ANALYTIC, CO-ANALYTIC AND PROJECTIVE SETS FROM BROUWER'S INTUITIONISTIC PERSPECTIVE

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Abstract. We study projective subsets of Baire space $\mathcal{N}$ from Brouwer's intuitionistic point of view, using his Thesis on Bars and his continuity axioms. Note that, also intuitionistically, $\mathcal{N}$ is homeomorphic to $\mathcal{N} \times \mathcal{N}$. A subset of $\mathcal{N}$ is *projective* if it results from a closed or an open subset of $\mathcal{N} \times \mathcal{N} = \mathcal{N}$ by a finite number of applications of the two operations of *projection* and *universal projection* or: *co-projection*. We first study the *analytic* subsets of $\mathcal{N}$; these are the projections of the closed subsets of $\mathcal{N}$. We consider a number of examples and discover a fine structure in the class of the analytic subsets of $\mathcal{N}$ that fail to be positively Borel. We then turn to the strictly analytic subsets of $\mathcal{N}$. A subset of $\mathcal{N}$ is called *strictly analytic* if it coincides with the range of a continuous function from $\mathcal{N}$ to $\mathcal{N}$. We explain the importance of the difference between strictly analytic and analytic subsets of $\mathcal{N}$. Using Brouwer's Thesis on Bars we prove separation and boundedness theorems for strictly analytic subsets of $\mathcal{N}$. We then examine *co-analytic* subsets of $\mathcal{N}$; these are the co-projections of the open subsets of $\mathcal{N} \times \mathcal{N}$. We give some examples showing that the classical duality between analytic and co-analytic sets either disappears or takes some very subtle forms. We show different ways to prove that some co-analytic sets are not analytic and that some analytic sets are not co-analytic. We consider the set of the codes of the closed and located subsets of $\mathcal{N}$ that are *almost-countable* as an example of a set that is a projection of a co-analytic set. We bring to light the collapse of the projective hierarchy: every (positively) projective set coincides with the projection of a co-analytic subset of $\mathcal{N}$.

I sat upon the shore
Fishing, with the arid plain behind me
Shall I at least set my lands in order?
London Bridge is falling down falling down falling down

T.S.Eliot, *The Waste Land*

1. Introduction

This is the third in a series of papers on intuitionistic descriptive set theory. We think it worthwhile to study the questions asked by Baire, Borel, Lebesgue, and the mathematicians who joined their enterprise, from Brouwer's intuitionistic point of view.

In [45] we proved an intuitionistic version of the Borel hierarchy theorem. In [46], we explored the remarkable fine structure of the intuitionistic Borel hierarchy, and, in particular, the fine structure of the class $\mathbf{\Sigma}^0_2$, consisting of the countable unions of closed subsets of $\mathcal{N}$. The earlier paper [41] already contains some important results concerning analytic and co-analytic subsets of $\mathcal{N}$.

This introductory section is divided into three parts. In the first part, we briefly present the basic assumptions of intuitionistic analysis. In the second part, we survey some of the results of the earlier papers. In the third part, we give a sketch of the further contents of the paper and the main results.

1.1. The basic assumptions of intuitionistic analysis.

The logical constants are used in their intuitionistic sense. A statement $P \vee Q$ is considered proven only if one either has a proof of P or a proof of Q. A statement $\exists x \in V[P(x)]$ is considered proven only if one is able to produce an element x of V with a proof of the fact that x has the property P.

In the following we list the axioms of intuitionistic analysis as we are going to use them, without arguing their plausibility. L.E.J. Brouwer was the first to use them, see [2, 3, 4, 5, 6], and the question how to state and defend them has been further discussed by others, see [12, 13, 16, 25, 31, 30, 32, 45].

Some of the axioms in our list imply other ones. The reason that we do not present the axioms in their strongest form only is that, the stronger the statement, the more one may have doubts.

We let $\mathbb{N}$ denote the set of the natural numbers and $\mathcal{N}$ the set of all infinite sequences of natural numbers, calling it *Baire space*.

We use $m, n, \ldots$ as variables over the set $\mathbb{N}$ and $\alpha, \beta, \ldots$ as variables over the set $\mathcal{N}$.

An element of $\mathcal{N}$ is thought of as a function from $\mathbb{N}$ to $\mathbb{N}$, and, given α, n, we denote the result of applying α to n by $\alpha(n)$.

For all α, β we define: $\alpha \# \beta$, *α is apart from β*, if and only if, for some n, $\alpha(n) \neq \beta(n)$, and $\alpha = \beta$, *α coincides with β*, if and only if, for all n, $\alpha(n) = \beta(n)$.

For each m in $\mathbb{N}$ we let $\underline{m}$ be the element of $\mathcal{N}$ satisfying: for all n, $\underline{m}(n) = m$.

Axiom 1 (First Axiom of Countable Choice)**.** *For every binary relation R on $\mathbb{N}$, if for every m there exists n such that mRn, then there exists α such that, for every m, $mR\alpha(m)$.*

We let J be a one-to-one function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. J is called a *pairing function* on $\mathbb{N}$.

Let R be a binary relation on $\mathbb{N}$. R is called a *decidable* relation on $\mathbb{N}$ if and only if there exists β in $\mathcal{N}$ such that, for all m, n, mRn if and only if $\beta\big(J(m,n)\big) \neq 0$.

Axiom 2 (Minimal Axiom of Countable Choice)**.** *For every* decidable *binary relation R on $\mathbb{N}$, if for every m there exists n such that mRn, then there exists α such that, for every m, $mR\alpha(m)$.*

Axiom 2 is a weak version of Axiom 1. Note that, in the case of a decidable relation on $\mathbb{N}$, one may define the promised α by: let, for each m, $\alpha(m)$ be the least number n such that mRn. The axiom guarantees that the set $\mathcal{N}$ of functions from $\mathbb{N}$ to $\mathbb{N}$ is closed under this operation, the operation of *minimalization* or *searching for the least number with a given decidable property.*

For every α, for every m, we define an element α^m of $\mathcal{N}$ by: for all n, $\alpha^m(n) = \alpha\big(J(m,n)\big)$. α^m is called the *m-th subsequence of α*.

Axiom 3 (Second Axiom of Countable Choice)**.** *For every binary relation $R \subseteq \mathbb{N} \times \mathcal{N}$, if for every m there exists α such that $mR\alpha$, then there exists α such that, for every m, $mR\alpha^m$.*

$\mathbb{N}^*$ is the set of all finite sequences of natural numbers. We let $\langle \quad \rangle$ be a one-to-one function from $\mathbb{N}^*$ *onto* $\mathbb{N}$. We call the number $\langle a(0), a(1), \ldots, a(n-1)\rangle$ the *code* of the finite sequence $\big(a(0), a(1), \ldots, a(n-1)\big)$. For each α, for each n, we define: $\overline{\alpha}n = \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1)$.

We do not mention the Axioms of Dependent Choice as we are not going to apply them in the arguments of this paper.

Axiom 4 (Brouwer's Continuity Principle)**.** *For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every α there exists n such that αRn, then, for every α there exist m, n such that, for every β, if $\overline{\beta}m = \overline{\alpha}m$, then βRn.*

Let X be a subset of $\mathcal{N}$. X is called a *spread* or a *located and closed subset of* $\mathcal{N}$ if and only if there exists β such that (i) for every s, $\beta(s) = 0$ if and only if, for some α in X, $s = \overline{\alpha}n$, and (ii) for every α, α belongs to X if and only if, for all n, $\beta(\overline{\alpha}n) = 0$.

It is a well-known fact that Brouwer's Continuity Principle generalizes to spreads:

Axiom 5 (Brouwer's Continuity Principle, generalized version)**.** *For every subset X of $\mathcal{N}$ that is a spread, for every binary relation $R \subseteq X \times \mathbb{N}$, if for every α in X there exists n such that $\alpha R n$, then, for every α in X there exist m, n such that, for every β in X, if $\overline{\beta}m = \overline{\alpha}m$, then $\beta R n$.*

Let γ be an element of $\mathcal{N}$. We define: γ codes *a (continuous) function from $\mathcal{N}$ to* $\mathbb{N}$, or: γ is *a (continuous) function from $\mathcal{N}$ to* $\mathbb{N}$ if and only if, for each α, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$.

For every γ coding a function from $\mathcal{N}$ to $\mathbb{N}$, for every α, we let $\gamma(\alpha)$ be the natural number p satisfying: there exists n such that $\gamma(\overline{\alpha}n) = p + 1$, and, for each $i < n$, $\gamma(\overline{\alpha}i) = 0$.

Axiom 6 (First Axiom of Continuous Choice)**.** *For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every α there exists n such that $\alpha R n$, then there exists γ coding a function from $\mathcal{N}$ to $\mathbb{N}$ such that, for every α, $\alpha R\big(\gamma(\alpha)\big)$.*

Let γ be an element of $\mathcal{N}$. We define: γ codes *or:* is *a (continuous) function from $\mathcal{N}$ to $\mathcal{N}$* if and only if, for each n, γ^n codes a function from $\mathcal{N}$ to $\mathbb{N}$.

For every γ coding a function from $\mathcal{N}$ to $\mathcal{N}$, for every α, we let $\gamma|\alpha$ be the element β of $\mathcal{N}$ satisfying: for each n, $\beta(n) = \gamma^n(\alpha)$.

Axiom 7 (Second Axiom of Continuous Choice)**.** *For every binary relation $R \subseteq \mathcal{N} \times \mathcal{N}$, if for every α there exists β such that $\alpha R \beta$, then there exists γ coding a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $\alpha R(\gamma|\alpha)$.*

Like Brouwer's Continuity Principle, Axiom 4, the First and the Second Axiom of Continuous Choice, Axioms 6 and 7 generalize to spreads. We refrain from writing down these generalized versions, as their formulation is straightforward.

We let $*$ denote the binary function on $\mathbb{N}$ corresponding to concatenation of finite sequences, so, for all s, t in $\mathbb{N}$, $s * t$ is the code number of the finite sequence one obtains by putting the finite sequence coded by t behind the finite sequence coded by s.

We assume that the code number of the empty sequence is 0, so: $\langle\, \rangle = 0$.

We also assume that the pairing function on $\mathbb{N}$ and the function coding the finite sequences of natural numbers by natural numbers are connected as follows: for all n, for all s, $J(n, s) = \langle n \rangle * s$.

Note that J maps $\mathbb{N} \times \mathbb{N}$ onto the set $\mathbb{N} \setminus \{0\}$.

Axiom 8 (On the existence of the set of stumps)**.** [1]

There exists a subset of $\mathcal{N}$, called the set **Stp** *of* stumps, *satisfying the following conditions:*

(i) *For every β, if $\beta(0) \neq 0$, then β belongs to* **Stp**.

(ii) *For every β, if $\beta(0) = 0$, and, for each n, β^n belongs to* **Stp**, *then β itself belongs to* **Stp**.

(iii) *For every subset Q of* **Stp**, *if*
- *For every β, if $\beta(0) \neq 0$, then β belongs to Q, and*
- *For every β, if $\beta(0) = 0$, and, for each n, β^n belongs to Q, then β itself belongs to Q,*

then **Stp** *coincides with Q.*

[1] There is a small difference between the set **Stp** as it is introduced here and the sets called **Stp** in [45], [46], respectively.

We shall follow the convention of using $\sigma, \tau, \ldots$ as variables over the set **Stp**.

If σ is a stump satisfying $\sigma(0) \neq 0$, we sometimes say: *σ is/codes the empty stump.*

Note that, for each σ, if $\sigma(0) = 0$, then, for each n, σ^n is a stump: *the n-th immediate substump of σ.*

We define relations $<, \leq$ on the set **Stp**, as follows, by simultaneous transfinite induction.

> For all stumps σ, τ: $\sigma \leq \tau$ if and only if either σ is the empty stump, or σ is non-empty and, for all n, $\sigma^n < \tau$ and: $\sigma < \tau$ if and only if τ is not the empty stump and there exists n such that $\sigma \leq \tau^n$.

Let Q be a subset of **Stp**. Q is an *inductive* subset of **Stp** if and only if every empty stump belongs to Q, and for every non-empty stump τ, τ belongs to Q if every immediate substump τ^n of τ belongs to Q. Q is a *hereditary* subset of **Stp** if and only if, for every stump τ, if every stump $\sigma < \tau$ belongs to Q, then τ itself belongs to Q.

The proof of the next theorem is an exercise in transfinite induction.

Theorem 1.1.

(i) *Every inductive subset Q of* **Stp** *coincides with* **Stp**.

(ii) *For all σ, τ, ρ in* **Stp**,
- $\sigma \leq \sigma$, *and not:* $\sigma < \sigma$,
- *if* $\sigma < \tau$, *then* $\sigma \leq \tau$,
- *if* $\sigma \leq \tau$ *and* $\tau < \rho$, *then* $\sigma < \rho$,
- *if* $\sigma < \tau$ *and* $\tau \leq \rho$, *then* $\sigma < \rho$, *and*
- *if* $\sigma \leq \tau$ *and* $\tau \leq \rho$, *then* $\sigma \leq \rho$.

(iii) *Every hereditary subset Q of* **Stp** *coincides with* **Stp**.

Let s belong to $\mathbb{N}$. We denote the length of the finite sequence of natural numbers coded by s by *length*(s).

Let s belong to $\mathbb{N}$ and let $n =$ *length*(s). s is thought of as a function from the set $\{0, 1, \ldots, n-1\}$ to $\mathbb{N}$. For each $i <$ *length*(s), we let $s(i)$ be the value s assumes at i. Note that $s = \langle s(0), s(1), \ldots, s(n-1)\rangle$.

For each s, for each $i \leq$ *length*(s), we let $\overline{s}i$ be the code number of the finite sequence of length i that is an initial part of s.

Let s be an element of $\mathbb{N}$ and let α belong to $\mathcal{N}$. We define: *α admits s* if and only if, for each i, if $i \leq$ *length*(s), then $\alpha(\overline{s}i) = 0$.

Let B be a subset of $\mathbb{N}$. B is called a *bar in $\mathcal{N}$* if and only if, for each α, there exists n such that $\overline{\alpha}n$ belongs to B.

Axiom 9 (Brouwer's Thesis on Bars)**.** *Let B be a subset of $\mathbb{N}$ that is a bar in $\mathcal{N}$. There exists a stump σ such that the set of all s in B that are admitted by σ is a bar in $\mathcal{N}$.*

Lemma 1.2. *Let σ be a stump, and let α belong to $\mathcal{N}$. If, for all β, there exists n such that $\overline{\beta}n$ is admitted by σ and $\alpha(\overline{\beta}n) \neq 0$, then α itself is a stump.*

Proof. The proof is by induction on the set of stumps. Note that there is nothing to prove if $\sigma(0) \neq 0$. Now assume $\sigma(0) = 0$ and the statement has been seen to hold for every immediate substump σ^n of σ. Assume that α belongs to $\mathcal{N}$ and that for all β, there exists n such that $\overline{\beta}n$ is admitted by σ and $\alpha(\overline{\beta}n) \neq 0$. If $\alpha(0) \neq 0$, α is a stump and we are done. If $\alpha(0) = 0$, then, for each p, for all β, there exists n such that $\overline{\beta}n$ is admitted by σ^p and $\alpha^p(\overline{\beta}n) \neq 0$, and therefore, by the induction hypothesis, α^p is a stump. It follows that α itself is a stump. □

We let A^1_1 be the set of all α such that, for each β, there exists n such that $\alpha(\overline{\beta}n) \neq 0$. Note that A^1_1 is the set of all α that code a function from $\mathcal{N}$ to $\mathbb{N}$. For this reason, A^1_1 is sometimes called **Fun**.

Also note that, for each α, α belongs to A^1_1 if and only if the set of all s that are not admitted by α is a bar in $\mathcal{N}$. For this reason, A^1_1 is sometimes called **Bar**.

Theorem 1.3.

(i) **Stp** *is a subset of* A^1_1.
(ii) A^1_1 *is a subset of* **Stp**.

Proof. (i) One proves this by induction on the set of stumps.

(ii) Suppose that α belongs to A^1_1. Applying Axiom 9 we find a stump σ such that, for every β, there exists n such that $\overline{\beta}n$ is admitted by σ and $\alpha(\overline{\beta}n) \neq 0$. Applying Lemma 1.2, we conclude that α is a stump. □

A subset C of $\mathbb{N}$ is called *monotone* if and only if, for each s, for each n, if s belongs to C, then $s * \langle n \rangle$ belongs to C.

A subset C of $\mathbb{N}$ is called *inductive* if and only if, for each s, if, for each n, $s * \langle n \rangle$ belongs to C, then s itself belongs to C.

Axiom 10 (Principle of Induction on Monotone Bars). *Let B be a monotone subset of $\mathbb{N}$ that is a bar in $\mathcal{N}$. Let C be an inductive subset of $\mathbb{N}$ that contains B. Then $0 = \langle\ \rangle$ belongs to C.*

Cantor space $\mathcal{C}$ is the set of all α in $\mathcal{N}$ such that, for each n, $\alpha(n) = 0$ or $\alpha(n) = 1$.

Axiom 11 (Unrestricted Fan Theorem). *Let B be a subset of $\mathbb{N}$ that is a bar in Cantor space $\mathcal{C}$. There exists a finite subset B' of B that is a bar in $\mathcal{C}$.*

For each β in $\mathcal{N}$ we let D_β be the set of all n in $\mathbb{N}$ such that $\beta(n) \neq 0$. The set D_β sometimes is called *the set decided by β*.

Axiom 12 ((Restricted) Fan Theorem). *Let β be an element of $\mathbb{N}$ such that D_β is a bar in $\mathcal{C}$. There exists a finite subset B' of D_β that is a bar in $\mathcal{C}$.*

Let X be a subset of $\mathcal{N}$. X is called a *finitary spread* or a *fan* or a *compact located and closed subset of $\mathcal{N}$* if and only if there exists β such that (i) for every s, $\beta(s) = 0$ if and only if, for some α in X, $s = \overline{\alpha}n$, and (ii) for every α, α belongs to X if and only if, for all n, $\beta(\overline{\alpha}n) = 0$, and (iii) for every s there exists n such that, for every m, if $m > n$, then $\beta(s * \langle m \rangle) \neq 0$.

Theorem 1.4 (Fan Theorem, general formulation).

(i) *(Using Axiom 11): Let X be a compact located and closed subset of $\mathcal{N}$. Let B be a subset of $\mathbb{N}$ that is a bar in X. There exists a finite subset B' of B that is a bar in X.*
(ii) *(Using Axiom 12): Let X be a compact located and closed subset of $\mathcal{N}$. Let β be a subset of $\mathcal{N}$ such that D_β is a bar in X. There exists a finite subset B' of D_β that is a bar in X.*

1.2. **Some results from the earlier papers.**

Let s be an element of $\mathbb{N}$ and let α belong to $\mathcal{N}$. We define: *α passes through s* or: *s is an initial part of α* if and only if, for some n, $\overline{\alpha}n = s$.

Let X be a subset of Baire space $\mathcal{N}$. X is *basic open* if and only if either X is empty or there exists s such that X consists of all α in $\mathcal{N}$ passing through s. X is *open* if and only if X is a countable union of basic open sets, that is, there exists γ in $\mathcal{N}$ such that, for every α, α belongs to X if and only if, for some n, $\gamma(n) > 1$ and α passes through $\gamma(n) - 1$.

For each β in $\mathcal{N}$, we let G_β be the set of all α such that, for some n, $\beta(\overline{\alpha}n) \neq 0$. One may prove that a subset X of $\mathcal{N}$ is open if and only if, for some β, X coincides with G_β.

We let E_1 be the set of all α in $\mathcal{N}$ such that, for some n, $\alpha(n) \neq 0$.

The following notion is very important. Let X, Y be subsets of $\mathcal{N}$ and let γ be (the code of) a continuous function from $\mathcal{N}$ to $\mathcal{N}$. γ *reduces the set* X *to the set* Y if and only if, for each α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y. X *reduces to* Y if and only if there exitsts a function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing X to Y.

One may prove that a subset X of $\mathcal{N}$ is open if and only if X reduces to E_1.

Let X be a subset of $\mathcal{N}$. X is *closed* if and only if there exists an open subset Y of $\mathcal{N}$ such that X is the set of all α in $\mathcal{N}$ that do not belong to Y.

Let α, β be elements of $\mathcal{N}$. We define: β *admits* α if and only if, for each n, $\beta(\overline{\alpha}n) = 0$.

For each β in $\mathcal{N}$, we let F_β be the set all α admitted by β. One may prove that a subset X of $\mathcal{N}$ is closed if and only if, for some β, X coincides with F_β. We sometimes say that β *is a code for the set* F_β.

We let A_1 be the set of all α in $\mathcal{N}$ such that, for each n, $\alpha(n) = 0$.

One may prove that a subset X of $\mathcal{N}$ is closed if and only if X reduces to A_1.

Let X be a subset of $\mathcal{N}$. X is *positively Borel* if and only if one may obtain X from open subsets of $\mathcal{N}$ by the repeated application of the operations of countable union and countable intersection.

We let 1^* be the element of $\mathcal{N}$ satisfying: $1^*(0) = 0$ and, for each $n > 0$, $1^*(n) = 1$. Note that 1^* is a stump. We define a subset of the set of stumps, the set of the *non-zero stumps* as follows, by induction: a stump σ is a non-zero stump if either $\sigma = 1^*$, or, $\sigma(0) =)$ and, for each n, $\sigma(\langle n \rangle) = 0$ and σ^n is a non-zero stump.

For each non-zero stump σ we define subsets E_σ and A_σ of $\mathcal{N}$, as follows, by induction on the set of stumps. $E_{1^*} := E_1$ and $A_{1^*} = A_1$, and, for each non-zero stump different from 1^*, for each α, α belongs to E_σ if and only if, for some n, α^n belongs to A_{σ^n}, and α belongs to A_σ if and only if, for all n, α^n belongs to E_{σ^n}.

We define a subset of the set of stumps, the hereditarily repetitive stumps, as follows, by induction: a stump σ is hereditarily repetitive if and only if either $\sigma(0) \neq 0$ or: $\sigma(0) = 0$ and for each m there exists $n > m$ such that $\sigma^n = \sigma^m$ and, for each n, σ^n is hereditarily repetitive.

