Ranking Functions for Loops with Disjunctive Exit-Conditions

Rody Kersten\textsuperscript{1}  Marko van Eekelen\textsuperscript{1,2}

\textsuperscript{1}Institute for Computing and Information Sciences (iCIS),
Radboud University Nijmegen

\textsuperscript{2}School for Computer Science, Open University of the Netherlands

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Ranking Function

• Decreases in every basic block
• Here: in every loop iteration
• Bounded by zero

```plaintext
1 while (i < 15) {
2   i++;
3 }
```

• Ranking function for the loop above is \(15 - i\)
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

1 \textbf{while} (i < 15) {
2 \hspace{1em} consumeResource();
3 \hspace{1em} i++;
4 \}

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- **Resource Analysis**

```c
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
Motivation and Aim

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```
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
Introduction

Basic Procedure

Piecewise Ranking Functions

Condition Jumping

Conclusions
Inference of Polynomial Loop Ranking Functions

O. Shkaravska, R. Kersten, M. van Eekelen.
Test-Based Inference of Polynomial Loop-Bound Functions.

PPPJ’10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java
Applicable Loops

- The basic method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \]
\[ sC := e_1 [\lt, \gt, \leq, \geq, =, \neq] e_2 \]

- where \( e_i \) are arithmetical expressions
- i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
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Quadratic Example

```java
public int m(int a, int b, int c) {
    int count=0;
    while (a > 0 && c <= b && c > 0) {
        if ( c == b ) { a--; c = 0; }
        c++;
        count++;
    }
    return count;
}
```

**Test runs**

1\textsuperscript{st} group: degree 2 NCA on plane
- \(a=1, b=1, c=1\) => \(\text{count}=1\)
- \(a=1, b=1, c=2\) => \(\text{count}=2\)
- \(a=1, b=1, c=3\) => \(\text{count}=3\)
- \(a=1, b=2, c=2\) => \(\text{count}=1\)
- \(a=1, b=2, c=3\) => \(\text{count}=2\)
- \(a=1, b=3, c=3\) => \(\text{count}=1\)

2\textsuperscript{nd} group: degree 1 NCA on plane
- \(a=2, b=1, c=1\) => \(\text{count}=2\)
- \(a=2, b=1, c=2\) => \(\text{count}=4\)
- \(a=2, b=2, c=2\) => \(\text{count}=3\)

3\textsuperscript{rd} group: degree 0 NCA on plane
- \(a=3, b=1, c=1\) => \(\text{count}=3\)

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:

\(RF(a, b, c) = a \times b - c + 1\)

Expected degree of polynomial (here: \(d=2\))
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a decreases clause

```jml
1  //@ decreases i < 15 ? 15 − i : 0;
2  while (i < 15) {
3      i++;
4  }
```
Presentation Outline

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Condition Jumping

Conclusions
Any loop ranking function is piecewise...

```
1 while (i < 15) {
2     i++;  
3 }
```

Its ranking function is actually:

\[
\begin{cases} 
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

1 \textbf{while} ((i>0 \&\& i<20) \textbf{||} i>50) \{ \\
2 \quad \textbf{if} (i>50) \text{ } i--; \\
3 \quad \textbf{else} \text{ } i++; \\
4 \}

It’s ranking function is non-trivially piecewise:

\[
\begin{cases} 
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```java
/*@ decreases (i>0&&i<20) ? 20−i : (i>50 ? i−50 : 0); 
while ((i>0 && i<20) || i>50) {
    if (i>50) i--; 
    else i++; 
}
```
Applicable Loops

- The extended method considers loops with conditions in the following form:

\[
C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \\
\text{sC} := e_1 \ [\text{\textless, \textgreater, \leq, \geq, \text{=} , \neq }] \ e_2
\]

- where \( e_i \) are arithmetical expressions
- i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

1. while ((i>0 && i<20) || i>50) {
   2. if (i>50) i--;
   3. else i++;
   4. }

1. Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20 \land \neg(i > 50)$
   - $i > 50 \land \neg(i > 0 \land i < 20)$
   - $i > 0 \land i < 20 \land i > 50$

2. Execute the basic procedure separately for each of the pieces.
### Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. **Split up the condition into disjunctive parts:**
   - \( i > 0 \land i < 20 \land \neg (i > 50) \)
   - \( i > 50 \land \neg (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
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4 }

1 Split up the condition into disjunctive parts:
   • \( i > 0 \land i < 20 \land \neg(i > 50) \)
   • \( i > 50 \land \neg(i > 0 \land i < 20) \)
   • \( i > 0 \land i < 20 \land i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
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1 Split up the condition into disjunctive parts:
   • $i > 0 \land i < 20$
   • $i > 50$

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Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }

\[
\begin{cases}
20- i & \text{if } (i > 0) \land (i < 20)  \\
i - 50 & \text{if } i > 50  \\
0 & \text{else}
\end{cases}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Presentation Outline

- Introduction
- Basic Procedure
- Piecewise Ranking Functions
- Condition Jumping
- Conclusions
Example

```c
1 while ((i>0 && i<20) || i>22) {
2    if (i>22) i--;
3    else i+=4;
4 }
```

\[
\begin{cases}
\left\lceil \frac{(20 - i)}{4} \right\rceil & \text{if } (i > 0) \land (i < 20) \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--; 
3   else i+=4;
4 }
```

\[
\begin{cases}
\lceil (20 - i)/4 \rceil + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
\lceil (20 - i)/4 \rceil & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
i - 22 & \text{if } i > 22 \\
0 & \text{else}
\end{cases}
\]
What happens...
Detection of Condition Jumping: Example

1 \textbf{while } ((i>0 \&\& i<20) \textbf{|| } i>22) \{ \\
2 \quad \textbf{if } (i>22) \; \; i--; \\
3 \quad \textbf{else } i+=4; \\
4 \} \\

\[ \text{next}_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22) 
\end{cases} \]

1 (declare-fun i () Int) \\
2 (define-fun nexti ((x Int)) Int \\
3 \quad (ite (> x 22) (\neg x 1) (+ x 4))) \\
4 (assert (and (and (> i 0) (< i 20)) \\
5 \quad (> (nexti i) 22))) \\
6 (check-sat)
Detection of Condition Jumping: Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }
```

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
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5   (> (nexti i) 22)))
6 (check-sat)
```
Detection of Condition Jumping: Example

```plaintext
while ((i>0 && i<20) || i>22) {
  if (i>22) i--;
  else i+=4;
}

next_i(i) = \{
  i - 1, \text{ if } i > 22 \\
  i + 4, \text{ if } \neg (i > 22)
\}
```

```plaintext
(declare-fun i () Int)
(define-fun nexti ((x Int)) Int
  (ite (> x 22) (− x 1) (+ x 4)))
(assert (and (and (> i 0) (< i 20))
  (> (nexti i) 22)))
(check-sat)
```
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a \texttt{next} function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:

$$\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$$
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:
   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes: $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Generic

$J$ is the set of models of which it is known that condition jumping occurs, $Q$ is a queue of models, find all models that jump from $b_1$ to $b_2$:

1. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land b_2(next(\bar{v})) \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 1.
   - UNSAT $\rightarrow$ Goto 2.

2. Q empty?
   - Yes $\rightarrow$ Done.
   - No $\rightarrow$ Goto 3.

3. Pop a model $\bar{q}$ off the queue $Q$. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land next(\bar{v}) = \bar{q} \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 3.
   - UNSAT $\rightarrow$ Goto 2.
We now know the set \( \{ i \mid i \mod 4 = 3 \land i > 0 \land i < 20 \} \) for which jumping occurs.

So, we can split the condition \( i > 0 \land i < 20 \) into two:

\( i > 0 \land i < 20 \land i \mod = 3 \) and \( i > 0 \land i < 20 \land i \mod \neq 3 \)

We can then apply the basic method separately to each of these disjunctive pieces.
Generating Ranking Functions: Generic

- We now know the set \( D_{1,2} \) for which jumping occurs
- So, we can split the condition \( b_1 \) into two: \( b_1(\bar{v}) \land \bar{v} \in D_{1,2} \) and \( b_1(\bar{v}) \land \bar{v} \notin D_{1,2} \)
- We can then apply the basic method to each of these disjunctive pieces
Multi-Jumping

1. DNF-split into $n$ conditions
2. For each $i$ and $j$, $1 \leq i < j \leq n$, detect jumping from $D_i$ to $D_j$. Build a list $J$ of jumping pairs $(D_x, D_y)$ for which condition jumping from $D_x$ to $D_y$ can occur.
3. If there are no more jumping pairs $(D_x, D_y)$ for which $D_x$ is unflagged, done! Else, goto 4.
4. Pop a jumping pair $(D_x, D_y)$ off $J$, for which $D_x$ is unflagged.
5. Find the set $D_{x,2}$ of all nodes in $D_x$ from which jumping to $D_y$ occurs and, dually, the set $D_{x,1}$ for which no jumping to $D_y$ occurs. Replace any condition pair $(D_x, D_z)$ in $J$ by $(D_{x,1}, D_z)$. Add $(D_{x,2}, D_y)$ to $J$.
   - If $D_{x,1} = \emptyset$, flag $D_{x,2}$ as complete, goto 3.
   - Else, for any jumping pair $(D_z, D_x)$ in $J$ (i.e. for which jumping from $D_z$ to $D_x$ can occur), unflag $D_z$, detect jumping into $D_{x,1}$ and $D_{x,2}$ and update $J$ accordingly. Goto 3.
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Conclusions

- Extension to the method presented at PPPJ’10, which can infer *polynomial* ranking functions:
  - Definition of Condition Jumping
  - Detection of Condition Jumping
  - Infer ranking functions for loops in which condition jumping occurs

- Ranking functions for loops can be used in the creation of a *global* ranking function in order to prove termination

- If the body of a loop with ranking function \( RF(\bar{v}) \) consumes \( n \) resources, then we know that the whole loop consumes \( RF(\bar{v}) \cdot n \) resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution