Finitely generated free Heyting algebras via Birkhoff duality and coalgebra

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Abstract. Algebras axiomatized entirely by rank 1 axioms are algebras for a functor and thus the free algebras can be obtained by a direct limit process. Dually, the final coalgebras can be obtained by an inverse limit process. In order to explore the limits of this method we look at Heyting algebras which have mixed rank 0-1 axiomatizations. We will see that Heyting algebras are special in that they are almost rank 1 axiomatized and can be handled by a slight variant of the rank 1 coalgebraic methods.

1 Introduction

Coalgebraic methods and techniques are becoming increasingly important in investigating non-classical logics [24]. In particular, logics axiomatized by rank 1 axioms allow coalgebraic representation as coalgebras for a functor [17, 23]. We recall that an equation is of rank 1 for an operation f if each variable occurring in the equation is under the scope of exactly one occurrence of f. As a result the algebras for these logics become algebras for a functor over the category of underlying algebras without the operation f. Consequently, free algebras are initial algebras in the category of algebras for this functor. This correspondence immediately gives a constructive description of free algebras for rank 1 logics [13, 1, 7]. Examples of rank 1 logics are the basic modal logic **K**, basic positive modal logic, graded modal logic, probabilistic modal logic, coalition logic and so on [23]. For a coalgebraic approach to the complexity of rank 1 logics we refer to [23]. On the other hand, rank 1 axioms are too simple—very few well-known logics are axiomatized by rank 1 axioms. Therefore, one would want to extend the existing coalgebraic techniques to non-rank 1 logics. However, as follows from [18], algebras for these logics cannot be represented as algebras for a functor and we cannot use the standard construction of free algebras in a straightforward way.

This paper is an extended version of [6]. However, unlike [6], here we give a complete solution to the problem of describing finitely generated free Heyting algebras in a systematic way using methods similar to those used for rank 1 logics. This paper together with [6] and [7] is a facet of a larger joint project with Alexander Kurz on coalgebraic treatment of modal logics beyond rank 1. We recall that an equation is of rank 0-1 for an operation f if each variable occurring in the equation is under the scope of at most

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one occurrence of f. With the ultimate goal of generalizing a method of constructing free algebras for varieties axiomatized by rank 1 axioms to the case of rank 0-1 axioms, we consider the case of Heyting algebras (intuitionistic logic, which is of rank 0-1 for $f = \to$). In particular, we construct free Heyting algebras. For an extension of coalgebraic techniques to deal with the finite model property of non-rank 1 logics we refer to [22].

Free Heyting algebras have been the subject of intensive investigation for decades. The one-generated free Heyting algebra was constructed by Rieger and Nishimura in the 50s. In the 70s Urquhart gave an algebraic characterization of finitely generated free Heyting algebras. A very detailed description of finitely generated free Heyting algebras in terms of their dual spaces was obtained in the 80s by Grigolia, Shehtman, Bellissima and Rybakov. This method is based on a description of the points of finite depth of the dual frame of the free Heyting algebra. For the details of this construction we refer to [11, Section 8.7] and [5, Section 3.2] and the references therein. Finally, Ghilardi [12] introduced a different method for describing free Heyting algebras. His technique builds the free Heyting algebra on a distributive lattice step by step by freely adding to the original lattice the implications of degree n, for each $n \in \omega$. Ghilardi [12] used this technique to show that every finitely generated free Heyting algebra is a bi-Heyting algebra. A more detailed account of Ghilardi's construction can be found in [9] and [14]. Ghilardi and Zawadowski [14], based on this method, derive a model-theoretic proof of Pitts' uniform interpolation theorem. In [3] a similar construction is used to describe free linear Heyting algebras over a finite distributive lattice and [21] uses the same method to construct high order cylindric Heyting algebras.

Our contribution is to derive Ghilardi's representation of finitely generated free Heyting algebras in a simple, transparent, and modular way which is based entirely on the ideas of the coalgebraic approach to rank 1 logics, though it uses these ideas in a non-standard way. We split the process into two parts. We first apply the initial algebra construction to weak and pre Heyting algebras—these are consecutive rank 1 approximants of Heyting algebras. We then use a non-standard colimit system based on the sequence of algebras for building free pre-Heyting algebras in the standard coalgebraic framework.

On the negative side, we use some properties particular to Heyting algebras, and thus our work does not yield a method that applies in general. Nevertheless, we expect that the approach, though it would have to be tailored, is likely to be successful in other instances as well. Obtained results allow us to derive a coalgebraic representation for weak and pre Heyting algebras and sheds new light on the very special nature of Heyting algebras.

The paper is organized as follows. In Section 2 we recall the so-called Birkhoff (discrete) duality for distributive lattices. We use this duality in Section 3 to build free weak Heyting algebras and in Section 4 to build free pre-Heyting algebras. Obtained results are applied in Section 5 for describing free Heyting algebras. In Section 6 we give a coalgebraic representation for weak and pre-Heyting algebras. We conclude the paper by listing some future work.

2 Discrete duality for distributive lattices

We recall that a non-zero element a of a distributive lattice D is called *join-irreducible* if for every $b, c \in D$ we have that $a \leq b \vee c$ implies $a \leq b$ or $a \leq c$. For each distributive lattice (DL for short) D let $J(D)$ denote the set of all join-irreducible elements of D. Let also $\leq$ be the restriction of the order of D to $J(D)$. Then $(J(D), \leq)$ is a poset. Recall also that for every poset X a subset $U \subseteq X$ is called a *downset* if $x \in U$ and $y \leq x$ imply $y \in U$. For each poset X we denote by $\mathcal{O}(X)$ the distributive lattice $(\mathcal{O}(X), \cap, \cup, \emptyset, X)$ of all downsets of X. Then every finite distributive lattice D is isomorphic to the lattice of all downsets of $(J(D), \leq)$ and vice versa, every poset X is isomorphic to the poset of join-irreducible elements of $\mathcal{O}(X)$. We call $(J(D), \leq)$ the *dual poset* of D and we call $\mathcal{O}(X)$ the *dual lattice* of X.

This duality can be extended to the duality of the category $\mathbf{DL}_{\mathrm{fin}}$ of finite bounded distributive lattices and bounded lattice morphisms and the category $\mathbf{Pos}_{\mathrm{fin}}$ of finite posets and order-preserving maps. In fact, if $h : D \to D'$ is a bounded lattice morphism, then the restriction of $h^\flat$, the lower adjoint of h, to $J(D')$ is an order-preserving map between $(J(D'), \leq')$ and $(J(D), \leq)$, and if $f : X \to X'$ is an order-preserving map between two posets X and X', then $f^\downarrow : \mathcal{O}(X) \to \mathcal{O}(X'),\ S \mapsto\ \downarrow f(S)$ is $\bigvee$-preserving and its upper adjoint $(f^\downarrow)^\sharp = f^{-1} : \mathcal{O}(X') \to \mathcal{O}(X)$ is a bounded lattice morphism. Moreover, injective bounded lattice morphisms (i.e. embeddings or, equivalently, regular monomorphisms) correspond to surjective order-preserving maps, and surjective lattice morphisms (homomorphic images) correspond to order embeddings (order-preserving and order-reflecting injective maps) that are in one-to-one correspondence with subposets of the corresponding poset.

We also recall that an element a, $a \neq 1$, of a distributive lattice D is called *meet-irreducible* if for every $b, c \in D$ we have that $b \wedge c \leq a$ implies $b \leq a$ or $c \leq a$. We let $M(D)$ denote the set of all meet-irreducible elements of D.

Proposition 2.1. *Let D be a finite distributive lattice. Then for each $p \in J(D)$, there exists $\kappa(p) \in M(D)$ such that $p \not\leq \kappa(p)$ and for each $a \in D$ we have*

$$p \leq a \text{ or } a \leq \kappa(p).$$

Proof. For $p \in J(D)$, let $\kappa(p) = \bigvee\{a \in D \mid p \not\leq a\}$. Then it is clear that the condition involving all $a \in D$ holds. Note that if $p \leq \kappa(p) = \bigvee\{a \in D \mid p \not\leq a\}$, then, applying the join-irreducibility of p, we get $a \in D$ with $p \not\leq a$ but $p \leq a$, which is clearly a contradiction. So it is true that $p \not\leq \kappa(p)$. Now we show that $\kappa(p)$ is meet irreducible. First note that since p is not below $\kappa(p)$, the latter cannot be equal to 1. Also, if $a, b \not\leq \kappa(p)$ then $p \leq a, b$ and therefore $p \leq a \wedge b$. Thus it follows that $a \wedge b \not\leq \kappa(p)$. This concludes the proof of the proposition. □

Proposition 2.2. *Let X be a finite set and $F_{DL}(X)$ the free distributive lattice over X. Then the poset $(J(F_{DL}(X)), \leq)$ of join-irreducible elements of $F_{DL}(X)$ is isomorphic to $(\mathcal{P}(X), \supseteq)$, where $\mathcal{P}(X)$ is the power set of X and each subset $S \subseteq X$ corresponds to the conjunction $\bigwedge S \in F_{DL}(X)$. Moreover, for $x \in X$ and $S \subseteq X$ we have*

$$\bigwedge S \leq x \text{ iff } x \in S.$$

Proof. This is equivalent to the disjunctive normal form representation for elements of $F_{DL}(X)$. □

3 Weak Heyting algebras

3.1 Freely adding weak implications

Definition 3.1. *[10] A pair $(A, \to)$ is called a* weak Heyting algebra[3] *if A is a bounded distributive lattice and $\to\colon A^2 \to A$ a* weak implication*, that is, a binary operation satisfying the following axioms for all $a, b, c \in A$:*
(1) $a \to a = 1$,
(2) $a \to (b \wedge c) = (a \to b) \wedge (a \to c)$.
(3) $(a \vee b) \to c = (a \to c) \wedge (b \to c)$.
(4) $(a \to b) \wedge (b \to c) \leq a \to c$.

