Abstract
We present the interatomic force constants and phonon dispersions of graphite and graphene from the LCBOPII empirical bond order potential. We find a good agreement with experimental results, particularly in comparison to other bond order potentials. From the flexural mode we determine the bending rigidity of graphene to be 0.69 eV at zero temperature. We discuss the large increase of this constant with temperature and argue that derivation of force constants from experimental values should take this feature into account. We examine also other graphitic systems, including multilayer graphene for which we show that the splitting of the flexural mode can provide a tool for characterization.

Keywords: phonons, graphene, graphite, force constants, LCBOPII, bending rigidity

1. Introduction
The phonon spectrum of a crystalline solid provides information on several important physical properties like sound velocities, thermal conductivity, heat capacity and thermal expansion. The phonon spectrum of graphite has been intensively studied experimentally [1–6] and theoretically [7–10] in the past and some models have also been shown to be accurate for the description of fullerenes [8] and of graphite slabs [10]. In the more recent past, Raman spectroscopy has proven to be of crucial importance also for the characterization of graphene and nanotubes as well as for graphitic nanostructures of lower symmetry, like bent tubes and graphene edges [11–13]. However, the many unusual structural aspects of graphite, like the observed ripples [14], negative thermal expansion [15, 16], edge reconstruction [17] and localized [18] and extended defects [19, 20] make it desirable to describe the energetics of carbon in different structural and bonding configurations beyond the harmonic approximation by means of a unique potential. Bond order potentials are a class of empirical interatomic potentials (EIPs) designed for this purpose [21–23] and the possible breaking and formation of bonds in structural phase transitions like the graphite to diamond transition where the character of the bonding changes from sp² to sp³ [24] or even to sp³d⁵ [25] [26]. In view of this larger and exacting scope it may be expected that the phonon spectra derived from these potentials are not as accurate as those derived from models meant to describe a single specific situation.

However they allow to study, without further adjustment of parameters, all carbon structures, including the effect of defects, edges and other structural changes, also as a function of temperature. The purpose of this paper is to evaluate the force constants and phonon spectrum of graphitic structures derived from the Long-range Carbon Bond Order Potential (LCBOPII) [27] and compare these results to experimental values, force constant models and to the Tersoff [28] and Brenner [22] bond order potentials for carbon.

The phonon dispersions of graphene and graphite have been measured experimentally [1–6, 29], determined from ab initio calculations [2, 30–32] and calculated from bond order EIPs [33, 34]. The ab initio results generally agree very well with experimental measurements whereas widely used EIPs such as the Tersoff [28] and Brenner [22] potentials give less accurate results [33], particularly in the optical region. One reason for this is that the range of interatomic interactions in EIPs is limited for computational efficiency whereas force constant models show that interatomic force constants (IFCs) up to fourth or even fifth nearest neighbours (NNs) must be included for accurate phonon dispersions [1, 31]. The second generation LCBOPII [27] EIP includes long-range interactions up to 6 Å which is well beyond fifth NNs in graphene and it is interesting to study their effect on the phonons in comparison to other approaches.

In Section 2 we describe the computational method with emphasis on the anomalous dispersion of the flexural mode. In Section 3 we present the LCBOPII phonon dispersion of graphene and graphite, compare our force constants to other models and examine the role of specific force constants on the phonon dispersions. We devote Section 4 to the analysis of the bending rigidity and its tem-
per temperature dependence. In Section 5 we show the phonons of (10,10) nanotubes and show the relevance of low-energy phonons of multilayer graphene for their characterization.

