IMAGES OF LOCALLY FINITE DERIVATIONS OF
POLYNOMIAL ALGEBRAS IN TWO VARIABLES

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Abstract. In this paper we show that the image of any locally
finite \(k\)-derivation of the polynomial algebra \(k[x, y]\) in two variables
over a field \(k\) of characteristic zero is a Mathieu subspace. We also
show that the two-dimensional Jacobian conjecture is equivalent
to the statement that the image \(\text{Im} D\) of every \(k\)-derivation \(D\) of
\(k[x, y]\) such that \(1 \in \text{Im} D\) and \(\text{div} D = 0\) is a Mathieu subspace of
\(k[x, y]\).

1. Introduction

Kernels of derivations have been studied in many papers. On the
other hand, only a few results are known concerning images of deriva-
tions.

In this paper we consider the question if the image of a derivation
of a polynomial algebra in two variables over a field \(k\) is a Mathieu
subspace of the polynomial algebra.

The notion of the Mathieu subspaces was introduced recently by the
third-named author in \([Z]\) in order to study the Mathieu conjecture
\([M]\), the image conjecture \([Z\text{I}]\) and the Jacobian conjecture (see \([BCW]\)
and \([E\text{I}]\)). We will recall its definition in Section 2 below.

Throughout this paper we fix the following notation: \(k\) is a field of
characteristic zero and \(x, y\) are two free commutative variables. We
denote by \(A\) the polynomial algebra \(k[x, y]\) over the field \(k\).

The contents of the paper are arranged as follows.

In Section 2 we recall some facts concerning Mathieu subspaces and
show that the image of a \(k\)-derivation of \(A\) needs not be a Mathieu
subspace (see Example 2.4).
In Section 3 we prove in Theorem 3.1 that for every locally finite $k$-derivation $D$ of $A$, the image $\text{Im} \ D$ is a Mathieu subspace. Finally in Section 4 we show in Theorem 4.3 that the two-dimensional Jacobian conjecture is equivalent to the following: if $D$ is a $k$-derivation of $A$ with $\text{div} \ D = 0$ such that $1 \in \text{Im} \ D$, then $\text{Im} \ D$ is a Mathieu subspace of $A$.

2. Preliminaries

We start with the following notion introduced in [Z2].

Definition 2.1. Let $R$ be any commutative $k$-algebra and $M$ a $k$-subspace of $R$. Then $M$ is a Mathieu subspace of $R$ if the following condition holds: if $a \in R$ is such that $a^m \in M$ for all $m \geq 1$, then for any $b \in R$, there exists an $N \in \mathbb{N}$ such that $ba^m \in M$ for all $m \geq N$.

Obviously every ideal of $R$ is a Mathieu subspace of $R$. However not every Mathieu subspace of $R$ is an ideal of $R$. Before we give some examples, we first recall the following simple lemma proved in Lemma 4.5, [Z2], which will be very useful for our later arguments. For the sake of completeness, we here also include a proof.

Lemma 2.2. If $M$ is a Mathieu subspace of $R$ and $1 \in M$, then $M = R$.

Proof: Since $1 \in M$, it follows that $1^m = 1 \in M$ for all $m \geq 1$. Then for every $a \in R$, $a = a1^m \in M$ for all large $m$. Hence $R \subseteq M$ and $R = M$. □

Example 2.3. Let $R := k[t, t^{-1}]$ be the algebra of Laurent polynomials in the variable $t$. For each $c \in k$, let $D_c$ be the differential operator $\frac{d}{dt} + ct^{-1}$ of $R$. Then $\text{Im} \ D_c := D_cR$ is a Mathieu subspace of $R$ if and only if $c \notin \mathbb{Z}$ or $c = -1$.

Note that the conclusion above follows directly by applying Lefschetz’s principle to Proposition 2.6 [Z2]. Since Proposition 2.6 in [Z2] is for multi-variable case and its proof is quite involved, we here include a self-contained proof for the one variable case.

Proof: Note first that for any $m \in \mathbb{Z}$, $D_c^m = (m + c)^m t^{m-1}$. So, if $c \notin \mathbb{Z}$, then $\text{Im} \ D_c = R$. Hence a Mathieu subspace of $R$.

If $c \in \mathbb{Z}$ but $c \neq -1$, then $D_c = (1 + c) \neq 0$. So $1 \in \text{Im} \ D_c$. Since $D_c t^{-c} = (-c + c) t^{-c-1} = 0$, it is easy to see that $t^{-c-1} \notin \text{Im} \ D_c$. Hence $\text{Im} \ D_c \neq R$. Then by Lemma 2.2, $\text{Im} \ D_c$ is not a Mathieu subspace of $R$. 

Finally, assume $c = -1$. Since $D_{-1} t^m = (m - 1) t^{m-1}$ for all $m \in \mathbb{Z}$, it is easy to see that $\text{Im} \, D_{-1}$ is the subspace of the Laurent polynomials in $R$ without constant term. Then by the Duistermaat-van der Kallen theorem [DK], $M$ is a Mathieu subspace of $R$. □

Note that when $c = -1$, $\text{Im} \, D_{-1}$ is a Mathieu subspace of $R$. But it clearly is not an ideal of $R$. For more examples of Mathieu subspaces which are not ideals, see Section 4 in [Z2].

When $c = 0$, we see that $\text{Im} \, \frac{d}{dt}$ is not a Mathieu subspace of $R$.

Now observe that $k[t, t^{-1}] \simeq k[x, y]/(xy - 1)$, where $t$ corresponds to the class of $x$ and $t^{-1}$ to the class of $y$. Then the derivation $\frac{d}{dt}$ of $R$ can be lifted to a $k$-derivation $D$ of $k[x, y]$, which maps $x$ to $\frac{d}{dt} t = 1$ and $y$ to $\frac{d}{dt} t^{-1} = -t^{-2}$, i.e., $-y^2$. This leads to the following example.

**Example 2.4.** Let $D = \partial_x - y^2 \partial_y$. Then $\text{Im} \, D$ is not a Mathieu subspace of $k[x, y]$.

**Proof:** Note that $1 = Dx \in \text{Im} \, D$. However $y \notin \text{Im} \, D$ since for any $g \in k[x, y]$ the $y$-degree of $Dg$ can not be 1. So by Lemma 2.2 $\text{Im} \, D$ is not a Mathieu subspace of $k[x, y]$. □

The following lemma will also be needed in Section 3.

**Lemma 2.5.** Let $R$ be any $k$-algebra, $L$ a field extension of $k$ and $M$ a $k$-subspace of $R$. Assume that $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$. Then $M$ is a Mathieu subspace of the $k$-algebra $R$.

**Proof:** We view $L \otimes_k R$ as a $k$-algebra in the obvious way. Since $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$, from Definition 2.1 it is easy to see that $L \otimes_k M$ (as a $k$-subspace) is also a Mathieu subspace of the $k$-algebra $L \otimes_k R$. Now we identify $R$ with the $k$-subalgebra $1 \otimes_k R$ of the $k$-algebra $L \otimes_k R$. Then from Definition 2.1 again, it is easy to check that the intersection $(L \otimes_k M) \cap R = M$ is a Mathieu subspace of $R$. □

Note that by the lemma above, when we prove that a $k$-subspace of a polynomial algebra over $k$ is a Mathieu subspace of the polynomial algebra, we may freely replace $k$ by any field extension of $k$. For instance, we may assume that $k$ is algebraically closed.

To conclude this section we recall a result from [EWZ] which will be used in Section 3 below.

Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $k[z, z^{-1}]$ the algebra of Laurent polynomials in $z_i$ ($1 \leq i \leq n$). For any non-zero $f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \in k[z, z^{-1}]$, we denote by $\text{Supp} \,(f)$ the support of
Theorem 2.6. ([EWZ]) Let \( 0 \neq f \in k[z, z^{-1}] \) and \( u \) any rational point, i.e., a point with all coordinates being rational, of \( \text{Poly}(f) \). Then there exists \( m \geq 1 \) such that \( (\mathbb{R}_+u) \cap \text{Supp}(f^m) \neq \emptyset \).

3. Images of Locally Finite Derivations of \( k[x, y] \)

Let \( D \) be any \( k \)-derivation of \( A(= k[x, y]) \). Then \( D \) is said to be locally finite if for every \( a \in A \) the \( k \)-vector space spanned by the elements \( D^i a \) (\( i \geq 1 \)) is finite dimensional.

The main result of this section is the following theorem.

Theorem 3.1. Let \( D \) be any locally finite \( k \)-derivation of \( A \). Then \( \text{Im} D \) is a Mathieu subspace of \( A \).

To prove this theorem, we need the following result, which is Corollary 4.7 in [E2].