It is easy to see that by (2) weak implication is order-preserving in the second coordinate and by (4) order-reversing in the first. The following lemma yields a useful equational property of wHAs.

Lemma 3.2. *Let $(A, \to)$ be a wHA. For each $a, b \in A$ we have*

$$a \to b = a \to (a \wedge b).$$

Proof. By (2) we have $a \to (a \wedge b) = (a \to a) \wedge (a \to b)$ and by (1) we have $a \to a = 1$ so we obtain $a \to b = a \to (a \wedge b)$. □

Let D and D' be distributive lattices. We let $\to(D \times D')$ denote the set $\{a \to b : a \in D \text{ and } b \in D'\}$. We stress that this is just a set bijective with $D \times D'$. The implication symbol is just a formal notation. For each distributive lattice D we also let $F_{DL}(\to(D \times D))$ denote the free distributive lattice over $\to(D \times D)$. Moreover, we let

$$H(D) = F_{DL}(\to(D \times D))/_{\approx}$$

where $\approx$ is the DL congruence generated by the axioms (1)–(4). We want to stress that we are not thinking of the axioms as a basis for an equational theory for a binary operation $\to$ here. The point of view is that of describing a bounded distributive lattice by generators and relations. That is, we want to find the quotient of the free bounded distributive lattice over the set $\to(D \times D)$ with respect to the lattice congruence generated by the pairs of elements of $F_{DL}(\to(D \times D))$ in (1)–(4) with a, b, c ranging over D. For an element $a \to b \in F_{DL}(\to(D \times D))$ we denote by $[a \to b]_{\approx}$ the $\approx$ equivalence class of $a \to b$.

The rest of the section will be devoted to showing that for each finite distributive lattice D the poset $(J(H(D)), \leq)$ is isomorphic to $(\mathcal{P}(J(D)), \subseteq)$. Below we give a dual proof of this fact. The dual proof, which relies on the fact that identifying two elements of an algebra simply corresponds to throwing out those points of the dual that are below one and not the other, is produced in a simple, modular, and systematic way that doesn't require any prior insight into the structure of these particular algebras.

[3] In [10] weak Heyting algebras are called 'weakly Heyting algebras'.

We start with a finite distributive lattice D and the free DL generated by the set

$$\to(D \times D) = \{a \to b \mid a, b \in D\}$$

of all formal arrows over D. As follows from Proposition 2.2, $J(F_{DL}(\to(D \times D)))$ is isomorphic to the power set of $\to(D \times D)$, ordered by reverse inclusion. Each subset of $\to(D \times D)$ corresponds to the conjunction of the elements in that subset; the empty set of course corresponds to 1. Now we want to take quotients of this free distributive lattice wrt various lattice congruences, namely the ones generated by the set of instances of the axioms of weak Heyting algebras.

The axiom $x \to x = 1$.

Here we want to take the quotient of $F_{DL}(\to(D \times D))$ with respect to the lattice congruence of $F_{DL}(\to(D \times D))$ generated by the set $\{(a \to a, 1) \mid a \in D\}$. By duality this quotient is given dually by the *subset*, call it P_1, of our initial poset $P_0 = J(F_{DL}(\to(D\times D)))$, consisting of those join-irreducibles of $F_{DL}(\to(D\times D))$ that do not violate this axiom. Thus, for $S \in J(F_{DL}(\to(D \times D)))$, S is admissible provided

$$\forall a \in D \qquad (\bigwedge S \leq 1 \quad \Longleftrightarrow \quad \bigwedge S \leq a \to a).$$

Since all join-irreducibles are less than or equal to 1, it follows that the only join-irreducibles that are admissible are the ones that are below $a \to a$ for all $a \in D$. That is, viewed as subsets of $\to(D \times D)$, only the ones that contain $a \to a$ for each $a \in D$:

$$P_1 = \{S \in P_0 \mid a \to a \in S \text{ for each } a \in D\}.$$

The axiom $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$.

We now want to take a further quotient and thus we want to keep only those join-irreducibles from P_1 that do not violate this second axiom. That is, $S \in P_1$ is admissible provided

$$\forall a, b, c \qquad (\bigwedge S \leq a \to (b \wedge c) \quad \Longleftrightarrow \quad \bigwedge S \leq a \to b \text{ and } \bigwedge S \leq a \to c).$$

which means

$$\forall a, b, c \qquad (a \to (b \wedge c) \in S \quad \Longleftrightarrow \quad a \to b \in S \text{ and } a \to c \in S).$$

Proposition 3.3. *The poset P_2 of admissible join-irreducibles at this stage is order isomorphic to the set*

$$Q_2 = \{f : D \to D \mid \forall a \in D \quad f(a) \leq a\}$$

ordered pointwise.

Proof. An admissible S from P_2 corresponds to the function $f_S : D \to D$ given by

$$f_S(a) = \bigwedge\{b \in D \mid a \to b \in S\}.$$

In the reverse direction a function in P_2 corresponds to the admissible set

$$S_f = \{a \to b \mid f(a) \le b\}.$$

The proof that this establishes an order isomorphism is a straightforward verification. $\square$

The axiom $(x \vee y) \to z = (x \to z) \wedge (y \to z)$.

We want the subposet of P_2 consisting of those f's such that

$$\forall a, b, c \qquad \big((a \vee b) \to c \in S_f \quad \iff \quad a \to c \in S_f \text{ and } b \to c \in S_f\big).$$

To this end notice that

$$\begin{aligned} & \forall a, b, c && \big((a \vee b) \to c \in S_f \iff (a \to c \in S_f \text{ and } b \to c \in S_f)\big) \\ \iff & \forall a, b, c && \big(f(a \vee b) \le c \quad \iff \quad (f(a) \le c \text{ and } f(b) \le c)\big) \\ \iff & \forall a, b && f(a \vee b) = f(a) \vee f(b). \end{aligned}$$

That is, the poset, P_3, of admissible join-irreducibles left at this stage is isomorphic to the set

$$Q_3 = \{f : D \to D \mid f \text{ is join-preserving and } \forall a \in D \quad f(a) \le a\}.$$

The axiom $(x \to y) \wedge (y \to z) \le x \to z$.

It is not hard to see that this yields, in terms of join-preserving functions $f : D \to D$,

$$\begin{aligned} Q_4 &= \{f \in Q_3 \mid \forall a \in D \; f(a) \le f(f(a))\} \\ &= \{f : D \to D \mid f \text{ is join-preserving and } \forall a \in D \; f(a) \le f(f(a)) \le f(a) \le a\} \\ &= \{f : D \to D \mid f \text{ is join-preserving and } \forall a \in D \; f(f(a)) = f(a) \le a\}. \end{aligned}$$

We note that the elements of Q_4 are nuclei [15] on the order-dual lattice of D. Since the f's in Q_4 are join and 0 preserving, they are completely given by their action on $J(D)$. The additional property shows that these functions have lots of fixpoints. In fact, we can show that they are completely described by their join-irreducible fixpoints.

Lemma 3.4. *Let $f \in Q_4$, then for each $a \in D$ we have*

$$f(a) = \bigvee \{r \in J(D) \mid f(r) = r \le a\}.$$

Proof. Clearly $\bigvee\{r \in J(D) \mid f(r) = r \le a\} \le f(a)$. For the converse, let r be maximal in $J(D)$ wrt the property that $r \le f(a)$. Now it follows that

$$r \le f(a) = f(f(a)) = \bigvee \{f(q) \mid J(D) \ni q \le f(a)\}.$$

Since r is join-irreducible, there is $q \in J(D)$ with $q \le f(a)$ and $r \le f(q)$. Thus $r \le f(q) \le q \le f(a)$ and by maximality of r we conclude that $q = r$. Now $r \le f(q)$

and $q = r$ yields $r \leq f(r)$. However, $f(r) \leq r$ as this holds for any element of D and thus $f(r) = r$. Since any element in a finite lattice is the join of the maximal join-irreducibles below it, we obtain

$$\begin{aligned} f(a) &= \bigvee\{r \in J(D) \mid r \text{ is maximal in } J(D) \text{ wrt } r \leq f(a)\} \\ &\leq \bigvee\{r \in J(D) \mid f(r) = r \leq f(a)\} \ \leq \ f(a). \end{aligned}$$

Finally, notice that if $f(r) = r \leq f(a)$ then as $f(a) \leq a$, we have $f(r) = r \leq a$. Conversely, if $f(r) = r \leq a$ then $r = f(r) = f(f(r)) \leq f(a)$ and we have proved the lemma. □

Proposition 3.5. *The set of functions in Q_4, ordered pointwise, is order isomorphic to the powerset of $J(D)$ in the usual inclusion order.*

Proof. The order isomorphism is given by the following one-to-one correspondence

$$\begin{array}{rcl} Q_4 & \leftrightarrows & \mathcal{P}(J(D)) \\ f & \mapsto & \{p \in J(D) \mid f(p) = p\} \\ f_T & \leftarrow\!\shortmid & T \end{array}$$

where $f_T : D \to D$ is given by $f_T(a) = \bigvee\{p \in J(D) \mid T \ni p \leq a\}$. Using the lemma, it is straightforward to see that these two assignments are inverse to each other. Checking that f_T is join preserving and satisfies $f^2 = f \leq id_D$ is also straightforward. Finally, it is clear that $f_T \leq f_S$ if and only if $T \subseteq S$. □

Theorem 3.6. *Let D be a finite distributive lattice and $X = (J(D), \leq)$ its dual poset. Then*

1. *The poset $(J(H(D)), \leq)$ is isomorphic to the poset $(\mathcal{P}(X), \subseteq)$ of all subsets of X ordered by inclusion.*
2. *$J(H(D)) = \{[\bigwedge_{q \notin T}(q \to \kappa(q))]_{\approx} \mid T \subseteq J(D)\}$, (where $\kappa(q)$ is the element defined in Proposition 2.1).*

Proof. As shown above, the poset $J(H(D))$, obtained from $J(F_{DL}(\to(D \times D)))$ by removing the elements that violate the congruence schemes (1)–(4), is isomorphic to the poset Q_4, and Q_4 is in turn isomorphic to $\mathcal{P}(J(D))$ ordered by inclusion, see Proposition 3.5.

