Ranking Functions for Loops with Disjunctive Exit-Conditions

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Presentation Outline

Introduction

Basic Procedure

Piecewise Ranking Functions

Condition Jumping

Conclusions
Ranking Function

- Decreases in every basic block
- Here: in every loop iteration
- Bounded by zero

```
1 while (i < 15) {
  2   i++;
  3 }
```

- Ranking function for the loop above is $15 - i$
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- Resource Analysis

```c
1 while (i < 15) {
2   consumeResource();
3   i++;
4 }
```
Motivation and Aim

- Prove termination
- Bounding runtime
- Compiler optimisations
- **Resource Analysis**

```plaintext
1 while (i < 15) {
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3    i++;
4 }
```
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- **Resource Analysis**

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Inference of Polynomial Loop Ranking Functions

O. Shkaravska, R. Kersten, M. van Eekelen.
Test-Based Inference of Polynomial Loop-Bound Functions.
PPPJ'10: Proceedings of the 8th International Conference on the Principles and Practice of Programming in Java
The basic method considers loops with conditions in the following form:

\[ C := sC \mid \mathbb{C}_1 \land \mathbb{C}_2 \]
\[ sC := e_1 [\langle, \rangle, \leq, \geq, =, \neq] e_2 \]

where \( e_i \) are arithmetical expressions

i.e. conjunctions over arithmetical (in)equalities
Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N^k_d = \binom{d+k}{k}$ input values satisfying NCA and the exit condition
3. Interpolate a polynomial from the results
Test-Based Approach

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2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying NCA and the exit condition

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Test-Based Approach

1. Instrument loop with a counter
2. Do test runs for a set of $N_d^k = \binom{d+k}{k}$ input values satisfying \textbf{NCA} and the exit condition
3. Interpolate a polynomial from the results
Quadratic Example

```
public int m(int a, int b, int c) {
    int count = 0;
    while (a > 0 && c <= b && c > 0) {
        if (c == b) { a--; c = 0; }
        c++;
        count++;
    }
    return count;
}
```

Test runs:

1st group: degree 2 NCA on plane
- a=1, b=1, c=1 => count=1
- a=1, b=1, c=2 => count=2
- a=1, b=1, c=3 => count=3
- a=1, b=2, c=2 => count=1
- a=1, b=2, c=3 => count=2
- a=1, b=3, c=3 => count=1

2nd group: degree 1 NCA on plane
- a=2, b=1, c=1 => count=2
- a=2, b=1, c=2 => count=4
- a=2, b=2, c=2 => count=3

3rd group: degree 0 NCA on plane
- a=3, b=1, c=1 => count=3

Find the interpolating polynomial and generate the method annotated with the corresponding ranking function:
RF(a, b, c) = a*b - c + 1

Expected degree of polynomial (here: d=2)
Soundness

- The procedure itself is unsound
- Use external prover to verify the inferred ranking functions
- KeY: http://www.key-project.org/
- Ranking function can be expressed in JML as a decreases clause

```java
//@ decreases i < 15 ? 15 - i : 0;
while (i < 15) {
  i++;
}
```
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Any loop ranking function is piecewise...

```
1 while (i < 15) {
2     i++;
3 }
```

Its ranking function is actually:

\[
\begin{cases}
15 - i & \text{if } (i < 15) \\
0 & \text{else}
\end{cases}
\]
Non-Trivial Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--; 
3   else i++; 
4 } 

It's ranking function is non-trivially piecewise:

\[
\begin{cases} 
20 - i & \text{if } (i > 0) \land (i < 20) \\
i - 50 & \text{if } i > 50 \\
0 & \text{else}
\end{cases}
\]
Expressing Piecewise Ranking Functions in JML

```java
//@ decreases (i>0&&i<20) ? 20−i : (i>50 ? i−50 : 0);
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;  
    else i++;  
}
```
The extended method considers loops with conditions in the following form:

\[ C := sC \mid C_1 \land C_2 \mid C_1 \lor C_2 \]
\[ sC := e_1 [<, >, \leq, \geq, =, \neq] e_2 \]

where \( e_i \) are arithmetical expressions

i.e. first-order propositional logic expressions over arithmetical (in)equalities
Extending the Basic Procedure: Example