Proposition 3.2. Let \( D \) be any locally finite \( k \)-derivation of \( A \). Then up to the conjugation by a \( k \)-automorphism of \( A \), \( D \) has one of the following forms:

i) \( D = (ax + by)\partial_x + (cx + dy)\partial_y \) for some \( a, b, c, d \in k \);

ii) \( D = \partial_x + by\partial_y \) for some \( b \in k \);

iii) \( D = ax\partial_x + (x^m + amy)\partial_y \) for some \( a \in k \) and \( m \geq 1 \);

iv) \( D = f(x)\partial_y \) for some \( f(x) \in k[x] \).

Lemma 3.3. With the same notations as in Proposition 3.2, the following statements hold.

(a) If \( D \) is of type ii), then \( D \) is surjective.

(b) If \( D \) is of type iii), then

\[
\text{Im} D = \begin{cases} 
(x^m) & \text{if } a = 0, \\
(x, y) & \text{if } a \neq 0.
\end{cases}
\]

(c) If \( D \) is of type iv), then \( \text{Im} D = (f(x)) \).

Proof: (a) is well-known, see [C] or [F] (p. 96). (c) is obvious, so it remains to prove (b).

If \( a = 0 \), then \( D = x^m\partial_y \), and hence \( \text{Im} D = (x^m) \). So assume \( a \neq 0 \). Replacing \( D \) by \( a^{-1} D \) (without changing the image \( \text{Im} D \)), we may assume that \( D = (x\partial_x + my\partial_y) + bx^m\partial_y \) for some nonzero \( b \in k \).

Observe that for any \( i, j \in \mathbb{N} \), we have

\[
D(x^iy^j) = (i + mj)x^iy^j + jbx^{m+i}y^{j-1}.
\]
Next we use induction on $j \geq 0$ to show that $x^iy^j \in \text{Im} \, D$ whenever $i + j > 0$.

First, assume $j = 0$. Then by Eq. (3.2), we have $Dx^i = ix^i$, and hence $x^i \in \text{Im} \, D$ for all $i \geq 1$.

Now assume $j \geq 1$. Since $m \geq 1$, we have $m + i \geq 1$ for all $i \geq 0$. Combining this fact with Eq. (3.2), we get $x^iy^j \in \text{Im} \, D$ since $i + mj \neq 0$ for all $i \geq 0$. Hence we have proved that $x^iy^j \in \text{Im} \, D$ if $i + j > 0$. Note that 1 does not lie in $\text{Im} \, D$ since this space is contained in the ideal generated by $x$ and $y$. Therefore we have $\text{Im} \, D = (x, y)$. □

**Lemma 3.4.** Let $z = (z_1, z_2, ..., z_n)$ be $n$ free commutative variables and $D := \sum_{i=1}^n a_iz_i\partial_{z_i}$ for some $a_i \in k \ (1 \leq i \leq n)$. Then $\text{Im} \, D$ is a Mathieu subspace of $k[z]$.

Note that $D$ in the lemma is a locally finite derivation of the polynomial algebra $k[z]$. To show the lemma, let’s first recall the following well-known results.

**Lemma 3.5.** For any polynomials $f, g \in k[z]$ and a positive integer $m \geq 1$, we have

\begin{align}
(3.3) & \quad \text{Poly} \, (fg) = \text{Poly} \, (f) + \text{Poly} \, (g), \\
(3.4) & \quad \text{Poly} \, (f^m) = m\text{Poly} \, (f),
\end{align}

where the sum in the first equation above denotes the Minkowski sum of polytopes.

*Proof:* Eq. (3.3) is well-known, which was first proved by A. M. Ostrowski [O1] in 1921 (see also Theorem VI, p. 226 in [O2] or Lemma 2, p. 11 in [Stu]). To show Eq. (3.4), one can first check easily that the polytope $m\text{Poly} \, (f)$ and the polytope obtained by taking the Minkowski sum of $m$ copies of $\text{Poly} \, (f)$ actually share the same set of extremal vertices, namely, the set of the vertices $mv_i$, where $v_i$ runs through all extremal vertices of $\text{Poly} \, (f)$. Consequently, these two polytopes coincide. Then from this fact and Eq. (3.3), we see that Eq. (3.4) follows. □

*Proof of Lemma 3.4:* If all $a_i$’s are zero, then $D = 0$ and $\text{Im} \, D = 0$. Hence the lemma holds in this case. So, we assume that not all $a_i$’s are zero.