In order to prove the second statement, let $q \in J(D)$, and consider $q \to \kappa(q) \in F_{DL}(\to (D \times D))$. If we represent $H(D)$ as the lattice of downsets $\mathcal{O}(J(H(D)))$, then the action of the quotient map on this element is given by

$$\begin{array}{rcl} F_{DL}(\to (D \times D)) & \to & H(D) \\ q \to \kappa(q) & \mapsto & \{T' \in \mathcal{P}(J(D)) \mid q \to \kappa(q) \in S_{T'}\}. \end{array}$$

Now

$$
\begin{aligned}
q \to \kappa(q) \in S_{T'} &\iff f_{T'}(q) \leq \kappa(q) \\
&\iff \bigvee(\downarrow q \cap T') \leq \kappa(q) \\
&\iff q \notin T'.
\end{aligned}
$$

The last equivalence follows from the fact that $a \leq \kappa(q)$ if and only if $q \nleq a$ and the only element of $\downarrow q$ that violates this is q itself. We now can see that for any $T \subseteq J(D)$ we have

$$
\begin{aligned}
F_{DL}(\to (D \times D)) &\quad\to\quad H(D) \\
[\bigwedge_{q \notin T} (q \to \kappa(q))]_{\approx} &\quad\mapsto\quad \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \notin T \;\Rightarrow\; q \to \kappa(q) \in S_{T'}\} \\
&\qquad = \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \notin T \;\Rightarrow\; q \notin T'\} \\
&\qquad = \{T' \in \mathcal{P}(J(D)) \mid \forall q \quad (q \in T' \;\Rightarrow\; q \in T\} \\
&\qquad = \{T' \in \mathcal{P}(J(D)) \mid T' \subseteq T\}.
\end{aligned}
$$

That is, under the quotient map $F_{DL}(\to (D \times D)) \;\to\; H(D)$, the elements $\bigwedge_{q \notin T}(q \to \kappa(q))$ are mapped to the principal downsets $\downarrow T$, for each $T \in \mathcal{P}(J(D)) = J(H(D))$. Since these principal downsets are exactly the join-irreducibles of $\mathcal{O}(J(H(D))) = H(D)$, we have that $\{\ [\bigwedge_{q \notin T}(q \to \kappa(q))]_{\approx} \ \mid\ T \subseteq J(D)\ \} = J(H(D))$. □

Next we will prove a useful technical lemma that will be applied often throughout the remainder of the paper. Let D be a finite distributive lattice. For $a, b \in D$ and $T \subseteq J(D)$ we write $T \leq a \to b$ if $\bigwedge S_T \leq a \to b$ in $H(D)$, where $S_T = S_{f_T} = \{a \to b : f_T(a) \leq b\}$.

Lemma 3.7. *Let D be a finite distributive lattice. For each $a, b \in D$, $a \to b \in H(D)$ and $T \subseteq J(D)$ we have*

$$T \leq a \to b \quad \textit{iff} \quad \forall p \in T \ (p \leq a \textit{ implies } p \leq b)$$

Proof.

$$
\begin{aligned}
T \leq a \to b &\iff \bigwedge S_T \leq a \to b \\
&\iff a \to b \in S_T \\
&\iff f_T(a) \leq b \\
&\iff \bigvee(\downarrow a \cap T) \leq b \\
&\iff \forall p \in T \ (p \leq a \text{ implies } p \leq b).
\end{aligned}
$$

□

It follows from Theorem 3.6(1) that if two finite distributive lattices D and D' have an equal number of join-irreducible elements, then $H(D)$ is isomorphic to $H(D')$. To see this, we note that if $|J(D)| = |J(D')|$, then $(\mathcal{P}(J(D)), \subseteq)$ is isomorphic to

$(\mathcal{P}(J(D')), \subseteq)$. This, by Theorem 3.6(1), implies that $H(D)$ is isomorphic to $H(D')$. In particular, any two non-equivalent orders on any finite set give rise to two non-isomorphic distributive lattices with isomorphic H-images.

Remark 3.8. All the results in this section for finite distributive lattices can be generalized to the infinite case. In the infinite case instead of finite posets we would need to work with Priestley spaces and instead of the finite powerset we need to work with the Vietoris space (see Section 6). Since for our purposes finite distributive lattices suffice, for now, we will stick with the finite case.

3.2 Free weak Heyting algebras

In the coalgebraic approach to generating the free algebra, it is a fact of central importance that H as described here is actually a functor. That is, for a DL homomorphism $h : D \to E$ one can define a DL homomorphism $H(h) : H(D) \to H(E)$ so that H becomes a functor on the category of DLs. To see this, we only need to note that H is defined by rank 1 axioms. We recall that for an operator f (in our case f is the weak implication $\to$) an equation is of *rank 1* if each variable in the equation is under the scope of exactly one occurrence of f and an equation is of *rank 0-1* if each variable in the equation is under the scope of at most one occurrence of f. It is easy to check that axioms (1)-(4) for weak Heyting algebras are rank 1. Therefore, H gives rise to a functor $H : \mathbf{DL} \to \mathbf{DL}$ [2, 18]. Moreover, the category of weak Heyting algebras is isomorphic to the category $Alg(H)$ of the algebras for the functor H. For the details of such correspondences we refer to [2, 1, 13, 7, 18]. We would like to give a concrete description of how H applies to DL homomorphisms. We describe this in algebraic terms here and give the dual construction via Birkhoff duality.

Let $h : D \to E$ be a DL homomorphism. Recall that the dual map from $J(E)$ to $J(D)$ is just the lower adjoint $h^\flat$ with domain and codomain properly restricted. By abuse of notation we will just denote this map by $h^\flat$, leaving it to the reader to decide what the proper domain and codomain is. Now $H(D) = F_{DL}(\to (D \times D))/{<}Ax(D){>}$, where ${<}Ax(D){>}$ is the DL congruence generated by $Ax(D)$ and $Ax(D)$ is the set of all instances of the axioms (1)-(4) with $a, b, c \in D$. Also let q_D be the quotient map corresponding to mod'ing out by ${<}Ax(D){>}$. The map $h : D \to E$ yields a map $h \times h : D \times D \longrightarrow E \times E$ and this of course yields a lattice homomorphism $F_{DL}(h \times h) : F_{DL}(\to (D \times D)) \longrightarrow F_{DL}(\to (E \times E))$. Now the point is that $F_{DL}(h \times h)$ carries elements of $Ax(D)$ to elements of $Ax(E)$ and thus in particular to elements of ${<}Ax(E){>}$ (it is an easy verification and only requires h to be a homomorphism for axiom schemes (2) and (3)). This is equivalent to saying that $Ax(D) \subseteq Ker(q_E \circ F_{DL}(h \times h))$ and thus ${<}Ax(D){>} \subseteq Ker(q_E \circ F_{DL}(h \times h))$, or equivalently that there is a unique map $H(h) : H(D) \to H(E)$ that makes the following diagram commute

$$\begin{array}{ccc} F_{DL}(\to (D \times D)) & \xrightarrow{F_{DL}(h \times h)} & F_{DL}(\to (E \times E)) \\ \downarrow{\scriptstyle q_D} & & \downarrow{\scriptstyle q_E} \\ H(D) & \overset{H(h)}{\dashrightarrow} & H(E). \end{array}$$

The dual diagram is

$$\begin{array}{ccc}
\mathcal{P}(D \times D) & \xleftarrow{(h\times h)^{-1}} & \mathcal{P}(E \times E) \\
{\scriptstyle e_D}\uparrow & & \uparrow{\scriptstyle e_E} \\
\mathcal{P}(J(D)) & \xleftarrow{\mathcal{P}(h^\flat)} & \mathcal{P}(J(E))
\end{array}$$

The map $e_D : \mathcal{P}(D) \hookrightarrow \mathcal{P}(D \times D)$ is the embedding, via Q_4 and so on into P_0 as obtained above. That is, $e_D(T) = \{a \to b \mid \forall p \in T\ (p \leq a \Rightarrow p \leq b\}$. Now in this dual setting, the fact that there is a map $\mathcal{P}(h^\flat)$ is equivalent to the fact that $(h \times h)^{-1} \circ e_E$ maps into the image of the embedding e_D. This is easily verified:

$$\begin{aligned}
(h \times h)^{-1}(e_E(T)) &= \{a \to b \mid \forall q \in T\ (q \leq h(a) \Rightarrow q \leq h(b)\} \\
&= \{a \to b \mid \forall q \in T\ (h^\flat(q) \leq a \Rightarrow h^\flat(q) \leq b\} \\
&= \{a \to b \mid \forall p \in h^\flat(T)\ (p \leq a \Rightarrow p \leq b\} \\
&= e_D(h^\flat(T)).
\end{aligned}$$

Thus we can read off directly what the map $\mathcal{P}(h^\flat)$ is: it is just forward image under $h^\flat$. That is, if we call the dual of $h : D \to E$ by the name $f : J(E) \to J(D)$, then $\mathcal{P}(f) = f[\,]$ where $f[\,]$ is the lifted forward image mapping subsets of $J(E)$ to subsets of $J(D)$. Finally, we note that $\mathcal{P}$ satisfies $\mathcal{P}(f)$ is an embedding if and only if f is injective, and $\mathcal{P}(f)$ is surjective if and only if f is surjective.

