IMAGES OF LOCALLY FINITE DERIVATIONS OF POLYNOMIAL ALGEBRAS IN TWO VARIABLES

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Abstract. In this paper we show that the image of any locally finite \( k \)-derivation of the polynomial algebra \( k[x, y] \) in two variables over a field \( k \) of characteristic zero is a Mathieu subspace. We also show that the two-dimensional Jacobian conjecture is equivalent to the statement that the image \( \text{Im} D \) of every \( k \)-derivation \( D \) of \( k[x, y] \) such that \( 1 \in \text{Im} D \) and \( \text{div} D = 0 \) is a Mathieu subspace of \( k[x, y] \).

1. Introduction

Kernels of derivations have been studied in many papers. On the other hand, only a few results are known concerning images of derivations.

In this paper we consider the question if the image of a derivation of a polynomial algebra in two variables over a field \( k \) is a Mathieu subspace of the polynomial algebra.

The notion of the Mathieu subspaces was introduced recently by the third-named author in [Z2] in order to study the Mathieu conjecture [M], the image conjecture [Z1] and the Jacobian conjecture (see [BCW] and [El]). We will recall its definition in Section 2 below.

Throughout this paper we fix the following notation: \( k \) is a field of characteristic zero and \( x, y \) are two free commutative variables. We denote by \( A \) the polynomial algebra \( k[x, y] \) over the field \( k \).

The contents of the paper are arranged as follows.

In Section 2 we recall some facts concerning Mathieu subspaces and show that the image of a \( k \)-derivation of \( A \) needs not be a Mathieu subspace (see Example 2.4).
In Section 3 we prove in Theorem 3.1 that for every locally finite \( k \)-derivation \( D \) of \( A \), the image \( \text{Im} \, D \) is a Mathieu subspace. Finally in Section 4 we show in Theorem 4.3 that the two-dimensional Jacobian conjecture is equivalent to the following: if \( D \) is a \( k \)-derivation of \( A \) with \( \text{div} \, D = 0 \) such that \( 1 \in \text{Im} \, D \), then \( \text{Im} \, D \) is a Mathieu subspace of \( A \).

2. Preliminaries

We start with the following notion introduced in [Z2].

**Definition 2.1.** Let \( R \) be any commutative \( k \)-algebra and \( M \) a \( k \)-subspace of \( R \). Then \( M \) is a Mathieu subspace of \( R \) if the following condition holds: if \( a \in R \) is such that \( a^m \in M \) for all \( m \geq 1 \), then for any \( b \in R \), there exists an \( N \in \mathbb{N} \) such that \( ba^m \in M \) for all \( m \geq N \).

Obviously every ideal of \( R \) is a Mathieu subspace of \( R \). However not every Mathieu subspace of \( R \) is an ideal of \( R \). Before we give some examples, we first recall the following simple lemma proved in Lemma 4.5, [Z2], which will be very useful for our later arguments. For the sake of completeness, we here also include a proof.

**Lemma 2.2.** If \( M \) is a Mathieu subspace of \( R \) and \( 1 \in M \), then \( M = R \).

**Proof:** Since \( 1 \in M \), it follows that \( 1^m = 1 \in M \) for all \( m \geq 1 \). Then for every \( a \in R \), \( a = a1^m \in M \) for all large \( m \). Hence \( R \subseteq M \) and \( R = M \).

**Example 2.3.** Let \( R := k[t, t^{-1}] \) be the algebra of Laurent polynomials in the variable \( t \). For each \( c \in k \), let \( D_c \) be the differential operator \( \frac{d}{dt} + ct^{-1} \) of \( R \). Then \( \text{Im} \, D_c := D_c R \) is a Mathieu subspace of \( R \) if and only if \( c \not\in \mathbb{Z} \) or \( c = -1 \).

Note that the conclusion above follows directly by applying Lefschetz’s principle to Proposition 2.6 [Z2]. Since Proposition 2.6 in [Z2] is for multi-variable case and its proof is quite involved, we here include a self-contained proof for the one variable case.

**Proof:** Note first that for any \( m \in \mathbb{Z} \), \( D_c t^m = (m + c)t^{m-1} \). So, if \( c \not\in \mathbb{Z} \), then \( \text{Im} \, D_c = R \). Hence a Mathieu subspace of \( R \).

If \( c \in \mathbb{Z} \) but \( c \neq -1 \), then \( D_c t = (1 + c) \neq 0 \). So \( 1 \in \text{Im} \, D_c \). Since \( D_c t^{-c} = (-c + t) t^{-c-1} = 0 \), it is easy to see that \( t^{-c-1} \not\in \text{Im} \, D_c \). Hence \( \text{Im} \, D_c \neq R \). Then by Lemma 2.2 \( \text{Im} \, D_c \) is not a Mathieu subspace of \( R \).
Finally, assume $c = -1$. Since $D_{-1} t^m = (m - 1) t^{m-1}$ for all $m \in \mathbb{Z}$, it is easy to see that $\text{Im} \ D_{-1}$ is the subspace of the Laurent polynomials in $R$ without constant term. Then by the Duistermaat-van der Kallen theorem [DK], $M$ is a Mathieu subspace of $R$. □

Note that when $c = -1$, $\text{Im} \ D_{-1}$ is a Mathieu subspace of $R$. But it clearly is not an ideal of $R$. For more examples of Mathieu subspaces which are not ideals, see Section 4 in [Z2].

When $c = 0$, we see that $\text{Im} \ \partial_x - y^2 \partial_y$ is not a Mathieu subspace of $k[x, y]$. This leads to the following example.

**Example 2.4.** Let $D = \partial_x - y^2 \partial_y$. Then $\text{Im} \ D$ is not a Mathieu subspace of $k[x, y]$.

**Proof:** Note that $1 = Dx \in \text{Im} \ D$. However $y \not\in \text{Im} \ D$ since for any $g \in k[x, y]$ the $y$-degree of $Dg$ can not be 1. So by Lemma 2.2 $\text{Im} \ D$ is not a Mathieu subspace of $k[x, y]$. □

The following lemma will also be needed in Section 3.

**Lemma 2.5.** Let $R$ be any $k$-algebra, $L$ a field extension of $k$ and $M$ a $k$-subspace of $R$. Assume that $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$. Then $M$ is a Mathieu subspace of the $k$-algebra $R$.

**Proof:** We view $L \otimes_k R$ as a $k$-algebra in the obvious way. Since $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$, from Definition 2.1 it is easy to see that $L \otimes_k M$ (as a $k$-subspace) is also a Mathieu subspace of the $k$-algebra $L \otimes_k R$.

Now we identify $R$ with the $k$-subalgebra $1 \otimes_k R$ of the $k$-algebra $L \otimes_k R$. Then from Definition 2.1 again, it is easy to check that the intersection $(L \otimes_k M) \cap R = M$ is a Mathieu subspace of $R$. □

Note that by the lemma above, when we prove that a $k$-subspace of a polynomial algebra over $k$ is a Mathieu subspace of the polynomial algebra, we may freely replace $k$ by any field extension of $k$. For instance, we may assume that $k$ is algebraically closed.

To conclude this section we recall a result from [EWZ] which will be used in Section 3 below.

Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $k[z, z^{-1}]$ the algebra of Laurent polynomials in $z_i$ ($1 \leq i \leq n$). For any non-zero $f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_{\alpha} z^\alpha \in k[z, z^{-1}]$, we denote by $\text{Supp}(f)$ the support of
$f(z)$, i.e., the set of all $\alpha \in \mathbb{Z}^n$ such that $c_\alpha \neq 0$, and $\text{Poly}(f)$ the (Newton) polytope of $f(z)$, i.e., the convex hull of $\text{Supp}(f)$ in $\mathbb{R}^n$.

**Theorem 2.6.** ([EWZ]) Let $0 \neq f \in k[z, z^{-1}]$ and $u$ any rational point, i.e., a point with all coordinates being rational, of $\text{Poly}(f)$. Then there exists $m \geq 1$ such that $(\mathbb{R}_+u) \cap \text{Supp}(f^m) \neq \emptyset$.

### 3. Images of Locally Finite Derivations of $k[x, y]$

Let $D$ be any $k$-derivation of $A(= k[x, y])$. Then $D$ is said to be **locally finite** if for every $a \in A$ the $k$-vector space spanned by the elements $D^i a$ ($i \geq 1$) is finite dimensional.

The main result of this section is the following theorem.

**Theorem 3.1.** Let $D$ be any locally finite $k$-derivation of $A$. Then $\text{Im} D$ is a Mathieu subspace of $A$.

To prove this theorem, we need the following result, which is Corollary 4.7 in [E2].

**Proposition 3.2.** Let $D$ be any locally finite $k$-derivation of $A$. Then up to the conjugation by a $k$-automorphism of $A$, $D$ has one of the following forms:

i) $D = (ax + by)\partial_x + (cx + dy)\partial_y$ for some $a, b, c, d \in k$;

ii) $D = \partial_x + by\partial_y$ for some $b \in k$;

iii) $D = ax\partial_x + (x^m + amy)\partial_y$ for some $a \in k$ and $m \geq 1$;

iv) $D = f(x)\partial_y$ for some $f(x) \in k[x]$.

**Lemma 3.3.** With the same notations as in Proposition 3.3, the following statements hold.

(a) If $D$ is of type ii), then $D$ is surjective.

(b) If $D$ is of type iii), then

$$
\text{Im} D = \begin{cases} 
(x^m) & \text{if } a = 0, \\
(x, y) & \text{if } a \neq 0.
\end{cases}
$$

(c) If $D$ is of type iv), then $\text{Im} D = (f(x))$.

**Proof:** (a) is well-known, see [C] or [F] (p. 96). (c) is obvious, so it remains to prove (b).

If $a = 0$, then $D = x^m\partial_y$, and hence $\text{Im} D = (x^m)$. So assume $a \neq 0$. Replacing $D$ by $a^{-1}D$ (without changing the image $\text{Im} D$), we may assume that $D = (x\partial_x + my\partial_y) + bx^m\partial_y$ for some nonzero $b \in k$. Observe that for any $i, j \in \mathbb{N}$, we have

$$
D(x^iy^j) = (i + mj)x^iy^j + jbx^{m+i}y^{j-1}.
$$

(3.2)
Next we use induction on $j \geq 0$ to show that $x^i y^j \in \text{Im} D$ whenever $i + j > 0$.

First, assume $j = 0$. Then by Eq. (3.2), we have $D x^i = i x^i$, and hence $x^i \in \text{Im} D$ for all $i \geq 1$.

Now assume $j \geq 1$. Since $m \geq 1$, we have $m + i \geq 1$ for all $i \geq 0$. Then by the induction assumption, $j b x^{m+i} y^{j-1} \in \text{Im} D$ for all $i \geq 0$. Combining this fact with Eq. (3.2), we get $x^i y^j \in \text{Im} D$ since $i + mj \neq 0$ for all $i \geq 0$. Hence we have proved that $x^i y^j \in \text{Im} D$ if $i + j > 0$. Note that 1 does not lie in $\text{Im} D$ since this space is contained in the ideal generated by $x$ and $y$. Therefore we have $\text{Im} D = (x, y)$. □

**Lemma 3.4.** Let $z = (z_1, z_2, \ldots, z_n)$ be $n$ free commutative variables and $D := \sum_{i=1}^{n} a_i z_i \partial_{z_i}$ for some $a_i \in k$ $(1 \leq i \leq n)$. Then $\text{Im} D$ is a Mathieu subspace of $k[z]$.

Note that $D$ in the lemma is a locally finite derivation of the polynomial algebra $k[z]$. To show the lemma, let’s first recall the following well-known results.

**Lemma 3.5.** For any polynomials $f, g \in k[z]$ and a positive integer $m \geq 1$, we have

\[(3.3) \quad \text{Poly} (fg) = \text{Poly} (f) + \text{Poly} (g),\]
\[(3.4) \quad \text{Poly} (f^m) = m \text{Poly} (f),\]

where the sum in the first equation above denotes the Minkowski sum of polytopes.

**Proof:** Eq. (3.3) is well-known, which was first proved by A. M. Ostrowski [O1] in 1921 (see also Theorem VI, p. 226 in [O2] or Lemma 2.2, p. 11 in [Stu]). To show Eq. (3.4), one can first check easily that the polytope $m \text{Poly} (f)$ and the polytope obtained by taking the Minkowski sum of $m$ copies of $\text{Poly} (f)$ actually share the same set of extremal vertices, namely, the set of the vertices $mv_i$, where $v_i$ runs through all extremal vertices of $\text{Poly} (f)$. Consequently, these two polytopes coincide. Then from this fact and Eq. (3.3), we see that Eq. (3.4) follows. □

**Proof of Lemma 3.4:** If all $a_i$’s are zero, then $D = 0$ and $\text{Im} D = 0$. Hence the lemma holds in this case. So, we assume that not all $a_i$’s are zero.

Let $S$ be the set of integral solutions $\beta \in \mathbb{Z}^n$ of the linear equation $\sum_{i=1}^{n} a_i \beta_i = 0$. Note that $S \neq \emptyset$ (since $0 \in S$) and is a finitely generated $\mathbb{Z}$-module. Let $V$ be the subspace of $\mathbb{R}^n$ spanned by elements of $S$ over
$\mathbb{R}$. Then $V$ is a $\mathbb{R}$-subspace of $\mathbb{R}^n$ with $r := \dim_\mathbb{R} V < n$. Furthermore, $V$ can be described as the set of common solutions of some linear equations with rational coefficients, since clearly the $\mathbb{Q}$-vector space generated by the $\mathbb{Z}$-generators of $S$ can.

Note also that for any $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^n$, we have $D z^\beta = (\sum_{i=1}^n a_i \beta_i) z^\beta$. Hence, for any $\beta \in \mathbb{N}^n$, the monomial $z^\beta \in \text{Im} D$ iff $\beta \not\in S$, or equivalently, $\beta \not\in V$. Consequently, for any $0 \neq h(z) \in \mathbb{C}[z]$, we have

$$h(z) \in \text{Im} D \iff \text{Supp} (h) \cap V = \emptyset. \quad (3.5)$$

Now, let $0 \neq f(z) \in \mathbb{C}[z]$ such that $f^m \in \text{Im} D$ for all $m \geq 1$. We claim $\text{Poly} (f) \cap V = \emptyset$.

Assume otherwise. Since all vertices of the polytope $\text{Poly} (f)$ are rational (actually integral), every face of $\text{Poly} (f)$ can be described as the set of common solutions of some linear equations with rational coefficients. Since this is also the case for $V$ (as pointed above) and $\text{Poly} (f) \cap V \neq \emptyset$ (by our assumption), it is easy to see that there exists at least one rational point $u \in \text{Poly} (f) \cap V$. Then by Theorem 2.6 there exists $m \geq 1$ such that $(\mathbb{R} u) \cap \text{Supp} (f^m) \neq \emptyset$, and by Eq. (3.5), $f^m \not\in \text{Im} D$. Hence, we get a contradiction. Therefore, the claim holds.

Finally, we show that $\text{Im} D$ is a Mathieu subspace as follows.

Let $f(z)$ be as above and $d$ the distance between $V$ and $\text{Poly} (f)$. Then by the claim above and the fact that $\text{Poly} (f)$ is a compact subset of $\mathbb{R}^n$, we have $d > 0$. Furthermore, for any $m \geq 1$, by Eq. (3.4) we have $\text{Poly} (f^m) = m \text{Poly} (f)$. Hence, the distance between $V$ and $\text{Poly} (f^m)$ is given by $dm$.

Now let $h(z)$ be an arbitrary element of $k[z]$. Note that by Eqs. (3.3) and (3.4) we have $\text{Poly} (f^m h) = m \text{Poly} (f) + \text{Poly} (h)$ for all $m \geq 1$. Hence, for large enough $m$, the distance between $V$ and $\text{Poly} (f^m h)$ is positive, whence $\text{Poly} (f^m h) \cap V = \emptyset$. In particular, $\text{Supp} (f^m h) \cap V = \emptyset$, and by Eq. (3.5), $f^m h \in \text{Im} D$ when $m \gg 0$. Then by Definition 2.1 we see that $\text{Im} D$ is indeed a Mathieu subspace of $k[z]$. \hfill $\Box$

Now we can prove the main theorem of this section as follows.

**Proof of Theorem 3.1.** First, by Proposition 3.2, we only need to show that $\text{Im} D$ is a Mathieu subspace of $A$ in each of the four cases in Proposition 3.2. Furthermore, by Lemma 3.3 it only remains to prove case i). So assume $D = (ax + by) \partial_x + (cx + dy) \partial_y$ for some $a, b, c, d \in k$.

Second, by Lemma 2.5 we may assume that $k$ is algebraically closed.

Third, note that $D$ preserves the subspace $H := kx + ky \subset A$, so its restriction $D|_H$ on $H$ is a linear endomorphism of $H$. Since $k$ is
algebraically closed, there exists a linear automorphism $\sigma$ of $H$ such that the conjugation $\sigma(D|_H)\sigma^{-1}$ gives the Jordan form of $D|_H$. Let $\tilde{\sigma}$ be the unique extension of $\sigma$ to an automorphism of $A$. Then it is easy to see that $\tilde{\sigma}D\tilde{\sigma}^{-1}$ is also a $k$-derivation of $A$.

Note that $\text{Im} \tilde{\sigma}D\tilde{\sigma}^{-1} = \tilde{\sigma}\text{(Im} D)$ and in general Mathieu subspaces are preserved by $k$-algebra automorphisms. Therefore, we may replace $D$ by $\tilde{\sigma}D\tilde{\sigma}^{-1}$, if necessary, and assume that $D = a(x\partial_x + y\partial_y) + x\partial_y$ (in case that the Jordan form of $D|_H$ is an $2 \times 2$ Jordan block) or $D = ax\partial_x + by\partial_y$ (in case that the Jordan form of $D|_H$ is diagonal).

For the former case, by Lemma 3.3 (b) with $m = 1$, we see that $\text{Im} D$ is an ideal, and hence a Mathieu subspace of $A$. For the latter case, it follows from Lemma 3.4 that $\text{Im} D$ also a Mathieu subspace of $A$. Therefore, the theorem holds. □

4. Connection with the Two-Dimensional Jacobian Conjecture

In the previous section we showed that the image of every locally finite $k$-derivation of $A$ is a Mathieu subspace of $A$. However, as we have shown in Example 2.4.1, $\text{Im} D$ needs not to be a Mathieu subspace of $A$ for every $k$-derivation $D$ of $A$. This leads to the question of which $k$-derivations $D$ of $A$ have the property that $\text{Im} D$ is a Mathieu subspace of $A$. More precisely, we can ask

**Question 4.1.** Let $D$ be any $k$-derivation of $A$ such that $\text{div} D = 0$, where for any $D = p\partial_x + q\partial_y$ ($p, q \in A$), $\text{div} D := \partial_x p + \partial_y q$. Is $\text{Im} D$ a Mathieu subspace of $A$?

Adding one more condition, we get

**Question 4.2.** Let $D$ be any $k$-derivation of $A$ such that $\text{div} D = 0$. If $1 \in \text{Im} D$, is $\text{Im} D$ a Mathieu subspace of $A$?

Note that by Lemma 2.2, this question is equivalent to asking if $D$ is surjective under the further condition $1 \in \text{Im} D$.

The motivation of the two questions above come from the following theorem.

**Theorem 4.3.** Question 4.2 has an affirmative answer iff the two dimensional Jacobian conjecture is true.

**Proof:** ($\Rightarrow$) Assume that Question 4.2 has an affirmative answer. Let $F = (f, g) \in k[x, y]^2$ with $\det JF = 1$. Consider the $k$-derivation $D := g_y\partial_x - g_x\partial_y$. Then $\text{div} D = 0$ and $1 = \det JF = Df \in \text{Im} D$. Since by our hypothesis $\text{Im} D$ is a Mathieu subspace of $A$, it follows
from Lemma 2.2 that $\text{Im} \ D = A$, i.e., $D$ is surjective. Then it follows from a theorem of Stein [Ste] (see also [C]) that $D$ is locally nilpotent.

Since $D = \partial/\partial f$, $\ker D = \ker \partial/\partial f = k[g]$ by Proposition 2.2.15 in [E1]. Since $D$ has a slice $f$, it follows that $A = k[g][f]$, i.e., $F$ is invertible over $k$. So the two-dimensional Jacobian conjecture is true.

($\Leftarrow$) Assume that the two-dimensional Jacobian conjecture is true. Let $D = p\partial_x + q\partial_y \ (p, q \in A)$ be a $k$-derivation of $A$ such that $\text{div} \ D = 0$ and $1 \in \text{Im} \ D$.

Since $\text{div} \ D = 0$, we have $\partial_x p = \partial_y (-q)$. Then by Poincaré's lemma, there exists $g \in A$ such that $p = \partial_y g$ and $q = -\partial_x g$. So $D = g_y \partial_x - g_x \partial_y$.

Since $1 \in \text{Im} \ D$, we get $1 = Df$ for some $f \in A$. Let $F := (f, g) \in k[x, y]^2$. Then we have $\det JF = Df = 1$. Since by our hypothesis $F$ is invertible, it follows that $k[x, y] = k[f, g]$. Hence, we have

\[
\text{Im} \ D = \text{Im} \left( \frac{\partial}{\partial f} \right) = \frac{\partial}{\partial f}(k[f, g]) = k[f, g] = A.
\]

In particular, $\text{Im} \ D$ is a Mathieu subspace of $A$. \qed

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