Let $S$ be the set of integral solutions $\beta \in \mathbb{Z}^n$ of the linear equation $\sum_{i=1}^n a_i\beta_i = 0$. Note that $S \neq \emptyset$ (since $0 \in S$) and is a finitely generated $\mathbb{Z}$-module. Let $V$ be the subspace of $\mathbb{R}^n$ spanned by elements of $S$ over
Then $V$ is a $\mathbb{R}$-subspace of $\mathbb{R}^n$ with $r := \dim_{\mathbb{R}} V < n$. Furthermore, $V$ can be described as the set of common solutions of some linear equations with rational coefficients, since clearly the $\mathbb{Q}$-vector space generated by the $\mathbb{Z}$-generators of $S$ can.

Note also that for any $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^n$, we have $D z^\beta = (\sum_{i=1}^n a_i \beta_i) z^\beta$. Hence, for any $\beta \in \mathbb{N}^n$, the monomial $z^\beta \in \text{Im} D$ iff $\beta \notin S$, or equivalently, $\beta \notin V$. Consequently, for any $0 \neq h(z) \in \mathbb{C}[z]$, we have

$$h(z) \in \text{Im} D \iff \text{Supp (} h \text{)} \cap V = \emptyset.$$  \hspace{1cm} (3.5)

Now, let $0 \neq f(z) \in \mathbb{C}[z]$ such that $f^m \in \text{Im} D$ for all $m \geq 1$. We claim $\text{Poly (} f \text{)} \cap V = \emptyset$.

Assume otherwise. Since all vertices of the polytope $\text{Poly (} f \text{)}$ are rational (actually integral), every face of $\text{Poly (} f \text{)}$ can be described as the set of common solutions of some linear equations with rational coefficients. Since this is also the case for $V$ (as pointed above) and $\text{Poly (} f \text{)} \cap V \neq \emptyset$ (by our assumption), it is easy to see that there exists at least one rational point $u \in \text{Poly (} f \text{)} \cap V$. Then by Theorem 2.6 there exists $m \geq 1$ such that $(\mathbb{R} u) \cap \text{Supp (} f^m \text{)} \neq \emptyset$, and by Eq. (3.5), $f^m \notin \text{Im} D$. Hence, we get a contradiction. Therefore, the claim holds.

Finally, we show that $\text{Im} D$ is a Mathieu subspace as follows.

Let $f(z)$ be as above and $d$ the distance between $V$ and $\text{Poly (} f \text{)}$. Then by the claim above and the fact that $\text{Poly (} f \text{)}$ is a compact subset of $\mathbb{R}^n$, we have $d > 0$. Furthermore, for any $m \geq 1$, by Eq. (3.4) we have $\text{Poly (} f^m \text{)} = m \text{Poly (} f \text{)}$. Hence, the distance between $V$ and $\text{Poly (} f^m \text{)}$ is given by $dm$.

Now let $h(z)$ be an arbitrary element of $k[z]$. Note that by Eqs. (3.3) and (3.4) we have $\text{Poly (} f^m h \text{)} = m \text{Poly (} f \text{)} + \text{Poly (} h \text{)}$ for all $m \geq 1$. Hence, for large enough $m$, the distance between $V$ and $\text{Poly (} f^m h \text{)}$ is positive, whence $\text{Poly (} f^m h \text{)} \cap V = \emptyset$. In particular, $\text{Supp (} f^m h \text{)} \cap V = \emptyset$, and by Eq. (3.5), $f^m h \in \text{Im} D$ when $m \gg 0$. Then by Definition 2.1 we see that $\text{Im} D$ is indeed a Mathieu subspace of $k[z]$. □

Now we can prove the main theorem of this section as follows.

**Proof of Theorem 3.1.** First, by Proposition 3.2, we only need to show that $\text{Im} D$ is a Mathieu subspace of $A$ in each of the four cases in Proposition 3.2. Furthermore, by Lemma 3.3 it only remains to prove case i). So assume $D = (ax + by) \partial_x + (cx + dy) \partial_y$ for some $a, b, c, d \in k$.

Second, by Lemma 2.5, we may assume that $k$ is algebraically closed.

Third, note that $D$ preserves the subspace $H := k x + k y \subset A$, so its restriction $D|_H$ on $H$ is a linear endomorphism of $H$. Since $k$ is
algebraically closed, there exists a linear automorphism \( \sigma \) of \( H \) such that the conjugation \( \sigma(D|_H)\sigma^{-1} \) gives the Jordan form of \( D|_H \). Let \( \tilde{\sigma} \) be the unique extension of \( \sigma \) to an automorphism of \( A \). Then it is easy to see that \( \tilde{\sigma}D\tilde{\sigma}^{-1} \) is also a \( k \)-derivation of \( A \).