Remark 3.9. It follows from Theorem 3.6(1) that the functor H can be represented as a composition of two functors. Let $B : \mathbf{DL}_{\text{fin}} \to \mathbf{BA}_{\text{fin}}$ be the functor from the category of finite distributive lattices to the category of finite Boolean algebras which maps every finite distributive lattice to its free Boolean extension—the (unique) Boolean algebra generated by this distributive lattice. It is well known [19] that the dual of the functor B is the forgetful functor from the category of finite posets to the category of finite sets which maps every finite poset to its underlying set. Further, let also $H_B : \mathbf{BA}_{\text{fin}} \to \mathbf{DL}_{\text{fin}}$ be the functor H restricted to Boolean algebras. That is, given a Boolean algebra A we define $H_B(A)$ as the free DL over $\to (B, B)$ modded out by the axioms (1)-(4) of wHAs. Then the functor which is dual to H_B maps each finite set X to $(\mathcal{P}(X), \subseteq)$ and therefore $H : \mathbf{DL}_{\text{fin}} \to \mathbf{DL}_{\text{fin}}$ is the composition of B with H_B.

Since weak Heyting algebras are the algebras for the functor H, we can make use of coalgebraic methods for constructing free weak Heyting algebras. Similarly to [7], where free modal algebras and free distributive modal algebras were constructed, we construct finitely generated free weak Heyting algebras as initial algebras of $Alg(H)$. That is, we have a sequence of bounded distributive lattices, each embedded in the next:

$$\begin{array}{lcl}
n & \longrightarrow & F_{DL}(n), \text{ the free bounded distributive lattice on } n \text{ generators} \\
D_0 & = & F_{DL}(n) \\
D_{k+1} & = & D_0 + H(D_k), \text{ where } + \text{ is the coproduct in } \mathbf{DL} \\
i_0 \ : \ D_0 \to & & D_0 + H(D_0) \ = D_1 \text{ the embedding given by coproduct} \\
i_k \ : \ D_k \to D_{k+1} & & \text{where } i_k = id_{D_0} + H(i_{k-1})
\end{array}$$

For $a, b \in D_k$, we denote by $a \to_k b$ the equivalence class $[a \to b]_\approx \in H(D_k) \subseteq D_{k+1}$. Now, by applying the technique of [2], [1], [13], [7] to weak Heyting algebras, we arrive at the following theorem.

Theorem 3.10. *The direct limit $(D_\omega, (D_k \to D_\omega)_k)$ in $\mathbf{DL}$ of the system $(D_k, i_k : D_k \to D_{k+1})_k$ with the binary operation $\to_\omega$: $D_\omega \times D_\omega \to D_\omega$ defined by $a \to_\omega b = a \to_k b$, for $a, b \in D_k$ is the free n-generated weak Heyting algebra when we embed n in D_ω via $n \to D_0 \to D_\omega$.*

Now we will look at the dual of $(D_\omega, \to_\omega)$. Let $X_0 = \mathcal{P}(n)$ be the dual of D_0 and let

$$X_{k+1} = X_0 \times \mathcal{P}(X_k)$$

be the dual of D_{k+1}.

Theorem 3.11. *The sequence $(X_k)_{k<\omega}$ with maps $\pi_k : X_0 \times \mathcal{P}(X_k) \to X_k$ defined by*

$$\pi_k = id_{X_0} \times \mathcal{P}(\pi_{k-1}) \text{ i.e. } \pi_k(x, A) = (x, \pi_{k-1}[A])$$

is dual to the sequence $(D_k)_{k<\omega}$ with maps $i_k : D_k \to D_{k+1}$. In particular, the π_k's are surjective.

Proof. The dual of D_0 is $X_0 = \mathcal{P}(n)$, and since $D_{k+1} = D_0 + H(D_k)$, it follows that $X_{k+1} = X_0 \times \mathcal{P}(X_k)$ as sums go to products and as H is dual to $\mathcal{P}$. For the maps, $\pi_0 : X_0 \times \mathcal{P}(X_0) \to X_0$ is just the projection onto the first coordinate since i_0 is the injection given by the sum construction. We note that π_0 is surjective. Now the dual $\pi_k : X_{k+1} = X_0 \times \mathcal{P}(X_k) \to X_k = X_0 \times \mathcal{P}(X_{k-1})$ of $i_k = id_{D_0} + H(i_{k-1})$ is $id_{X_0} \times \mathcal{P}(\pi_{k-1})$ which is exactly the map given in the statement of the theorem. Note that a map of the form $X \times Y \to X \times Z$ given by $(x, y) \mapsto (x, f(y))$ where $f : Y \to Z$ is surjective if and only the map f is. Also, as we saw above $\mathcal{P}(\pi_k)$ is surjective if and only if π_k is. Thus by induction, all the π_k's are surjective. □

4 Pre-Heyting algebras

In this section we define pre-Heyting algebras which form a subvariety of weak Heyting algebras and describe free pre-Heyting algebras. We first note that, for any weak Heyting algebra A, the map $A \to A$ given by $a \mapsto (1 \to a)$ is meet-preserving and also preserves 1 by virtue of the first two axioms of weak Heyting algebras. For the same reason, the map from a distributive lattice D to $H(D)$ mapping each element a of D to $1 \to a$ also is meet-preserving and preserves 1. For Heyting algebras more is true: the map $H \to H$ given by $a \mapsto (1 \to a)$ is just the identity map and thus, in particular, it is a lattice homomorphism. In other words, Heyting algebras satisfy additional rank 1 axioms beyond those of weak Heyting algebras.

Definition 4.1. *A weak Heyting algebra $(A, \to)$ is called a* pre-Heyting algebra, PHA *for short, if the following additional axioms are satisfied for all $a, b \in A$:*

(5) $1 \to 0 = 0$,

(6) $(1 \to a) \vee (1 \to b) = 1 \to (a \vee b)$.

Since these are again rank 1 axioms, we can obtain a description of the free finitely generated pre-Heyting algebras using the same method as for weak Heyting algebras. Accordingly, for a finite distributive lattice D, similarly to what we did in the previous section, we let

$$K(D) = F_{DL}(\to(D \times D))/_{\approx}$$

where $\approx$ is the DL congruence generated by the axioms (1)–(6) viewed as relational schemas. This of course means we can just proceed from where we left off in Section 3 and identify the further quotient of $H(D)$ obtained by the schema $(1 \to a) \vee (1 \to b) \approx 1 \to (a \vee b)$ for a and b ranging over the elements of D and $1 \to 0 \approx 0$. That is, we need to calculate

$$K(D) = H(D)/_{\approx}$$

where $\approx$ is the DL congruence generated by the relation schema given by axioms (5)-(6).

We say that a subset S of a poset $(X, \leq)$ is *rooted* if there exists $p \in S$ such that $q \leq p$ for each $q \in S$. Note that it follows from the definition that a rooted subset is necessarily non-empty. We denote by $\mathcal{P}_r(X)$ the set of all rooted subsets of X.

Theorem 4.2. *Let D be a finite distributive lattice and $X = (J(D), \leq)$ its dual poset. Then*

1. *The poset $(J(K(D)), \leq)$ is isomorphic to the poset $(\mathcal{P}_r(X), \subseteq)$ of all rooted subsets of X ordered by inclusion.*
2. $J(K(D)) = \{[(1 \to x) \wedge (\bigwedge_{q<x, q \notin T'}(q \to \kappa(q))]_{\approx} \mid T' \subseteq {\downarrow}x \setminus \{x\}, x \in X\}$.
3. *The map $D \to K(D)$ given by $a \mapsto (1 \to a)$ is an injective bounded lattice homomorphism whose dual is the surjective order-preserving map $root : \mathcal{P}_r(X) \to X$ sending each rooted subset of X to its root.*

Proof. (1) By Theorem 3.6(1), $(J(H(D)), \leq)$ is isomorphic to $(\mathcal{P}(X), \subseteq)$. Thus, we need to show that the rooted subsets of X are exactly the subsets which are admissible with respect to axioms (5) and (6). For the axiom (5), it may be worth clarifying the meaning of this axiom: the 0 (and 1) on the left side are elements of D, and the expression $1 \to 0$ is one of the generators of $K(D)$, whereas the 0 on the right of the equality is the bottom of the bounded lattice $K(D)$ — we will denote it by $0_{K(D)}$ for now. An $S \subseteq X$ is admissible for (5) provided

$$S \leq 1 \to 0 \iff S \leq 0_{K(D)}.$$

Now by Lemma 3.7, $S \leq 1 \to 0$ if and only if, for all $p \in S$, $p \leq 1$ implies $p \leq 0$. Since the former is true for every $p \in X$ and the latter is false for all $p \in X$, the only $S \in \mathcal{P}(X)$ satisfying this condition is $S = \emptyset$. On the other hand, as in any lattice, no join-irreducible in $K(D)$ is below $0_{K(D)}$. Thus (5) eliminates $S = \emptyset$.

Weak Heyting implication is meet-preserving and thus order-preserving in the second coordinate so that we have that $1 \to (a \vee b) \geq (1 \to a) \vee (1 \to b)$ already in $H(D)$

for every D. Therefore, a set $S \subseteq X$ is admissible with respect to (5) and (6) iff $S \neq \emptyset$ and

$$S \leq 1 \to (a \vee b) \text{ implies } S \leq 1 \to a \text{ or } S \leq 1 \to b.$$

Now by Lemma 3.7, $S \leq 1 \to x$ iff $S \subseteq {\downarrow}x$, so for non-empty S we need $S \subseteq {\downarrow}(a \vee b)$ to imply that $S \subseteq {\downarrow}a$ or $S \subseteq {\downarrow}b$ for all $a, b \in D$. This is easily seen to be equivalent to rootedness: If $S \subseteq X$ is rooted with root p. Then $S \subseteq {\downarrow}(a \vee b)$ implies $p \leq a \vee b$ and thus $p \leq a$ or $p \leq b$ so that $S \subseteq {\downarrow}a$ or $S \subseteq {\downarrow}b$. Conversely, if S is admissible then $S \neq \emptyset$ and, as it is finite, every element of S is below a maximal element of S. If $p \in S$ is maximal but not the maximum of S then $S \subseteq {\downarrow}(p \vee a)$, where $a = \bigvee(S \setminus \{p\})$, but $S \not\subseteq {\downarrow}p$ and $S \not\subseteq {\downarrow}a$.

(2) The proof is similar to to the proof of Theorem 3.6(2). Recall that for any $T \subseteq X$ we have that the join-irreducible $\{T' \in \mathcal{P}(X) \mid T' \subseteq T\} = {\downarrow}T$ in $\mathcal{O}(\mathcal{P}(X))) \cong K(D)$ is equal to $[\bigwedge_{q \notin T}(q \to \kappa(q))]_{\approx}$. Also $1 \to x$ is join-irreducible and corresponds to ${\downarrow}x \subseteq X$. That is,

$$[1 \to x]_{\approx} = [\bigwedge_{q \not\leq x}(q \to \kappa(q))]_{\approx}.$$

Thus, in particular, for $T \subseteq X$ rooted with root x and $T' = T \setminus \{x\}$, the join-irreducible corresponding to T is given by

$$\begin{aligned} [\bigwedge_{q \notin T}(q \to \kappa(q)]_{\approx} &= [(\bigwedge_{q \not\leq x}(q \to \kappa(q)) \wedge (\bigwedge_{q \leq x, q \notin T}(q \to \kappa(q))]_{\approx} \\ &= [(1 \to x) \wedge (\bigwedge_{q < x, q \notin T'}(q \to \kappa(q))]_{\approx}. \end{aligned}$$

Since this is the case in $H(D)$, it is certainly also true in the further quotient $K(D)$ and in (1) we have shown that all join-irreducibles of $K(D)$ correspond to rooted subsets of X, thus the statement follows.

(3) The map $D \to K(D)$ given by $a \mapsto (1 \to a)$ is clearly a homomorphism since we have quotiented out by all the necessary relations: $[1 \to 1]_{\approx} = 1_{K(D)}$ by (1), $[1 \to 0]_{\approx} = 0_{K(D)}$ by (5), and the map is meet and join preserving by (2) and (6), respectively. As we saw in Section 2, the dual of a homomorphism between finite lattices is the restriction to join-irreducibles of its lower adjoint, that is, our homomorphism is dual to the map $r : \mathcal{P}_r(X) \to X$ given by

$$\forall T \in \mathcal{P}_r(X) \, \forall a \in D \quad (r(T) \leq a \quad \iff \quad T \leq (1 \to a)).$$

For $T \in \mathcal{P}_r(X)$ we have $T \leq (1 \to a) = \bigvee_{x \in X, x \leq a}(1 \to x)$ if and only if there is an $x \in X$ with $x \leq a$ and $T \leq (1 \to x)$. Furthermore $T \leq (1 \to x)$ if and only if $T \subseteq {\downarrow} x$ if and only if $root(T) \leq x$. That is, $r(T) \leq a$ if and only if $root(T) \leq a$ so that, indeed, $root(T) = r(T)$. It is clear that the map $root$ is surjective and thus the dual homomorphism $D \hookrightarrow K(D)$ is injective. □

The following technical proposition will be used for obtaining some important results in the next section of the paper.

Proposition 4.3. *Let D be a finite distributive lattice and $X = (J(D), \leq)$ its dual poset. Let $S \in \mathcal{P}_r(X)$ and let $x \in X$. Identifying $\mathcal{P}_r(X)$ with $J(K(D))$ we have the following equivalences*

$$\begin{aligned} & root(S) = x \\ \iff & S \leq 1 \to x \text{ but } S \nleq 1 \to \kappa(x) \\ \iff & \text{ it is not the case that } (S \leq 1 \to x \implies S \leq 1 \to \kappa(x)). \end{aligned}$$

Proof. We first assume that $S \leq 1 \to x$ and $S \nleq 1 \to \kappa(x)$. Then $S \subseteq {\downarrow}x$ and $S \nsubseteq {\downarrow}\kappa(x)$. It follows that for each $s \in S$ we have $s \leq x$ and there is $t \in S$ with $t \nleq \kappa(x)$. Therefore, we have $x \leq t$. Since $t \in S$, we obtain $t = x$. So $x \in S$. This implies that x is the root of S, which means that $root(S) = x$. Conversely, suppose $root(S) = x$. Then $S \subseteq {\downarrow}x$ and $x \in S$. So $S \leq 1 \to x$. On the other hand, we know that $y \nleq \kappa(y)$, for each $y \in J(D)$. Therefore, $x \nleq \kappa(x)$ and thus $S \nsubseteq {\downarrow}\kappa(x)$. This implies that $S \nleq 1 \to \kappa(x)$. That ($S \leq 1 \to x$ and $S \nleq 1 \to \kappa(x)$) is equivalent to (it is not the case that $(S \leq 1 \to x \implies S \leq 1 \to \kappa(x))$) is obvious.

Since pre-Heyting algebras are the algebras for the functor K, we can construct free pre-Heyting algebras from the functor K similarly to how we constructed free weak Heyting algebras from the functor H in the previous section. Given an order-preserving map $f : X \to X'$ between two finite posets X and X' we define $\mathcal{P}_r(f) : \mathcal{P}_r(X) \to \mathcal{P}_r(X')$ by setting $\mathcal{P}_r(f) = f[\,]$. It is easy to see that this is the action of the functor $\mathcal{P}_r$ dual to K. Then we will have the analogues of Theorems 3.10 and 3.11 for free pre-Heyting algebras.

We consider the following sequence of bounded distributive lattices:

$$\begin{aligned} D_0 &= F_{DL}(n) \\ D_{k+1} &= D_0 + K(D_k), \end{aligned}$$

$i_0 : D_0 \to D_0 + K(D_0) = D_1$ the embedding given by coproduct

$i_k : D_k \to D_{k+1}$ where $i_k = id_{D_0} + K(i_{k-1})$

In the same way as for weak Heyting algebras we have the following description of free pre-Heyting algebras.

Theorem 4.4. *The direct limit $(D_\omega, (D_k \to D_\omega)_k)$ in $\mathbf{DL}$ of the system $(D_k, i_k : D_k \to D_{k+1})_k$ with the binary operation $\to_\omega$: $D_\omega \times D_\omega \to D_\omega$ defined by $a \to_\omega b = a \to_k b$, for $a, b \in D_k$ is the free n-generated pre-Heyting algebra.*

Let X_0 be the dual of D_0 and let

$$X_{k+1} = X_0 \times \mathcal{P}_r(X_k)$$

be the dual of D_{k+1}.

Theorem 4.5. *The sequence $(X_k)_{k<\omega}$ with maps $\pi_k : X_0 \times \mathcal{P}_r(X_k) \to X_k$ defined by*

$$\pi_k = id_{X_0} \times \mathcal{P}_r(\pi_{k-1}) \text{ i.e. } \pi_k(x, A) = (x, \pi_{k-1}[A])$$

is dual to the sequence $(D_k)_{k<\omega}$ with maps $i_k : D_k \to D_{k+1}$. In particular, the π_k's are surjective.

Proof. The proof is analogues to the proof of Theorem 3.11. □

5 Heyting algebras

In this section we will apply the technique of building free weak and pre-Heyting algebras to describe free Heyting algebras. We recall the following definition of Heyting algebras relative to weak Heyting algebras.

Definition 5.1. *[15] A weak Heyting algebra* $(A, \to)$ *is called a* Heyting algebra, HA *for short, if the following two axioms are satisfied for all* $a, b \in A$*:*

(i) $b \leq a \to b$,
(ii) $a \wedge (a \to b) \leq b$.

Let D be a finite distributive lattice. We have seen how to build the free weak Heyting algebra and the free pre-Heyting algebra over D incrementally. Let $F_{HA}(D)$ denote the free HA freely generated by the bounded distributive lattice D. Further let $F^n_{HA}(D)$ denote the elements of $F_{HA}(D)$ of $\to$-rank less than or equal n. Then each $F^n_{HA}(D)$ is a distributive lattice, the bounded lattice reduct of $F_{HA}(D)$ is the direct limit (union) of the chain

$$D = F^0_{HA}(D) \subseteq F^1_{HA}(D) \subseteq F^2_{HA}(D) \ldots$$

and the implication on $F_{HA}(D)$ is given by the maps $\to$: $(F^n_{HA}(D))^2 \to F^{n+1}_{HA}(D)$ with $(a, b) \mapsto (a \to b)$. Further, since any Heyting algebra is a pre-Heyting algebra and the inclusion $F^n_{HA}(D) \subseteq F^{n+1}_{HA}(D)$ may be seen as given by the mapping $a \mapsto (1 \to a)$, the natural maps sending generators to generators make the following colimit diagrams commute

$$\begin{array}{ccccccc}
D & \xhookrightarrow{i_0} & K(D) & \xhookrightarrow{i_1} & K(K(D)) & \xhookrightarrow{i_2} & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\
D & \xhookrightarrow{j_0} & F_{HA}(D) & \xhookrightarrow{j_1} & F^2_{HA}(D) & \xhookrightarrow{j_2} & \cdots
\end{array}$$

Notice that under the assignment $a \mapsto 1 \to a$, the equation (i) becomes $1 \to b \leq a \to b$ which is true in any pre HA by (2), and (ii) becomes $(1 \to a) \wedge (a \to b) \leq (1 \to b)$ which is true in any pre HA by virtue of (4). So these equations are already satisfied in the steps of the upper sequence. However, an easy calculation shows that for D the three-element lattice $1 \to (u \to 0)$ and $(1 \to u) \to (1 \to 0)$ are not equal (where u is the middle element) thus the implication is not well-defined on the limit of the upper sequence. We remedy this by taking a quotient with respect to the relational scheme

$$1 \to (a \to b) = (1 \to a) \to (1 \to b)$$

in the second iteration of the functor K and onwards. We proceed, as we've done throughout this paper by identifying the dual correspondent of this equation.

Proposition 5.2. *Let* D *be a finite distributive lattice and* $a, b \in D$*. The inequality*

$$1 \to (a \to b) \leq (1 \to a) \to (1 \to b)$$

holds in $K(K(D))$*.*

Proof. By axiom (3) it follows that $\to$ is order reversing in the first coordinate so that we have $1 \to (a \to b) \leq (1 \to a) \to (a \to b)$ since $1 \geq 1 \to a$. Also by Lemma 3.2, we have $(1 \to a) \to (a \to b) = (1 \to a) \to [(1 \to a) \wedge (a \to b)]$. Now using (4) we have $(1 \to a) \wedge (a \to b) \leq 1 \to b$ so that $(1 \to a) \to (a \to b) \leq (1 \to a) \to (1 \to b)$. By transitivity of the order we have the desired result. $\square$

Proposition 5.3. *Let D be a finite distributive lattice and $X = (J(D), \leq)$ its dual poset. Further, let θ be a congruence on $K(K(D))$. Then the following are equivalent:*

1. *For all $a, b \in D$ the inequality $(1 \to a) \to (1 \to b) \leq 1 \to (a \to b)$ holds in $K(K(D))/\theta$;*
2. *For all $x, y \in X$ the inequality $(1 \to x) \to (1 \to \kappa(y)) \leq 1 \to (x \to \kappa(y))$ holds in $K(K(D))/\theta$*

Proof. (i) implies (ii) is clear since (ii) is a special case of (i). We prove that (ii) implies (i).

$$\begin{aligned}(1 \to a) \to (1 \to b) &= \Big(\bigvee_{X \ni x \leq a} (1 \to x)\Big) \to \Big(\bigwedge_{X \ni y \not\leq b} (1 \to \kappa(y))\Big) \\ &= \bigwedge_{\substack{X \ni x \leq a \\ X \ni y \not\leq b}} ((1 \to x) \to (1 \to \kappa(y))) \\ &\leq \bigwedge_{\substack{X \ni x \leq a \\ X \ni y \not\leq b}} (1 \to (x \to \kappa(y))) \\ &= 1 \to \bigwedge_{\substack{X \ni x \leq a \\ X \ni y \not\leq b}} (x \to \kappa(y)) \\ &= 1 \to (a \to b)).\end{aligned}$$

$\square$

We are now ready to translate this into a dual property which we will call (G) after Ghilardi who introduced it in [12].

Proposition 5.4. *Let X be a finite poset. The following conditions are equivalent:*

1. *$\forall x, y \in X$ the inequalities*

   $(1 \to x) \to (1 \to \kappa(y)) \leq 1 \to (x \to \kappa(y))$ *hold in* $\mathcal{O}(P_r(P_r(X))$;

2. *$\forall \tau \in P_r(P_r(X))\ \forall T \in \tau\ \forall S \in P_r(X)$*

   $$(S \leq T \implies \exists T' \in \tau\ (T' \leq T \text{ and } root(S) = root(T'))\qquad (G)$$

Proof. First we prove that (1) implies (2). To this end suppose (1) holds and let $T \in \tau \in P_r(P_r(X))$ and $S \in P_r(X)$. Suppose that for all $T' \in \tau$ either $T' \not\leq T$ or

$root(S) \neq root(T')$. Now consider $\tau_T = {\downarrow}T \cap \tau$. We obviously have that ${\downarrow}T \cap \tau$ is a rooted subset of $\mathcal{P}_r(X)$ and therefore

$$\tau_T = ({\downarrow}T \cap \tau) \in \mathcal{P}_r(\mathcal{P}_r(X)) \qquad\qquad (\star)_2.$$

We have $root(S) \neq root(T')$ for all $T' \in \tau_T$, and thus, letting $x = root(S)$ and using the observation in Proposition 4.3, we have

$$\begin{aligned}
& \forall T' \in \tau_T \qquad root(T') \neq x \\
\iff & \forall T' \in \tau_T \quad (T' \leq 1 \to x \implies T' \leq 1 \to \kappa(x)) \\
\iff & \tau_T \leq (1 \to x) \to (1 \to \kappa(x)) \\
\implies & \tau_T \leq 1 \to (x \to \kappa(x)) \\
\iff & \forall T' \in \tau_T \qquad T' \leq x \to \kappa(x) \\
\iff & \forall T' \in \tau_T \; \forall y \in T' \quad (y \leq x \implies y \leq \kappa(x)) \\
\iff & \forall T' \in \tau_T \qquad x \notin T' \\
\implies & \qquad x \notin T.
\end{aligned}$$

The two implications come from the fact that we assume that (1) holds and because, in particular, $T \in \tau_T$, respectively. Now we have $x = root(S) \in S$ but $x \notin T$ so $S \nleq T$ and we have proved (2) by contraposition.

Now suppose (2) holds, let $x, y \in X$, and let $\tau \in \mathcal{P}_r(\mathcal{P}_r(X))$ with $\tau \leq (1 \to x) \to (1 \to \kappa(y))$.

$$\begin{aligned}
& \tau \leq (1 \to x) \to (1 \to \kappa(y)) \\
\iff & \forall T \in \tau \quad (T \leq 1 \to x \implies T \leq 1 \to \kappa(y)) \\
\iff & \forall T \in \tau \quad (T \leq {\downarrow}x \implies T \leq {\downarrow}\kappa(y)).
\end{aligned}$$

We want to show that $T \leq x \to \kappa(y)$ for each $T \in \tau$. That is, that for all $z \in T$ we have $z \leq x$ implies $z \leq \kappa(y)$. So let $z \in T$ with $z \leq x$. We obviously have that ${\downarrow}z \cap T$ is a rooted subset of X and therefore

$$T_z = ({\downarrow}z \cap T) \in \mathcal{P}_r(X) \qquad\qquad (\star)_1.$$

Since $T_z \leq T$ it follows by (2) that

$$\exists T' \in \tau \qquad (T' \leq T \text{ and } z = root(T_z) = root(T')).$$

Now $x \geq z = root(T_z) = root(T')$ implies that $T' \leq {\downarrow}x$ and thus we have $T' \leq {\downarrow}\kappa(y)$. In particular, $z = root(T') \leq \kappa(y)$. That is, we have shown that for all $z \in T$, if $z \leq x$ then $z \leq \kappa(y)$ as required. □

Our strategy in building the free n-generated Heyting algebra will be to start with D, the free n-generated distributive lattice, embed it in $K(D)$, and then this in a quotient of $K(K(D))$ obtained by modding out by $1 \to (a \to b) = (1 \to a) \to (1 \to b)$ for

$a, b \in D$. For the further iterations of K this identification is iterated. The following is the general situation that we need to consider, viewed dually:

$$\begin{array}{ccccc} X_0 & \xleftarrow{root} & \mathcal{P}_r(X_0) & \xleftarrow{root} & \mathcal{P}_r(\mathcal{P}_r(X_0)) \\ & & \uparrow & & \uparrow \\ & & X_1 & \xleftarrow{root} & \mathcal{P}_r(X_1) \\ & & & & \uparrow \\ & & & & X_2 \end{array}$$

For this reason the inductive step deals with a quotient of a quotient and we need to refine the Proposition 5.4 above. We note that it holds not only for $\mathcal{P}_r(X)$ and $\mathcal{P}_r(\mathcal{P}_r(X))$, but also for any subsets $X_1 \subseteq \mathcal{P}_r(X)$ and $X_2 \subseteq \mathcal{P}_r(X_1)$ satisfying $(\star)_1$ and $(\star)_2$, respectively. Indeed, these are the only specific properties of $\mathcal{P}_r(X)$ and $\mathcal{P}_r(\mathcal{P}_r(X))$ that we used in the proof of the proposition. Therefore, we have the following corollary.

Corollary 5.5. *Let X_0 be a finite poset, X_1 a sub-poset of $\mathcal{P}_r(X_0)$, and X_2 a sub-poset of $\mathcal{P}_r(X_1)$. For $i = 1$ and 2, let $(\star)_i$ be the condition*

$$x \in T \in X_i \implies T_x = (\downarrow x \cap T) \in X_i$$

If the conditions $(\star)_1$ and $(\star)_2$ both hold then the following are equivalent:

1. $\forall x, y \in X_0$ *the inequalities*

   $(1 \to x) \to (1 \to \kappa(y)) \leq 1 \to (x \to \kappa(y))$ *hold in* $\mathcal{O}(X_2)$;

2. $\forall \tau \in X_2 \; \forall T \in \tau \; \forall S \in X_1$

   $(S \leq T \implies \exists T' \in \tau \; (T' \leq T$ *and* $root(S) = root(T')).$ $\quad (G)$

Proof. The proof is exactly the same as the proof of Proposition 5.4 we just need to replace $\mathcal{P}_r(X)$ by X_1 and $\mathcal{P}_r(\mathcal{P}_r(X))$ by X_2. We also note that for $(1) \Rightarrow (2)$ direction we use just the condition $(\star)_2$ and for $(2) \Rightarrow (1)$ we use only $(\star)_1$. ☐

We now consider the following sequence of finite posets

$$\begin{aligned} X_0 &= J(F_{DL}(n))(= \mathcal{P}(n)) \\ X_1 &= \mathcal{P}_r(X_0) \end{aligned}$$

For $n \geq 1 \quad X_{n+1} = \{\tau \in \mathcal{P}_r(X_n) \mid \forall T \in \tau \; \forall S \in X_n$
$\qquad (S \leq T \implies \exists T' \in \tau \; (T' \leq T \text{ and } root(S) = root(T'))\}.$

We denote by ∇ the sequence

$$X_0 \xleftarrow{root} X_1 \xleftarrow{root} X_2 \; \ldots$$

For $n \geq 1$, we say that ∇ satisfies $(\star)_n$ if

$$x \in T \in X_n \implies T_x = (\downarrow x \cap T) \in X_n.$$

Lemma 5.6. *∇ satisfies $(\star)_n$ for each $n \geq 1$ and the root maps $root : X_{n+1} \to X_n$ are surjective for each $n \geq 0$.*

Proof. X_1 consists of all rooted subsets of X_0 and thus $(\star)_1$ is clearly satisfied. Now let $n \geq 2$. We assume that $T \in \tau \in X_n$ and we show that $\tau_T = \downarrow T \cap \tau$ also belongs to X_n. So let $U \in \tau_T$, $S \in X_n$ and $S \leq U$. Then since $U \in \tau$, there exists $U' \in \tau$ such that $U' \leq U$ and $root(S) = root(U')$. But $U \in \tau_T$ implies that $U \leq T$. Therefore we have $U' \leq T$ and so $U' \in \tau_T$. Thus, $\tau_T \in X_n$ and ∇ satisfies $(\star)_n$, for each $n \geq 1$. Finally, we show that all the $root$ maps are surjective. To see this, assume $U \in X_n$. We show that $\downarrow U \in X_{n+1}$. Suppose $T \in \downarrow U$ and for some $S \in X_n$ we have $S \leq T$. Then $S \in \downarrow U$ and by setting $T' = S$ we easily satisfy the condition (G). Finally, note that $root(\downarrow U) = U$ and thus $root : X_{n+1} \to X_n$ is surjective. □

Let Δ be the system

$$D_0 \xhookrightarrow{i_0} D_1 \xhookrightarrow{i_1} D_2 \ \ldots$$

of distributive lattices dual to ∇. For each $n \geq 0$, $i_n : D_n \to D_{n+1}$, is a lattice homomorphism dual to $root$. By Theorem 4.2(3) $i_n(a) = 1 \to a$, for $a \in D_n$. By Lemma 5.6, $root$ is surjective, so each i_n is injective. Each $X_{n+1} \subseteq \mathcal{P}_r(X_n)$ so that each D_{n+1} is a quotient of $K(D_n)$ and thus, for each n, we also have partial implication operations:

$$\begin{aligned} \to_n \colon \ D_n \times D_n &\to \ D_{n+1} \\ (a, b) &\mapsto [a \to b]. \end{aligned}$$

Here $[a \to b]$ is the equivalence class of $a \to b \in K(D_n)$ as an element in D_{n+1}. Let D_ω be the limit of Δ in the category of distributive lattices then D_ω is naturally turned into a Heyting algebra.

Lemma 5.7. *The operations $\to_n$ can be extended to an operation $\to_\omega$ on D_ω and the algebra $(D_\omega, \to_\omega)$ is a Heyting algebra.*

Proof. The colimit D_ω of Δ may be constructed as the union of the D_ns with D_n identified with the image of $i_n : D_n \hookrightarrow D_{n+1}$. It is then clear that the partial operations $\to_n\colon \ D_n \times D_n \to \ D_{n+1}$ yield a total, well-defined binary operation provided, for all $n \geq 0$ and all $a, b \in D_n$, we have $i_{n+1}(\to_n (a, b)) = \to_{n+1} (i_n(a), i_n(b))$. But this is exactly

$$1 \to_{n+1} (a \to_n b) = (1 \to_n a) \to_{n+1} (1 \to_n b).$$

As we've shown in Corollary 5.5 and Lemma 5.6, the sequence ∇, and thus the dual sequence Δ have been defined exactly so that this holds. It remains to show that the algebra $(D_\omega, \to)$ is a Heyting algebra. Let $a \in D_\omega$, then there is some $n \geq 0$ with $a \in D_n$. Now $a \to_\omega a = a \to_n a \in D_{n+1}$. Since D_{n+1} is a further quotient of $K(D_n)$ and $a \to_n a = 1$ already in $K(D_n)$, this is certainly also true in D_{n+1} and $1_{D_{n+1}} = 1_{D_\omega}$ so the equation (1) of weak Heyting algebras is satisfied in $(D_\omega, \to_\omega)$. Similarly each of the equations (2)-(4) are satisfied in $(D_\omega, \to_\omega)$ so that it is a weak Heyting algebra. But the two last equations, (i) and (ii) are also satisfied as explained in the discussion at the beginning of this section: Let $a, b \in D_\omega$. Then there exist $k, n \geq 0$ such that $a \in D_k$ and $b \in D_n$. Without loss of generality we may assume that $k \leq n$ and then, by identifying a with its image under the embedding of D_k into D_n, we obtain $a, b \in D_n$. Now $i_n(b) = 1 \to_n b \leq a \to_n b$ follows from the fact that $\to_n$ is a weak Heyting implication and a weak Heyting implication is order-reversing in the first coordinate. Thus, $i_n(b) \leq a \to_n b$, which means that $b \leq a \to b$ is satisfied in D_ω. Moreover, by axiom (4) of weak Heyting algebras we have $(1 \to_n a) \wedge (a \to_n b) \leq 1 \to_n b$. Thus, $i_n(a) \wedge (a \to_n b) \leq i_n(b)$, which means that $a \wedge (a \to b) \leq b$ is satisfied in D_ω and $(D_\omega, \to)$ is a Heyting algebra. □

Corollary 5.8. *$(D_\omega, \to)$ is the n-generated free Heyting algebra.*

Proof. Let $F_{HA}(n)$ denote the free HA freely generated by n generators. This is of course the same as the free HA generated by D, $F_{HA}(D)$, where D is the free distributive lattice generated by n elements. As discussed at the beginning of this section this lattice is the colimit (union) of the chain

$$D = F^0_{HA}(D) \subseteq F^1_{HA}(D) \subseteq F^2_{HA}(D) \ldots$$

and the implication on $F_{HA}(D)$ is given by the maps $\to : (F^n_{HA}(D))^2 \to F^{n+1}_{HA}(D)$ with $(a, b) \mapsto (a \to b)$. Further, the natural maps sending generators to generators make the following colimit diagrams commute

$$\begin{array}{ccccccc}
D & \hookrightarrow & K(D) & \hookrightarrow & K(K(D)) & \hookrightarrow & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
D & \overset{j_0}{\hookrightarrow} & F_{HA}(D) & \overset{j_1}{\hookrightarrow} & F^2_{HA}(D) & \overset{j_2}{\hookrightarrow} & \cdots
\end{array}$$

Now, the system Δ is obtained from the upper sequence by quotienting out by the equations $1 \to_{n+1} (a \to_n b) = (1 \to_n a) \to_{n+1} (1 \to_n b)$ for each $n \geq 0$. Since these equations all hold for the lower sequence, it follows that the D_ns are intermediate quotients:

$$\begin{array}{ccccccc}
D & \hookrightarrow & K(D) & \hookrightarrow & K(K(D)) & \hookrightarrow & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
D_0 & \overset{i_0}{\hookrightarrow} & D_1 & \overset{i_1}{\hookrightarrow} & D_2 & \overset{i_2}{\hookrightarrow} & \cdots \\
\downarrow{\scriptstyle id} & & \downarrow & & \downarrow & & \\
D & \overset{j_0}{\hookrightarrow} & F_{HA}(D) & \overset{j_1}{\hookrightarrow} & F^2_{HA}(D) & \overset{j_2}{\hookrightarrow} & \cdots
\end{array}$$

Therefore, $F_{HA}(n)$ is a homomorphic image of D_ω and any map $f : n \to B$ with B a Heyting algebra defines a unique extension $\tilde{f} : F_{HA}(n) \to B$ so that $\tilde{f} \circ i = f$ where $i : n \to F_{HA}(n)$ is the injection of the free generators. Since i actually maps into the sublattice of $F_{HA}(n)$ generated by n, which is the initial lattice $D = D_0$ in our sequences, composition of $\tilde{f}$ with the quotient map from D_ω to $F_{HA}(n)$ shows that

D_ω also has the universal mapping property (without the uniqueness). The uniqueness follows since D_ω clearly is generated by n as HA (since D_0 is generated by n as a bounded lattice, D_1 is generated by D_0 using $\to_0$, and so on). Since the free HA on n generators is unique up to isomorphism and D_ω has its universal mapping property and is a Heyting algebra, it follows it is the free HA (and the quotient map from D_ω to $F_{HA}(n)$ is in fact an isomorphism). $\square$

6 A coalgebraic representation of wHAs and PHAs

In this section we discuss a coalgebraic semantics for weak and pre-Heyting algebras. A coalgebraic representation of modal algebras and distributive modal algebras can be found in [1], [16] and [20], [7], respectively.

We recall that a *Stone space* is a compact Hausdorff space with a basis of clopen sets. For a Stone space X, its *Vietoris space* $V(X)$ is defined as the set of all closed subsets of X, endowed with the topology generated by the subbasis

1. $\Box U = \{F \in V(X) : F \subseteq U\}$,
2. $\Diamond U = \{F \in V(X) : F \cap U \neq \emptyset\}$,

where U ranges over all clopen subsets of X. It is well known that X is a Stone space iff $V(X)$ is a Stone space. Let X and X' be Stone spaces and $f : X \to X'$ be a continuous map. Then $V(f) = f[\,]$ is a continuous map between $V(X)$ and $V(X')$. We denote by V the functor on Stone spaces that maps every Stone space X to its Vietoris space $V(X)$ and maps every continuous map f to $V(f)$. Every modal algebra $(B, \Box)$ can be represented as a coalgebra $(X, \alpha : X \to VX)$ for the Vietoris functor on Stone spaces [1, 16]. A coalgebraic representation of distributive modal algebras can be found in [20] and [7]. We note that modal algebras as well as distributive modal algebras are given by rank 1 axioms. Using the same technique as in Section 3, one can obtain a description of free modal algebras and free distributive modal algebras [1], [13], [7].

Our goal is to give a coalgebraic representation for weak Heyting algebras and pre-Heyting algebras. Recall that a *Priestley space* is a pair $(X, \leq)$ where X is a Stone space and $\leq$ is a reflexive, symmetric and transitive relation satisfying the *Priestley separation axiom*:

If $x, y \in X$ are such that $x \not\leq y$, then there exists a clopen downset U with $y \in U$ and $x \notin U$.

We denote by **PS** the category of Priestley spaces and order-preserving continuous maps. It is well known that every distributive lattice D can be represented as a lattice of all clopen downsets of the Priestley space of its prime filters. Given a Priestley space X, let $V_r(X)$ be a subspace of $V(X)$ of all closed rooted subsets of X. The same proof as for $V(X)$ shows that $V_r(X)$ is a Stone space.

Lemma 6.1. *Let X be a Priestley space. Then*

1. *$(V(X), \subseteq)$ is a Priestley space.*
2. *$(V_r(X), \subseteq)$ is a Priestley space.*

Proof. (1) As we mentioned above $V(X)$ is a Stone space. Let $F, F' \in V(X)$ and $F \not\subseteq F'$. Then there exists $x \in F$ such that $x \notin F'$. Since every compact Hausdorff space is normal, there exists a clopen set U such that $F' \subseteq U$ and $x \notin U$. Thus, $F' \in \Box U$ and $F \notin \Box U$. All we need to observe now is that for each clopen U of X, the set $\Box U$ is a clopen downset of $V(X)$. But this is obvious.

(2) the proof is the same as for (1). □

Let $(X, \leq)$ and $(X', \leq')$ be Priestley spaces and $f : X \to X'$ a continuous order-preserving map. Then it is easy to check that $V(f) = f[\,]$ is a continuous order-preserving map between $(V(X), \subseteq)$ and $(V(X'), \subseteq)$, and $V_r(f) = f[\,]$ is a continuous order-preserving map between $(V_r(X), \subseteq)$ and $(V_r(X'), \subseteq)$. Thus, V and V_r define functors on the category of Priestley spaces.

Definition 6.2. *(Celani and Jansana [10]) A weak Heyting space is a triple $(X, \leq, R)$ such that $(X, \leq)$ is a Priestley space and R is a binary relation on X satisfying the following conditions:*

1. *$R(x) = \{y \in X : xRy\}$ is a closed set, for each $x \in X$.*
2. *For each $x, y, z \in X$ if $x \leq y$ and xRz, then yRz.*
3. *For each clopen set $U \subseteq X$ the sets $[R](U) = \{x \in X : R(x) \subseteq U\}$ and $\langle R\rangle(U) = \{x \in U : R(x) \cap U \neq \emptyset\}$ are clopen.*

Let $(X, \leq, R)$ and $(X', \leq', R')$ be two weak Heyting spaces. We say that $f : X \to X'$ is a weak Heyting morphism if f is continuous, $\leq$-preserving and R-bounded morphism (i.e., for each $x \in X$ we have $fR(x) = R'f(x)$). Then the category of weak Heyting algebras is dually equivalent to the category of weak Heyting spaces and weak Heyting morphisms [10]. We will quickly recall how the dual functors are defined on objects. Given a weak Heyting algebra $(A, \to)$ we take a Priestley dual X_A of A and define R_A on X_A by setting: for each $x, y \in X_A$, xR_Ay if for each $a, b \in A$, $a \to b \in x$ and $b \in x$ imply $b \in y$. Conversely, if $(X, \leq, R)$ is a weak Heyting space, then we take the distributive lattice of all clopen downsets of X and for clopen downsets $U, V \subseteq X$ we define $U \to V = \{x \in X : R(x) \cap U \subseteq V\}$.

Remark 6.3. In fact, [10] works with clopen upsets instead of downsets and the inverse of the relation R. We chose working with downsets to be consistent with the previous parts of the paper.

Theorem 6.4. *The category of weak Heyting spaces is isomorphic to the category of Vietoris coalgebras on the category of Priestley spaces.*

Proof. Given a weak Heyting space $(X, \leq, R)$. We consider a coalgebra $(X, R(.) : X \to V(X))$. The map $R(.)$ is well defined by Definition 6.2(1). It is order-preserving by Definition 6.2(2) and is continuous by Definition 6.2(3). Thus, $(X, R(.) : X \to V(X))$ is a V-coalgebra. Conversely, let $(X, \alpha : X \to V(X))$ be a V-coalgebra. Then (X, R_α), where $xR_\alpha y$ iff $y \in \alpha(x)$, is a weak Heyting space. Indeed, R being well defined and order-preserving imply conditions (1) and (2) of Definition 6.2, respectively. Finally, α being continuous implies condition (3) of Definition 6.2. That this correspondence can be lifted to the isomorphism of categories is easy to check. □

We say that a weakly Heyting space $(X, \leq, R)$ is *pre-Heyting space* if for each $x \in X$ the set $R(x)$ is rooted.

Theorem 6.5.

1. *The category of pre-Heyting algebras is dually equivalent to the category of pre-Heyting spaces.*
2. *The category of pre-Heyting spaces is isomorphic to the category of V_r-coalgebras on the category of Priestley spaces.*

Proof. (1) By the duality of weak Heyting algebras and weak Heyting spaces it is sufficient to show that a weak Heyting algebra satisfies conditions (5)-(6) of Definition 4.1 iff $R(x)$ is rooted. We will show that, as in Theorem 4.2, the axiom (5) is equivalent to $R(x) \neq \emptyset$, for each $x \in X$, while axiom (6) is equivalent to $R(x)$ having a unique maximal element. Assume a weak Heyting space $(X, \leq, R)$ validates axiom (5). Then in the weak Heyting algebra of all clopen downsets of X we have $X \to \emptyset = \emptyset$. Thus for each $x \in X$ we have $R(x) \subseteq \emptyset$ iff $x \in \emptyset$. Thus, for each $x \in X$ we have $R(x) \neq \emptyset$. Now suppose for each clopen downsets $U, V \subseteq X$ the following holds $X \to (U \cup V) \subseteq (X \to U) \cup (X \to V)$. Then we have that $R(x) \subseteq U \cup V$ implies $R(x) \subseteq U$ or $R(x) \subseteq V$. Since $R(x)$ is closed and X is a Priestley space, we have that every point of $R(x)$ is below some maximal point of $R(x)$. We assume that there exists more than one maximal point of $R(x)$. Then the same argument as in [4, Theorem 2.7(a)] shows that there are clopen downsets U and V such that $R(x) \subseteq U \cup V$, but $R(x) \nsubseteq U$, $R(x) \nsubseteq V$. This is a contradiction, so $R(x)$ is rooted, On the other hand, it is easy to check that if $R(x)$ is rooted for each $x \in X$, then (5)-(6) are valid. Finally, a routine check shows that this correspondence can be lifted to an isomorphism of the categories of pre-Heyting algebras and pre-Heyting spaces.

(2) The proof is similar to the proof of Theorem 6.4. The extra condition on pre-Heyting spaces obviously implies that a map $R(.) : X \to V_r(X)$ is well defined and conversely $(X, \alpha : X \to V_r(X))$ being a coalgebra implies that $R_\alpha(x)$ is rooted for each $x \in X$. The rest of the proof is a routine check. □

Thus, we obtained a coalgebraic semantics/representation of weak and pre-Heyting algebras.

7 Conclusions and future work

In this paper we described finitely generated free (weak, pre) Heyting algebras using an initial algebra-like construction. The main idea is to split the axiomatization of Heyting algebras into its rank 1 and non-rank 1 parts. The rank 1 approximants of Heyting algebras are weak and pre-Heyting algebras. For weak and pre-Heyting algebras we applied the standard initial algebra construction and then adjusted it for Heyting algebras. We used Birkhoff duality for finite distributive lattices and finite posets to obtain the dual characterization of the finite posets that approximate the duals of free algebras. As a result we obtained Ghilardi's representation of these posets in a more systematic and transparent way. For weak and pre-Heyting algebras we also introduced a neat coalgebraic representation.

There are a few possible directions for further research. As we mentioned in the introduction, although we considered Heyting algebras (intuitionistic logic), this method could be applied to other non-classical logics. More precisely, the method is available if a signature of the algebras for this logic can be obtained by adding an extra operator to a locally finite variety. Thus, various non-rank 1 modal logics such as $\mathbf{S4}$, $\mathbf{K4}$ and other more complicated modal logics, as well as distributive modal logics, are the obvious candidates. On the other hand, one cannot always expect to have such a simple representation of free algebras. The algebras corresponding to other many-valued logics such as MV-algebras, l-groups, BCK-algebras and so on, are other examples where this method could lead to interesting representations. The recent work [8] that connects ontologies with free distributive algebras with operators shows that such representations of free algebras are not only interesting from a theoretical point of view, but could have very concrete applications.

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