The following theorem is the main result of [45], see Theorems 4.9, 7.9 and 7.10 of [45].

Theorem 1.5 (Borel Hierarchy Theorem)**.**

(i) *For each non-zero stump σ, every element of A_σ is apart from every element of E_σ.*

(ii) *For each subset X of $\mathcal{N}$, X is positively Borel if and only if there exists a hereditarily repetitive non-zero stump σ such that X reduces to E_σ.*

(iii) *For all non-zero stumps σ, τ, if $\sigma < \tau$, then both A_σ and E_σ reduce to both A_τ and E_τ.*

(iv) *For every hereditarily repetitive non-zero stump σ, A_σ* positively fails to reduce to *E_σ, that is, for every continuous function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps A_σ into E_σ, then there exists α in E_σ such that $\gamma|\alpha$ belongs to E_σ, and E_σ* positively fails to reduce to *A_σ, that is, for every continuous function γ from $\mathcal{N}$ to $\mathcal{N}$, if γ maps E_σ into A_σ, then there exists α in A_σ such that $\gamma|\alpha$ belongs to A_σ.*

For all subsets X, Y of $\mathcal{N}$, we let $D(X, Y)$ be the set of all α such that either α^0 belongs to X or α^1 belongs to Y. The set $D(X, Y)$ is called the *disjunction* of the sets X and Y. One may prove that a subset Z of $\mathcal{N}$ reduces to the set $D(X, Y)$ if and only if there exists subsets Z_0, Z_1 of Z such that $Z = Z_0 \cup Z_1$ and Z_0 reduces to X and Z_1 reduces to Y.

For each α in $\mathcal{N}$ we let $S(\alpha)$ be the element of $\mathcal{N}$ satisfying: for each n, $\big(S(\alpha)\big)^n = \alpha$ and $\big(S(\alpha)\big)(0) = 0$.

Note that, if σ is a (non-zero) (hereditarily repetitive) stump, then so is $S(\sigma)$. $S(\sigma)$ is called: the *successor* of σ.

Let σ be a non-empty stump. σ is called *weakly comparative* if and only if, for all m, n there exists p such that $\sigma^m \le \sigma^p$ and $\sigma^n \le \sigma^p$. This notion has been introduced in [45], page 39.

The following result is Theorem 8.8 of [45].

Theorem 1.6 (The persisting difficulty of disjunction)**.** *For each hereditarily repetitive and weakly comparative non-zero stump σ, the set $D(A_1, A_\sigma)$ does not reduce to the set $A_{S(\sigma)}$.*

Recall that $\underline{0}$ is the element of $\mathcal{N}$ satisfying: for each n, $\underline{0}(n) = 0$.

We define a function associating to every stump a subset CB_σ of $\mathcal{N}$, called the *Cantor-Bendixson set associated to* σ, as follows, by induction.

1. For each stump σ, if $\sigma(0) \neq 0$, then $CB_\sigma = \emptyset$.
2. For each non-empty stump σ, $CB_\sigma = \{\underline{0}\} \cup \bigcup\limits_{n\in\mathbb{N}} \overline{\underline{0}}n * \langle 1\rangle * CB_{\sigma^n}$.

We want to mention some of the results proven for these sets in [46].

Theorem 1.7 (The Cantor-Bendixson hierarchy, part 1)**.**

- (i) *For each non-empty stump σ, the set CB_σ is an enumerable subset of $\mathcal{N}$.*
- (ii) *For all hereditarily iterative stumps σ, τ, if $\sigma \le \tau$, then the set CB_σ reduces to the set CB_τ.*
- (iii) *For all hereditarily iterative stumps σ, τ, if $\sigma < \tau$, then the set CB_τ does not reduce to the set CB_σ.*

Following in Cantor's footsteps, we define, for every subset X of $\mathcal{N}$, a function associating to each stump σ a subset $Der(\sigma, X)$ of $\mathcal{N}$, called the *σ-th Cantor-Bendixson derivative set of* X, as follows, by induction.

1. For each stump σ, if $\sigma(0) \neq 0$, then $Der(\sigma, X) = X$.
2. For each non-empty stump σ, $Der(\sigma, X)$ is the set of all α such that, for each each n, there exists β in $\bigcap\limits_{i\in\mathbb{N}} Der(\sigma^i, X)$ such that β passes through $\overline{\alpha}n$ and $\beta \# \alpha$.

Theorem 1.8 (The Cantor-Bendixson hierarchy, part 2)**.**

- (i) *For all stumps σ, τ, if $\sigma \le \tau$, then $Der(\tau, CB_\sigma) = \emptyset$.*
- (ii) *For all stumps σ, τ, if $\tau < \sigma$, then $\underline{0}$ belongs to $Der(\tau, CB_\sigma)$.*

We also define, for every subset X of $\mathcal{N}$, a function associating to each stump σ a subset $\mathbb{P}(\sigma, X)$ of $\mathcal{N}$, called the *σ-th perhapsive extension of* X, as follows, by induction.

1. For each stump σ, if $\sigma(0) \neq 0$, then $\mathbb{P}(\sigma, X) = X$
2. For each non-empty stump σ, $\mathbb{P}(\sigma, X)$ is the set of all α, such that, for some β in X, if $\alpha \# \beta$, then there exists n such that α belongs to $\mathbb{P}(\sigma^n, X)$.

Theorem 1.9 (The Cantor-Bendixson hierarchy, part 3)**.**

- (i) *For every inhabited subset X of $\mathcal{N}$, for all hereditarily iterative stumps σ, τ, if $\sigma \le \tau$, then $\mathbb{P}(\sigma, X)$ is a subset of $\mathbb{P}(\tau, X)$.*
- (ii) *For all hereditarily iterative stumps σ, the set $\overline{CB_\sigma}$ is a subset of the set $\mathbb{P}(\sigma, CB_\sigma)$.*
- (iii) *For all hereditarily iterative stumps σ, τ, if the set $\overline{CB_\sigma}$ is a subset of the set $\mathbb{P}(\tau, CB_\sigma)$, then $\sigma \le S(\tau)$.*

For each n, for all subsets $X_0, X_1, \ldots X_n$ of $\mathcal{N}$ we let $D^n_{i=0}(X_i)$ be the set of all α in $\mathcal{N}$ such that, for some $i \le n$, α^i belongs to X_i, and we let $C^n_{i=0}(X_i)$ be the set of all α in $\mathcal{N}$ such that, for all $i \le n$, α^i belongs to X_i.

We define a function associating to every stump σ a subset DCB_σ of $\mathcal{N}$, called the *disjunctive Cantor-Bendixson set associated to* σ, as follows, by induction.

(1) For each stump σ, if $\sigma(0) \neq 0$, then $DCB_\sigma = \{\underline{0}\}$.
(2) For each non-empty stump σ, $DCB_\sigma = \{\underline{0}\} \cup \bigcup_{n\in\mathbb{N}} \underline{\overline{0}}n * \langle 1 \rangle * D^n_{i=0}(DCB_{\sigma^i})$.

We also define a function associating to every stump σ a subset CCB_σ of $\mathcal{N}$, called the *conjunctive Cantor-Bendixson set associated to* σ, as follows, by induction.

(1) For each stump σ, if $\sigma(0) \neq 0$, then $CCB_\sigma = D^2(\{\underline{0}\})$.
(2) For each non-empty stump σ, $CCB_\sigma = \{\underline{0}\} \cup \bigcup_{n\in\mathbb{N}} \underline{\overline{0}}n * \langle 1 \rangle * C^n_{i=0}(CCB_{\sigma^i})$.

For each n, for each subset X of $\mathcal{N}$, we let $D^n(X)$ be the set of all α such that, for some $i < n$, α^i belongs to X, and we let $C^n(X)$ be the set of all α such that, for all $i < n$, α^i belongs to X.

Let X be a subset of $\mathcal{N}$. X is called *disjunctively productive* if and only if, for each n, the set $D^{n+1}(X)$ does not reduce to the set $D^n(X)$, and X is called *conjunctively productive* if and only if, for each n, the set $C^{n+1}(X)$ does not reduce to the set $C^n(X)$.

The following theorem is a main result of [46].

Theorem 1.10 (Two more hierarchies within $\mathbf{\Sigma}^0_2$)**.**

(i) *For each stump σ, the sets DCB_σ and CCB_σ are countable unions of closed sets and thus belong to the class $\mathbf{\Sigma}^0_2$.*
(ii) *For all hereditarily repetitive stumps σ, τ, if $\sigma \leq \tau$, then the set DCB_σ reduces to the set DCB_τ, and the set CCB_σ reduces to the set CCB_τ.*
(iii) *For all hereditarily repetitive stumps σ, τ, if $\sigma < \tau$, then the set DCB_τ does not reduce to the set DCB_σ, and the set CCB_τ does not reduce to the set CCB_σ.*
(iv) *For each hereditarily repetitive stump σ, the set DCB_σ is disjunctively productive, and the set CCB_σ is conjunctively productive.*

The paper [41] contains some results that are very important in connection with the subject of this paper.

First, the set **Fin**, consisting of all α in Cantor space $\mathcal{C}$ such that there exists n such that for all m, if $m > n$, then $\alpha(m) = 0$ is introduced. This set belongs to the class $\mathbf{\Sigma}^0_2$ and it is contrasted with the set **Almost*****Fin** consisting of all α in $\mathcal{C}$ such that, for every *strictly increasing* γ, there exists n such that $\alpha\big(\gamma(n)\big) = 0$. It is not difficult to see that **Almost*****Fin** is a co-projection of an open subset of $\mathcal{N}$ and thus a co-analytic subset of $\mathcal{N}$. The perhapsive extensions of the set **Fin**, that is, the sets $\mathbb{P}(\sigma, \mathbf{Fin})$, where σ is a stump, are called, in this paper, the *Borel approximations of the set* **Almost*****Fin**. Note that, for each stump σ, the set $\mathbb{P}(\sigma, \mathbf{Fin})$ is a positively Borel subset of $\mathcal{N}$.

For each subset X of $\mathcal{N}$ containing **Fin** the set X^+ is defined as the set of all α such that there exists β in **Fin** with the property: *if $\alpha \# \beta$, then α belongs to X.*

The following results are proven in [41].

Theorem 1.11 (A *simple* co-analytic subset of $\mathcal{N}$ that is not positively Borel)**.**

(i) *For every stump σ, the set $\mathbb{P}(\sigma, \mathbf{Fin})$ is a subset of the set* **Almost*****Fin**.
(ii) *For all stumps σ, τ, if $\sigma < \tau$, then $\mathbb{P}(\sigma, \mathbf{Fin})$ is a proper subset of $\mathbb{P}(\tau, \mathbf{Fin})$.*
(iii) *For every stump σ, the set $\big(\mathbb{P}(\sigma, \mathbf{Fin})\big)^+$ coincides with the set $\mathbb{P}(S(\sigma), \mathbf{Fin})$, and the set $\mathbb{P}(\sigma, \mathbf{Fin})$is a proper subset of $\big(\mathbb{P}(\sigma, \mathbf{Fin})\big)^+$.*
(iv) *The set* (**Almost*****Fin**)$^+$ *coincides with the set* **Almost*****Fin**.
(v) *For every positively Borel set X that is a subset of* **Almost*****Fin** *and contains* **Fin** *there exists a positively Borel set Y that is a subset of* **Almost*****Fin** *and contains X such that Y does not reduce to X, (and thus does not coincide with X).*

The second set that is subjected to close inspection in [41] is the set $\mathbf{MonPath}_{01}$ consisting of all α with the the property that there exists γ such that, for all n, $\gamma(n) \leq$

$\gamma(n+1) \leq 1$ and $\alpha(\overline{\gamma}n) = 0$. $\mathbf{MonPath_{01}}$ is the projection of a closed subset of $\mathcal{N}$ and, therefore, an analytic subset of $\mathcal{N}$, but $\mathbf{MonPath_{01}}$ is not positively Borel. (From a classical point of view, $\mathbf{MonPath_{01}}$ is a closed subset of $\mathcal{N}$.)

Note that, for each α, for each s, $\alpha^0(s) = \alpha(\langle 0 \rangle * s)$.

Recall that $\underline{1}$ is the element of $\mathcal{N}$ satisfying: for each n, $\underline{1}(n) = 1$.

For every subset X of $\mathcal{N}$ we let X^- be the set of all α such that either α admits $\underline{1}$ or α^0 belongs to X.

The class $\mathcal{B}$ of the *Borel-approximations of the set* $\mathbf{MonPath_{01}}$ is defined inductively, as follows.

The set $(\mathbf{MonPath_{01}})^{\neg\neg}$ is the *initial Borel-approximation of the set* $\mathbf{MonPath_{01}}$. Note that, for each α, α belongs to the closed set $(\mathbf{MonPath_{01}})^{\neg\neg}$ if and only if, for each n, there exists s admitted by α such that $\mathit{length}(s) = n$, and, for each $i < \mathit{length}(s) - 1$, $s(i) \leq s(i+1) \leq 1$.

For each subset X of $\mathcal{N}$, X belongs to the class $\mathcal{B}$ if and only if either X coincides with $(\mathbf{MonPath_{01}})^{\neg\neg}$, or there exists Y in $\mathcal{B}$ such that X coincides with Y^- or there exists an infinite sequence $Y_0, Y_1, \ldots$ of elements of $\mathcal{B}$ such that X coincides with $\bigcap_{n \in \mathbb{N}} Y_n$.

Theorem 1.12 (A *simple* analytic subset of $\mathcal{N}$ that is not positively Borel)**.**

(i) *Every set in the class $\mathcal{B}$ is positively Borel.*

(ii) *The set* $\mathbf{MonPath_{01}}$ *is a subset of every set in the class $\mathcal{B}$.*

(iii) *For every X in $\mathcal{B}$ there exists a function from $\mathcal{N}$ to X that is not a function from $\mathcal{N}$ to* $\mathbf{MonPath_{01}}$*, so X does not coincide with* $\mathbf{MonPath_{01}}$.

(iv) *For each positively Borel set X containing the set* $\mathbf{MonPath_{01}}$ *there exists a set Y in the class $\mathcal{B}$ properly contained in X.*

1.3. **The main results of this paper.** Apart from this introductory Section, the paper contains Sections numbered 2 to 9.

In Section 2, we discover that the set of the *positively ill-founded trees*, that is, the set of all α that admit every element of an infinite sequence of (codes of) finite sequences decreasing in the sense of the Kleene-Brouwer-ordering, is not a complete element of the class $\mathbf{\Sigma}^1_1$ of the analytic sets. We also explain that the set of codes of the located and closed subsets of $\mathcal{N}$ that are positively uncountable is a complete element of the class $\mathbf{\Sigma}^1_1$, and that the same holds for the set of codes of the closed subsets of $\mathcal{N}$ that contain an element α of $\mathcal{N}$ satisfying $\forall m \exists n > m[\alpha(n) = 1]$.

In Section 3, we find a hierarchy of analytic sets. The basic idea is, to consider, for each stump σ, two sets: first, the set $\mathit{Share}(CB_\sigma)$ consisting of all β that admit a member of the Cantor-Bendixson-set CB_σ and, secondly, the set $\mathit{Share}(\overline{CB_\sigma})$ consisting of all β that admit a member of the closure $\overline{CB_\sigma}$ of the CB_σ. Each set $\mathit{Share}(CB_\sigma)$ belongs to the class $\mathbf{\Sigma}^0_2$ and each set $\mathit{Share}(\overline{CB_\sigma})$ is an analytic set, and for all stumps σ, τ, if $\sigma \leq \tau$ then the set $\mathit{Share}(CB_\sigma)$ reduces to the set $\mathit{Share}(CB_\tau)$ and the set $\mathit{Share}(\overline{CB_\sigma})$ reduces to the set $\mathit{Share}(\overline{CB_\tau})$. It turns out to be useful to introduce so-called *hereditarily increasing stumps*. For all hereditarily increasing stumps σ, τ, if $\sigma < \tau$, then the set $\mathit{Share}(CB_\tau)$ does not reduce to the set $\mathit{Share}(CB_\sigma)$ and the set $\mathit{Share}(\overline{CB_\tau})$ does not reduce to the set $\mathit{Share}(\overline{CB_\sigma})$. Tne fact that, for almost every stump σ, the set $\mathit{Share}(\overline{CB_\sigma})$ is not positively Borel, is a consequence of Theorem 1.12.

In Section 4, we study the set $\mathit{Share}(\mathcal{C})$, consisting of all α in $\mathcal{N}$ that admit an element of Cantor space $\mathcal{C}$. The class of the subsets of $\mathcal{N}$ reducing to the set $\mathit{Share}(\mathcal{C})$ is a proper extension of the class of the closed subsets of $\mathcal{N}$ although, in classical mathematics, the two classes coincide: the set $\mathbf{MonPath_{01}}$ reduces to the set $\mathit{Share}(\mathcal{C})$ but is not a closed subset of $\mathcal{N}$ as it fails to be positively Borel, see Theorem 1.12. The class of the closed subsets of $\mathcal{N}$ is closed under the operation of countable intersection but not under the operation of finite union. The class of all subsets of $\mathcal{N}$ reducing to the set $\mathit{Share}(\mathcal{C})$, however, is closed under both operations. The set $D^2(A_1)$ is another and easier example of a set that is not closed but reduces to *Share*$(\mathcal{C})$.

The Section also contains some observations on the assumption that the set E_1 reduces to the set *Share*$(\mathcal{C})$.

Section 5 is devoted to the important distinction between analytic and strictly analytic subsets of $\mathcal{N}$. A subset X of $\mathcal{N}$ is *strictly analytic* if there exists γ in $\mathcal{N}$ coding a continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of the function, that is, for each α, α belongs to X if and only if, for some β, $\gamma|\beta = \alpha$.

In Section 6, we give constructive proofs of the Lusin Separation Theorem and the Novikov Separation Theorem. These theorems are proven for strictly analytic subsets of $\mathcal{N}$. The proofs depend on Brouwer's Thesis on Bars. It is also shown that the positively Borel set $D^2(A_1)$ positively fails to be the range of a strongly one-to-one function from $\mathcal{N}$ to $\mathcal{N}$. The set $D^2(A_1)$ thus turns out to be a counterexample to (the constructive version of) a famous characterization of positively Borel subsets of $\mathcal{N}$ due to Lusin.

In Section 7, we consider co-analytic subsets of $\mathcal{N}$. It turns out that the set of codes of the *well-founded trees*, that is the set of all α such that every infinite sequence either positively fails to be a sequence of (codes) of finite sequences decreasing in the sense of the Kleene-Brouwer ordering or has a member that is not admitted by α, coincides with the set $A^1_1 = \mathbf{Bar}$. The proof depends on Brouwer's Thesis on Bars, Axiom 9. The set *Sink*$(\mathbf{Almost^*Fin})$ consisting of all β satisfying the condition that every α admitted by β belongs to the set $\mathbf{Almost^*Fin}$ is another complete element of the class $\mathbf{\Pi}^1_1$ of the co-analytic subsets of $\mathcal{N}$. We also study the set $E^1_1!$ consisting of those β that admit exactly one element of $\mathcal{N}$. Every co-analytic subset of $\mathcal{N}$ reduces to the set $E^1_1!$, but $E^1_1!$ itself is not co-analytic.

In Section 8, we prove that complete elements of the class of the analytic subsets of $\mathcal{N}$ positively fail to be co-analytic and that complete elements of the class of the co-analytic sets are not analytic. We also show that complete elements of either class positively fail to be positively Borel. Using Brouwer's Thesis on bars, Axiom 9, we prove Souslin's theorem saying that every subset of $\mathcal{N}$ that is both strictly analytic and co-anlytic is positively Borel.

In Section 9, we study co-projections of analytic sets and projections of co-analytic sets. We study the set of all β coding a located and closed subset of $\mathcal{N}$ that is, in a sense to be defined, *almost-countable* as an example of a set that is a projection of a co-analytic set. We prove that the Second Axiom of Continuous Choice, Axiom 7, implies the collapse of the projective hierarchy.

2. Analytic subsets of $\mathcal{N}$: some examples

2.1. **The class of the analytic subsets of $\mathcal{N}$.** We let $\langle\ ,\ \rangle$ denote the following pairing function on $\mathcal{N}$. We define, for each α, for each β, for each n, $\langle\alpha,\beta\rangle(2n) = \alpha(n)$ amd $\langle\alpha,\beta\rangle(2n+1) = \beta(n)$.

Conversely, for each α, we define elements α_I and α_{II} of $\mathcal{N}$ by: for each n, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$.

Note that, for each α, $\alpha = \langle\alpha_I, \alpha_{II}\rangle$.

For every subset X of $\mathcal{N}$, we let *the (existential) projection* of X, notation $Ex(X)$, be the set of all α such that, for some β, $\langle\alpha,\beta\rangle$ belongs to X.

A subset X of $\mathcal{N}$ is called *analytic* if and only if there exists a closed subset Y of $\mathcal{N}$ such that X coincides with $Ex(Y)$. The class of the analytic subsets of $\mathcal{N}$ is denoted by $\mathbf{\Sigma}^1_1$.

One may parametrize the class $\mathbf{\Pi}^0_1$ of the closed subsets of $\mathcal{N}$ as follows. As we defined in Section1, for each β, F_β is the set of all α in $\mathcal{N}$ such that, for each n, $\beta(\overline{\alpha}n) = 0$. A subset X of $\mathcal{N}$ is closed if and only there exists β such that X coincides with F_β.

One may parametrize the class $\mathbf{\Sigma}^1_1$ of the analytic subsets of $\mathcal{N}$ as follows. For each β, let A_β be the set $Ex(F_\beta)$, that is the set of all α in $\mathcal{N}$ such that, for some γ, the pair $\langle\alpha,\gamma\rangle$ belongs to F_β. A subset X of $\mathcal{N}$ is analytic if and only there exists β such that X coincides with A_β.

We let US^1_1 be the set of all α such that α_{II} belongs to A_{α_I}.

For each subset X of $\mathcal{N}$, for each β, we let $X \upharpoonright \beta$ be the set of all α such that $\langle\beta,\alpha\rangle$ belongs to X.

For each class $\mathcal{K}$ of subsets of $\mathcal{N}$, for each subset X of $\mathcal{N}$, X is called a *cataloguing* or a *universal* element of the class $\mathcal{K}$ if and only if X belongs to $\mathcal{K}$, and, for every element Y of $\mathcal{K}$, there exists β such that Y coincides with $X \upharpoonright \beta$.

We use S^* to denote the successor-function from $\mathbb{N}$ to $\mathbb{N}$, and $\circ$ to denote the operation of composition on $\mathcal{N}$. So, for each α,β in $\mathcal{N}$, for each n in $\mathbb{N}$, $\alpha\circ\beta(n) = \alpha\big(\beta(n)\big)$.

For each α, for each i, for each j, we define: $\alpha^{i,j} := (\alpha^i)^j$.

Theorem 2.1.

(i) *The set US^1_1 is a universal element of the class $\mathbf{\Sigma}^1_1$.*
(ii) *For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, X_n belongs to $\mathbf{\Sigma}^1_1$, then both $\bigcup_{n\in\mathbb{N}} X_n$ and $\bigcap_{n\in\mathbb{N}} X_n$ belong to $\mathbf{\Sigma}^1_1$.*
(iii) *For every subset X of $\mathcal{N}$, if X is positively Borel, then X is analytic.*
(iv) *For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Sigma}^1_1$, then $Ex(X)$ belongs to $\mathbf{\Sigma}^1_1$.*
(v) *For all subsets X, Y of $\mathcal{N}$, if X reduces to Y and Y is analytic, then X is analytic.*

Proof. (i) Note that, for each α, α belongs to US^1_1 if and only if α_{II} belongs to A_{α_I} if and only if, for some γ, $\langle\alpha_{II},\gamma\rangle$ belongs to F_{α_I} if and only if, for some γ, for every n, $\alpha_I(\overline{\langle\alpha_{II},\gamma\rangle}n) = 0$. We thus see that US^1_1 belongs to the class $\mathbf{\Sigma}^1_1$. The fact that, for each element Y of $\mathbf{\Sigma}^1_1$, there exists β such that Y coincides with $US^1_1 \upharpoonright \beta$ follows from the observations preceding this theorem.

(ii) Let $Y_0, Y_1, \ldots$ be a sequence of closed subsets of $\mathcal{N}$ such that, for each n, $X_n = Ex(Y_n)$. Using the Second Axiom of Countable Choice, Axiom 3, find β in $\mathcal{N}$ such that, for each n, $Y_n = F_{\beta^n}$ and $X_n = Ex(Y_n) = A_{\beta^n}$. Define subsets Z_0, Z_1 of $\mathcal{N}$, as follows. Z_0 is the set of all α such that $\langle\alpha^0, \alpha^1\circ S^*\rangle$ belongs to $Y_{\alpha^1(0)}$ and Z_1 is the set of all α such that, for all n, $\langle\alpha^0,\alpha^{1,n}\rangle$ belongs to Y_n. The sets Z_0, Z_1 are closed subsets of $\mathcal{N}$: one may determine γ in $\mathcal{N}$ satisfying: for each α, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$ if and only if there exists n such that $\beta^{\alpha^1(0)}(\overline{\langle\alpha^0,\alpha^1\circ S^*\rangle}n) \neq 0$ and one may determine δ satisfying: for each α, there exists n such that $\delta(\overline{\alpha}n) \neq 0$ if and only if there exist i,j such that $i \le n$ and $j \le n$ and $\beta^j\overline{\langle\alpha^0,\alpha^{1,j}\rangle}i) \neq 0$.

Note that the set Z_0 coincides with the set F_γ and that the set Z_1 coincides with the set F_δ.

Note that, for each α, α belongs to $\bigcup_{n\in\mathbb{N}} X_n$ if and only if there exist n,β such that $\langle\alpha,\beta\rangle$ belongs to X_n if and only if there exists β such that $\langle\alpha,\beta\circ S^*\rangle$ belongs to $X_{\beta(0)}$ if and only if α belongs to $Ex(Z_0)$. It follows that the set $\bigcup_{n\in\mathbb{N}} X_n$ coincides with the set $Ex(Z_0) = Ex(F_\gamma) = A_\gamma$.

Also note that, for each α, α belongs to $\bigcap_{n\in\mathbb{N}} X_n$ if and only if, for each n, there exists β such that $\langle\alpha,\beta\rangle$ belongs to X_n if and only if, by the Second Axiom of Countable Choice, there exists β such that, for each n, $\langle\alpha,\beta^n\rangle$ belongs to X_n if and only there exists β such that α, β belongs to Z_1. It follows that the set $\bigcap_{n \in \mathbb{N}} X_n$ coincides with the set $Ex(Z_1) = Ex(F_\delta) = A_\delta$.

(iii) follows easily from (ii), by induction on the class of the positively Borel sets.

(iv) Suppose that X belongs to $\mathbf{\Sigma}^1_1$. Let Y be a closed subset of $\mathcal{N}$ such that $X = Ex(Y)$. Observe that for every α, α belongs to $Ex(X)$ if and only if, for some β, $\langle \alpha, \beta \rangle$ belongs to X if and only if, for some β, for some γ, the sequence $\langle \langle \alpha, \beta \rangle, \gamma \rangle$ belongs to Y. Let Z be the set of all α such that $\langle \langle \alpha^0, \alpha^{1,0} \rangle, \alpha^{1,1} \rangle$ belongs to Y. Observe that Z is a closed subset of $\mathcal{N}$ and $Ex(X)$ coincides with $Ex(Z)$, and, therefore, $Ex(X)$ belongs to $\mathbf{\Sigma}^1_1$.

(v) Let X, Y be subsets of $\mathcal{N}$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to Y.

Assume that Y is analytic. We have to show that also X is analytic. Let Z be a closed subset of $\mathcal{N}$ such that Y coincides with $Ex(Z)$. Observe that for every α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y if and only if, for some β, $\langle \gamma|\alpha, \beta \rangle$ belongs to Z. Let V be the set of all α such that $\langle \gamma|\alpha^0, \alpha^1 \rangle$ belongs to Z. Observe that V is closed and X coincides with $Ex(V)$, and, therefore, X is analytic. □

2.2. **The set E^1_1 = Path and the set PIF of all α admitting each element of an infinite $<_{KB}$-decreasing sequence.**

Recall that, for every class $\mathcal{K}$ of subsets of $\mathcal{N}$, for every subset X of $\mathcal{N}$, X is a *complete* element of $\mathcal{K}$ if and only if X belongs to $\mathcal{K}$, and every subset Y of $\mathcal{N}$ that belongs to $\mathcal{K}$ reduces to X.

We let E^1_1 be the set of all α in $\mathcal{N}$ such that for some β, for all n, $\alpha(\overline{\beta}n) = 0$. E^1_1 might be described as the set of all α that admit *an infinite path* in Baire space, and, for this reason, we sometimes call it **Path**. We shall prove, in the next theorem, that E^1_1 is a complete element of the class $\mathbf{\Sigma}^1_1$, or, as we will say, a *complete analytic set*.

We now define the *Kleene-Brouwer ordering* $<_{KB}$ on $\mathbb{N}$. Sometimes, this ordering is called the *Lusin-Sierpiński ordering*, see [15], Section 2.G, p. 11.

> For all s, t in $\mathbb{N}$, $s <_{KB} t$ if and only if *either* t is (or: codes a finite sequence of natural numbers that is) a strict initial part of (the finite sequence of natural numbers coded by) s, *or* there exists i such that $i < length(s)$ and $i < length(t)$, and $\overline{s}i = \overline{t}i$ while $s(i) < t(i)$, that is, $s \perp t$ and (the finite sequence coded by) s comes lexicografically before (the finite sequence coded by) t.

$<_{KB}$ is a well-known decidable and linear ordering of $\mathbb{N}$.

We also define: for all s, t in $\mathbb{N}$, $s \sqsubseteq t$, *s is an initial part of t*, if and only if there exists $i \leq length(t)$ such that $s = \overline{t}i$, and $s \sqsubset t$, *s is a proper initial part of t*, if and only if there exists $i < length(t)$ such that $s = \overline{t}i$

Let α belong to $\mathcal{N}$. Recall that an element s of $\mathbb{N}$ is said to be *admitted by* α if and only if, for every t, if $t \sqsubseteq s$, then $\alpha(t) = 0$.

We let T_α, the *tree determined by* α, be the set of all s that are admitted by α.

We let **PIF** the set of all α in $\mathcal{N}$ such that the tree T_α is *positively ill-founded with respect to* $<_{KB}$, that is, **PIF** is the set of all α in $\mathcal{N}$ such that, for some β, for every n, $\beta(n)$ belongs to T_α, that is, α admits $\beta(n)$, and, for every n, $\beta(n+1) <_{KB} \beta(n)$.

We shall prove, in the next theorem, that **PIF** is *not* a complete analytic set. In classical mathematics, the sets E^1_1 and **PIF** coincide.

Let A, B be subsets of $\mathcal{N}$. *A is a subset of B* if only if every element of A is an element of B. *A is a proper subset of B* if A is a subset of B and, in addition, B is not a subset of A, that is, the assumption that B is a subset of A leads to a contradiction. Note that, if A is a proper subset of B, it may be true that the assumption that there is an element of B that does not belong to A also leads to a contradiction. For instance, let A be the real interval $[-1, 1]$ and let B be the set of all x in $[-1, 1]$ such that either x is really-apart from 0 or x really-coincides with 0. It follows from Brouwer's Continuity Principle that B is a proper subset of A. On the other hand, there is no number x such that x does not belong to B, as, for such a number x, one would have both: x is not really-apart from 0, that is: x really-coincides with 0 and: x does not really-coincide with 0.

Theorem 2.2.

(i) *The set* E^1_1 *is a complete analytic set.*
(ii) *The set* E^1_1 *is a proper subset of the set* **PIF**.
(iii) *The set* $D^2(A_1)$ *does not reduce to the set* **PIF**.
(iv) *The set* **PIF** *is not a complete analytic set.*

Proof. (i) Let X belong to $\mathbf{\Sigma}^1_1$ and let Y be a closed subset of $\mathcal{N}$ such that $X = Ex(Y)$ and let C be a decidable subset of $\mathbb{N}$ such that, for every β, β belongs to Y if and only if, for each n, $\overline{\beta}n$ belongs to C. Observe that for every α, α belongs to X if and only if, for some β, for every n, $\overline{\langle\alpha, \beta\rangle}n$ belongs to C. Define a function f from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, β in $\mathcal{N}$, for every n in $\mathbb{N}$, $(f|\alpha)(\overline{\beta}n) = 0$ if and only if $\overline{\langle\alpha, \beta\rangle}n$ belongs to C and observe that f reduces X to E^1_1.

(ii) First, we prove that every element of E^1_1 belongs to **PIF**. Let α belong to E^1_1. Find β such that, for all n, $\alpha(\overline{\beta}n) = 0$, and define γ in $\mathcal{N}$ such that, for each n, $\gamma(n) = \overline{\beta}n$. Note that, for each n, $\gamma(n)$ belongs to T_α and $\gamma(n + 1) <_{KB} \gamma(n)$, so α belongs to **PIF**.

We now observe that there are elements of **PIF** we can not prove to belong to E^1_1:

Let γ be an element of $\mathcal{N}$. Let α be an element of $\mathcal{N}$ such that, for every s, $\alpha(s) = 0$ if and only if *either* there exists n such that $s = \overline{\underline{0}}n$ or $s = \overline{\underline{1}}n$ and there is no $i \le n$ such that $\gamma(i) > 0$, *or* there exists n such that $s = \overline{\underline{0}}n$ and there is $i \le n$ such that $\gamma(i) > 0$ and the least such i is odd, *or* there exists n such that $s = \overline{\underline{1}}n$ and there is $i \le n$ such that $\gamma(i) > 0$ and the least such i is even. Note that α admits $\underline{0}$ if and only if: if there exists i such that $\gamma(i) \ne 0$, then the first such i is odd, and: α admits $\underline{1}$ if and only if: if there exists i such that $\gamma(i) \ne 0$, then the first such i is even.

We claim that α belongs to **PIF**. Let β be an element of $\mathcal{N}$ such that, for each n, (i) if there is no $i \le n$ such that $\gamma(i) > 0$, then $\beta(n) = \overline{\underline{1}}n$, and (ii) if there is $i \le n$ such that $\gamma(i) > 0$ and the least such i is even, then $\beta(n) = \overline{\underline{1}}n$, and (iii) if there exists $i \le n$ such that $\gamma(i) > 0$ and the least such i is odd, then $\beta(n) = \overline{\underline{0}}n$.
Note that, for each n, $\beta(n + 1) <_{KB} \beta(n)$ and $\alpha\big(\beta(n)\big) = 0$, so α belongs to **PIF**.

Now assume that α belongs to E^1_1 and find β in $\mathcal{N}$ such that, for each n, $\alpha(\overline{\beta}n) = 0$. Note that, if $\beta(0) = 0$, then, for each n, if n is the least i such that $\gamma(i) > 0$, then n is odd, and if $\beta(0) = 1$, then, for each n, if n is the least i such that $\gamma(i) > 0$, then n is even.

Suppose we define an infinite sequence γ as follows.

> Let d be the element of $\mathcal{N}$ that is the decimal expansion of π, that is, that is, $\pi = 3 + \sum_{n\in\mathbb{N}} d(n).10^{-n-1}$. Let γ satisfy the following requirement: for each i, if, for each $j < 99$, $d(i + j) = 9$, then $\gamma(i) = 1$, and, if for some $j < 99$, $d(i + j) \ne 9$, then $\gamma(i) = 0$.

For this γ, we have no proof of the statement: for each n, if n is the least i such that $\gamma(i) > 0$, then n is odd, and we also have no proof of the statement: for each n, if n is the least i such that $\gamma(i) > 0$, then n is even. If α results from this particular γ in the above way, then, clearly, we have no proof of the statement: "*α belongs to E^1_1*", and we must say that the statement: "*α belongs to E^1_1*", is a *reckless* one.

One obtains a contradiction from the assumption: "**PIF** *is a subset of* E^1_1" in the following way:

Assume **PIF** is a subset of E^1_1. Then, for every γ, *either* for each n, if n is the least i such that $\gamma(i) > 0$, then n is odd, *or* for each n, if n is the least i such that $\gamma(i) > 0$, then n is even. Applying Brouwer's Continuity Principle, Axiom 4, we determine m such that *either*, for each γ passing through $\overline{\underline{0}}m$, for each n, if n is the least i such that $\gamma(i) > 0$, then n is odd, *or*, for each γ passing through $\overline{\underline{0}}m$, for each n, if n is the least i such that $\gamma(i) > 0$, then n is even. Considering the two sequences $\overline{\underline{0}}(2m) * \langle 1 \rangle * \underline{0}$ and $\overline{\underline{0}}(2m+1) * \langle 1 \rangle * \underline{0}$, we see that this conclusion is false.

(iii) Assume that γ is an element of $\mathcal{N}$ coding a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(A_1)$ to **PIF**. We claim that γ will map every element of the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ into **PIF**. We then must conclude that the set $D^2(A_1)$ coincides with its closure $\overline{D^2(A_1)}$, and this is false.

Note that, for every α, (1): α belongs to $D^2(A_1)$ if and only if, *either*, for each n, $\alpha^0(n) = 0$, *or*, for each n, $\alpha^1(n) = 0$, and (2): α belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ if and only if, for each n, either $\overline{\alpha^0}n = \overline{\underline{0}}n$ or $\overline{\alpha^1}n = \overline{\underline{0}}n$.

Let α belong to $\mathcal{N}$. We define α_0, α_1 in $\mathcal{N}$ as follows. For each n, $(\alpha_0)^0(n) = 0$ and, for each m, if there is no i such that $m = \langle 0, i \rangle$, then $\alpha_0(m) = \alpha(m)$, and, for each n, $(\alpha_1)^1(n) = 0$ and, for each m, if there is no i such that $m = \langle 1, i \rangle$, then $\alpha_1(m) = \alpha(m)$.

Note that α belongs to $D^2(A_1)$ if and only if either $\alpha = \alpha_0$ or $\alpha = \alpha_1$. Also note that, for each $i < 2$, if α belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ and $\alpha \# \alpha_i$, then $\alpha = \alpha_{1-i}$.

Next, observe that, as both α_0 and α_1 belong to $D^2(A_1)$, both $\gamma|\alpha_0$ and $\gamma|\alpha_1$ belong to **PIF**. We determine δ_0, δ_1 in $\mathcal{N}$ such that, for each $i < 2$, for each n, $\delta_i(n+1) <_{KB} \delta_i(n)$, and for each t, if $t \sqsubseteq \delta_i(n)$, then $(\gamma|\alpha_i)(t) = 0$.

We now define ε in $\mathcal{N}$ such that, for each n, $\varepsilon(n) = \delta_0(n)$ if $\delta_1(n) \leq_{KB} \delta_0(n)$ and $\varepsilon(n) = \delta_1(n)$ if $\delta_0(n) <_{KB} \delta_1(n)$. Note that, for each n, both $\delta_1(n+1) <_{KB} \varepsilon(n)$ and $\delta_0(n+1) <_{KB} \varepsilon(n)$, and, therefore, $\varepsilon(n+1) <_{KB} \varepsilon(n)$.

Suppose we find n, t in $\mathbb{N}$ such that $t \sqsubseteq \varepsilon(n)$ and $(\gamma|\alpha)(t) \neq 0$. Using the continuity of the function coded by γ, we find m such that, for every β, if β passes through $\overline{\alpha}m$, then $(\gamma|\beta)(t) = (\gamma|\alpha)(t) \neq 0$. Observe that, if $\varepsilon(n) = \delta_0(n)$, then $\overline{\alpha}m \neq \overline{\alpha_0}m$, and, if $\varepsilon(n) = \delta_1(n)$, then $\overline{\alpha}m \neq \overline{\alpha_1}m$.

We define ζ in $\mathcal{N}$ such that, for each n, *if* for each $j \leq n$, f or each $t \sqsubseteq \varepsilon(j)$, $(\gamma|\alpha)(t) = 0$, *then* $\zeta(n) = \varepsilon(n)$, and *if* there exists $j \leq n$, such that, for some $t \sqsubseteq \varepsilon(i)$, $(\gamma|\alpha)(t) \neq 0$, and j_0 is the least such i, *then either* $\varepsilon(j_0) = \delta_0(j_0)$ and $\zeta(n) = \delta_1(n)$ *or* $\varepsilon(j_0) \neq \delta_0(j_0)$ and $\zeta(n) = \delta_0(n)$.

Note that, for each n, $\zeta(n+1) <_{KB} \zeta(n)$.

Now assume that α belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$. We claim that, for each n, for each t, if $t \sqsubseteq \zeta(n)$, then $(\gamma|\alpha)(t) = 0$. We prove this claim as follows.

Let n be a natural number. We have two cases to consider.

Case (1). For each $j \leq n$, for each t, if $t \sqsubseteq \varepsilon(j)$, then $(\gamma|\alpha)(t) = 0$. As, in this case, $\zeta(n) = \varepsilon(n)$, we conclude that, for each t, if $t \sqsubseteq \zeta(n)$, then $(\gamma|\alpha)(t) = 0$.

Case (2). There exists $j \leq n$ such that, for some t, $t \sqsubseteq \varepsilon(j)$ and $(\gamma|\alpha)(t) \neq 0$ and j_0 is the least such j. Note that if $\varepsilon(j_0) = \delta_0(j_0)$, then $\alpha \# \alpha_0$, and, therefore, as α belongs to $\overline{D^2(A_1)}$, $\alpha = \alpha_1$, and thus, for each k, for each t, if $t \sqsubseteq \delta_1(k)$, then $(\gamma|\alpha)(t) = 0$, and, in particular, as $\zeta(n) = \delta_1(n)$, for each t, if $t \sqsubseteq \zeta(n)$, then $(\gamma|\alpha)(t) = 0$.

Similarly , if $\varepsilon(j_0) \neq \delta_0(j_0)$, then $\alpha \# \alpha_1$, and, therefore, as α belongs to $\overline{D^2(A_1)}$, $\alpha = \alpha_0$, and thus, for each k, for each t, if $t \sqsubseteq \delta_0(k)$, then $(\gamma|\alpha)(t) = 0$, and, in particular, as $\zeta(n) = \delta_0(n)$, for each t, if $t \sqsubseteq \zeta(n)$, then $(\gamma|\alpha)(t) = 0$.

This ends the proof of our claim.

We must conclude that, for each α, if α belongs to $\overline{D^2(A_1)}$, then $\gamma|\alpha$ belongs to **PIF**. As we are assuming that γ reduces the set $D^2(A_1)$ to the set **PIF**, we find that the sets $\overline{D^2(A_1)}$ and $D^2(A_1)$ coincide. As the set $\overline{D^2(A_1)}$ is a spread containing $\underline{0}$, we apply Brouwer's Continuity Principle and find m such that *either* for every α in $\overline{D^2(A_1)}$, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $\alpha^0 = \underline{0}$, *or* for every α in $\overline{D^2(A_1)}$, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $\alpha^1 = \underline{0}$. Both alternatives are false.

(iv) As the set $D^2(A_1)$ is an analytic set, the conclusion follows from (iii). □

2.3. **The set of all β coding a positively uncountable located and closed subset of $\mathcal{N}$.**

We intend to give another example of a complete analytic subset of $\mathcal{N}$.

Let X be a subset of $\mathcal{N}$. We say that X is a *perfect spread* if and only if X is a closed subset of $\mathcal{N}$ and there exist β in $\mathcal{N}$ satisfying the following conditions:

(i) for each s, $\beta(s) = 0$ if and only if s contains an element of X, and
(ii) for each s, if $\beta(s) = 0$, then there exist t, u such that $s \sqsubseteq t$ and $s \sqsubseteq u$ and $t \perp u$ and $\beta(t) = \beta(u) = 0$.

Note that Cantor space $\mathcal{C}$, the set of all α in $\mathcal{N}$ such that, for each n, $\alpha(n) = 0$ or $\alpha(n) = 1$, is a perfect spread.

Let γ be an element of $\mathcal{N}$. γ *codes* or: *is a (continuous) function from Cantor space $\mathcal{C}$ to the set $\mathbb{N}$ of the natural numbers* if and only if, for each α in $\mathcal{C}$, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$.

Let γ be a function from $\mathcal{C}$ to $\mathbb{N}$ and let α belong to $\mathcal{C}$. We let $\gamma(\alpha)$, *the value that the function γ assumes at the argument α*, be the natural number p such that, for some i, $\gamma(\overline{\alpha}i) = p + 1$, while, for each j, if $j < i$, then $\gamma(\overline{\alpha}j) = 0$.

The set of all functions from $\mathcal{C}$ to $\mathbb{N}$ will sometimes be denoted by $\mathbb{N}^{\mathcal{C}}$.

Let γ be an element of $\mathcal{N}$. γ *codes* or: *is a (continuous) function from Cantor space $\mathcal{C}$ to Baire space $\mathcal{N}$* if and only if, for each n, γ^n is a function from $\mathcal{C}$ to $\mathbb{N}$.

Let γ be a function from $\mathcal{C}$ to $\mathcal{N}$ and let α belong to $\mathcal{C}$. We let $\gamma|\alpha$, *the value that the function γ assumes at the argument α*, be the infinite sequence β in $\mathcal{N}$ such that, for each n, $\beta(n) = \gamma^n(\alpha)$.

We let **Bin** be the set of all natural numbers coding a finite binary sequence, that is, for all s, s belongs to **Bin** if and only if, for all i, if $i < length(s)$, then $s(i) = 0$ or $s(i) = 1$.

Let γ be a function from $\mathcal{C}$ to $\mathcal{N}$ and let s belong to **Bin**. We let $\gamma|s$ be the largest number t such that $length(t) \leq length(s)$ and, for each j, if $j < length(t)$, then there exists p such that $p \leq length(s)$ and $\gamma^j(\overline{s}p) = t(j) + 1$ and, for every k, if $k < p$, then $\gamma^j(\overline{s}k) = 0$. Observe that $length(\gamma|s) \leq length(s)$.

Also observe that, for every n, for every α in $\mathcal{C}$, $\gamma|(\overline{\alpha}n) \sqsubseteq \overline{(\gamma|\alpha)}n$, that is, for all s in **Bin** for all t, if $\gamma|s = t$, then γ maps every infinite sequence passing through s onto an infinite sequence passing through t.

Note that, for all α in $\mathcal{C}$, for all β in $\mathcal{N}$, $\gamma|\alpha = \beta$ if and only if, for each n, there exists m such that $\overline{\beta}n \sqsubseteq \gamma|\overline{\alpha}m$.

The set of all functions from $\mathcal{C}$ to $\mathcal{N}$ will sometimes be denoted by $\mathcal{N}^{\mathcal{C}}$.

Let γ be a function from $\mathcal{C}$ to $\mathcal{N}$. γ is called a *strongly injective function from $\mathcal{C}$ to $\mathcal{N}$* if and only if, for all α in $\mathcal{C}$, for all β in $\mathcal{C}$, if $\alpha \# \beta$, then $\gamma|\alpha \# \gamma|\beta$.

Lemma 2.3.

(i) *For every subset X of $\mathcal{N}$, if X is a perfect spread, then there exists a strongly injective function from $\mathcal{C}$ to X.*
(ii) *For every strongly injective function γ from $\mathcal{C}$ to $\mathcal{N}$, the set $Range(\gamma) = \{\gamma|\alpha \mid \alpha \in \mathcal{C}\}$ is a perfect spread.*

(iii) *For all* α*, the set* $\{\alpha, \underline{0}\}$ *is a closed subset of* $\mathcal{N}$ *if and only if either* $\alpha = \underline{0}$ *or* $\alpha \# \underline{0}$*. The assumption that, for all* α*, the set* $\{\alpha, \underline{0}\}$ *is a closed subset of* $\mathcal{N}$*, leads to a contradiction.*
(iv) *The assumption that, for every function* γ *from* $\mathcal{C}$ *to* $\mathcal{N}$*, Range*(γ) *is a closed subset of* $\mathcal{N}$*, leads to a contradiction.*
(v) *The set* $\mathbb{N}^{\mathcal{C}}$ *is an open subset of* $\mathcal{N}$ *and a complete element of the class* $\mathbf{\Sigma}^0_1$*.*
(vi) *The set* $\mathcal{N}^{\mathcal{C}}$ *is a countable intersection of open subsets of* $\mathcal{N}$ *and a complete element of the class* $\mathbf{\Pi}^0_2$*.*
(vii) *The set of all strongly injective functions from* $\mathcal{C}$ *to* $\mathcal{N}$ *is a countable intersection of open subsets of* $\mathcal{N}$ *and a complete element of the class* $\mathbf{\Pi}^0_2$*.*

Proof. (i) Let X be a subset of $\mathcal{N}$ and a perfect spread. Find β in $\mathcal{N}$ such that, for each s, $\beta(s) = 0$ if and only if s contains an element of X. We now define a mapping F from the set **Bin** of all natural numbers coding a finite binary sequence to the set $\mathbb{N}$, as follows:

(i) $F(\langle\,\rangle) = \langle\,\rangle$.
(ii) Suppose s belongs to **Bin** and $F(s)$ has been defined already and $\beta\big(F(s)\big) = 0$. We then search for the least number u such that $s \sqsubseteq u(0)$ and $s \sqsubseteq u(1)$ and $\beta\big(u(0)\big) = \beta\big(u(1)\big) = 0$ and $u(0) \perp u(1)$ and we define: $F(s * \langle 0\rangle) := u(0)$ and $F(s * \langle 1\rangle) := u(1)$.

Note that, for every s in **Bin**, for every i in $\{0,1\}$, $F(s)$ is a proper initial part of $F(s * \langle i\rangle)$.

We now let γ be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for every α in $\mathcal{C}$, for every n, $\gamma|\alpha$ passes through $F(\overline{\alpha}n)$. It will be clear that γ is a strongly injective function from $\mathcal{C}$ to X.

(ii) Suppose that γ is a strongly injective function from $\mathcal{C}$ to $\mathcal{N}$. Note that, for every α in $\mathcal{C}$, for every n, there exists m such that $\mathit{length}\big(\gamma|(\overline{\alpha}m)\big) \geq n$. It follows from the (Restricted) Fan Theorem, Axiom 12, that, for each n, there exists m such that, for every α in $\mathcal{C}$, $\mathit{length}\big(\gamma|(\overline{\alpha}m)\big) \geq n$. Using the Minimal Axiom of Countable Choice, 2, we find δ in $\mathcal{N}$ such that, for each n, for every s in **Bin**, if $\mathit{length}(s) \geq \delta(n)$, then $\mathit{length}(\gamma|s) \geq n$.

Note that, for every t in $\mathbb{N}$, there exists α in $\mathcal{C}$ such that $\gamma|\alpha$ passes through t if and only if there exists s in **Bin** such that $\mathit{length}(s) = \delta\big(\mathit{length}(t)\big)$ and $t \sqsubseteq \gamma|s$.

As, for each n, the set of all s in **Bin** such that $\mathit{length}(s) = n$ is a finite set, one may decide, for every t in $\mathbb{N}$, whether or not there exists α in $\mathcal{C}$ such that $\gamma|\alpha$ passes through t.

We define β in $\mathcal{N}$ such that, for every t, $\beta(t) = 0$ if and only if there exists α in $\mathcal{C}$ such that $\gamma|\alpha$ passes through t.

We claim that *Range*(γ) coincides with the closed set F_β. It suffices to prove that F_β is a subset of *Range*(γ).

Let n belong to $\mathbb{N}$. Note that, for every α in $\mathcal{C}$, for every β in $\mathcal{C}$, if $\overline{\alpha}n \neq \overline{\beta}n$, then there exists p such that $\gamma|(\overline{\alpha}p) \perp \gamma|(\overline{\beta}p)$. Using the (Restricted) Fan Theorem, Axiom 12, we find p such that, for every α in $\mathcal{C}$, for every β in $\mathcal{C}$, if $\overline{\alpha}n \neq \overline{\beta}n$, then $\gamma|(\overline{\alpha}p) \perp \gamma|(\overline{\beta}p)$. Note that, for all s in **Bin**, $\mathit{length}(\gamma|s) \leq \mathit{length}(s)$, and conclude that, for all s, t in **Bin**, if $\mathit{length}(s) = \mathit{length}(t) = p$ and $\overline{s}n \neq \overline{t}n$, then $\gamma|s \perp \gamma|t$. Using the Minimal Axiom of Countable Choice, Axiom 2, we find η in $\mathcal{N}$ such that, for every n, for all s, t in **Bin**, if $\mathit{length}(s) = \mathit{length}(t) = \eta(n)$ and $\overline{s}n \neq \overline{t}n$, then $\gamma|s \perp \gamma|t$.

It follows that, for every n, for every α in $\mathcal{C}$, for every β in $\mathcal{C}$, if $\overline{\alpha}n \neq \overline{\beta}n$, then $\gamma|\big(\overline{\alpha}(\eta(n))\big) \perp \gamma|\big(\overline{\beta}(\eta(n))\big)$, and, therefore, as $\gamma|\big(\overline{\alpha}(\eta(n))\big) \sqsubseteq \overline{(\gamma|\alpha)}\big(\eta(n)\big)$ and $\gamma|\big(\overline{\beta}(\eta(n))\big) \sqsubseteq \overline{(\gamma|\beta)}\big(\eta(n)\big)$, also $\overline{(\gamma|\alpha)}\big(\eta(n)\big) \neq \overline{(\gamma|\beta)}\big(\eta(n)\big)$.

Now suppose that ε belongs to F_β. Note that, for each n, $\beta(\overline{\varepsilon}n) = 0$, and, therefore, there exists α in $\mathcal{C}$ such that $\gamma|\alpha$ passes through $\overline{\varepsilon}n$, and, therefore, there exists s in

Bin such that $\overline{\varepsilon}n \sqsubseteq \gamma|s$. We define ζ in $\mathcal{N}$ such that, for each n, $\zeta(n)$ is the least s such that s belongs to **Bin** and and $\overline{\varepsilon}n \sqsubseteq \gamma|s$.

Note that, for every n, for every p, if $p > \eta(n)$, then there is exactly one s in **Bin** such that $length(s) = n$ and $\gamma|s \sqsubseteq \overline{\varepsilon}p$. It follows that, for every n, for every p, if $p > \eta(n)$, then $\overline{(\zeta(p))}(n) = \overline{(\zeta(p+1))}(n)$. Let α be the element of $\mathcal{C}$ such that, for every n, $\alpha(n) = \zeta\big(\eta(n+1)\big)(n)$. Observe that α belongs to $\mathcal{C}$ and $\gamma|\alpha = \varepsilon$, so ε belongs to $Range(\gamma)$.

(iii) Let α belong to $\mathcal{N}$. Let β be an element of $\mathcal{N}$ such that, for every s, $\beta(s) = 0$ if and only if either α passes through s or $\underline{0}$ passes through s. Note that F_β coincides with the closure $\overline{\{\alpha, \underline{0}\}}$ of the set $\{\alpha, \underline{0}\}$. Also note that F_β is a spread.

Observe that, if either $\alpha \# \underline{0}$ or $\alpha = \underline{0}$, then F_β coincides with $\{\alpha, \underline{0}\}$.

Now assume that F_β coincides with the set $\{\alpha, \underline{0}\}$. Then, for every γ in the spread F_β, either $\gamma = \alpha$ or $\gamma = \underline{0}$. Applying Brouwer's Continuity Principle, we find m such that, for every γ in F_β, if $\overline{\gamma}m = \overline{\underline{0}}m$ then $\gamma = \underline{0}$. If $\overline{\alpha}m = \overline{\underline{0}}m$, then $\alpha = \underline{0}$, and, if not, then $\alpha \# \underline{0}$.

(iv) Note that, for every α in $\mathcal{N}$, there exists a continuous function from Cantor space $\mathcal{C}$ to $\mathcal{N}$ such that, for every β in $\mathcal{C}$, if $\beta(0) = 0$, then $\gamma|\beta = \alpha$ and, if $\beta(0) = 1$, then $\gamma|\beta = \underline{0}$, and, therefore, $Range(\gamma)$ coincides with the set $\{\alpha, \underline{0}\}$. Applying (iii), we find that it is not true that, for every function γ from $\mathcal{C}$ to $\mathcal{N}$, $Range(\gamma)$ is a closed subset of $\mathcal{N}$.

(v) Note that, for every γ in $\mathcal{N}$, γ is a function from Cantor space $\mathcal{C}$ to $\mathbb{N}$ if and only if for every α in $\mathcal{C}$, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$. Using the (Restricted) Fan Theorem, Axiom 12, we conclude that, for every γ in $\mathcal{N}$, γ is a function from Cantor space $\mathcal{C}$ to $\mathbb{N}$ if and only if there exist b, n such that (1) for every i, if $i < length(b)$, then $\gamma\big(b(i)\big) \neq 0$ and (2) for every s in **Bin**, if $length(s) = n$, then there exists $i < length(b)$ such that $b(i) \sqsubseteq s$. Thus we see that the set $\mathbb{N}^{\mathcal{C}}$, consisting all functions γ from $\mathcal{C}$ to $\mathbb{N}$, is an open subset of $\mathcal{N}$.

In order to prove that the set $\mathbb{N}^{\mathcal{C}}$ is a complete element of the class $\mathbf{\Sigma}^0_1$, we construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every s in **Bin**, $(\delta|\alpha)(s) = \alpha\big(length(s)\big)$. Clearly, γ reduces the set $E_1 = \{\alpha \in \mathcal{N} | \exists n[\alpha(n) \neq 0\}$ to the set $\mathbb{N}^{\mathcal{C}}$. As the set E_1 is a complete element of the class $\mathbf{\Sigma}^0_1$, also the set $\mathbb{N}^{\mathcal{C}}$ is a complete element of the class $\mathbf{\Sigma}^0_1$.

(vi) For every γ in $\mathcal{N}$, γ belongs to the set $\mathcal{N}^{\mathcal{C}}$ if and only if, for each n, γ^n belongs to the set $\mathbb{N}^{\mathcal{C}}$. It follows that the set $\mathcal{N}^{\mathcal{C}}$ is a countable intersection of open subsets of $\mathcal{N}$ and belongs to the class $\mathbf{\Pi}^0_2$. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set E_1 to the set $\mathbb{N}^{\mathcal{C}}$. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each n, $(\varepsilon|\alpha)^n = \delta|(\alpha^n)$. Clearly, ε reduces the set A_2 to the set $\mathcal{N}^{\mathcal{C}}$, and, therefore, the set $\mathcal{N}^{\mathcal{C}}$ is a complete element of the class $\mathbf{\Pi}^2_0$.

(vii) In our proof of (iii), we saw that, for every γ in the set $\mathcal{N}^{\mathcal{C}}$, γ is strongly injective if and only if, for each n, there exists p such that $p > n$ and, for all s in **Bin**, for all t in **Bin**, if $length(s) = length(t) = p$ and $\overline{s}n \neq \overline{t}n$, then $\gamma|s \perp \gamma|t$. As, for each p, the set of all s in **Bin** such that $length(s) = p$ is a finite set, and the set $\mathcal{N}^{\mathcal{C}}$ itself has been shown to belong to the class $\mathbf{\Pi}^0_2$, we conclude that the set of all γ in $\mathcal{N}^{\mathcal{C}}$ that are strongly injective also belongs to the class $\mathbf{\Pi}^0_2$.

Let A_2 be the set of all α in $\mathcal{N}$ with the property that, for each m, there exists n such that $\alpha^m(n) \neq 0$. The set A_2 is a complete element of the class $\mathbf{\Pi}^0_2$, and, therefore, a subset X of $\mathcal{N}$ that belongs to the class $\mathbf{\Pi}^0_2$ is a complete element of the class $\mathbf{\Pi}^0_2$ if and only if the set A_2 reduces to the set X.

Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ satisfying the condition that, for each α, for each m, for each s in **Bin**, $(\delta|\alpha)^m(s) = s(m)+1$ *if* $length(s) > m$ and there exists $p < length(s)$ such that $\alpha^m(p) \neq 0$, and $(\delta|\alpha)^m(s) = 0$ *if* one of these conditions fails to hold.

Note that, for every α in $\mathcal{N}$, if α belongs to the set A_2, then $\delta|\alpha$ will be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for every β in $\mathcal{C}$, $(\delta|\alpha)|\beta$ coincides with β, and, conversely, if $\delta|\alpha$ is a strongly injective function from $\mathcal{C}$ to $\mathcal{N}$, then for every β in $\mathcal{C}$, $(\delta|\alpha)|\beta$ coincides with β, and α belongs to A_2. It follows that δ reduces the set A_2 to the set of all strongly injective functions from $\mathcal{C}$ to $\mathcal{N}$. □

Let X be a subset of $\mathcal{N}$.

We call the set X *strongly (positively) uncountable* if and only if, for each α, there exists β in X such that, for all n, $\beta \# \alpha^n$.

We call the set X *weakly (positively) uncountable* if and only if X is inhabited and, for each α, if, for each n, α^n belongs to X, then there exists β in X such that, for all n, $\beta \# \alpha^n$.

For each β in $\mathcal{N}$ we have introduced F_β, the *closed set parametrized by* β as the set of all α in $\mathcal{N}$ such that, for every n, $\beta(\overline{\alpha}n) = 0$.

We now introduce, for each β in $\mathcal{N}$, the set A_β, the *analytic set paramatrized by* β as the set of all α such that, for some γ, for all n, $\beta(\langle\overline{\alpha}n, \overline{\gamma}n\rangle) = 0$. Note that A_β coincides with $Ex(F_\beta)$, the existential projection of the closed set F_β.

We let **UNC** the set of all β in $\mathcal{N}$ such that (i) β is a spread-law admitting at least one element of $\mathcal{N}$, that is: $\beta(\langle\ \rangle) = 0$ and, for each s, $\beta(s) = 0$ if and only, for some n, $\beta(s * \langle n\rangle) = 0$, and (ii) the set F_β, that is, the set consisting of all α in $\mathcal{N}$ satisfying, for all n, $\beta(\overline{\alpha}n) = 0$, is weakly positively uncountable.

We let **UNC′** the set of all β in $\mathcal{N}$ such that the (closed) set F_β, that is, the set consisting of all α in $\mathcal{N}$ satisfying, for all n, $\beta(\overline{\alpha}n) = 0$ is strongly positively uncountable.

We let **UNC″** the set of all β in $\mathcal{N}$ such that the (analytic) set A_β, that is, the set consisting of all α in $\mathcal{N}$ with the property that, for some γ in $\mathcal{N}$, for all n, $\beta(\langle\overline{\alpha}n, \overline{\gamma}n\rangle) = 0$, is strongly positively uncountable.

The second item in the following theorem has been proven in [10].

The classical result corresponding to the fourth item of the following theorem is due to W. Hurewicz, see [15], Theorem 27.5. The proof of this theorem in [15] is not constructive.

Theorem 2.4.

(i) *For every subset X of $\mathcal{N}$, if X is an inhabited spread, and X is weakly positively uncountable, then X is strongly positively uncountable.*

(ii) *For every subset X of $\mathcal{N}$, if X is strongly positively uncountable, then there exists a strongly injective function from $\mathcal{C}$ to X.*

(iii) *For every subset X of $\mathcal{N}$, if X has a subset that is a perfect spread, then X is strongly positively uncountable.*

(iv) *The sets* **UNC**, **UNC′** *and* **UNC″** *are complete analytic sets.*

Proof. (i) Let X be a subset of $\mathcal{N}$ that is an inhabited spread and let γ be (the code of) a continuous function from $\mathcal{N}$ to $\mathcal{N}$ with the property that, for all α, for all n, if $\overline{\alpha}n$ does contain an element of X, then $\overline{(\gamma|\alpha)}n = \overline{\alpha}n$, and, if $\overline{\alpha}n$ does not contain an element of X, then $(\gamma|\alpha)(n)$ is the least k such that $\overline{(\gamma|\alpha)}n * \langle k\rangle$ does contain an element of X. Note that γ *retracts* $\mathcal{N}$ onto X, that is, for every α in $\mathcal{N}$, $\gamma|\alpha$ belongs to X, and, for every α in X, $\gamma|\alpha = \alpha$. Suppose that X is weakly positively uncountable. Now let α belong to $\mathcal{N}$ and consider the sequence β satisfying, for each n, $\beta^n = \gamma|(\alpha^n)$. Note that, for each n, $\gamma|(\alpha^n)$ belongs to X and find δ in X such that, for each n, $\delta \# \beta^n$. Note that, for each n, either $\delta \# \alpha^n$ or $\gamma|(\alpha^n) = \beta^n \# \alpha^n$. In the latter case, there exists m such that $\overline{\alpha^n}m$ does not contain an element of X, and therefore, as $\overline{\delta}m$ does contain an element of X, also $\delta \# \alpha^n$. Thus we see that, given any α, we may determine δ in X such that, for each n, $\delta \# \alpha^n$, that is, X is strongly positively uncountable.

(ii) Let X be a strongly positively uncountable subset of $\mathcal{N}$. Using the Second Axiom of Continuous Choice we find γ in $\mathcal{N}$ such that γ is the code of a continuous function from $\mathcal{N}$ to $\mathcal{N}$ with the property: for all α in $\mathcal{N}$, $\gamma|\alpha$ belongs to X and, for all n, $\gamma|\alpha \# \alpha^n$. We now claim the following:

> For all s in $\mathbb{N}$ there exist t, u in $\mathbb{N}$ such that $s \sqsubseteq t$ and $s \sqsubseteq u$ and $(\gamma|s) \perp (\gamma|t)$.

We prove this claim as follows. Let δ be the infinite sequence $s * \underline{0}$ and let ε be an infinite sequence passing through s such that ε^k coincides with $\gamma|\delta$, where $k = \mathit{length}(s)$, and, for each n, if $n > \mathit{length}(s)$ and there is no i such that $n = J(k, i)$, then $\varepsilon(n) = 0$. Note that $\gamma|\varepsilon$ is apart from $\varepsilon^k = \gamma|\delta$. Find n such that $\overline{(\gamma|\delta)}n \neq \overline{(\gamma|\varepsilon)}n$ and then find m such that $\overline{\delta}n \sqsubseteq \gamma|(\overline{\delta}m)$ and $\overline{\varepsilon}n \sqsubseteq \gamma|(\overline{\varepsilon}m)$. Now define: $t := \overline{\delta}m$ and $u := \overline{\varepsilon}m$.

Recall that **Bin** be the set of all natural numbers s coding a finite binary sequence, that is, with the property: for each $i < \mathit{length}(s)$, $s(i) = 0$ or $s(i) = 1$.

We now define a mapping F from the set **Bin** to the set $\mathbb{N}$, as follows:

(i) $F(\langle\,\rangle) = \langle\,\rangle$.

(ii) Let s belong to **Bin** and assume $F(s)$ has been defined already. We search for the least natural number u such that $F(s) \sqsubseteq u(0)$ and $F(s) \sqsubseteq u(1)$ and $\big(\gamma|u(0)\big) \perp \big(\gamma|u(1)\big)$, and we define $F(s * \langle 0\rangle) = u(0)$ and $F(s * \langle 1\rangle) = u(1)$.

Note that, for each s in **Bin**, $F(s) \sqsubseteq F(s*\langle 0\rangle)$ and $F(s) \sqsubseteq F(s*\langle 1\rangle)$ and $F(s*\langle 0\rangle) \perp F(s*\langle 1\rangle)$. It follows that, for each s in **Bin**, $\mathit{length}\big(F(s)\big) \geq \mathit{length}(s)$. We let δ be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for each α in $\mathcal{C}$, for each n, $\delta|\alpha$ passes through $F(\overline{\alpha}n)$. We let ε be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for each α in $\varepsilon|\alpha = \gamma|(\delta|\alpha)$. Note that, for all α in $\mathcal{C}$, for all β in $\mathcal{C}$, if $\alpha \# \beta$, then, for some n, $\overline{\alpha}n \neq \overline{\beta}n$, and, therefore, $F(\overline{\alpha}n) \perp F(\overline{\beta}n)$, and, therefore, $\gamma|\big(F(\overline{\alpha}n)\big) \perp \gamma|\big(F(\overline{\beta}n)\big)$, and, therefore, $\varepsilon|\alpha \# \varepsilon|\beta$. Clearly, ε is a strongly injective function from $\mathcal{C}$ to X.

(iii) Assume that X has a subset Y that is a perfect spread. Find γ in $\mathcal{N}$ such that, for each s, $\gamma(s) = 0$ if and only if s contains an element of Y. Let α belong to $\mathcal{N}$. We have to find an element β of X such that, for each n, $\beta \# \alpha^n$. To this end, we define an element δ of $\mathcal{N}$, as follows. We let $\delta(0)$ be $u(0)$ where u is the least natural number u such that $u(0) \perp u(1)$ and $\gamma(u(0)) = \gamma(u(1)) = 0$ and α^0 does not pass through $u(0)$. For each n, we let $\delta(n+1)$ be $u(0)$ where u is the least natural number u such that $\delta(n) \sqsubseteq u(0)$ and $\delta(n) \sqsubseteq u(1)$ and $u(0) \perp u(1)$ and $\gamma(u(0)) = \gamma(u(1)) = 0$ and α^{n+1} does not pass through $u(0)$. Find β such that, for each n, β passes through $\delta(n)$ and observe that, for each n, $\beta \# \alpha^n$ and $\gamma(\overline{\beta}n) = 0$. It follows that β belongs to Y and, therefore, to X. Clearly, X is strongly positively uncountable.

(iv) Using Lemma 2.3, we conclude from (ii) and (iii) that the set **UNC** coincides with the set of all β such that, $\beta(\langle\,\rangle) = 0$ and $\forall s[\beta(s) = 0 \leftrightarrow \exists n[\beta(s * \langle n\rangle) = 0]]$ and $\exists\gamma[\forall n[\gamma(n) = 0 \rightarrow \beta(n) = 0] \wedge \forall s[\gamma(s) = 0 \rightarrow \exists t \exists u[s \sqsubseteq t \wedge s \sqsubseteq u \wedge t \perp u \wedge \gamma(t) = \gamma(u) = 0]]]$. The set **UNC** thus is seen to be an analytic subset of $\mathcal{N}$.

We now want to prove that the set E^1_1 reduces to the set **UNC**. It then follows that **UNC**, like E^1_1 itself, is a complete analytic set.

We have to define a function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing the set E^1_1 to the set **UNC**. Our idea is the following. Suppose α belongs to E^1_1. Find β such that α *admits* β, that is, for all n, $\alpha(\overline{\beta}n) = 0$. We intend to make sure that $\gamma|\alpha$ admits every infinite sequence δ with the property that, for each n, either $\delta(n) = 2\overline{\beta}(n) + 1$ or $\delta(n) = 2\overline{\beta}(n) + 2$.

In order to achieve this goal, we let γ be (the code of) a continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if, for all $i < \mathit{length}(s)$, either $s(i) = 0$ or $s(i) = 2t$ or $s(i) = 2t + 1$ where t is a natural number satisfying the following conditions: $\mathit{length}(t) = i + 1$ and for all $j < i$, $\alpha(\overline{t}(j + 1)) = 0$ and: either $s(j) = 2\overline{t}(j + 1) + 1$ or $s(j) = 2\overline{t}(j + 1) + 2$.

Note that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if $(\gamma|\alpha)(s * \langle 0\rangle) = 0$.

Also note that, for each α, for each s, for each n, if $(\gamma|\alpha)(s) = 0$ and $n < \mathit{length}(s) - 1$ and $s(n) = 0$, then $s(n + 1) = 0$.

We *claim* that, for each α, α belongs to E^1_1 if and only if $\gamma|\alpha$ belongs to **UNC**.

In order to prove this claim, let us first assume that α belongs to E^1_1. Find β such that α admits β. We define ε in $\mathcal{N}$ such that, for each s, $\varepsilon(s) = 0$ if and only if, for each n, if $n < \mathit{length}(s)$, then either $s(n) = \overline{\beta}(n + 1) + 1$, or $s(n) = \overline{\beta}(n + 1) + 2$. Note that F_ε is a perfect spread and a subset of the spread $F_{\gamma|\alpha}$, and thus, by (iii), the set $F_{\gamma|\alpha}$ is strongly positively uncountable, and, therefore, $\gamma|\alpha$ belongs to **UNC**.

Let us now assume that the spread $F_{\gamma|\alpha}$ is strongly positively uncountable. Using (ii), we find δ such that F_δ is a perfect spread and a subset of $F_{\gamma|\alpha}$. Note that, for each s, for each n, if $(\gamma|\alpha)(s) = 0$ and $n < \mathit{length}(s)$ and $s(n) = 0$, then, for all t, if $(\gamma|\alpha)(t) = 0$ and $s \sqsubseteq t$, then there exists k such that $t = s * \underline{0}k$. It follows that, for each s, for each n, if $\delta(s) = 0$ and $n < \mathit{length}(s)$, then $s(n) \neq 0$. Now find β such that, for each n, $\beta(n)$ is the least k such that $\delta(\overline{\beta}n * \langle k\rangle) = 0$. Let ε be an element of $\mathcal{N}$ such that, for each n, either $\beta(n) = 2\varepsilon(n) + 1$ or $\beta(n) = 2\varepsilon(n) + 2$. Note that, for each n, $\alpha(\varepsilon(n)) = 0$ and $\mathit{length}(\varepsilon(n)) = n + 1$ and $\varepsilon(n) \sqsubseteq \varepsilon(n + 1)$. Find η such that, for each n, η passes through $\beta(n)$, and note that, for each n, $\alpha(\overline{\eta}n) = 0$, so α belongs to E^1_1.

By (ii) and (iii), the set $\mathbf{UNC}'$ coincides with the set of all β such that $\exists\gamma[\forall s[\gamma(s) = 0 \rightarrow \forall t[t \sqsubseteq s \rightarrow \beta(t) = 0]] \wedge \forall s[\gamma(s) = 0 \rightarrow \exists t\exists u[s \sqsubseteq t \wedge s \sqsubseteq u \wedge t \perp u \wedge \gamma(t) = \gamma(u) = 0]]]$ and thus is seen to be an analytic subset of $\mathcal{N}$. Note the function γ we just defined in order to reduce the set E^1_1 to the set **UNC**, also reduces the set E^1_1 to the set $\mathbf{UNC}'$. We may conclude that the set $\mathbf{UNC}'$ is a complete analytic set.

Again, by (ii) and (iii), and the Second Axiom of Countable Choice, the set $\mathbf{UNC}''$ coincides with the set of all β such that $\exists\gamma\exists\delta[\forall n[\gamma(s) = 0 \rightarrow (\mathit{length}(s) = \mathit{length}\big(\delta(s)\big) \wedge \forall i \leq \mathit{length}(s)[\beta(\langle\overline{s}i, \overline{\delta(s)}i\rangle) = 0]] \wedge \forall s\forall i[\delta(s) \sqsubseteq \delta\big(s * \langle i\rangle\big)] \wedge \forall s[\gamma(s) = 0 \rightarrow \exists t\exists u[s \sqsubseteq t \wedge s \sqsubseteq u \wedge t \perp u \wedge \gamma(t) = \gamma(u) = 0]]]$ and thus is seen to be an analytic subset of $\mathcal{N}$.

In order to see that also $\mathbf{UNC}''$ is a complete analytic set, we remind ourselves of the fact that every closed subset of $\mathcal{N}$ is an analytic subset of $\mathcal{N}$. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all β, for all s, t, if $\mathit{length}(s) = \mathit{length}(t)$, then $(\delta|\beta)(\langle s, t\rangle) = 0$ if and only if $\beta(s) = 0$. Note that, for every β, for every α, for every γ, for all n, $(\delta|\beta)(\langle\overline{\alpha}n, \overline{\gamma}n\rangle) = 0$ if and only if $\beta(\overline{\alpha}n) = 0$. It follows that, for every β, for every α, α belongs to the analytic $A_{\delta|\beta}$ if and only if α belongs to the closed set F_β, that is, F_β coincides with $A_{\delta|\beta}$.

It now easily follows that, for every β, β belongs to $\mathbf{UNC}'$ if and only if $\delta|\beta$ belongs to $\mathbf{UNC}''$, that is, δ reduces the set $\mathbf{UNC}'$ to the set $\mathbf{UNC}''$. It follows that $\mathbf{UNC}''$, like $\mathbf{UNC}'$, is a complete analytic set. □

2.4. **The set of all β admitting an element of Inf.**

Let X be a subset of $\mathcal{N}$.

As in [36], we let $\mathit{Share}(X)$ be the set of all β in $\mathcal{N}$ such that the closed set F_β contains an element of X, that is, *shares* a member with X.

We let $\mathit{Share}^*(X)$ be the set of all β in $\mathcal{N}$ such that β is a spread-law and β belongs to $\mathit{Share}(X)$.

We let $\mathit{Share}^*_{01}(X)$ be the set of all β in $\mathcal{N}$ such that β is a spread-law defining a subfan of Cantor space $\mathcal{C}$ and β belongs to $\mathit{Share}(X)$. Note that β is a spread-law defining a subfan of Cantor space $\mathcal{C}$ if and only if, for each s, $\beta(s) = 0$ if and only if either $\beta(s * \langle 0 \rangle) = 0$ or $\beta(s * \langle 1 \rangle) = 0$ and, for all $i > 1$, $\beta(s * \langle i \rangle) \neq 0$.

We let **Inf** be the set of all α in Cantor space $\mathcal{C}$ such that, for each m, there exists n such that $n > m$ and $\alpha(n) = 1$. Note that, for all α in $\mathcal{C}$, α belongs to **Inf** if and only if α is the characteristic function of a decidable and positively infinite subset of $\mathbb{N}$.

The next result corresponds to a well-known fact in classical descriptive set theory, see [15], page 209, Exercise 27, or [28], page 137, Exercise 4.2.3.

Theorem 2.5. *The sets* $\mathit{Share}(\mathbf{Inf})$*,* $\mathit{Share}^*(\mathbf{Inf})$ *and* $\mathit{Share}^*_{01}(\mathbf{Inf})$*, are, each of them, complete analytic sets.*

Proof. The proof that the sets $\mathit{Share}(\mathbf{Inf})$, $\mathit{Share}^*(\mathbf{Inf})$ and $\mathit{Share}^*_{01}(\mathbf{Inf})$ are analytic subsets of $\mathcal{N}$ is easy and left to the reader.

We now show that the set E^1_1 reduces to the set $\mathit{Share}(\mathbf{Inf})$. It then follows from Theorem 2.2 that the latter set is a complete analytic set.

We let δ be an element of $\mathcal{N}$ such that $\delta(\langle\ \rangle) = \langle\ \rangle$ and, for each s, for each n, $\delta(s * \langle n \rangle) = \delta(s) * \underline{0}n * \langle 1 \rangle$.

We now let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if there exist t, n such that $s = \delta(t) * \underline{0}n$ and $\alpha(t) = 0$. We claim that the function γ reduces the set E^1_1 to the set $\mathit{Share}(\mathbf{Inf})$, and we prove this claim as follows.

First, let α belong to E^1_1. Find β such that, for all n, $\alpha(\overline{\beta}n) = 0$. Note that, for each n, for each t, if $t \sqsubset \delta(\overline{\beta}n)$, then $(\gamma|\alpha)(t) = 0$. Let ε be the element of $\mathcal{N}$ such that, for each n, ε passes through $\delta(\overline{\beta}n)$. Note that ε belongs to Cantor space $\mathcal{C}$ and that, for each n, $\varepsilon\big(n + \sum_{i=0}^{i=n} \beta(i)\big) = 1$, and that, for each n, $(\gamma|\alpha)(\overline{\beta}n) = 0$. It follows that $\gamma|\alpha$ belongs to $\mathit{Share}(\mathbf{Inf})$.

Conversely, assume that $\gamma|\alpha$ belongs to $\mathit{Share}(\mathbf{Inf})$. Find β in **Inf** such that, for each n, $(\gamma|\alpha)(\overline{\beta}n) = 0$. Define ε in $\mathcal{N}$ such that $\varepsilon(0) =$ the least i such that $\beta(i) = 1$ and, for each n, $\varepsilon(n+1) =$ the least i such that $\varepsilon(\beta(n) + i + 1) = 1$. Note that, for each n, β passes through $\delta(\overline{\varepsilon}n)$. It follows that, for each n, $\alpha(\overline{\varepsilon}n) = 0$, that is, α belongs to E^1_1.

This ends the proof of our claim.

Note that the function γ also reduces the set E^1_1 to the set $\mathit{Share}^*(\mathbf{Inf})$ and to the set $\mathit{Share}^*_{01}(\mathbf{Inf})$. It follows that also these sets are complete analytic sets. □

2.5. **The Suslin operation.** We want to conclude this Section with an observation concerning the famous Souslin operation.

A *Souslin system on* $\mathcal{N}$ is a mapping $s \mapsto P_s$ that associates to very s in $\mathbb{N}$ a subset P_s of $\mathcal{N}$.

The *Souslin operation* applied to such a system produces the set

$$\mathcal{A}_s P_s = \bigcup_{\alpha \in \mathcal{N}} \bigcap_{n \in \mathbb{N}} P_{\overline{\alpha}n}.$$

The following theorem theorem shows that, if one applies the Souslin operation to a Souslin system consisting of analytic sets, the result is again an analytic set.

Recall that, given any β in $\mathcal{N}$, we let A_β be the (analytic) subset of $\mathcal{N}$ consisting of all α such that, for some γ, for every n, $\beta(\langle \overline{\alpha}n, \overline{\gamma}n \rangle) = 0$.

Theorem 2.6. *For each* β *in* $\mathcal{N}$*, the set* $\mathcal{A}_s A_{\beta^s}$ *is an analytic subset of* $\mathcal{N}$

Proof. Note that, for each β, for each α, $\alpha \in \mathcal{A}_s A_{\beta^s}$ if and only if $\exists\gamma\forall n[\alpha \in A_{\beta^{\overline{\gamma}n}}$ if and only if $\exists\gamma\forall n\exists\delta[\langle\alpha, \delta\rangle \in F_{\beta^{\overline{\gamma}n}}]$ if and only if $\exists\gamma\exists\delta\forall n[\langle\alpha, \delta^n\rangle \in F_{\beta^{\overline{\gamma}n}}]$ if and only if $\exists\gamma\forall n[\langle\alpha, \gamma^{1,n}\rangle \in F_{\beta^{\overline{\gamma^0}n}}]$.

□

3. Some analytic subsets of $\mathcal{N}$ that fail to be positively Borel

3.1. **Introduction.** In classical descriptive set theory, one may prove, if one assumes that, for every subset X of $\mathcal{N}$ in the Boolean algebra generated by the analytic subsets of $\mathcal{N}$, the usual game for two players I, II in $\mathcal{N}$ with X as the payoff set is determined in the sense that either player I or player II has a winning strategy: *every analytic subset of $\mathcal{N}$ that is not a Borel subset of $\mathcal{N}$ is a complete element of the class $\mathbf{\Sigma}^1_1$*, see [15], Theorem 26.4. In intuitionistic mathematics, however, the set $\mathbf{MonPath}_{01}$, consisting of all α in $\mathcal{N}$ with the property that, for some γ in $\mathcal{N}$, for each n $\gamma(n) \le \gamma(n+1) \le 1$ and, for each n, $\alpha(\overline{\gamma}n) = 0$, is an example of a "simple" analytic set that is not a positively Borel subset of $\mathcal{N}$, see Theorem 1.12.

We now intend to fulfil a promise made in [41], Subsection 2.26, and partially fulfilled already in [32], Sections 11.21-26: we explain there exist analytic subsets of $\mathcal{N}$ that fail to be positively Borel belonging to many different degrees of reducibility.

3.2. **The set** *Share*(A_1) **is disjunctively productive.** Let α, β belong to $\mathcal{N}$. We define: *β admits α* if and only if α belongs to the closed set F_β, that is, for all n, $\beta(\overline{\alpha}n) = 0$. The set F_β is the set of all α in $\mathcal{N}$ that are admitted by β.

Let X be a subset of $\mathcal{N}$. As in the previous Section, we let *Share*(X) be the set of all β in $\mathcal{N}$ such that the closed set F_β contains an element of X, that is, *shares* a member with X.

Note that the set E^1_1 coincides with the the set *Share*$(\mathcal{N})$.

Note that the $\mathbf{MonPath}_{01}$ coincides with the set *Share*(S_{01mon}) where S_{01mon} is the set of all γ in $\mathcal{N}$ satisfying the condition: $\forall n[\gamma(n) \le \gamma(n+1) \le 1]$.

For every n, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let *the (disjoint) sum of the finite sequence* $(X_0, X_1, \ldots, X_{n-1})$, notation: $\bigoplus_{i<n} X_i$, be the set $\bigcup_{i<n} \langle i \rangle * X_i$.

For every n, for every subset X of $\mathcal{N}$, we let *the n-fold multiple of the set X*, notation: $n \cdot X$, be the set $\bigcup_{i<n} \langle i \rangle * X$.

For every n, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let *the disjunction of the finite sequence* $(X_0, X_1, \ldots, X_{n-1})$, notation: $D^{i=n-1}_{i=0}(X_i)$ or: $D_{i<n}(X_i)$, be the set of all α in $\mathcal{N}$ such that, for some $i < n$, α^i belongs to X_i.

For every n, for every subset X of $\mathcal{N}$, we let *the n-fold disjunction of the set X*, notation: $D^n(X)$, be the set of all α such that, for some $i < n$, α^i belongs to X.

For every n, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let *the conjunction of the finite sequence* $(X_0, X_1, \ldots, X_{n-1})$, notation:$C^{i=n-1}_{i=0}(X_i)$ or: $C_{i<n}(X_i)$, be the set of all α in $\mathcal{N}$ such that, for all $i < n$, α^i belongs to X_i.

For every n, for every subset X of $\mathcal{N}$, we let *the n-fold conjunction of the set X*, notation: $C^n(X)$, be the set of all α such that, for all $i < n$, α^i belongs to X.

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $(\gamma|\alpha)^0 = (\gamma|\alpha)^1 = \alpha^0$ and, for each $n > 0$, $(\gamma|\alpha)^n = \alpha^{n-1}$. Observe that, for each subset X of $\mathcal{N}$, for each $n > 0$, the function γ reduces the set $D^n(X)$ to the set $D^{n+1}(X)$ and the set $C^n(X)$ to the set $C^{n+1}(X)$.

For every subset X of $\mathcal{N}$ such that there exists α not belonging to X, the set $\emptyset = D^0(X)$ reduces to the set X and to $D^1(X)$. For every subset X of $\mathcal{N}$ such that there exists α in X, the set $\mathcal{N} = C^0(X)$ reduces to the set X and to $C^1(X)$.

A subset X of $\mathcal{N}$ satisfying the condition: for all n, $D^{n+1}(X)$ does not reduce to the set $D^n(X)$, is called *disjunctively productive*, see [32], Section 12.7, and [45], Subsection 6.3.3.

A subset X of $\mathcal{N}$ satisfying the condition: for all n, $C^{n+1}(X)$ does not reduce to the set $C^n(X)$, is called *conjunctively productive*, see [32], Section 12.7, and [45], Theorem 6.5(ii).

We let A_1 be the subset of $\mathcal{N}$ such that, for all α, α belongs to A_1 if and only if, for all n, $\alpha(n) = 0$. The set A_1 has the infinite sequence $\underline{0}$ as its one and only member. A subset X of $\mathcal{N}$ is closed if and only if X reduces to A_1.

A subset X of $\mathcal{N}$ reduces to the set $D^2(A_1)$ if and only if there exist closed subsets Y, Z of $\mathcal{N}$ such that $X = Y \cup Z$.

The set A_1 is disjunctively productive and the set $D^2(A_1)$ is both disjunctively and conjunctively productive as has been shown in [32] and [45].

For each α, for each m, for each n, we let $c(\alpha, m, n)$ be the number of elements of the set of all $j < m$ such that, for all $i < n$, $\alpha^j(\overline{\underline{0}}i) = 0$.

This notion will play a role in the proof of the sixth item of next theorem, and also in the statement and proofs of Theorem 3.4 and 3.5.

The seventh item of the next lemma states a result that is easily concluded from the fifth item and the well-known and just-mentioned fact that the set A_1 is disjunctively productive, see [32] and [45], but it seems useful, in view of further developments, to prove it in the way it is proven here.