1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }

1 Split up the condition into disjunctive parts:
   - i > 0 ∧ i < 20 ∧ ¬(i > 50)
   - i > 50 ∧ ¬(i > 0 ∧ i < 20)
   - i > 0 ∧ i < 20 ∧ i > 50

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--;
    else i++;
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \land \neg(i > 50) \)
   - \( i > 50 \land \neg(i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```
while ((i > 0 && i < 20) || i > 50) {
    if (i > 50) i--;
    else i++;  
}
```

1. Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \lor (i > 50) \)
   - \( i > 50 \lor (i > 0 \land i < 20) \)
   - \( i > 0 \land i < 20 \land i > 50 \)

2. Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

1 \textbf{while} ((i>0 \&\& i<20) || i>50) \{ \\
2 \quad \textbf{if} (i>50) \; i--; \\
3 \quad \textbf{else} \; i++; \\
4 \} \\

1 Split up the condition into disjunctive parts:
   - \( i > 0 \land i < 20 \)
   - \( i > 50 \)

2 Execute the basic procedure separately for each of the pieces.
Extending the Basic Procedure: Example

```c
while ((i>0 && i<20) || i>50) {
    if (i>50) i--; 
    else i++; 
}
```

1. Split up the condition into disjunctive parts:
   - $i > 0 \land i < 20$
   - $i > 50$

2. Execute the basic procedure separately for each of the pieces
Extending the Basic Procedure: Example

```
1 while ((i>0 && i<20) || i>50) {
2   if (i>50) i--;
3   else i++;
4 }
```

\[
\begin{align*}
20 - i & \quad \text{if } (i > 0) \land (i < 20) \\
i - 50 & \quad \text{if } i > 50 \\
0 & \quad \text{else}
\end{align*}
\]
Extending the Basic Procedure: Generic

1. Put the condition in Disjunctive Normal Form
2. Split up the condition into its disjunctive pieces
3. Execute the basic procedure separately for each of the pieces
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Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

\[
\begin{cases}
  \left\lceil \frac{(20 - i)}{4} \right\rceil & \text{if } (i > 0) \land (i < 20) \\
  i - 22 & \text{if } i > 22 \\
  0 & \text{else}
\end{cases}
\]
```
Example

```plaintext
1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--; 
3   else i+=4;
4 }
```

\[
\begin{cases}
    \left\lceil \frac{(20-i)}{4} \right\rceil + 1 & \text{if } (i > 0) \land (i < 20) \land i \mod 4 = 3 \\
    \left\lfloor \frac{(20-i)}{4} \right\rfloor & \text{if } (i > 0) \land (i < 20) \land i \mod 4 \neq 3 \\
    i - 22 & \text{if } i > 22 \\
    0 & \text{else}
\end{cases}
\]
What happens...
Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2   if (i>22) i--;
3   else i+=4;
4 }

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
  i + 4 & \text{if } \neg(i > 22)
\end{cases}
\]

1 (declare-fun i () Int)
2 (define-fun nexti ((x Int)) Int
3   (ite (> x 22) (- x 1) (+ x 4)))
4 (assert (and (and (> i 0) (< i 20))
5   (> (nexti i) 22)))
6 (check-sat)
Detection of Condition Jumping: Example

1 while ((i>0 && i<20) || i>22) {
2     if (i>22) i--;
3     else i+=4;
4 }
Detection of Condition Jumping: Example

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while (i>0 && i<20) || i>22) {
  if (i>22) i--;
  else i+=4;
}
```

\[
next_i(i) = \begin{cases} 
  i - 1 & \text{if } i > 22 \\
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\end{cases}
\]