Note that \( \text{Im } \tilde{\sigma}D\tilde{\sigma}^{-1} = \tilde{\sigma}(\text{Im } D) \) and in general Mathieu subspaces are preserved by \( k \)-algebra automorphisms. Therefore, we may replace \( D \) by \( \tilde{\sigma}D\tilde{\sigma}^{-1} \), if necessary, and assume that \( D = a(x\partial_x + y\partial_y) + x\partial_y \) (in case that the Jordan form of \( D|_H \) is an \( 2 \times 2 \) Jordan block) or \( D = ax\partial_x + by\partial_y \) (in case that the Jordan form of \( D|_H \) is diagonal).

For the former case, by Lemma 3.3 (b) with \( m = 1 \), we see that \( \text{Im } D \) is an ideal, and hence a Mathieu subspace of \( A \). For the latter case, it follows from Lemma 3.4 that \( \text{Im } D \) is a Mathieu subspace of \( A \). Therefore, the theorem holds.

4. Connection with the Two-Dimensional Jacobian Conjecture

In the previous section we showed that the image of every locally finite \( k \)-derivation of \( A \) is a Mathieu subspace of \( A \). However, as we have shown in Example 2.4, \( \text{Im } D \) needs not to be a Mathieu subspace of \( A \) for every \( k \)-derivation \( D \) of \( A \). This leads to the question of which \( k \)-derivations \( D \) of \( A \) have the property that \( \text{Im } D \) is a Mathieu subspace of \( A \). More precisely, we can ask

**Question 4.1.** Let \( D \) be any \( k \)-derivation of \( A \) such that \( \text{div } D = 0 \), where for any \( D = p\partial_x + q\partial_y \ (p, q \in A) \), \( \text{div } D := \partial_x p + \partial_y q \). Is \( \text{Im } D \) a Mathieu subspace of \( A \)?

Adding one more condition, we get

**Question 4.2.** Let \( D \) be any \( k \)-derivation of \( A \) such that \( \text{div } D = 0 \). If \( 1 \in \text{Im } D \), is \( \text{Im } D \) a Mathieu subspace of \( A \)?

Note that by Lemma 2.2, this question is equivalent to asking if \( D \) is surjective under the further condition \( 1 \in \text{Im } D \).

The motivation of the two questions above come from the following theorem.

**Theorem 4.3.** Question 4.2 has an affirmative answer iff the two dimensional Jacobian conjecture is true.

**Proof:** \((\Rightarrow)\) Assume that Question 4.2 has an affirmative answer. Let \( F = (f, g) \in k[x, y]^2 \) with \( \det JF = 1 \). Consider the \( k \)-derivation \( D := g\partial_x - f\partial_y \). Then \( \text{div } D = 0 \) and \( 1 = \det JF = Df \in \text{Im } D \). Since by our hypothesis \( \text{Im } D \) is a Mathieu subspace of \( A \), it follows
from Lemma 2.2 that $\text{Im } D = A$, i.e., $D$ is surjective. Then it follows from a theorem of Stein [Ste] (see also [C]) that $D$ is locally nilpotent. Since $D = \partial / \partial f$, $\ker D = \ker \partial / \partial f = k[g]$ by Proposition 2.2.15 in [E1]. Since $D$ has a slice $f$, it follows that $A = k[g][f]$, i.e., $F$ is invertible over $k$. So the two-dimensional Jacobian conjecture is true. 

(⇐) Assume that the two-dimensional Jacobian conjecture is true. Let $D = p \partial_x + q \partial_y$ ($p, q \in A$) be a $k$-derivation of $A$ such that $\text{div } D = 0$ and $1 \in \text{Im } D$.

Since $\text{div } D = 0$, we have $\partial_x p = \partial_y (-q)$. Then by Poincaré’s lemma, there exists $g \in A$ such that $p = \partial_y g$ and $q = -\partial_x g$. So $D = g \partial_x - g_x \partial_y$.

Since $1 \in \text{Im } D$, we get $1 = Df$ for some $f \in A$. Let $F := (f, g) \in k[x, y]^2$. Then we have $\det JF = Df = 1$. Since by our hypothesis $F$ is invertible, it follows that $k[x, y] = k[f, g]$. Hence, we have

$$\text{Im } D = \text{Im } \frac{\partial}{\partial f} = \frac{\partial}{\partial f}(k[f, g]) = k[f, g] = A.$$ 

In particular, $\text{Im } D$ is a Mathieu subspace of $A$. □

REFERENCES


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