Lemma 3.1.

(i) *For every subset X of $\mathcal{N}$, the set X reduces to the set Share(X).*
(ii) *For every subset X of $\mathcal{N}$, if X is a closed subset of $\mathcal{N}$, then Share(X) is an analytic subset of $\mathcal{N}$.*
(iii) *For every n, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, the set $D_{i<n}\big(Share(X_i)\big)$ coincides with the set $Share(\bigoplus_{i<n} X_i)$.*
(iv) *For every n, for every subset X of $\mathcal{N}$, the set $D^n\big(Share(X)\big)$ coincides with the set $Share(n \cdot X)$.*
(v) *The set Share(A_1) reduces to the set A_1 and the set A_1 reduces to the set Share(A_1).*
(vi) *For each m, for each γ, if γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^m\big(Share(A_1)\big)$ to itself, then, for each $k < m$, for each α:*
if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > 0$.
(vii) *The set Share(A_1) is disjunctively productive.*
(viii) *The set Share$(\{\underline{0}, \underline{1}\})$ does not reduce to the set $\{\underline{0}, \underline{1}\}$.*

Proof. (i) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if α passes through s. Note that, for each α, the set $F_{\gamma|\alpha}$ coincides with the set $\{\alpha\}$. Clearly, for each subset X of $\mathcal{N}$, the function γ reduces the set X to the set *Share*(X).

(ii) Find β such that X with F_β. Note that *Share*(X) coincides with the set of all α such that, for some γ in $\mathcal{N}$, for all n, $\beta(\overline{\gamma}n) = \alpha(\overline{\gamma}n) = 0$. Clearly, *Share*$(X)$ is an analytic subset of $\mathcal{N}$.

(iii) Let n belong to $\mathbb{N}$ and assume $X_0, X_1, \ldots, X_{n-1}$ are subsets of $\mathcal{N}$. Note that, for each α, α belongs to $D_{i<n}\big(Share(X_i)\big)$ if and only if, for some $i < n$, α^i belongs to *Share*(X_i) if and only if, for some $i < n$, for some β in X_i, for each n, $\alpha^i(\overline{\beta}n) = 0$ if and only if, for some $i < n$, for some β in X_i, for each n, $\alpha(\langle i \rangle * \overline{\beta}n) = 0$ if and only if α belongs to $Share(\bigoplus_{i<n} X_i)$.

(iv) This is an easy consequence of (iii).

(v) Clearly, the set *Share*(A_1) is a closed subset of $\mathcal{N}$ and, therefore, the set *Share*(A_1) reduces to the set A_1.

According to (i), the set A_1 also reduces to the set *Share*(A_1).

(vi) Let m be a natural number and suppose γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^m\big(Share(A_1)\big)$ to the set $D^m\big(Share(A_1)\big)$ itself. We claim that, for all α, for all $k < m$,

if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$.

We establish this claim by induction and first consider the case $k = 0$. Suppose: for each n, $c(\alpha, m, n) > 0$ and we find p such that $c(\gamma|\alpha, m, p) = 0$. Using the continuity of the function γ, we determine k such that, for every β passing through $\overline{\alpha}k$, for every $i < m$, for every $q \le p$, $(\gamma|\beta)^i(\overline{\underline{0}}q) = (\gamma|\alpha)^i(\overline{\underline{0}}q)$, and, therefore, $c(\gamma|\beta, m, p) = 0$. Find $i < m$ such that, for all $j \le k$, $\alpha^i(\overline{\underline{0}}j) = 0$. Observe that $(\overline{\alpha}k * \underline{0})^i$ admits $\underline{0}$. It follows that that $\overline{\alpha}k * \underline{0}$ belongs to the set $D^m\big(Share(A_1)\big)$, and $\gamma|(\overline{\alpha}k * \underline{0})$ does not, so we have a contradiction.

Now assume $k+1 < m$ and we have seen: for all α, that, if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$. Suppose: for each n, $c(\alpha, m, n) > k + 1$ and we find p such that $c(\gamma|\alpha, m, p) = k + 1$. Using the continuity of the function γ, we determine l such that, for every β passing through $\overline{\alpha}l$, for every $i < m$, for every $q \le p$, $(\gamma|\beta)^i(\overline{\underline{0}}q) = (\gamma|\alpha)^i(\overline{\underline{0}}q)$, and, therefore, $c(\gamma|\beta, m, p) = k+1$. Without loss of generality, one may assume that, for all $i \le k + 1$, for all $q \le l$, $\alpha^i(\overline{\underline{0}}q) = 0$, and, for all i, if $k + 1 \le i < m$, then there exists $q \le p$ such that $(\gamma|\alpha)^i(\overline{\underline{0}}q) \neq 0$. We now let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β,

(1) $\overline{(\delta|\beta)}l = \overline{\alpha}l$, and
(2) for every $i \le k + 1$, for every $q \le l$, $(\delta|\beta)^i(\overline{\underline{0}}q) = \alpha^i(\overline{\underline{0}}q) = 0$ and, for each n, $(\delta|\beta)^i(\overline{\underline{0}}(l + n + 1) = \beta^i(n)$, and
(3) for every i, if $k + 1 < i < m$, then, for all $n > l$, $(\delta|\beta)^i(n) = 1$.

Note that, for every β, $\delta|\beta$ belongs to $D^m\big(Share(A_1)\big)$ if and only if there exists $i \le k+1$ such that $(\delta|\beta)^i$ admits $\underline{0}$ if and only if there exists $i \le k+1$ such that $\beta^i = \underline{0}$.

Now let T be the set of all β in $\mathcal{C}$ such that, for all m, n, if $\beta(m) \neq 0$ and $\beta(n) \neq 0$, then $m = n$. The set T thus consists of all elements β of Cantor space $\mathcal{C}$ that assume the value 1 at most one time. Note that the set T is a spread.

Note that, for every β in T, for every n, the number of elements of the set $\{i \le k + 1 | \overline{\beta^i}n = \overline{\underline{0}}n\}$ is at least $k + 1$. It follows that, for every β in T, for every n, $c(\delta|\beta, m, n) \ge k + 1$. The induction hypothesis now enables us to conclude: for every β in T, for every n, $c\big(\gamma|(\delta|\beta), m, n\big) \ge k + 1$, and, therefore, for all $i \le k + 1$, $\big(\gamma|(\delta|\beta)\big)^i$ admits $\underline{0}$, and, therefore, $\gamma|(\delta|\beta)$ belongs to $D^m\big(Share(A_1)\big)$, and, therefore, $\delta|\beta$ belongs to $D^m\big(Share(A_1)\big)$, and, therefore, there exists $i \le k + 1$ such that $\beta^i = \underline{0}$.

Applying Brouwer's Continuity Principle, we find q, i such that, for every β in T, if β passes through $\overline{\underline{0}}q$, $\beta^i = \underline{0}$. This is false, of course, as there exists β in T passing through $\overline{\underline{0}}q$ such that, for some r, $\beta^i(r) = 1$.

We must conclude: for each α, if, for each n, $c(\alpha, m, n) > k + 1$, then, for each n, $c(\gamma|\alpha, m, n) > k + 1$.

This establishes our claim.

(vii) Let m be a natural number and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^{m+1}\big(Share(A_1)\big)$ to the set $D^m\big(Share(A_1)\big)\big)$.

We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each i, if $i < m$, then $(\delta|\alpha)^i = (\gamma|\alpha)^i$, and $(\delta|\alpha)^m = \underline{1}$. Note that the function δ reduces the set $D^{m+1}\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $D^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ as well as the function γ. The function δ, however, also reduces the set $D^{m+1}\big(Share(A_1)\big)$ to the set $D^{m+1}\big(Share(A_1)\big)$ itself. On the other hand, for each n, $c(\underline{0}, m + 1, n) = m + 1$, and, for each n, $c(\delta|\underline{0}, m, n) \le m$. This contradicts the conclusion of (v).

We must conclude that, for each m, the set $D^{m+1}\big(Share(A_1)\big)$ does not reduce to the set $D^m\big(Share(A_1)\big)$, that is, the set $Share(A_1)$ is disjunctively productive.

(viii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $(\gamma|\alpha)(\overline{\underline{0}}n) = 0$ if and only if $\overline{\alpha^0}n = \overline{\underline{0}}n$ and $(\gamma|\alpha)(\overline{\underline{1}}n) = 0$ if and only if $\overline{\alpha^1}n = \overline{\underline{0}}n$. Note that the function γ reduces the set $D^2(A_1)$ to the set *Share*$(\{\underline{0}, \underline{1}\})$. It follows that the set *Share*$(\{\underline{0}, \underline{1}\})$, like the set $D^2(A_1)$ itself, is not closed and does not reduce to the closed set $\{\underline{0}, \underline{1}\}$. □

3.3. **Preserving the property of disjunctive productivity.** Let n be a natural number and let $X_0, X_1, \ldots, X_{n-1}$ be subsets of $\mathcal{N}$. We let $S_{i<n}(X_i)$ be the set $\bigcup\limits_{i<n} \overline{\underline{0}}i * \langle 1 \rangle * X_i$.

Lemma 3.2.

(i) *For every n, for all subsets $X_0, X_1, \ldots, X_{n-1}$ of $\mathcal{N}$, the set Share$\big(S_{i<n}(X_i)\big)$ reduces to the set Share$(\bigoplus\limits_{i<n} X_i)$ and the set Share$(\bigoplus\limits_{i<n} X_i)$ reduces to the set Share$\big(S_{i<n}(X_i)\big)$.*

(ii) *For every n, for all subsets $X_0, X_1, \ldots, X_{n-1}$ of $\mathcal{N}$, the set Share$\big(S_{i<n}(X_i)\big)$ reduces to the set $D_{i<n}\big(Share(X_i)\big)$ and the set $D_{i<n}\big(Share(X_i)\big)$ reduces to the set Share$\big(S_{i<n}(X_i)\big)$.*

Proof. (i) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $(\gamma|\alpha)(\langle\ \rangle) = \alpha(\langle\ \rangle)$ and, for each i, $(\gamma|\alpha)(\langle i \rangle) = 0$ if and only if, for each $j \le i$, $\alpha(\overline{\underline{0}}j) = 0$, and, for each i, for each s, $(\gamma|\alpha)(\langle i \rangle * s) = \alpha(\overline{\underline{0}}i * \langle 1 \rangle * s)$. Clearly, γ reduces the set *Share*$\big(S_{i<n}(X_i)\big)$ reduces to the set *Share*$(\bigoplus\limits_{i<n} X_i)$.

Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $(\delta|\alpha)(\langle\ \rangle) = \alpha(\langle\ \rangle)$ and, for each i, $(\delta|\alpha)(\overline{\underline{0}}i) = 0$ if and only if $i < n$, and, for each i, for each s, $(\delta|\alpha)(\overline{\underline{0}}i * \langle 1 \rangle * s) = \alpha(\langle i \rangle * s)$. Clearly, δ reduces the set *Share*$(\bigoplus\limits_{i<n} X_i)$ to the set *Share*$\big(S_{i<n}(X_i)\big)$.

(ii) Use (i) and Lemma 2.1(ii).

□

Let $X_0, X_1, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$. We define two subsets of $\mathcal{N}$, $S^0_{i\in\mathbb{N}}(X_i)$ and $S^1_{i\in\mathbb{N}}(X_i)$, as follows:

(i) For each α, α belongs to $S^0_{i\in\mathbb{N}}(X_i)$ if and only if *either* $\alpha = \underline{0}$ *or* there exist n, β such that $\alpha = \overline{\underline{0}}n * \langle 1 \rangle * \beta$ and β belongs to X_n.

(ii) For each α, α belongs to $S^1_{i\in\mathbb{N}}(X_i)$ if and only if, *if* $\alpha \# \underline{0}$, *then* there exist n, β such that $\alpha = \overline{\underline{0}}n * \langle 1 \rangle * \beta$ and β belongs to X_n.

A subset X of $\mathcal{N}$ is called *inhabited* if and only if there exists α such that α belongs to X.

A subset X of $\mathcal{N}$ is called *enumerable* if and only if there exists α such that, for all β, β belongs to X if and only if, for some n, $\beta = \alpha^n$.

Lemma 3.3.

Let $X_0, X_1, \ldots$ be an infinite sequence of inhabited subsets of $\mathcal{N}$.

(i) *The set $S^0_{i\in\mathbb{N}}(X_i)$ is a subset of the set $S^1_{i\in\mathbb{N}}(X_i)$.*

(ii) *The set $S^1_{i\in\mathbb{N}}(X_i)$ is not a subset of the set $S^0_{i\in\mathbb{N}}(X_i)$.*

(iii) *If, for each n, X_n is a closed subset of $\mathcal{N}$, then $S^1_{i\in\mathbb{N}}(X_i)$ is a closed subset of $\mathcal{N}$.*

(iv) *If, for each n, X_n is a finitary spread, then $S^1_{i\in\mathbb{N}}(X_i)$ is a finitary spread.*

(v) *If, for each n, X_n is an enumerable subset of $\mathcal{N}$, then $S^0_{i\in\mathbb{N}}(X_i)$ is an enumerable subset of $\mathcal{N}$.*

Proof. Leaving the proof of the other items to the reader, we only prove (ii).

Using the Second Axiom of Countable Choice, we determine α such that, for each n, α^n belongs to X_n. We let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, if $\beta \# \underline{0}$, and n is the the least i such that $\beta(i) \neq 0$, then $\gamma|\beta = \overline{\underline{0}}n * \langle 1 \rangle * \alpha^n$. Note that $\gamma|\underline{0} = \underline{0}$ and that γ maps $\mathcal{N}$ into $S^1_{i\in\mathbb{N}}(X_i)$. Assume that, for every β, $\gamma|\beta$ belongs to $S^0_{i\in\mathbb{N}}(X_i)$, that is, for every β, *either* $\gamma|\beta = \underline{0}$, *or* there exist n such that $\gamma|\beta = \overline{\underline{0}}n * \langle 1 \rangle * \alpha^n$. Using Brouwer's Continuity Principle, we find m such that *either*, for every β, if $\overline{\beta}m = \overline{\underline{0}}m$, then $\gamma|\beta = \underline{0}$, *or*, for every β, if $\overline{\beta}m = \overline{\underline{0}}m$, then there exist n such that $\gamma|\beta = \overline{\underline{0}}n*\langle 1\rangle*\alpha^n$, and, therefore, *either*, for every β, if $\overline{\beta}m = \overline{\underline{0}}m$, then $\beta = \underline{0}$, *or*, for every β, if $\overline{\beta}m = \overline{\underline{0}}m$, then $\beta \# \underline{0}$. Both alternatives are absurd. □

We defined the following notion just before Lemma 3.1 and intend to use it again in the proof of the next theorem.

For each α, for each m, for each n, we let $c(\alpha, m, n)$ be the number of elements of the set of all $j < m$ such that, for all $i < n$, $\alpha^j(\overline{\underline{0}}i) = 0$.

Theorem 3.4.

Let $X_0, X_1, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$ such that, for each n, the set $Share(X_n)$ is disjunctively productive and the set $Share(X_n)$ reduces to the set $Share(X_{n+1})$.

(i) *For each n, the set $Share(X_n)$ reduces to the set $Share\big(S_{i<n+1}(X_i)\big)$.*

(ii) *For each $n > 0$, the set $Share\big(S_{i<n}(X_i)\big)$ reduces to the set $Share\big(S_{i<n+1}(X_i)\big)$ and, for both $j < 2$, the set $Share\big(S_{i<n}(X_i)\big)$ reduces to the set $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$.*

(iii) *For each m, for each p, the set $D^m\big(Share(S_{i<p}(X_i))\big)$ reduces to the set $Share\big(S_{i<m\cdot p}(X_i)\big)$.*

(iv) *For each $n > 0$, the set $D_{i<n+1}\big(Share(X_{n+i})\big)$ does not reduce to the set $D_{i<n}\big(Share(X_i)\big)$.*

(v) *For both $j < 2$, for each $n > 0$, the set $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ does not reduce to the set $Share\big(S_{i<n}(X_i)\big)$.*

(vi) *For both $j < 2$, for each m, for each γ, if γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $D^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ itself, then, for each $k < m$, for each α, if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$.*

(vii) *For both $j < 2$, the set $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ is disjunctively productive.*

(viii) *For both $j < 2$, the set $C\big(D^2(A_1), Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ does not reduce to the set $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$.*

Proof. (i) Let n be a natural number. Note that the set $Share(X_n)$ reduces to the set $D_{i<n+1}\big(Share(X_i)\big)$ and use Lemma 3.2(ii).

(ii) Let n be a positive natural number. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, if there exists $i < n$ such that either $s \sqsubseteq \overline{\underline{0}}i * \langle 1 \rangle$ or $\overline{\underline{0}}i * \langle 1 \rangle \sqsubseteq s$, then $(\gamma|\alpha)(s) = \alpha(s)$, and, if not, then $(\gamma|\alpha)(s) = 1$.

Clearly, γ reduces the set $Share\big(S_{i<n}(X_i)\big)$ to the set $Share\big(S_{i<n+1}(X_i)\big)$, and both to the set $Share\big(S^0_{i\in\mathbb{N}}(X_i)\big)$ and to the set $Share\big(S^1_{i\in\mathbb{N}}(X_i)\big)$.

(iii) Let m, p be natural numbers. Note that, by Lemmas 3.1 and 3.2 the set $D^m\big(Share(S_{i<p}(X_i))\big)$ reduces to the set $D^m\big(D_{i<p}(Share(X_i))\big)$. We let $Y_0, Y_1, \ldots, Y_{m\cdot p-1}$ be a finite sequence of subsets of $\mathcal{N}$ of length $m \cdot p$ such that, for each $j < m$, for each $i < p$, $Y_{j\cdot p+i}$ coincides with X_i. Note that the set $D^m\big(Share(S_{i<p}(X_i))\big)$ reduces to the set $D_{i<m\cdot p}\big(Share(Y_i)\big)$. As, for each n, the set $Share(X_n)$ reduces to the set $Share(X_{n+1})$, one may conclude that, for each $i < m \cdot p$, the set $Share(Y_i)$ reduces to the set $Share(X_i)$. It then follows that the set $D_{i<m\cdot p}\big(Share(Y_i)\big)$ reduces to the set $D_{i<m\cdot p}\big(Share(X_i)\big)$.

(iv) Let n be a positive natural number. Note that, for each $i < n$, the set $Share(X_i)$ reduces to the set $Share(X_{n-1})$. It follows that the set $D_{i<n}\big((Share(X_i)\big)$ reduces to the set $D^n\big(Share(X_{n-1})\big)$. Also observe that, for each $i \leq n$, the set $Share(X_{n-1})$ reduces to the set $Share(X_{n+i})$. It follows that the set $D^{n+1}\big(Share(X_{n-1})\big)$ reduces to

the set $D_{i<n+1}\big(\mathit{Share}(X_{n+i})\big)$. As the set $D^{n+1}\big(\mathit{Share}(X_{n-1})\big)$ does not reduce to the set $D^n\big(\mathit{Share}(X_{n-1})\big)$, it follows that the set $D_{i<n+1}\big(\mathit{Share}(X_{n+i})\big)$ does not reduce to the set $D_{i<n}\big((\mathit{Share}(X_i)\big)$.

(v) One verifies easily that, for each n, the set $D_{i<n+1}\big(\mathit{Share}(X_{n+i})\big)$ reduces to the set $\mathit{Share}\big(S_{i<2n+1}(X_i)\big)$, and therefore, by (ii), for both $j<2$, for each n, the set $D_{i<n+1}\big(\mathit{Share}(X_{n+i})\big)$ reduces to the set $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$. It now follows from (iii) that, for both $j<2$, for each n, the set $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ does not reduce to the set $\mathit{Share}\big(S_{i<n}(X_i)\big)$.

(vi) Let j,m be natural numbers such that $j<2$, and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to itself. We have to prove:

> for each $k<m$, for each α, if, for each n, $c(\alpha,m,n)>k$, then, for each n, $c(\gamma|\alpha,m,n)>k$.

Let us first consider the case $k=0$. Suppose α belongs to $\mathcal{N}$ and, for all n, $c(\alpha,m,n)>0$, and we find p such that $c(\gamma|\alpha,m,p)=0$. Using the continuity of the function γ, we determine k such that, for every β passing through $\overline{\alpha}k$, for every $j<m$, for every $q\le p$, $(\gamma|\beta)^j(\underline{\overline{0}}q)=(\gamma|\alpha)^i(\underline{\overline{0}}q)$, and, therefore, $c(\gamma|\beta,m,p)=0$. We define $l:=\max(m\cdot p,k)$. We then determine $i_0<m$ such that, for all $i\le 2l$, $\alpha^{i_0}(\underline{\overline{0}}i)=0$. It does not harm the generality of the argument to assume: $i_0=0$. We now construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β,

(1) $\overline{(\delta|\beta)}k=\overline{\alpha}k$, and
(2) for all i, if $i\le 2l$, then $(\delta|\beta)^0(\underline{\overline{0}}i)=\alpha^0(\underline{\overline{0}}i)=0$, and, for all i, if $i<l+1$, then $\underline{\overline{0}}^{(l+i)*\langle 1\rangle}\big((\delta|\beta)^0\big)=\beta^i$, and
(3) $(\delta|\beta)^0\big(\underline{\overline{0}}(2l+1)\big)=1$ and, for all i, if $i<l$, then, for all $t>l$, $(\delta|\beta)^0(\underline{\overline{0}}i*\langle 1\rangle*t)=1$, and $(\delta|\beta)^0\big(\underline{\overline{0}}(2l+1)\big)=1$, and
(4) for all $i>0$, for all $t>l$, $(\delta|\beta)^i(t)=1$.

Note that, for each β, β belongs to $D_{i<l+1}\big(\mathit{Share}(X_{l+i})\big)$ if and only if $(\delta|\beta)^0$ belongs to $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ if and only if $\delta|\beta$ belongs to $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$. As, for each β, $\overline{(\delta|\beta)}k=\overline{\alpha}k$, one may conclude: for each β, β belongs to $D_{i<l+1}\big(\mathit{Share}(X_{l+i})\big)$ if and only if $\delta|\beta$ belongs to $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ if and only if $\gamma|(\delta|\beta)$ belongs to $D^m\big(\mathit{Share}\big(S_{i<p}(X_i)\big)\big)$.

Recall that the set $D^m\big(\mathit{Share}\big(S_{i<p}(X_i)\big)\big)$ reduces to the set $\mathit{Share}\big(S_{i<m\cdot p}(X_i)\big)$ and that $m\cdot p\le l$. Using Lemma 2.2, we thus see that the set $D_{i<l+1}\big(\mathit{Share}(X_{l+i})\big)$ reduces to the set $D_{i<l}\big(\mathit{Share}(X_i)\big)$, and this contradicts (iv).

We must conclude: for each α, if, for each n, $c(\alpha,m,n)>0$, then, for each n, $c(\gamma|\alpha,m,n)>0$.

Let us now assume that $k+1<m$ and that we have proven:

> for each α, if, for each n, $c(\alpha,m,n)>k$, then, for each n, $c(\gamma|\alpha,m,n)>k$.

We intend to show:

> for each α, if, for each n, $c(\alpha,m,n)>k+1$, then, for each n, $c(\gamma|\alpha,m,n)>k+1$.

We do so as follows. Suppose α belongs to $\mathcal{N}$ and, for all n, $c(\alpha,m,n)>k+1$, and we find p such that $c(\gamma|\alpha,m,p)=k+1$. Using the continuity of the function γ, we determine l such that, for every β, if $\overline{\beta}l=\overline{\alpha}l$, then $c(\gamma|\beta,m,p)=c(\gamma|\alpha,m,p)$. Without loss of generality we may assume that, for all $i\le k+1$, for all $q\le l$, $\alpha^i(\underline{\overline{0}}q)=0$, and that, for all i such that $k+1\le i<m$, for every β passing through $\overline{\alpha}l$, there exists $q\le p$ such that $(\gamma|\beta)^i(\underline{\overline{0}}q)\neq 0$. We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β,

(1) $\overline{(\delta|\beta)}l=\overline{\alpha}l$, and

(2) for every $i \leq k+1$, for every $q \leq l$, $(\delta|\beta)^i(\overline{\underline{0}}q) = 0$ and, for each n, $(\delta|\beta)^i\big((\overline{\underline{0}}(q+n+1)\big) = \beta^i(n)$, and, for all s, if $s > l$ and $\underline{0}$ does not pass through s, then $(\delta|\beta)^i(s) = 1$, and
(3) for every i, if $k+1 < i < m$, then, for all $n > l$, $(\delta|\beta)^i(n) = 1$.

Note that, for every β, $\delta|\beta$ belongs to $D^m\big(\mathit{Share}(S^j_{i\in\mathbb{N}}(X_i))\big)$ if and only if there exists $i \leq k+1$ such that $(\delta|\beta)^i$ admits $\underline{0}$ if and only if there exists $i \leq k+1$ such that $\beta^i = \underline{0}$.

Now let T be the set of all β in $\mathcal{C}$ such that, for all m, n, if $\beta(m) \neq 0$ and $\beta(n) \neq 0$, then $m = n$. The set T consists of all elements β of Cantor space $\mathcal{C}$ that assume the value 1 at most one time. The set T is a spread.

Note that, for every β in T, for every n, the number of elements of the set $\{i \leq k+1 | \overline{\beta^i}n = \overline{\underline{0}}n\}$ is at least $k+1$. It follows that, for every β in k, for every n, $c(\delta|\beta, m, n) \geq k+1$. The induction hypothesis now enables us to conclude: for every β in T, for every n, $c\big(\gamma|(\delta|\beta), m, n\big) \geq k+1$, and, therefore, for all $i \leq k+1$, $\big(\gamma|(\delta|\beta)\big)^i$ admits $\underline{0}$, and, therefore, $\gamma|(\delta|\beta)$ belongs to $D^m\big(\mathit{Share}(S^j_{i\in\mathbb{N}}(X_i)\big)$, and, therefore, $\delta|\beta$ belongs to $D^m\big(\mathit{Share}(S^j_{i\in\mathbb{N}}(X_i)\big)$, and, therefore, there exists $i \leq k+1$ such that $\beta^i = \underline{0}$.

Applying Brouwer's Continuity Principle, we find q, i such that, for every β in T, if β passes through $\overline{\underline{0}}q$, $\beta^i = \underline{0}$. This is false, of course, as there exists β in T passing through $\overline{\underline{0}}q$ such that, for some r, $\beta^i(r) = 1$.

We must conclude: for each α, if, for each n, $c(\alpha, m, n) > k+1$, then, for each n, $c(\gamma|\alpha, m, n) > k+1$.

(vii) Let j, m be natural numbers such that $j < 2$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^{m+1}\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$.

We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each i, if $i < m$, then $(\delta|\alpha)^i = (\gamma|\alpha)^i$, and $(\delta|\alpha)^m = \underline{1}$. Note that the function δ reduces the set $D^{m+1}\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ as well as the function γ. The function δ, however, also reduces the set $D^{m+1}\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $D^{m+1}\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ itself. On the other hand, for each n, $c(\underline{0}, m+1, n) = m+1$, and, for each n, $c(\delta|\underline{0}, m, n) \leq m$. This contradicts the conclusion of (v).

We must conclude that, for each m, the set $D^{m+1}\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ does not reduce to the set $D^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, that is, the set $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ is disjunctively productive.

(viii) Let $j < 2$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C\big(D^2(A_1), \mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$. We claim that, for all α, if α^0 belongs to the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$, (that is, for each n, either $\overline{\alpha^{0,0}}n = \overline{\underline{0}}n$ or $\overline{\alpha^{0,1}}n = \overline{\underline{0}}n$), and α^1 admits $\underline{0}$, then $\gamma|\alpha$ admits $\underline{0}$.

We prove this claim as follows. Let α be an element of $\mathcal{N}$ such that α^0 belongs to $\overline{D^2(A_1)}$ and α^1 admits $\underline{0}$. Suppose we find a natural number n such that $(\gamma|\alpha)(\overline{\underline{0}}n) \neq 0$. Using the continuity of γ, find p such that, for every β, if β passes through $\overline{\alpha}p$, then $(\gamma|\beta)(\overline{\underline{0}}n) = (\gamma|\alpha)(\overline{\underline{0}}n)$. We define $q := \max(n, p)$. We know that, either, for each i, if $\langle 0, 0, i\rangle < q$, then $\alpha(\langle 0, 0, i\rangle) = 0$ or, for each i, if $\langle 0, 1, i\rangle < q$, then $\alpha(\langle 0, 1, i\rangle) = 0$, and without loss of generality, we may assume that the first of these two alternatives is true.

We now construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β,

(1) $\overline{(\delta|\beta)}p = \overline{\alpha}p$, and
(2) $(\delta|\beta)^{0,0} = \underline{0}$, and
(3) for each i, if $i < q$, then $(\delta|\beta)^1(\overline{\underline{0}}i) = 0$ and, for all s, if $s > q$, then $(\delta|\beta)^1(\overline{\underline{0}}i * \langle 1\rangle * s) = 1$, and
(4) for each i, if $q \leq i \leq 2q$, then $(\delta|\beta)^1(\overline{\underline{0}}i) = \beta(\overline{\underline{0}}i)$ and ${}^{\overline{\underline{0}}(i)*\langle 1\rangle}\big((\delta|\beta)^1\big) = \beta^i$, and
(5) $(\delta|\beta)^1\big(\overline{\underline{0}}(2q+1)\big) = 1$.

Note that, for each β, β belongs to $D_{i\le q}\big(Share((X_{q+i})\big)$ if and only if $(\delta|\beta)^1$ belongs to $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ if and only if $\delta|\beta$ belongs to $C\big(D^2(A_1), Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ if and only if $\gamma|(\delta|\beta)$ belongs to $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ if and only if $\gamma|(\delta|\beta)$ belongs to $Share\big(S_{i<q}(X_i)\big)\big)$. As, by Lemmas 2.1 and 2.2, $Share\big(S_{i<q}(X_i)\big)\big)$ reduces to the set the set $D_{i<q}\big(Share(X_i)\big)$. We thus see that the set $D_{i\le q}\big(Share((X_{q+i})\big)$ reduces to the set $D_{i<q}\big(Share(X_i)\big)$ and this conclusion contradicts the conclusion of (iii).

This establishes our claim that, for all α, if α^0 belongs to $\overline{D^2(A_1)}$ and α^1 admits $\underline{0}$, then $\gamma|\alpha$ admits $\underline{0}$.

We now construct a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, $(\varepsilon|\beta)^0 = \beta$, and $(\varepsilon|\beta)^1 = \underline{0}$.

Note that, for each β, if β belongs to $\overline{D^2(A_1)}$, then $(\varepsilon|\beta)^0$ belongs to $\overline{D^2(A_1)}$ and $(\varepsilon|\beta)^1$ admits $\underline{0}$, and, therefore, $\gamma|(\varepsilon|\beta)$ admits $\underline{0}$, so, $\gamma|(\varepsilon|\beta)$ belongs to $Share\big(S_{i\in\mathbb{N}}(X_i)\big)$, and, therefore, $\varepsilon|\beta$ belongs to $C\big(D^2(A_1), Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$, and, in particular, $(\varepsilon|\beta)^0$ belongs to $D^2(A_1)$, so β belongs to $D^2(A_1)$.

It follows that $\overline{D^2(A_1)}$ is a subset of $D^2(A_1)$ and this is false.

We must conclude that the set $C\big(D^2(A_1), Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$ does not reduce to the to the set $Share\big(S_{i\in\mathbb{N}}(X_i)\big)$. □

3.4. **Preserving the property of conjunctive productivity.** Let $Y_0, Y_1, \dots$ be an infinite sequence of subsets of $\mathcal{N}$. The sequence $Y_0, Y_1, \dots$ is called *conjunctively closed* if and only if, for each m, for each n, there exists p such that $C(Y_m, Y_n)$ reduces to Y_p.

Theorem 3.5.

Let $X_0, X_1, \dots$ be an infinite sequence of subsets of $\mathcal{N}$ such that $X_)$ is inhabited and, for each n, the set $Share(X_n)$ is disjunctively productive and the set $Share(X_n)$ reduces to the set $Share(X_{n+1})$ and the infinite sequence $Share(X_0), Share(X_1), Share(X_2), \dots$ is conjunctively closed.

(i) *For each m, for each n, there exists p such that the set $C\big(Share\big(S_{i<m}(X_i)\big), Share\big(S_{i<n}(X_i)\big)\big)$ reduces to the set $Share\big(S_{i<p}(X_i)\big)$.*
(ii) *For each m, for each γ, if γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$ to the set $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$ itself, then,*
for each $k < m$, for each α in $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$,
if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$.
(iii) *For each m, the set $C\big(D^2(A_1), C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)\big)$ does not reduce to the set $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$.*
(iv) *The set $Share\big(S_{i\in\mathbb{N}}(X_i)\big)$ is conjunctively productive.*

Proof. (i) Using Lemma 3.2(ii), observe that the set $C\big(Share\big(S_{i<m}(X_i)\big), Share\big(S_{i<n}(X_i)\big)\big)$ reduces to the set $C\big(D_{i<m}Share(X_i), D_{i<n}Share(X_i)\big)$. For each $i < m$, for each $j < n$, we determine a number $p(i,j)$ in $\mathbb{N}$ such that $C\big(Share(X_i), Share(X_j)\big)$ reduces to $Share(X_{p(i,j)})$. Using the fact that, for each n, the set $Share(X_n)$ reduces to the set $Share(X_{n+1})$, one may determine, for each $i < m$, for each $j < n$, a number $q(i,j)$ in $\mathbb{N}$ such that the set $C\big(Share(X_i), Share(X_j)\big)$ reduces to the set $Share(X_{q(i,j)})$, and, for each $i, i' < m$, for each $j, j' < n$, if either $i \neq i'$ or $j \neq j'$, then $q(i,j) \neq q(i',j')$. Let p be an element of $\mathbb{N}$ such that, for each $i < m$, for each $j < n$, $q(i,j) < p$. Note that the set $D_{i<n}\big(Share(X_i)\big)$ reduces to the set $D_{i<m}\big(D_{j<n}(Share(X_{p(i,j)})\big)$ and, therefore, also to the set $D_{i<p}\big(Share(X_i)\big)$ and, according to Lemma 3.2(ii), also to the set $Share\big(S_{i<p}(X_i)\big)$.

(ii) Let j, m be natural numbers such that $j < 2$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ to itself. We have to prove:

for each $k < m$, for each α in $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$, if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$.

Let us first consider the case $k = 0$. Suppose α belongs to $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$ and, for all n, $c(\alpha, m, n) > 0$, and we find p such that $c(\gamma|\alpha, m, p) = 0$. Using the continuity of the function γ, we determine k such that, for every β passing through $\overline{\alpha}k$, for every $i < m$, for every $q \leq p$, $(\gamma|\beta)^i(\underline{\overline{0}}q) = (\gamma|\alpha)^i(\underline{\overline{0}}q)$, and, therefore, $c(\gamma|\beta, m, p) = 0$. Using (i), we determine l such that the set $C^m\big(Share\big(S_{i<p}(X_i)\big)\big)$ reduces to the set $Share\big(S_{i<l}(X_i)\big)$. We define $q := \max(n, l)$. We then determine $i_0 < m$ such that, for all $j \leq q$, $\alpha^{i_0}(\underline{\overline{0}}j) = 0$. It does not harm the generality of the argument to assume: $i_0 = 0$. We now construct a function δ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β,

(1) $\overline{(\delta|\beta)}k = \overline{\alpha}k$, and
(2) for each i, if $i < q$, then $(\delta|\beta)^0(\underline{\overline{0}}i) = \alpha^0(\underline{\overline{0}}i) = 0$ and, for all s, if $s > q$, then $(\delta|\beta)^0(\underline{\overline{0}}i * \langle 1\rangle * s) = 1$, and
(3) for each i, if $i \leq q$, then $(\delta|\beta)^0\big((\underline{\overline{0}}(q+i)\big) = 0$, and $^{\underline{\overline{0}}(q+i)*\langle 1\rangle}\big((\delta|\beta)^0\big) = \beta^i$, and
(4) $(\delta|\beta)^0\big(\underline{\overline{0}}(2q+1)\big) = 1$, and
(5) for all $i > 0$, $(\delta|\beta)^i = \alpha^i$.

Note that, for each β, β belongs to $D_{i\leq q}\big(Share(X_{q+i})\big)$ if and only if $(\delta|\beta)^0$ belongs to $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ if and only if, (here we are using the fact that α belongs to $C^m\big(Share\big(S_{i<p}(X_i)\big)\big)$), $\delta|\beta$ belongs to $C^m\big(Share\big(S_{i\in\mathbb{N}}(X_i)\big)\big)$. As, for each β, $\overline{(\delta|\beta)}k = \overline{\alpha}k$, one may conclude: for each β, β belongs to $D_{i\leq q}\big(Share(X_{q+i})\big)$ if and only if $\gamma|(\delta|\beta)$ belongs to $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ if and only if $\gamma|(\delta|\beta)$ belongs to $C^m\big(Share\big(S_{i<p}(X_i)\big)\big)$.

By our choice of q, It follows that the set $D_{i<q+1}\big(Share(X_{q+i})\big)$ reduces to the set $Share\big(S_{i<q}(X_i)\big)$ and this contradicts Theorem 3.4(iii).

Let us now assume that $k + 1 < m$ and that we have proven:

for each α in $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, if, for each n, $c(\alpha, m, n) > k$, then, for each n, $c(\gamma|\alpha, m, n) > k$.

We intend to show:

for each α in $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, if, for each n, $c(\alpha, m, n) > k + 1$, then, for each n, $c(\gamma|\alpha, m, n) > k + 1$.

We do so as follows. Suppose α belongs to $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ and, for all n, $c(\alpha, m, n) > k + 1$, and we find p such that $c(\gamma|\alpha, m, p) = k$. Using the continuity of the function γ, we determine l such that, for every β, if $\overline{\beta}l = \overline{\alpha}l$, then $c(\gamma|\beta, m, p) = c(\gamma|\alpha, m, p)$. Without loss of generality we may assume that, for all $i \leq k + 1$, for all $r \leq l$, $\alpha^i(\underline{\overline{0}}r) = 0$, and that, for all i such that $k + 1 \leq i < m$, for every β passing through $\overline{\alpha}l$, there exists $r \leq p$ such that $(\gamma|\beta)^i(\underline{\overline{0}}r) \neq 0$. Using (i), we determine t such that the set $C^{m-k-1}\big(Share\big(S_{i<p}(X_i)\big)\big)$ reduces to the set $Share\big(S_{i<t}(X_i)\big)$ We define $q := \max(l, p, t)$. We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β,

(1) $\overline{(\delta|\beta)}l = \overline{\alpha}l$, and
(2) for every i, if $i \leq k$, then $(\delta|\beta)^i$ admits $\underline{0}$, and
(3) for every i, if $i < q$, then $(\delta|\beta)^{k+1}(\underline{\overline{0}}i) = \alpha^{k+1}(\underline{\overline{0}}i) = 0$, and, for all s, if $s > q$, then $(\delta|\beta)^{k+1}(\underline{\overline{0}}i * \langle 1\rangle * s) = 1$, and
(4) for every i, if $i \leq q$, then $(\delta|\beta)^{k+1}\big(\underline{\overline{0}}(q+i)\big) = 0$, and $^{\underline{\overline{0}}(q+i)*\langle 1\rangle}\big((\delta|\beta)^{k+1}\big) = \beta^i$, and
(5) $(\delta|\beta)^{k+1}(\underline{\overline{0}}(2q+1) = 1$, and
(6) for every i, if $k + 1 < i < m$, then $(\delta|\beta)^i = \alpha^i$.

First observe that, for each β, β belongs to $D_{i\leq q}\big(Share(X_{q+i})\big)$ if and only if $(\delta|\beta)^{k+1}$ belongs to $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ if and only if $\delta|\beta$ belongs to $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$.

Then note that, for each β, $\delta|\beta$ passes through $\overline{\alpha}l$, and therefore, if β belongs to $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$, then, for each i, if $k+1 \leq i < m$, then $\big((\gamma|(\delta|\beta)\big)^i$ belongs to $\mathit{Share}\big(S^j_{r\in\mathbb{N}}(X_r)\big)$ and to $\mathit{Share}\big(S_{r<p}(X_r)\big)$.

Next observe that, for each β, for each n, $c(\delta|\beta, m, n) > k$, and therefore, by the induction hypothesis, if β belongs to $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$, then, for each n, $c(\gamma|(\delta|\beta), m, n) > k$, and, therefore, for each i, if $i < k+1$, then $\big(\gamma|(\delta|\beta)\big)^i$ admits $\underline{0}$.

Also note that, for each β, if for each i, if $i < k+1$, then $\big(\gamma|(\delta|\beta)\big)^i$ admits $\underline{0}$, and if $k+1 \leq i < m$, then $\big((\gamma|(\delta|\beta)\big)^i$ belongs to $\mathit{Share}\big(S^j_{r\in\mathbb{N}}(X_r)\big)$, then $\gamma|(\delta|\beta)$ belongs to $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, and, therefore, $\delta|\beta$ itself belongs to $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ and β belongs to $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$.

Summing up, we see that, for each β, β belongs to $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$ if and only if for each i, if $k+1 \leq i < m$, then $\big(\gamma|(\delta|\beta)\big)^i$ belongs to $\mathit{Share}\big(S^j_{r\in\mathbb{N}}(X_r)\big)$ if and only if, for each i, if $i < k+1$, then $\big(\gamma|(\delta|\beta)\big)^i$ admits $\underline{0}$, and, if $k+1 \leq i < m$, then $\big((\gamma|(\delta|\beta)\big)^i$ belongs to to $\mathit{Share}\big(S_{r<p}(X_r)\big)$.

We thus see that the set $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$ reduces to the set $C\big(C^{k+1}(\mathit{Share}(A_1), \mathit{Share}\big(S_{r<p}(X_r)\big))\big)$.

It now follows from our definition of the numbers t and q that the set $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$ reduces to the set
$C\big(C^{k+1}(\mathit{Share}(A_1), \mathit{Share}\big(S_{i<q}(X_i)\big))\big)$.

We now need the observation that, for every subset Y of $\mathcal{N}$, the set $C(\mathit{Share}(A_1), \mathit{Share}(Y))$ reduces to the set $\mathit{Share}(Y)$.

(Define a function η from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\eta|\alpha)(s) = \max\big(\alpha^0(s), \alpha^1(s)\big)$. For every subset Y of $\mathcal{N}$,
the function η reduces the set $C(\mathit{Share}(A_1), \mathit{Share}(Y))$ to the set $\mathit{Share}(Y)$.)

It follows that the set $D_{i\leq q}\big(\mathit{Share}(X_{q+i})\big)$ reduces to the set $\mathit{Share}\big(S_{i<q}(X_i)\big)$, and this, together with Lemmas 2.1 and 2.2, contradicts Theorem 2.4(iii).

We must conclude: for each α in $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, if, for each n, $c(\alpha, m, n) > k+1$, then, for each n, $c(\gamma|\alpha, m, n) > k+1$.

(iii) Let j, m be natural numbers such that $j < 2$ and and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C\big(D^2(A_1), C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)\big)$ to the set $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$. Note that, for each α, if α^0 belongs to $D^2(A_1)$, then: α^1 belongs to $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ if and only if $\gamma|\alpha$ belongs to $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$. It now follows from (ii) that, for each α, if α^0 belongs to $D^2(A_1)$ and, for each $i < m$, $\alpha^{1,i}$ admits $\underline{0}$, then, for each $i < m$, $(\gamma|\alpha)^i$ admits $\underline{0}$. Using the continuity of the function γ, we may conclude that, for each α, if α^0 belongs to the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$, and, for each $i < m$, $\alpha^{1,i}$ admits $\underline{0}$, then, for each $i < m$, $(\gamma|\alpha)^i$ admits $\underline{0}$. We now let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, $(\delta|\beta)^0 = \beta$ and $(\delta|\beta)^1 = \underline{0}$. Note that, for each β, if β belongs to $\overline{D^2(A_1)}$, then $(\delta|\beta)^0$ does so, and, in addition, for each $i < m$, $(\delta|\beta)^{1,i}$ admits $\underline{0}$, and, therefore, $\gamma|(\delta|\beta)$ belongs to $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$, and $\delta|\beta$ itself belongs to $C\big(D^2(A_1), C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)\big)$, and $(\delta|\beta)^0 = \beta$ belongs to $D^2(A_1)$. We thus see that $\overline{D^2(A_1)}$ is a subset of $D^2(A_1)$, and this is false.