```plaintext
(declare-fun i () Int)
(define-fun nexti ((x Int)) Int
  (ite (> x 22) (− x 1) (+ x 4)))
(assert (and (and (> i 0) (< i 20))
  (> (nexti i) 22)))
(check-sat)
```
Detection of Condition Jumping: Generic

- Symbolically execute the loop body to find a next function for each program variable
- Use SMT-solver to search for a model that satisfies one piece first and another after execution of the loop body
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition \( i > 0 \land i < 20 \) into the piece with condition \( i > 22 \). Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: \( \{19\} \)
2. Find all nodes that can jump to \( \{19\} \), \( \{3, 7, 11, 15\} \) and add them to the list of jumping nodes:

\[
\{3, 7, 11, 15, 19\} = \{i \mid i \text{ mod } 4 = 3 \land i > 0 \land i < 20\}
\]
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$
2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:
   
   $\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$
Finding Models for Condition Jumping: Example

Find all nodes that jump from the piece with condition $i > 0 \land i < 20$ into the piece with condition $i > 22$. Using an SMT-solver:

1. Find all nodes that jump directly into the other piece: $\{19\}$

2. Find all nodes that can jump to $\{19\}$, $\{3, 7, 11, 15\}$ and add them to the list of jumping nodes:

$$\{3, 7, 11, 15, 19\} = \{i \mid i \mod 4 = 3 \land i > 0 \land i < 20\}$$
Finding Models for Condition Jumping: Generic

$J$ is the set of models of which it is known that condition jumping occurs, $Q$ is a queue of models, find all models that jump from $b_1$ to $b_2$:

1. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land b_2(\text{next}(\bar{v})) \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 1.
   - UNSAT $\rightarrow$ Goto 2.

2. Q empty?
   - Yes $\rightarrow$ Done.
   - No $\rightarrow$ Goto 3.

3. Pop a model $\bar{q}$ off the queue $Q$. Is there a model $\bar{v}$ for which $b_1(\bar{v}) \land \text{next}(\bar{v}) = \bar{q} \land \bar{v} \notin J$?
   - SAT $\rightarrow$ Add $\bar{v}$ to $J$ and $Q$, goto 3.
   - UNSAT $\rightarrow$ Goto 2.
• We now know the set \{i | i \mod 4 = 3 \land i > 0 \land i < 20\} for which jumping occurs

• So, we can split the condition $i > 0 \land i < 20$ into two: $i > 0 \land i < 20 \land i \mod = 3$ and $i > 0 \land i < 20 \land i \mod \neq 3$

• We can then apply the basic method separately to each of these disjunctive pieces
We now know the set $D_{1,2}$ for which jumping occurs.

So, we can split the condition $b_1$ into two:

- $b_1(\bar{v}) \land \bar{v} \in D_{1,2}$
- $b_1(\bar{v}) \land \bar{v} \notin D_{1,2}$

We can then apply the basic method to each of these disjunctive pieces.
Multi-Jumping

1. DNF-split into \( n \) conditions

2. For each \( i \) and \( j \), \( 1 \leq i < j \leq n \), detect jumping from \( D_i \) to \( D_j \). Build a list \( J \) of jumping pairs \( (D_x, D_y) \) for which condition jumping from \( D_x \) to \( D_y \) can occur.

3. If there are no more jumping pairs \( (D_x, D_y) \) for which \( D_x \) is unflagged, done! Else, goto 4.

4. Pop a jumping pair \( (D_x, D_y) \) off \( J \), for which \( D_x \) is unflagged.

5. Find the set \( D_{x,2} \) of all nodes in \( D_x \) from which jumping to \( D_y \) occurs and, dually, the set \( D_{x,1} \) for which no jumping to \( D_y \) occurs. Replace any condition pair \( (D_x, D_z) \) in \( J \) by \( (D_{x,1}, D_z) \). Add \( (D_{x,2}, D_y) \) to \( J \).
   - If \( D_{x,1} = \emptyset \), flag \( D_{x,2} \) as complete, goto 3.
   - Else, for any jumping pair \( (D_z, D_x) \) in \( J \) (i.e. for which jumping from \( D_z \) to \( D_x \) can occur), unflag \( D_z \), detect jumping into \( D_{x,1} \) and \( D_{x,2} \) and update \( J \) accordingly. Goto 3.
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• Extension to the method presented at PPPJ’10, which can infer *polynomial* ranking functions:
  • Definition of Condition Jumping
  • Detection of Condition Jumping
  • Infer ranking functions for loops in which condition jumping occurs
• Ranking functions for loops can be used in the creation of a *global* ranking function in order to prove termination
• If the body of a loop with ranking function $RF(\overline{v})$ consumes $n$ resources, then we know that the whole loop consumes $RF(\overline{v}) \cdot n$ resources
Implementation: ResAna

http://resourceanalysis.cs.ru.nl/resana

- The basic procedure and DNF-splitting (minus removal of unsatisfiable pieces) have been implemented
- Future work: implement condition jumping solution