext
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;  
3   else i++;  
4 }
```

1 Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;  
3   else i++;  
4 }  
```

\[
\begin{align*}
20 - i & \quad \text{if } (i > 0) \land (i < 20) \\
i - 50 & \quad \text{if } i > 50 \\
0 & \quad \text{else}
\end{align*}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```c
while ((i>0 && i<20) || i>22) {
    if (i>22) i--;    
    else i+=4;
}
```

\[
\begin{cases}
\left\lceil \frac{(20 - i)}{4} \right\rceil & \text{if } (i > 0) \land (i < 20) \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;  
3   else i+=4;
4 }

\[
\begin{cases}
\left\lfloor \frac{20-i}{4} \right\rfloor + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\left\lfloor \frac{20-i}{4} \right\rfloor & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
What happens...
Detection of Condition Jumping: Example

```plaintext
while ((i>0 && i<20) || i>22) {
  if (i>22) i--;
  else i+=4;
}
```

Next function:

```
inext_i(i) = \{
  i - 1             if i > 22 \\
i + 4              if \neg (i > 22)
\}
```

```
declare-fun i () Int
define-fun nexti ((x Int)) Int
  (ite (> x 22) (− x 1) (+ x 4)))
(assert (and (and (> i 0) (< i 20))
  (> (nexti i) 22)))
(check-sat)
```
Detection of Condition Jumping: Example

```latex
\begin{align*}
1 & \textbf{while } ((i>0 && i<20) || i>22) \{ \\
2 & \quad \textbf{if } (i>22) \ i\leftarrow\; ; \\
3 & \quad \textbf{else } \ i\leftarrow i+4; \\
4 & \} \\
\end{align*}
```

\[ next_i(i) = \begin{cases} 
    i - 1 & \text{if } i > 22 \\
    i + 4 & \text{if } \neg(i > 22) 
\end{cases} \]

```
1 (declare-fun i () Int) 
2 (define-fun nexti ((x Int)) Int 
3 \quad (ite (> x 22) (- x 1) (+ x 4))) 
4 (assert (and (and (> i 0) (< i 20)) 
5 \quad (> (nexti i) 22))) 
6 (check-sat) 
```
Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22) 
\end{cases}

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3   (ite (> x 22) (− x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5     (> (nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a `next` function for each program variable.
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body.
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \{19\}
2. Find all nodes that can jump to \{19\}, \{3, 7, 11, 15\} and add them to the list of jumping nodes: 
\[
\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
\]
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$

2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:
   
   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
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Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:

$$\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$$
Finding Models for Condition Jumping: Generic

\( J \) is the set of models of which it is known that condition jumping occurs, \( Q \) is a queue of models, find all models that jump from \( b_1 \) to \( b_2 \):

1. Is there a model \( \overline{v} \) for which \( b_1(\overline{v}) \land b_2(\text{next}(\overline{v})) \land \overline{v} \notin J \)?
   - SAT → Add \( \overline{v} \) to \( J \) and \( Q \), goto 1.
   - UNSAT → Goto 2.

2. Q empty?
   - Yes → Done.
   - No → Goto 3.

3. Pop a model \( \overline{q} \) off the queue \( Q \). Is there a model \( \overline{v} \) for which \( b_1(\overline{v}) \land \text{next}(\overline{v}) = \overline{q} \land \overline{v} \notin J \)?
   - SAT → Add \( \overline{v} \) to \( J \) and \( Q \), goto 3.
   - UNSAT → Goto 2.
Generating Ranking Functions: Example

- We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.
- So, we can split the condition \( i > 0 \land i < 20 \) into two:
  \( i > 0 \land i < 20 \land i \mod = 3 \) and \( i > 0 \land i < 20 \land i \mod \neq 3 \)
- We can then apply the basic method separately to each of these disjunctive pieces.
Generating Ranking Functions: Generic

- We now know the set $D_{1,2}$ for which jumping occurs
- So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \land \bar{v} \notin D_{1,2}$
- We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_{x,2}$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_{x,1}$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by $(D_{x,1}, D_z)$. Add $(D_{x,2}, D_y)$ to $J$.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update $J$ accordingly. Goto 3.
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Conclusions

• Extension to the method presented at PPPJ’10, which can infer polynomial ranking functions:
  • Definition of Condition Jumping
  • Detection of Condition Jumping
  • Infer ranking functions for loops in which condition jumping occurs

• Ranking functions for loops can be used in the creation of a global ranking function in order to prove termination

• If the body of a loop with ranking function $RF(\bar{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution