Ranking Functions for Loops with Disjunctive Exit-Conditions

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2nd International Workshop on the Foundational and Practical Aspects of Resource Analysis (FOPARA’11), Madrid

May 19, 2011
Presentation Outline

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Condition Jumping

Conclusions
### Ranking Function

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```plaintext
1 while (i < 15) {
2    i++;
3 }
```

- Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```java
while (i < 15) {
    consumeResource();
    i++;
}
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```c
while (i < 15) {
    consumeResource();
    i++;
}
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```plaintext
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
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Conclusions
O. Shkaravska, R. Kersten, M. van Eekelen.

Test-Based Inference of Polynomial Loop-Bound Functions.

PPPJ’10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java
Applicable Loops

- The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]
\[ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 \]

- where \( e_i \) are arithmetical expressions
- i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. **Instrument loop with a counter**
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N^k_d = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
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3. Interpolate a polynomial from the results
Quadratic Example

public int m(int a, int b, int c) {
    int count=0;
    while (a > 0 && c <= b && c > 0) {
        if ( c == b ) { a--; c = 0; }
        c++;
        count++;
    }
    return count;
}

Test runs
1\textsuperscript{st} group: degree 2 NCA on plane
a=1, b=1, c=1 => count=1
a=1, b=1, c=2 => count=2
a=1, b=1, c=3 => count=3
a=1, b=2, c=2 => count=1
a=1, b=2, c=3 => count=2
a=1, b=3, c=3 => count=1

2\textsuperscript{nd} group: degree 1 NCA on plane
a=2, b=1, c=1 => count=2
a=2, b=1, c=2 => count=4
a=2, b=2, c=2 => count=3

3\textsuperscript{rd} group: degree 0 NCA on plane
a=3, b=1, c=1 => count=3

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:
RF(a, b, c) = a*b − c + 1

Expected degree of polynomial (here: d=2)
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a `decreases` clause

```plaintext
1  //@ decreases i < 15 ? 15 - i : 0;
2  while (i < 15) {
3     i++;
4  }
```
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Any loop ranking function is piecewise...

while (i < 15) {
    i++;
}

Its ranking function is actually:

\[
\begin{cases} 
15 - i & \text{if (} i < 15 \text{)} \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

It's ranking function is non-trivially piecewise:

\[
\begin{cases}
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```java
//@ decreases (i>0&&i<20) ? 20−i : (i>50 ? i−50 : 0);
while ((i>0 && i<20) || i>50) {
   if (i>50) i--;   // Decreases
   else i++;         // Decreases
}
```
The extended method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \]
\[ sC := e_1 [\lt, \gt, \leq, \geq, =, \neq] e_2 \]

- where \( e_i \) are arithmetical expressions
- i.e. first-order propositional logic expressions over arithmetical (in)equality
Extending the Basic Procedure: Example

\begin{verbatim}
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }
\end{verbatim}

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```c
1 while ((i>0 && i<20) || i>50) {
2    if (i>50) i--;
3    else i++;
4 }
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg(i > 50) \)
   - \( i > 50 \land \neg(i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

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Extending the Basic Procedure: Example

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

while ((i > 0 && i < 20) || i > 50) {
    if (i > 50) i--;
    else i++;
}

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \)
   - \( i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2     if (i>50) i--;
3     else i++;
4 }

1 Split up the condition into disjunctive parts:
   • i > 0 \land i < 20
   • i > 50

2 Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```plaintext
while ((i>0 && i<20) || i>50) {
  if (i>50) i--;
  else i++;
}

\[
\begin{align*}
20 - i & \quad \text{if } (i > 0) \land (i < 20) \\
i - 50 & \quad \text{if } i > 50 \\
0 & \quad \text{else}
\end{align*}
\]
```
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```python
while ((i > 0) && i < 20) || i > 22) {
    if (i > 22) i--;  
    else i += 4;
}
```

\[
\begin{cases}
  \lceil (20 - i)/4 \rceil & \text{if } (i > 0) \land (i < 20) \\
  i - 22 & \text{if } i > 22 \\
  0 & \text{else}
\end{cases}
\]
Example

```
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;  
3   else i+=4;  
4 }
```

\[
\begin{align*}
\lceil (20 - i)/4 \rceil + 1 & \quad \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\lceil (20 - i)/4 \rceil & \quad \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \quad \text{if } i > 22 \\
0 & \quad \text{else}
\end{align*}
\]
What happens...
Detection of Condition Jumping: Example

```plaintext
1 while ((i > 0 && i < 20) || i > 22) {
2   if (i > 22) i --;
3   else i += 4;
4 }

next_i(i) = \{ 
  i - 1   if i > 22 \\
  i + 4   if \neg(i > 22) 
\}

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int 
  (ite (> x 22) (− x 1) (+ x 4)))
3 (assert (and (and (> i 0) (< i 20)) 
  (> (nexti i) 22)))
4 (check-sat)
```
Detection of Condition Jumping: Example

\[
\text{while } ((i > 0 && i < 20) || i > 22) \{
\text{if } (i > 22) i--; \\
\text{else } i += 4;
\}
\]

\[\text{next}_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22)
\end{cases}\]

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3 (ite (> x 22) (− x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5 (>(nexti i) 22))
6 (check-sat)
Detection of Condition Jumping: Example

1 \textbf{while} ((i>0 \&\& i<20) || i>22) \{ \\
2 \quad \textbf{if} (i>22) i--; \\
3 \quad \textbf{else} i+=4; \\
4 \} \\

\[ next_i(i) = \begin{cases} 
    i - 1 & \text{if } i > 22 \\
    i + 4 & \text{if } \neg(i > 22)
\end{cases} \]

1 (\texttt{declare-fun i () Int}) \\
2 (\texttt{define-fun nexti ((x Int)) Int} \ \\
3 \quad (\texttt{ite (> x 22) (− x 1) (+ x 4)})) \\
4 (\texttt{assert (and (and (> i 0) (< i 20))} \\
5 \quad (> (nexti i) 22)))) \\
6 (\texttt{check-sat})
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a $\text{next}$ function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \{19\}

2. Find all nodes that can jump to \{19\}, \{3, 7, 11, 15\} and add them to the list of jumping nodes:
\[ \{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\} \]
Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. **Find all nodes that jump directly into the other piece:** $\{19\}$

2. **Find all nodes that can jump to** $\{19\}$, $\{3, 7, 11, 15\}$ **and add** them **to the list of jumping nodes:**

   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \( \{19\} \)
2. Find all nodes that can jump to \( \{19\}, \{3, 7, 11, 15\} \) and add them to the list of jumping nodes:
   \[
   \{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}
   \]
Finding Models for Condition Jumping: Generic

$J$ is the set of models of which it is known that condition jumping occurs, $Q$ is a queue of models, find all models that jump from $b_1$ to $b_2$:

\begin{enumerate}
\item Is there a model $\overline{v}$ for which $b_1(\overline{v}) \land b_2(\text{next}(\overline{v})) \land \overline{v} \not\in J$?
  \begin{itemize}
  \item SAT $\rightarrow$ Add $\overline{v}$ to $J$ and $Q$, goto 1.
  \item UNSAT $\rightarrow$ Goto 2.
  \end{itemize}
\item Q empty?
  \begin{itemize}
  \item Yes $\rightarrow$ Done.
  \item No $\rightarrow$ Goto 3.
  \end{itemize}
\item Pop a model $\overline{q}$ off the queue $Q$. Is there a model $\overline{v}$ for which $b_1(\overline{v}) \land \text{next}(\overline{v}) = \overline{q} \land \overline{v} \not\in J$?
  \begin{itemize}
  \item SAT $\rightarrow$ Add $\overline{v}$ to $J$ and $Q$, goto 3.
  \item UNSAT $\rightarrow$ Goto 2.
  \end{itemize}
\end{enumerate}
Generating Ranking Functions: Example

- We now know the set \( \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\} \) for which jumping occurs.
- So, we can split the condition \( i > 0 \land i < 20 \) into two:
  \( i > 0 \land i < 20 \land i \mod = 3 \) and \( i > 0 \land i < 20 \land i \mod \neq 3 \)
- We can then apply the basic method separately to each of these disjunctive pieces.
• We now know the set $D_{1,2}$ for which jumping occurs
• So, we can split the condition $b_1$ into two: $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$ and $b_1(\bar{v}) \land \bar{v} \not\in D_{1,2}$
• We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_{x,2}$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_{x,1}$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by $(D_{x,1}, D_z)$. Add $(D_{x,2}, D_y)$ to $J$.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update $J$ accordingly. Goto 3.
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Conclusions

- Extension to the method presented at PPPJ’10, which can infer polynomial ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs

- Ranking functions for loops can be used in the creation of a global ranking function in order to prove termination

- If the body of a loop with ranking function $RF(\bar{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\bar{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution