# IMAGES OF LOCALLY FINITE DERIVATIONS OF POLYNOMIAL ALGEBRAS IN TWO VARIABLES

ARNO VAN DEN ESSEN, DAVID WRIGHT AND WENHUA ZHAO

ABSTRACT. In this paper we show that the image of any locally finite $k$-derivation of the polynomial algebra $k[x,y]$ in two variables over a field $k$ of characteristic zero is a Mathieu subspace. We also show that the two-dimensional Jacobian conjecture is equivalent to the statement that the image $\operatorname{Im} D$ of every $k$-derivation $D$ of $k[x,y]$ such that $1 \in \operatorname{Im} D$ and $\operatorname{div} D = 0$ is a Mathieu subspace of $k[x,y]$.

## 1. Introduction

Kernels of derivations have been studied in many papers. On the other hand, only a few results are known concerning images of derivations.

In this paper we consider the question if the image of a derivation of a polynomial algebra in two variables over a field $k$ is a Mathieu subspace of the polynomial algebra.

The notion of the Mathieu subspaces was introduced recently by the third-named author in [Z2] in order to study the Mathieu conjecture [M], the image conjecture [Z1] and the Jacobian conjecture (see [BCW] and [E1]). We will recall its definition in Section 2 below.

Throughout this paper we fix the following notation: $k$ is a field of characteristic zero and $x, y$ are two free commutative variables. We denote by $A$ the polynomial algebra $k[x,y]$ over the field $k$.

The contents of the paper are arranged as follows.

In Section 2 we recall some facts concerning Mathieu subspaces and show that the image of a $k$-derivation of $A$ needs not be a Mathieu subspace (see Example 2.4).

*Date*: December 10, 2010.

2000 *Mathematics Subject Classification.* 13N15, 14R10, 14R15.

*Key words and phrases.* The Mathieu subspaces, locally finite derivations, the Jacobian conjecture.

The third-named author has been partially supported by the NSA Grant H98230-10-1-0168.

In Section 3 we prove in Theorem 3.1 that for every locally finite $k$-derivation $D$ of $A$, the image $\operatorname{Im} D$ is a Mathieu subspace. Finally in Section 4 we show in Theorem 4.3 that the two-dimensional Jacobian conjecture is equivalent to the following: *if $D$ is a $k$-derivation of $A$ with $\operatorname{div} D = 0$ such that $1 \in \operatorname{Im} D$, then $\operatorname{Im} D$ is a Mathieu subspace of $A$.*

## 2. **Preliminaries**

We start with the following notion introduced in [Z2].

**Definition 2.1.** *Let $R$ be any commutative $k$-algebra and $M$ a $k$-subspace of $R$. Then $M$ is a Mathieu subspace of $R$ if the following condition holds: if $a \in R$ is such that $a^m \in M$ for all $m \geq 1$, then for any $b \in R$, there exists an $N \in \mathbb{N}$ such that $ba^m \in M$ for all $m \geq N$.*

Obviously every ideal of $R$ is a Mathieu subspace of $R$. However not every Mathieu subspace of $R$ is an ideal of $R$. Before we give some examples, we first recall the following simple lemma proved in Lemma 4.5, [Z2], which will be very useful for our later arguments. For the sake of completeness, we here also include a proof.

**Lemma 2.2.** *If $M$ is a Mathieu subspace of $R$ and $1 \in M$, then $M = R$.*

*Proof:* Since $1 \in M$, it follows that $1^m = 1 \in M$ for all $m \geq 1$. Then for every $a \in R$, $a = a1^m \in M$ for all large $m$. Hence $R \subseteq M$ and $R = M$. $\square$

**Example 2.3.** *Let $R := k[t, t^{-1}]$ be the algebra of Laurent polynomials in the variable $t$. For each $c \in k$, let $D_c$ be the differential operator $\frac{d}{dt} + ct^{-1}$ of $R$. Then $\operatorname{Im} D_c := D_c R$ is a Mathieu subspace of $R$ if and only if $c \notin \mathbb{Z}$ or $c = -1$.*

Note that the conclusion above follows directly by applying Lefschetz's principle to Proposition 2.6 [Z2]. Since Proposition 2.6 in [Z2] is for multi-variable case and its proof is quite involved, we here include a self-contained proof for the one variable case.

*Proof:* Note first that for any $m \in \mathbb{Z}$, $D_c t^m = (m + c)t^{m-1}$. So, if $c \notin \mathbb{Z}$, then $\operatorname{Im} D_c = R$. Hence a Mathieu subspace of $R$.

If $c \in \mathbb{Z}$ but $c \neq -1$, then $D_c t = (1 + c) \neq 0$. So $1 \in \operatorname{Im} D_c$. Since $D_c t^{-c} = (-c + c)t^{-c-1} = 0$, it is easy to see that $t^{-c-1} \notin \operatorname{Im} D_c$. Hence $\operatorname{Im} D_c \neq R$. Then by Lemma 2.2, $\operatorname{Im} D_c$ is not a Mathieu subspace of $R$.

Finally, assume $c = -1$. Since $D_{-1}t^m = (m-1)t^{m-1}$ for all $m \in \mathbb{Z}$, it is easy to see that $\operatorname{Im} D_{-1}$ is the subspace of the Laurent polynomials in $R$ without constant term. Then by the Duistermaat-van der Kallen theorem [DK], $M$ is a Mathieu subspace of $R$. $\square$

Note that when $c = -1$, $\operatorname{Im} D_{-1}$ is a Mathieu subspace of $R$. But it clearly is not an ideal of $R$. For more examples of Mathieu subspaces which are not ideals, see Section 4 in [Z2].

When $c = 0$, we see that $\operatorname{Im} d/dt$ is not a Mathieu subspace of $R$. Now observe that $k[t, t^{-1}] \simeq k[x, y]/(xy - 1)$, where $t$ corresponds to the class of $x$ and $t^{-1}$ to the class of $y$. Then the derivation $d/dt$ of $R$ can be lifted to a $k$-derivation $D$ of $k[x, y]$, which maps $x$ to $\frac{d}{dt}t = 1$ and $y$ to $\frac{d}{dt}t^{-1} = -t^{-2}$, i.e., $-y^2$. This leads to the following example.

**Example 2.4.** *Let $D = \partial_x - y^2\partial_y$. Then $\operatorname{Im} D$ is not a Mathieu subspace of $k[x, y]$.*

*Proof:* Note that $1 = Dx \in \operatorname{Im} D$. However $y \notin \operatorname{Im} D$ since for any $g \in k[x, y]$ the $y$-degree of $Dg$ can not be 1. So by Lemma 2.2, $\operatorname{Im} D$ is not a Mathieu subspace of $k[x, y]$. $\square$

The following lemma will also be needed in Section 3.

**Lemma 2.5.** *Let $R$ be any $k$-algebra, $L$ a field extension of $k$ and $M$ a $k$-subspace of $R$. Assume that $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$. Then $M$ is a Mathieu subspace of the $k$-algebra $R$.*

*Proof:* We view $L \otimes_k R$ as a $k$-algebra in the obvious way. Since $L \otimes_k M$ is a Mathieu subspace of the $L$-algebra $L \otimes_k R$, from Definition 2.1 it is easy to see that $L \otimes_k M$ (as a $k$-subspace) is also a Mathieu subspace of the $k$-algebra $L \otimes_k R$.

Now we identify $R$ with the $k$-subalgebra $1 \otimes_k R$ of the $k$-algebra $L \otimes_k R$. Then from Definition 2.1 again, it is easy to check that the intersection $(L \otimes_k M) \cap R = M$ is a Mathieu subspace of $R$. $\square$

Note that by the lemma above, when we prove that a $k$-subspace of a polynomial algebra over $k$ is a Mathieu subspace of the polynomial algebra, we may freely replace $k$ by any field extension of $k$. For instance, we may assume that $k$ is algebraically closed.

To conclude this section we recall a result from [EWZ] which will be used in Section 3 below.

Let $z = (z_1, z_2, ..., z_n)$ be $n$ commutative free variables and $k[z, z^{-1}]$ the algebra of Laurent polynomials in $z_i$ $(1 \leq i \leq n)$. For any non-zero $f(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha \in k[z, z^{-1}]$, we denote by $\operatorname{Supp}(f)$ the *support* of

$f(z)$, i.e., the set of all $\alpha \in \mathbb{Z}^n$ such that $c_\alpha \neq 0$, and $\mathrm{Poly}\,(f)$ the (*Newton*) *polytope* of $f(z)$, i.e., the convex hull of $\mathrm{Supp}\,(f)$ in $\mathbb{R}^n$.

**Theorem 2.6.** ([EWZ]) *Let* $0 \neq f \in k[z, z^{-1}]$ *and* $u$ *any rational point, i.e., a point with all coordinates being rational, of* $\mathrm{Poly}\,(f)$*. Then there exists* $m \geq 1$ *such that* $(\mathbb{R}_+ u) \cap \mathrm{Supp}\,(f^m) \neq \emptyset$*.*

## 3. Images of Locally Finite Derivations of $k[x, y]$

Let $D$ be any $k$-derivation of $A(= k[x, y])$. Then $D$ is said to be *locally finite* if for every $a \in A$ the $k$-vector space spanned by the elements $D^i a$ $(i \geq 1)$ is finite dimensional.

The main result of this section is the following theorem.

**Theorem 3.1.** *Let $D$ be any locally finite $k$-derivation of $A$. Then* $\mathrm{Im}\, D$ *is a Mathieu subspace of $A$.*

To prove this theorem, we need the following result, which is Corollary 4.7 in [E2].

**Proposition 3.2.** *Let $D$ be any locally finite $k$-derivation of $A$. Then up to the conjugation by a $k$-automorphism of $A$, $D$ has one of the following forms:*

i) $D = (ax + by)\partial_x + (cx + dy)\partial_y$ *for some* $a, b, c, d \in k$*;*
ii) $D = \partial_x + by\partial_y$ *for some* $b \in k$*;*
iii) $D = ax\partial_x + (x^m + amy)\partial_y$ *for some* $a \in k$ *and* $m \geq 1$*;*
iv) $D = f(x)\partial_y$ *for some* $f(x) \in k[x]$*.*

**Lemma 3.3.** *With the same notations as in Proposition 3.2, the following statements hold.*

(a) *If $D$ is of type ii), then $D$ is surjective.*
(b) *If $D$ is of type iii), then*

$$\mathrm{Im}\, D = \begin{cases} (x^m) & \text{if } a = 0. \\ (x, y) & \text{if } a \neq 0. \end{cases} \tag{3.1}$$

(c) *If $D$ is of type iv), then* $\mathrm{Im}\, D = (f(x))$*.*

*Proof:* $(a)$ is well-known, see [C] or [F] (p. 96). $(c)$ is obvious, so it remains to prove $(b)$.

If $a = 0$, then $D = x^m \partial_y$, and hence $\mathrm{Im}\, D = (x^m)$. So assume $a \neq 0$. Replacing $D$ by $a^{-1}D$ (without changing the image $\mathrm{Im}\, D$), we may assume that $D = (x\partial_x + my\partial_y) + bx^m\partial_y$ for some nonzero $b \in k$. Observe that for any $i, j \in \mathbb{N}$, we have

$$D(x^i y^j) = (i + mj)x^i y^j + jbx^{m+i}y^{j-1}. \tag{3.2}$$

Next we use induction on $j \geq 0$ to show that $x^i y^j \in \operatorname{Im} D$ whenever $i + j > 0$.

First, assume $j = 0$. Then by Eq. (3.2), we have $Dx^i = ix^i$, and hence $x^i \in \operatorname{Im} D$ for all $i \geq 1$.

Now assume $j \geq 1$. Since $m \geq 1$, we have $m + i \geq 1$ for all $i \geq 0$. Then by the induction assumption, $jbx^{m+i}y^{j-1} \in \operatorname{Im} D$ for all $i \geq 0$. Combining this fact with Eq. (3.2), we get $x^i y^j \in \operatorname{Im} D$ since $i + mj \neq 0$ for all $i \geq 0$. Hence we have proved that $x^i y^j \in \operatorname{Im} D$ if $i + j > 0$. Note that 1 does not lie in $\operatorname{Im} D$ since this space is contained in the ideal generated by $x$ and $y$. Therefore we have $\operatorname{Im} D = (x, y)$. $\square$

**Lemma 3.4.** *Let $z = (z_1, z_2, \ldots, z_n)$ be $n$ free commutative variables and $D := \sum_{i=1}^n a_i z_i \partial_{z_i}$ for some $a_i \in k$ $(1 \leq i \leq n)$. Then $\operatorname{Im} D$ is a Mathieu subspace of $k[z]$.*

Note that $D$ in the lemma is a locally finite derivation of the polynomial algebra $k[z]$. To show the lemma, let's first recall the following well-known results.

**Lemma 3.5.** *For any polynomials $f, g \in k[z]$ and a positive integer $m \geq 1$, we have*

$$\operatorname{Poly}(fg) = \operatorname{Poly}(f) + \operatorname{Poly}(g), \tag{3.3}$$

$$\operatorname{Poly}(f^m) = m\operatorname{Poly}(f), \tag{3.4}$$

*where the sum in the first equation above denotes the Minkowski sum of polytopes.*

*Proof:* Eq. (3.3) is well-known, which was first proved by A. M. Ostrowski [O1] in 1921 (see also Theorem VI, p. 226 in [O2] or Lemma 2.2, p. 11 in [Stu]). To show Eq. (3.4), one can first check easily that the polytope $m\operatorname{Poly}(f)$ and the polytope obtained by taking the Minkowski sum of $m$ copies of $\operatorname{Poly}(f)$ actually share the same set of extremal vertices, namely, the set of the vertices $mv_i$, where $v_i$ runs through all extremal vertices of $\operatorname{Poly}(f)$. Consequently, these two polytopes coincide. Then from this fact and Eq. (3.3), we see that Eq. (3.4) follows. $\square$

*Proof of Lemma 3.4:* If all $a_i$'s are zero, then $D = 0$ and $\operatorname{Im} D = 0$. Hence the lemma holds in this case. So, we assume that not all $a_i$'s are zero.

Let $S$ be the set of integral solutions $\beta \in \mathbb{Z}^n$ of the linear equation $\sum_{i=1}^n a_i \beta_i = 0$. Note that $S \neq \emptyset$ (since $0 \in S$) and is a finitely generated $\mathbb{Z}$-module. Let $V$ be the subspace of $\mathbb{R}^n$ spanned by elements of $S$ over

$\mathbb{R}$. Then $V$ is a $\mathbb{R}$-subspace of $\mathbb{R}^n$ with $r := \dim_{\mathbb{R}} V < n$. Furthermore, $V$ can be described as the set of common solutions of some linear equations with rational coefficients, since clearly the $\mathbb{Q}$-vector space generated by the $\mathbb{Z}$-generators of $S$ can.

Note also that for any $\beta = (\beta_1, \beta_2, ..., \beta_n) \in \mathbb{N}^n$, we have $Dz^\beta = (\sum_{i=1}^n a_i\beta_i)z^\beta$. Hence, for any $\beta \in \mathbb{N}^n$, the monomial $z^\beta \in \operatorname{Im} D$ iff $\beta \notin S$, or equivalently, $\beta \notin V$. Consequently, for any $0 \neq h(z) \in \mathbb{C}[z]$, we have

$$h(z) \in \operatorname{Im} D \Leftrightarrow \operatorname{Supp}(h) \cap V = \emptyset. \tag{3.5}$$

Now, let $0 \neq f(z) \in \mathbb{C}[z]$ such that $f^m \in \operatorname{Im} D$ for all $m \geq 1$. We claim $\operatorname{Poly}(f) \cap V = \emptyset$.

Assume otherwise. Since all vertices of the polytope $\operatorname{Poly}(f)$ are rational (actually integral), every face of $\operatorname{Poly}(f)$ can be described as the set of common solutions of some linear equations with rational coefficients. Since this is also the case for $V$ (as pointed above) and $\operatorname{Poly}(f) \cap V \neq \emptyset$ (by our assumption), it is easy to see that there exists at least one rational point $u \in \operatorname{Poly}(f) \cap V$. Then by Theorem 2.6, there exists $m \geq 1$ such that $(\mathbb{R}_+u) \cap \operatorname{Supp}(f^m) \neq \emptyset$, and by Eq. (3.5), $f^m \notin \operatorname{Im} D$. Hence, we get a contradiction. Therefore, the claim holds.

Finally, we show that $\operatorname{Im} D$ is a Mathieu subspace as follows.

Let $f(z)$ be as above and $d$ the distance between $V$ and $\operatorname{Poly}(f)$. Then by the claim above and the fact that $\operatorname{Poly}(f)$ is a compact subset of $\mathbb{R}^n$, we have $d > 0$. Furthermore, for any $m \geq 1$, by Eq. (3.4) we have $\operatorname{Poly}(f^m) = m\operatorname{Poly}(f)$. Hence, the distance between $V$ and $\operatorname{Poly}(f^m)$ is given by $dm$.

Now let $h(z)$ be an arbitrary element of $k[z]$. Note that by Eqs. (3.3) and (3.4) we have $\operatorname{Poly}(f^m h) = m\operatorname{Poly}(f) + \operatorname{Poly}(h)$ for all $m \geq 1$. Hence, for large enough $m$, the distance between $V$ and $\operatorname{Poly}(f^m h)$ is positive, whence $\operatorname{Poly}(f^m h) \cap V = \emptyset$. In particular, $\operatorname{Supp}(f^m h) \cap V = \emptyset$, and by Eq. (3.5), $f^m h \in \operatorname{Im} D$ when $m \gg 0$. Then by Definition 2.1, we see that $\operatorname{Im} D$ is indeed a Mathieu subspace of $k[z]$. $\square$

Now we can prove the main theorem of this section as follows.

*Proof of Theorem 3.1:* First, by Proposition 3.2, we only need to show that $\operatorname{Im} D$ is a Mathieu subspace of $A$ in each of the four cases in Proposition 3.2. Furthermore, by Lemma 3.3 it only remains to prove case $i$). So assume $D = (ax + by)\partial_x + (cx + dy)\partial_y$ for some $a, b, c, d \in k$.

Second, by Lemma 2.5, we may assume that $k$ is algebraically closed.

Third, note that $D$ preserves the subspace $H := kx + ky \subset A$, so its restriction $D|_H$ on $H$ is a linear endomorphism of $H$. Since $k$ is algebraically closed, there exists a linear automorphism $\sigma$ of $H$ such that the conjugation $\sigma(D|_H)\sigma^{-1}$ gives the Jordan form of $D|_H$. Let $\tilde{\sigma}$ be the unique extension of $\sigma$ to an automorphism of $A$. Then it is easy to see that $\tilde{\sigma}D\tilde{\sigma}^{-1}$ is also a $k$-derivation of $A$.

Note that $\operatorname{Im}\tilde{\sigma}D\tilde{\sigma}^{-1} = \tilde{\sigma}(\operatorname{Im} D)$ and in general Mathieu subspaces are preserved by $k$-algebra automorphisms. Therefore, we may replace $D$ by $\tilde{\sigma}D\tilde{\sigma}^{-1}$, if necessary, and assume that $D = a(x\partial_x + y\partial_y) + x\partial_y$ (in case that the Jordan form of $D|_H$ is an $2 \times 2$ Jordan block) or $D = ax\partial_x + by\partial_y$ (in case that the Jordan form of $D|_H$ is diagonal).

For the former case, by Lemma 3.3, $(b)$ with $m = 1$, we see that $\operatorname{Im} D$ is an ideal, and hence a Mathieu subspace of $A$. For the latter case, it follows from Lemma 3.4 that $\operatorname{Im} D$ also a Mathieu subspace of $A$. Therefore, the theorem holds. $\square$

## 4. **Connection with the Two-Dimensional Jacobian Conjecture**

In the previous section we showed that the image of every locally finite $k$-derivation of $A$ is a Mathieu subspace of $A$. However, as we have shown in Example 2.4, $\operatorname{Im} D$ needs not to be a Mathieu subspace of $A$ for every $k$-derivation $D$ of $A$. This leads to the question of which $k$-derivations $D$ of $A$ have the property that $\operatorname{Im} D$ is a Mathieu subspace of $A$. More precisely, we can ask

**Question 4.1.** *Let $D$ be any $k$-derivation of $A$ such that* $\operatorname{div} D = 0$*, where for any $D = p\partial_x + q\partial_y$ $(p, q \in A)$,* $\operatorname{div} D := \partial_x p + \partial_y q$*. Is* $\operatorname{Im} D$ *a Mathieu subspace of $A$?*

Adding one more condition, we get

**Question 4.2.** *Let $D$ be any $k$-derivation of $A$ such that* $\operatorname{div} D = 0$*. If* $1 \in \operatorname{Im} D$*, is* $\operatorname{Im} D$ *a Mathieu subspace of $A$?*

Note that by Lemma 2.2, this question is equivalent to asking if $D$ is surjective under the further condition $1 \in \operatorname{Im} D$.

The motivation of the two questions above come from the following theorem.

**Theorem 4.3.** *Question 4.2 has an affirmative answer iff the two dimensional Jacobian conjecture is true.*

*Proof:* $(\Rightarrow)$ Assume that Question 4.2 has an affirmative answer. Let $F = (f, g) \in k[x, y]^2$ with $\det JF = 1$. Consider the $k$-derivation $D := g_y\partial_x - g_x\partial_y$. Then $\operatorname{div} D = 0$ and $1 = \det JF = Df \in \operatorname{Im} D$. Since by our hypothesis $\operatorname{Im} D$ is a Mathieu subspace of $A$, it follows from Lemma 2.2 that $\operatorname{Im} D = A$, i.e., $D$ is surjective. Then it follows from a theorem of Stein [Ste] (see also [C]) that $D$ is locally nilpotent.

Since $D = \partial/\partial f$, $\ker D = \ker \partial/\partial f = k[g]$ by Proposition 2.2.15 in [E1]. Since $D$ has a slice $f$, it follows that $A = k[g][f]$, i.e., $F$ is invertible over $k$. So the two-dimensional Jacobian conjecture is true.

($\Leftarrow$) Assume that the two-dimensional Jacobian conjecture is true. Let $D = p\partial_x + q\partial_y$ ($p, q \in A$) be a $k$-derivation of $A$ such that $\operatorname{div} D = 0$ and $1 \in \operatorname{Im} D$.

Since $\operatorname{div} D = 0$, we have $\partial_x p = \partial_y(-q)$. Then by Poincaré's lemma, there exists $g \in A$ such that $p = \partial_y g$ and $q = -\partial_x g$. So $D = g_y\partial_x - g_x\partial_y$.

Since $1 \in \operatorname{Im} D$, we get $1 = Df$ for some $f \in A$. Let $F := (f, g) \in k[x, y]^2$. Then we have $\det JF = Df = 1$. Since by our hypothesis $F$ is invertible, it follows that $k[x, y] = k[f, g]$. Hence, we have

$$\operatorname{Im} D = \operatorname{Im} \frac{\partial}{\partial f} = \frac{\partial}{\partial f}(k[f, g]) = k[f, g] = A.$$

In particular, $\operatorname{Im} D$ is a Mathieu subspace of $A$. $\square$

## References


[BCW] D. Bass, E. Connell, D. Wright, *The Jacobian Conjecture, Reduction of Degree and Formal Expansion of the Inverse.* Bull. Amer. Math. Soc. **7**, (1982), 287–330. [MR83k:14028].

[C] D. Cerveau, *Dérivations surjectives de l'anneau $C[x, y]$.* (French) [Surjective derivations of the ring $C[x, y]$] J. Algebra **195** (1997), no. 1, 320–335. [MR1468896].

[DK] J. J. Duistermaat and W. van der Kallen, *Constant Terms in Powers of a Laurent Polynomial.* Indag. Math. (N.S.) **9** (1998), no. 2, 221–231. [MR1691479].

[E1] A. van den Essen, *Polynomial Automorphisms and the Jacobian Conjecture.* Progress in Mathematics, 190. Birkhäuser Verlag, Basel, 2000. [MR1790619].

[E2] A. van den Essen, *Locally Finite and Locally Nilpotent Derivations with Applications to Polynomial Flows and Polynomial Morphisms.* Proc. AMS, **116**, No. 3 (1992), 861–871. [MR1111440].

[EWZ] A. van den Essen, R. Willems and W. Zhao, *Some Results on the Vanishing Conjecture of Differential Operators with Constant Coefficients.* arXiv:0903.1478v1 [math.AC]. Under submission.

[F] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations.* Encyclopaedia of Mathematical Sciences, **136**. Invariant Theory and Algebraic Transformation Groups, VII. Springer-Verlag, Berlin, 2006. [MR2259515].

[M] O. Mathieu, *Some Conjectures about Invariant Theory and Their Applications.* Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), 263–279, Sémin. Congr., 2, Soc. Math. France, Paris, 1997. [MR1601155].

[O1] A. M. Ostrowski, *Über die Bedeutung der Theorie der konvexen Polyeder für die formale Algebra*. Jahresberichte Deutsche Math. Verein **30** (1921), 98–99.

[O2] A. M. Ostrowski, *On multiplication and factorization of polynomials I. Lexicographic ordering and extreme aggregates of terms.* Aequationes Math. **13** (1975), no. 3, 201-228. [MR0399060].

[Ste] Y. Stein, *On the Density of Image of Differential Operators Generated by Polynomials.* J. Analyse Math. **52** (1989), 291–300. [MR0981505].

[Stu] B. Sturmfels, *Gröbner bases and convex polytopes*. University Lecture Series, **8**. American Mathematical Society, 1996. [MR1363949].

[Z1] W. Zhao, *Images of Commuting Differential Operators of Order One with Constant Leading Coefficients*. J. Alg. **324** (2010), no. 2, 231–247. See also arXiv:0902.0210 [math.CV].

[Z2] W. Zhao, *Generalizations of the Image Conjecture and the Mathieu Conjecture*. J. Pure Appl. Algebra. **214** (2010), 1200-1216. See also arXiv:0902.0212 [math.CV].

A. van den Essen, Department of Mathematics, Radboud University Nijmegen, Postbus 9010, 6500 GL Nijmegen, The Netherlands. E-mail: essen@math.ru.nl

D. Wright, Department of Mathematics, Washington University in St. Louis, St. Louis, MO. 63130, USA. E-mail: wright@math.wustl.edu

W. Zhao, Department of Mathematics, Illinois State University, Normal, IL 61790-4520, USA. E-mail: wzhao@ilstu.edu.