2. Methods

The phonon dispersions are calculated by means of standard lattice dynamics \[35\]. The interatomic force constants are evaluated by calculating, by central differences, the second derivatives (the IFCs) of the LCBOPII EIP with respect to atomic displacements around their equilibrium positions. The phonon frequencies at wavevector \( q \), \( \omega(q) \), are determined by diagonalizing the dynamical matrix

\[
D_{\alpha\beta}^{k,k'}(q) = \frac{1}{m_k m_{k'}} \sum_R \phi_{\alpha\beta}^{k,k'}(R) e^{i q \cdot R},
\]

where \( \phi_{\alpha\beta}^{k,k'}(R) \) is the force constant matrix, \( \alpha, \beta \) being Cartesian indices, for two atoms \( k \) and \( k' \) in unit cells separated by a lattice vector \( R \). In layered materials the lowest, out-of-plane, acoustic phonon branch (ZA) has a peculiar quadratic dispersion near the zone center with respect to the lattice parameters giving \( a \) the in-plane lattice parameter and \( m_C \) is the atomic mass of carbon. For graphene, the dispersion reduces to a purely quadratic form:

\[
\omega_{ZA}(q) = \sqrt{\frac{\kappa}{\rho_{2D}}} |q|^2,
\]

where \( \rho_{2D} = 4m_C/(\sqrt{3}a^2) \) is now a two-dimensional mass density.

3. Phonon dispersion

Minimization of the LCBOPII cohesive energy with respect to the lattice parameters gives \( a = \sqrt{3a_{CC}} = 2.4592 \) Å for graphene and \( a = \sqrt{3a_{CC}} = 2.4585 \) Å, \( c = 6.7344 \) Å for graphite. The phonon dispersions for graphene and \( ABAB \) graphite, calculated at these lattice parameters, are shown in Fig. 1. The branches are classified as follows: L stands for longitudinal in-plane, T for transverse in-plane and Z for transverse out-of-plane polarization at the \( \Gamma \) point. An A refers to acoustic modes and an O to optical modes. The O’ modes in graphite indicate out-of-phase oscillation of two equivalent atoms in neighboring layers. The phonon dispersions of graphite and graphene are very similar due to the weakness of the interlayer interactions compared to the strong covalent bonds binding the atoms in the layers. Consequently, most of the twelve branches in graphite are almost doubly degenerate with the exception of the out-of-plane branches below 400 cm\(^{-1}\).

Contrary to the two linear, in-plane acoustic LA and TA, modes the out-of-plane ZA mode has a quadratic dispersion near \( \Gamma \) which is typical of layered crystals \[37\].
The ZA mode is a bending mode, the two atoms in the unit cell move in phase in the z-direction, which, at long wavelengths, bends the surface resulting in rippling of the graphene sheet. The softness of this mode also means that it plays a dominant role at low temperatures. Also the optical out-of-plane, ZO mode has a considerably lower energy than the other optical branches due to the fact that atoms are much more free to move perpendicular to the plane than in the plane itself. At the \( K \)-point, the TO/LO and the ZA/ZO modes are degenerate by symmetry.

In Fig. 1 and Table 1 we compare the LCBOPII phonon spectrum to experimental results by high resolution electron energy-loss spectroscopy (HREELS) [3, 5, 29], inelastic x-ray scattering [1, 2] and inelastic neutron scattering [4]. The overall agreement with the experimental values is rather good, considering that the potential was not specifically fitted to reproduce the force constants of graphite.

LCBOPII performs very well compared to the popular Tersoff [28] and Brenner [22] EIPs, for which the phonon dispersions were recently published [33]. The Tersoff EIP overestimates the LO and TO branches by nearly 40% and both potentials show large discrepancies with experiments in the in-plane acoustic branches which are very well reproduced by LCBOPII. The latter modes are of particular importance for the thermal conductivity in graphene [38]. The only deviation occurs at the \( M \) point for the TA branch, where LCBOPII overestimates the experimental value from Ref. [1] by 13%. The measurements from Ref. [5] show even higher frequencies for this mode but these may have been obtained from a sample of poor quality as HREELS selection rules state that the TA mode should not be observable along the \( \Gamma - M \) line [1, 29]. Ab initio calculations also confirm the experimental results from Ref. [1]. From the slopes of the TA and LA modes we determine their respective sound velocities as 13.0 and 20.7 km/s which compare well to the experimental values

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(45^2), 465^3, 451^5, 485^8, 405^4, 482^5, 517^7, 530^8, 540^9
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constant matrix for the \(i\)-th NNs in graphene is

\[
\begin{pmatrix}
\phi_{od}^{(i)} & \phi_{od}^{(i)} & 0 \\
-\phi_{od}^{(i)} & \phi_{od}^{(i)} & 0 \\
0 & 0 & \phi_{od}^{(i)}
\end{pmatrix},
\]

where the off-diagonal, \(\phi_{od}^{(i)}\), elements for \(i = 1, 3, 5\) are equal to zero due to the hexagonal symmetry, \(\phi_{od}^{(2)} = 1.48\) eV/Å\(^2\) and \(\phi_{od}^{(5)} = O(10^{-6})\) eV/Å\(^2\) for LCBOPII. The IFCs from Ref. [1] are obtained by fitting a fourth NN force constants model to experimental values, those from Ref. [34] are derived from an extended Brenner EIP, also fitted to experimental values, and the IFCs from Ref. [31] are from a fourth NN force constant model including a nonzero \(\phi_{od}^{(2)}\) of -0.57 eV/Å\(^2\) fitted to the dispersion obtained \textit{ab initio} within the DFT-GGA approximation.

From the comparison of Table 2 we see that the first NN IFCs are very similar to the fitted force constants of the reference models. However the decay of the LCBOPII force constants beyond first NNs is too rapid compared to the sets of IFCs which reproduce the experimental values accurately.

To see how the in-plane phonon branches evolve if larger IFCs beyond first NNs are included we manually increase these force constants. We consecutively changed the in-plane IFCs from second to fifth NNs to match those from Ref. [1]. Since \(\phi_{2}^{1}\) is already 23% lower than the fitted values we changed the out-of-plane IFCs from first NNs. They were matched to those obtained in Ref. [34] since their model includes interactions up to fourth NNs only which better resembles LCBOPII.

The resulting phonon dispersions are shown in Fig. 2. The modification of the in-plane IFCs beyond first NNs clearly improves the phonon dispersion. Changes up to third NNs lower the frequencies of the transverse branches, particularly the TO branch along \(M - K\) and the TA branch at the \(M\)-point but perturbs the good agreement of the sound velocities of the linear modes. With in-plane IFCs changed up to fifth NNs the dispersion is in excellent agreement with experiments. This means that an improvement of only the long-range interactions of LCBOPII can considerably improve the accuracy of the potential for graphitic systems. However, this is not an easy task, since in the construction of LCBOPII short and long range interactions are fitted simultaneously. For the out-of-plane branches the optical ZO branch is greatly improved by the increase of the first NN IFC but the important quadratic behaviour of the ZA mode is lost. Interestingly the quadratic dispersion is recovered only once fourth NNs interaction is included.

4. Quadratic ZA dispersion and bending rigidity

The bending rigidity \(\kappa\) is the key quantity which characterizes the mechanical properties of membranes [31]. For a crystalline membrane like graphene it is intimately related to the quadratic ZA branch through Eq. (3). Reported val-
ues of $\kappa$ vary from 0.79 to 2.13 eV [24, 34, 42, 43]. Besides the different techniques used to calculate $\kappa$ and the different models of carbon interactions there might be other reasons for the confusing variety of reported values. First, when comparing the values of $\kappa$ for graphite and graphene, one should consider the bending rigidity per layer and not per unit cell since the latter results in a factor two difference. In fact, since graphite has two graphene layers in the unit cell, the coefficients of the $|\mathbf{q}|^4$ term in Eqs. (3) and (2) differ by a factor two while (see Fig. 1) the quadratic coefficients should be approximately equal. The second, more important reason, is that the bending rigidity of graphene has been found to be strongly temperature dependent in detailed Monte Carlo simulations [24, 42]. Contrary to liquid membranes [41], $\kappa$ increases with increasing temperature. This increase reaches roughly 40% already at room temperature [24, 42]. The temperature dependence of the bending rigidity implies that also the ZA phonon mode should depend on temperature which makes comparison of zero temperature dispersion as presented here to the room temperature experimental values non straightforward.

From the fit of Eq. (3) to the ZA dispersion along the $\Gamma - M$ line we obtain for the bending rigidity $\kappa = (0.69 \pm 0.02)$ eV for graphene. The same procedure for graphite with Eq. (2) yields $\kappa = (0.69 \pm 0.01)$ eV per layer and $C_{44} = (5.8 \pm 0.02) \times 10^8$ Pa. This value of $C_{44}$ is much lower than the experimental value of $5.03 \times 10^8$ Pa [59] due to the too small corrugation energy of LCBOPII which gives a difference of about 1.5 meV/atom against about 10 meV/atom [66, 17] between AA and AB graphite stacking. As a consequence, the transverse modes of graphite for wavevectors parallel to the c-axis shown in the inset of Fig. 1 have too low frequencies. The bending rigidities of graphite and graphene per layer are the same, which is in agreement with the results from Ref. [42] who find almost equal bending rigidities per layer for graphene and bilayer graphene.

The value of 0.69 eV is low compared to other studies and also lower than the 0.82 eV reported earlier for LCBOPII at zero temperature [24]. This value was obtained by evaluating the elastic energy per unit area $\mathcal{E}$ of carbon nanotubes. This energy is equal to $\mathcal{E} = 1/2 \kappa H^2$ where $H$ is the curvature of the nanotube. The apparent discrepancy with the result from the phonons is due to the fact that in forming a nanotube from graphene both elastic and torsional energy occur and it is only in the limit of very large nanotubes that the torsion energy can be neglected. We calculated the elastic energy, and the corresponding bending rigidity, for several nanotubes with radii from 11 to 70 Å with and without inclusion of the torsion term. The results in Fig. 3 clearly demonstrate that $\mathcal{E}$ is a linear function of $H^2$, without torsion, indicating a constant $\kappa = 0.69$ eV. With the inclusion of the torsion term the resulting $\kappa$ increases with the radius of the nanotube. For large nanotubes, with small torsion angles, the value of 0.69 eV found from the phonons is recovered. The value of 0.82 eV from Ref. [24] was indeed determined from a nanotube with a radius of approximately 11 Å.

5. Nanotubes and multilayer graphene

For completeness, we show in Fig. 4 also the phonons of a (10,10) nanotube that can be compared to Refs. [7, 82]. For nanotubes the differences between models are enhanced due to the complex folding of the bands.

Lastly, we examine the phonons of $n$-layer, AB stacked, graphene, going from a single graphene layer towards graphite. In this process, the low-frequency ZA mode splits into $n$ optical sub-branches as shown in Fig. 5 while for all other branches the splitting is much smaller ($\sim 2$ cm$^{-1}$). As shown in the left panel of Fig. 5, the frequency of these ‘breathing’ modes at $\Gamma$ for $n$-layer graphene are related to the longitudinal phonons of graphene along the $\Gamma - A$ line at wavevectors

$$q_n^m = \frac{2\pi m}{nc}, \quad (m = 0, \ldots, n - 1),$$

as if the modes were confined to an effective thickness of $nc/2$. This length is an interplanar distance larger than the actual thickness of the $n$-layer graphene. Interestingly, by extrapolating to a single layer, $n = 1$, we get an effective thickness equal to an interplanar distance, as suggested in Ref. [88]. Since the number and frequency of these low lying ZA modes is univocally determined by the number of graphene layers, their observation can be used for the characterization of multilayer graphene as a complement to the analysis relying on the 2D band done in Refs. [12, 13].
6. Conclusions

Empirical potentials are desirable for their simplicity and transferability to calculate the phonon frequencies of complex systems. The phonon dispersions of graphite and graphene are an important test for the accuracy of these EIPs. We have shown that LCBOPII gives good results for graphitic crystals particularly in comparison to other EIPs. We have analyzed the reasons for the remaining discrepancies, suggesting that the potential could be improved considerably by modification of the long-range interactions. The quadratic ZA bending mode plays a key role in the graphene structure at finite temperatures and we have discussed how this fact might influence the fitting of force constants models to experimental values measured at room temperature. Lastly, we point out that multilayer graphene is characterized by several low frequency breathing modes at Γ, that are univocally related to the number of layers and could be used for their characterization.

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