We must conclude that, for each m, the set $C\big(D^2(A_1), C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)\big)$ does not reduce to the set $C^m\big(\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$.

(iv) As the set X_0 is inhabited we may determine an element δ of X_0. We now prove that, for both $j < 2$ the set $D^2(A_1)$ reduces to the set $\mathit{Share}\big(S^j_{i\in\mathbb{N}}(X_i)\big)$. To this end, we construct a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if either $\langle 1\rangle * \delta$ passes through s, and, for each $i < \mathit{length}(s)$, $\alpha^0(i) = 0$, or $\underline{0}$ passes through s, and, for each $i < \mathit{length}(s)$, $\alpha^1(i) = 0$. One easily verifies that γ reduces the set $D^2(A_1)$ to the set $\mathit{Share}\big(S_{i\in\mathbb{N}}(X_i)\big)$.

It now easily follows from (iii) that, for both $j < 2$, for each m,
the set $C^{m+1}\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$ does not reduce to the set $C^m\big(Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)\big)$. □

Lemma 3.6.

Let $X_0, X_1, \dots$ and $Y_0, Y_1, \dots$ be infinite sequences of subsets of $\mathcal{N}$ such that, for each n, the set $Share(X_n)$ reduces to the set $Share(X_{n+1})$ and the set $Share(Y_n)$ reduces to the set $Share(Y_{n+1})$.

If, for each m, there exists n such that the set $Share(X_m)$ reduces to the set $Share(Y_n)$, then, for both $j < 2$, the set $Share\big(S^j_{i\in\mathbb{N}}(X_i)\big)$ reduces to the set $Share\big(S^j_{i\in\mathbb{N}}(Y_i)\big)$.

Proof. Let $X_0, X_1, \dots$ and $Y_0, Y_1, \dots$ be infinite sequences of subsets of $\mathcal{N}$ satisfying the conditions of the theorem. Using the First Axiom of Countable Choice, we determine δ in $\mathcal{N}$ such that, for each n, $\delta(n) < \delta(n+1)$ and the set $Share(X_n)$ reduces to the set $Share(Y_{\delta(n)})$. Using the Second Axiom of Countable Choice, we then determine ε in $\mathcal{N}$ such that, for each n, ε^n is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $Share(X_n)$ to the set $Share(Y_{\delta(n)})$.

We then define a function ζ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, the following conditions are satisfied:

(1) for each n, if n is the least i such that $\alpha(\overline{0}i) \neq 0$, then, $\delta(n)$ is the least i such that $(\zeta|\alpha)(\overline{0}i) \neq 0$, and,

(2) for each k, if there is no n such that $\delta(n) = k$, then $(\zeta|\alpha)(\overline{0}k * \langle 1\rangle) = 1$, and,

(3) for each n, if, for all $i \le n$, $\alpha(\overline{0}i) = 0$, then $^{\overline{0}(\delta(n))*\langle 1\rangle}(\zeta|\alpha) = \varepsilon^n|\big(^{\overline{0}n*\langle 1\rangle}\alpha\big)$.

We first prove that the function ζ reduces the set $Share\big(S^0_{i\in\mathbb{N}}(X_i)\big)$
to the set $Share\big(S^0_{i\in\mathbb{N}}(Y_i)\big)$.

Suppose α belongs to $Share\big(S^0_{i\in\mathbb{N}}(X_i)\big)$. Find λ in $S^0_{i\in\mathbb{N}}(X_i)$ such that α admits λ and distinguish two cases.

Case (a): $\lambda = \underline{0}$. Note that also $\zeta|\alpha$ admits $\underline{0}$ and $\underline{0}$ belongs to $S^0_{i\in\mathbb{N}}(Y_i)$. Therefore, $\zeta|\alpha$ belongs to $Share\big(S^0_{i\in\mathbb{N}}(Y_i)\big)$.

Case (b): $\lambda \# \underline{0}$. Find n, η such that $\lambda = \overline{0}n * \langle 1\rangle * \eta$ and η belongs to X_n. Conclude that $^{\overline{0}n*\langle 1\rangle}\alpha$ belongs to $Share(X_n)$, and therefore, $^{\overline{0}(\delta(n))*\langle 1\rangle}(\zeta|\alpha) = \varepsilon^n|\big(^{\overline{0}n*\langle 1\rangle}\alpha\big)$ belongs to $Share(Y_{\delta(n)})$. Find θ in $Y_{\delta(n)}$ such that $^{\overline{0}(\delta(n))*\langle 1\rangle}(\zeta|\alpha)$ admits θ and note: $\zeta|\alpha$ admits $\overline{0}\big(\delta(n)\big) * \langle 1\rangle * \theta$ and $\overline{0}\big(\delta(n)\big) * \langle 1\rangle * \theta$ belongs to $S^0_{i\in\mathbb{N}}(Y_i)$. Therefore, $\zeta|\alpha$ belongs to $Share\big(S^0_{i\in\mathbb{N}}(Y_i)\big)$.

Conversely, suppose $\zeta|\alpha$ belongs to $Share\big(S^0_{i\in\mathbb{N}}(Y_i)\big)$. Find λ in $S^0_{i\in\mathbb{N}}(X_i)$ such that $\zeta|\alpha$ admits λ and distinguish two cases.

Case (a): $\lambda = \underline{0}$. Note that also α admits $\underline{0}$ and $\underline{0}$ belongs to $S^0_{i\in\mathbb{N}}(X_i)$. Therefore, α belongs to $Share\big(S^0_{i\in\mathbb{N}}(X_i)\big)$.

Case (b): $\lambda \# \underline{0}$. Find n, η such that $\lambda = \overline{0}n * \langle 1\rangle * \eta$ and η belongs to Y_n. Conclude that $^{\overline{0}n*\langle 1\rangle}(\zeta|\alpha)$ belongs to $Share(Y_n)$. Find k such that $\delta(k) = n$ and note: $^{\overline{0}n*\langle 1\rangle}(\zeta|\alpha) = \varepsilon^k|\big(^{\overline{0}k*\langle 1\rangle}\alpha\big)$ belongs to $Share(Y_{\delta(n)})$, and, therefore, $^{\overline{0}k*\langle 1\rangle}\alpha$ belongs to $Share(X_k)$. Find θ in X_k such that $^{\overline{0}k*\langle 1\rangle}(\alpha)$ admits θ and note: α admits $\overline{0}k * \langle 1\rangle * \theta$ and $\overline{0}k * \langle 1\rangle * \theta$ belongs to $S^0_{i\in\mathbb{N}}(X_i)$. Therefore, α belongs to $Share\big(S^0_{i\in\mathbb{N}}(X_i)\big)$.

We now prove that the function ζ reduces the set $Share\big(S^1_{i\in\mathbb{N}}(X_i)\big)$
to the set $Share\big(S^1_{i\in\mathbb{N}}(Y_i)\big)$.

Suppose α belongs to $Share\big(S^1_{i\in\mathbb{N}}(X_i)\big)$. Find λ in $S^1_{i\in\mathbb{N}}(X_i)$ such that α admits λ. Note that, if $\lambda \# \underline{0}$, then there exist n, η such that $\lambda = \overline{0}n * \langle 1\rangle * \eta$ and η belongs to X_n. We now define an element μ of $\mathcal{N}$, as follows, step by step. For each n, if $\overline{\lambda}n = \overline{0}n$ then $\overline{\mu}\big(\delta(n)\big) = \overline{0}\big(\delta(n)\big)$. As soon as we discover n such that n is the least i such that $\lambda(i) \neq 0$, then, observing that $^{\overline{0}n*\langle 1\rangle}\alpha$ belongs to $Share(X_n)$, we conclude that

$\overline{0}\big(\delta(n)\big)*\langle 1\rangle(\zeta|\alpha) = \varepsilon^n|\big(\overline{0}n*\langle 1\rangle\alpha\big)$ belongs to $Share(Y_n)$ and we determine κ in $Y_{\delta(n)}$ such that $\overline{0}\big(\delta(n)\big)*\langle 1\rangle(\zeta|\alpha)$ admits κ, and we define: $\mu := \underline{0}\big(\delta(n)\big) * \langle 1\rangle * \kappa$. Clearly, μ belongs to $S^1_{i\in\mathbb{N}}(Y_i)$ and is admitted by $\zeta|\alpha$: therefore, $\zeta|\alpha$ belongs to $Share\big(S^1_{i\in\mathbb{N}}(Y_i)\big)$.

Now assume $\zeta|\alpha$ belongs to $Share\big(S^1_{i\in\mathbb{N}}(Y_i)\big)$. Find λ in $S^1_{i\in\mathbb{N}}(X_i)$ such that $\zeta|\alpha$ admits λ. We define an element μ of $\mathcal{N}$, as follows, step by step. For each n, if $\overline{\lambda}n = \overline{0}n$, then, for each k, if $n < \delta(k)$, then $\overline{\mu}k = \overline{0}k$. As soon as we discover n such that n is the least i such that $\lambda(i) \neq 0$, we determine k such that $n = \delta(k)$ and, observing that $\overline{0}n*\langle 1\rangle(\zeta|\alpha)$ belongs to $Share(Y_n)$ and that $\overline{0}n*\langle 1\rangle(\zeta|\alpha) = \varepsilon^k|\big(\overline{0}k*\langle 1\rangle\alpha\big)$, we conclude that $\overline{0}k*\langle 1\rangle\alpha$ belongs to $Share(X_n)$ and we determine κ in X_k such that $\overline{0}\big(\delta(n)\big)*\langle 1\rangle(\zeta|\alpha)$ admits κ, and we define: $\mu := \overline{0}k*\langle 1\rangle*\kappa$. Clearly, μ belongs to $S^1_{i\in\mathbb{N}}(X_i)$ and is admitted by α: therefore, α belongs to $Share\big(S^1_{i\in\mathbb{N}}(X_i)\big)$. □

3.5. **Building two hierarchies using hereditarily increasing stumps.** Theorem 3.4 and Lemma 3.6 enable us to build two hierarchies, the first one consisting of sets that belong to the class $\mathbf{\Sigma}^0_2$, and the second one consisting of analytic sets that fail to be positively Borel.

Slightly adapting a definition given in [46], Section 3, and mentioned in the introductory Section of this paper, just before Theorem 1.7, we introduce, for every stump σ, a subset CB^*_σ of $\mathcal{N}$ by means of the following inductive definition:

(i) $CB^*_{\underline{0}} := \{\underline{0}\}$.
(ii) For every nonempty stump σ, $CB^*_\sigma := S^0_{n\in\mathbb{N}}(CB^*_{\sigma^n}) = \{\underline{0}\} \cup \bigcup\limits_{n\in\mathbb{N}} \overline{0}n*\langle 1\rangle*CB^*_{\sigma^n}$.

The sets CB^*_σ are called *special Cantor-Bendixson-sets*. Note that every special Cantor-Bendixson-set is a Cantor-Bendixson-set in the sense of the earlier definition. It follows from [46], Theorem 3.3 that, for each stump σ, if CB^*_σ is an infinite set, then CB^*_σ is a proper subset of its sequential closure $\overline{CB^*_\sigma}$. Note that, for every stump σ, if σ is non-empty, then CB^*_σ is infinite and, therefore, not a closed subset of $\mathcal{N}$.

We find it useful to introduce a special kind of stumps, the so-called *hereditarily increasing stumps*.

First, we define, for all α, β in $\mathcal{C}$: *α is a subset of β*, notation: $\alpha \subseteq \beta$, if and only if, for each s, if $\alpha(s) = 1$, then $\beta(s) = 1$.

We now introduce the set **His** of the hereditarily increasing stumps by the following inductive definition.

(i) $\underline{0}$ is an element of **His**.
(ii) For every non-empty stump σ, σ belongs to **His** if and only if, for each n, σ^n belongs to **His**, and, for each n, σ^n is a subset of σ^{n+1}.

For all α, β in Cantor space $\mathcal{C}$, we let $\alpha \cup \beta$, the *union of α and β*, be the element of Cantor space satisfying: for all n, $\alpha \cup \beta(n) = 1$ if and only if $\alpha(n) = \beta(n) = 1$.

For all s, t in $\mathbb{N}$, we define: $s \leq^* t$ if and only if $length(s) = length(t)$ and, for all $i < length(s)$, $s(i) \leq t(i)$.

For every stump σ we let $\mathbb{H}(\sigma)$ be the element of Cantor space $\mathcal{C}$ satisfying: for each t, $\big(\mathbb{H}(\sigma)\big)(t) = 1$ if and only if there exists s such that $s \leq^* t$ and $\sigma(s) = 1$.

Lemma 3.7.

(i) *For every stump σ, σ is a subset of $\mathbb{H}(\sigma)$.*
(ii) *] For all stumps σ, σ is a hereditarily increasing stump if and only if for each s, for each t, if $\sigma(s) = 1$ and $s \leq^* t$, then $\sigma(t) = 1$.*
(iii) *For all stumps σ, τ, the infinite sequence $\sigma \cup \tau$ is again a stump, and, if both σ and τ are hereditarily increasing, then also $\sigma \cup \tau$ is hereditarily increasing.*

(iv) *For every stump σ, $\mathbb{H}(\sigma)$ is a hereditarily increasing stump.*
(v) *For every stump σ, for every hereditarily increasing stump τ, if σ is a subset of τ, then also $\mathbb{H}(\sigma)$ is a subset of τ.*
(vi) *For every non-empty stump σ, the set $\overline{CB^*_\sigma}$ coincides with the set $S^1_{n\in\mathbb{N}}(\overline{CB^*_{\sigma^n}})$.*

Proof. (i) This is obvious.

(ii) We first prove: for every hereditarily increasing stump σ, for each s, for each t, is $\sigma(s) = 1$ and $s \leq^* t$, then $\sigma(t) = 1$. The proof is by induction on the set of stumps.

First, note that the statement is trivially true if $\sigma = \underline{0}$.

Now assume that that σ is a non-empty stump and that the statement holds for every immediate substump σ^n of σ. Let s, t be elements of $\mathbb{N}$ such that $\sigma(s) = 1$ and $s \leq^* t$. One may assume: *length*$(s) > 0$. Find u, v such that $s = \langle s(0)\rangle * u$ and $t = \langle t(0)\rangle * v$. As $\sigma^{s(0)}$ is a subset of $\sigma^{t(0)}$ and $\sigma^{s(0)}(u) = 1$, also $\sigma^{t(0)}(v) = 1$. As $u \leq^* v$, also $\sigma^{t(0)}(v) = 1$ and $\sigma(t) = 1$.

We now prove: for every stump σ, if σ satisfies the condition: *for each s, for each t, if $\sigma(s) = 1$ and $s \leq^* t$, then $\sigma(t) = 1$*, then σ is hereditarily increasing. We use again induction on the set of stumps.

First, note that the statement is trivially true if $\sigma = \underline{0}$.

Now assume that σ is a non-empty stump and: for each s, for each t, if $\sigma(s) = 1$ and $s \leq^* t$, then $\sigma(t) = 1$. Note that, for each n, for each t, if $\sigma(\langle n\rangle * t) = 1$, then $\sigma(\langle n+1\rangle * t) = 1$, so σ^n is a subset of σ^{n+1}. Also, for each n, for each s, for each t, if $s \leq^* t$, then, if $\sigma(\langle n\rangle * s) = 1$, then $\sigma(\langle n\rangle * t) = 1$, that is: if $\sigma^n(s) = 1$, then $\sigma^n(t) = 1$. It follows from the induction hypothesis that, for each n, σ^n is hereditarily increasing. Clearly, also σ itself is hereditarily increasing.

(iii) We have to prove that, for all stumps σ,

for all stumps τ, the infinite sequence $\sigma \cup \tau$ is again a stump,

and we do so by induction on the set of stumps.

Note that, for each stump τ, $\underline{0} \cup \tau = \tau$ is again a stump. Now assume σ is a non-empty stump, and for all n, for every stump τ, $\sigma^n \cup \tau$ is a stump. Note that $\sigma \cup \underline{0} = \sigma$ is a stump.

Now assume τ is a non-empty stump. Note that, by the induction hypothesis, for each n, $(\sigma \cup \tau)^n = \sigma^n \cup \tau^n$ is a stump. It follows that $\sigma \cup \tau$ itself is a stump.

The observation that, if both σ and τ are hereditarily increasing, also $\sigma \cup \tau$ is hereditarily increasing, is an easy consequence of (ii).

(iv) We have to prove: for every stump σ, the infinite sequence $\mathbb{H}(\sigma)$ is again a stump, and we do so by induction on the set of stumps.

Note that $\mathbb{H}(\underline{0}) = \underline{0}$ is a stump indeed.

Now assume σ is a non-empty stump. Note that $\big(\mathbb{H}(\sigma)\big)(\langle\ \rangle) = 1$, and, for each n, $\big(\mathbb{H}(\sigma)\big)^n = \bigcup\limits_{i\leq n} \mathbb{H}(\sigma^i)$ and conclude, using (iii) and the induction hypothesis, that $\big(\mathbb{H}(\sigma)\big)^n$ is a stump. It follows that $\mathbb{H}(\sigma)$ itself is a stump.

(v) This easily follows from (ii).

(vi) The proof is left to the reader. □

Theorem 3.8.

(i) *For all hereditarily increasing stumps σ, τ, if $\sigma \leq \tau$, then the set Share(CB^*_σ) reduces to the set Share(CB^*_τ), and the set Share$(\overline{CB^*_\sigma})$ reduces to the set Share$(\overline{CB^*_\tau})$.*
(ii) *For each stump σ, the set Share(CB^*_σ) belongs to the class $\mathbf{\Sigma}^0_2$ and, if σ is non-empty, then the set Share$(\overline{CB^*_\sigma})$ is an analytic set that fails to be positively Borel.*
(iii) *For each hereditarily increasing stump σ, the sets Share(CB^*_σ) and Share$(\overline{CB^*_\sigma})$ are disjunctively productive.*

(iv) *For all hereditarily increasing stumps σ, τ, if $\sigma < \tau$, then the set Share(CB^*_τ) does not reduce to the set Share(CB^*_σ) and the set Share$(\overline{CB^*_\tau})$ does not reduce to the set Share$(\overline{CB^*_\sigma})$.*

Proof. (i) One proves this by induction on the set of stumps, using Lemma 2.6.

(ii) Using Lemma 2.3(iv), one may prove, by induction on the set of stumps, that, for every stump σ, the set CB^*_σ is an enumerable subset of $\mathcal{N}$. Note that, for every subset X of $\mathcal{N}$, if X is enumerable, then the set *Share*(X) belongs to the class $\mathbf{\Sigma}^0_2$: suppose that β is an element of $\mathcal{N}$ such that, for every α, α belongs to X if and only if, for some n, $\alpha = \beta^n$; then, for every α, α belongs to *Share*(X) if and only if, for some n, for all m, $\alpha(\overline{\beta^n}m) = 0$ and thus the set *Share*(X) is seen to be a member of the class $\mathbf{\Sigma}^0_2$.

As we know from Theorem 1.12, a result proven in [41], the set $\mathbf{MonPath}_{01}$, consisting of all α such that, for some γ, for each n, $\gamma(n) \le \gamma(n+1) \le 1$ and, for each n, $\alpha(\overline{\gamma}n) = 0$, is not positively Borel. Now consider $\overline{CB^*_{1^*}}$ and note that, for each γ, γ belongs to $\overline{CB^*_{1^*}} = \overline{\{\underline{0}\} \cup \{\overline{\underline{0}}n * \langle 1\rangle * \underline{0} | n \in \mathbb{N}\}}$ if and only if, for all m, n, if $\gamma(m) \neq 0$ and $\gamma(n) \neq 0$, then $\gamma(m) = 1$ and $m = n$. It is not difficult to see that the sets *Share*$(\overline{CB^*_{1^*}})$ and $\mathbf{MonPath}_{01}$ reduce to each other. Note that, for each non-empty stump σ, the set *Share*$(\overline{CB^*_{1^*}})$ reduces to the set *Share*$(\overline{CB^*_\sigma})$ and, therefore, the set *Share*$(\overline{CB^*_\sigma})$ is not positively Borel.

(iii) One proves this by induction on the set of hereditarily increasing stumps, using Theorem 3.4.

(iv) It follows from (iii) and Theorem 3.4(iv) that, for each non-empty hereditarily increasing stump τ, for each n, the set *Share*(CB^*_τ) does not reduce to the set *Share*$\big(S_{i \le n}(CB^*_{\tau^i})\big)$ and, therefore, by Theorem 3.4(i), also not to the set *Share*$(CB^*_{\tau^n})$. Now assume σ, τ are hereditarily increasing stumps such that $\sigma < \tau$. Find n such that $\sigma \le \tau^n$. Using (i), one easily concludes that the set *Share*(CB^*_τ) does not reduce to the set *Share*(CB^*_σ). The proof that, for all hereditarily increasing stumps σ, τ, the set *Share*$(\overline{CB^*_\tau})$ does not reduce to the set *Share*$(\overline{CB^*_\sigma})$ is similar. ☐

Let s belong to $\mathbb{N}$ and α to $\mathcal{N}$. We define: b *forbids* α if and only if there exist m, n such that $\overline{\alpha}m <$ *length*(b) and $b(\overline{\alpha}m) \neq 0$. We define: b *admits* α if and only if b does not forbid α.

For every α, δ in $\mathcal{N}$, we let α_δ be the element β of $\mathcal{N}$ satisfying: for each n, $\beta(\overline{\delta}n) = 0$ and, for each s, if δ does not pass through s, then $\beta(s) = \alpha(s)$.

3.6. **An isolated but intriguing question.** The following theorem reports a difficult but somewhat solitary fact. We conjecture but were unable to prove the more general result that, for every non-empty hereditarily increasing stump σ, the set *Share*(CB^*_σ) does not reduce to the set *Share*$(\overline{CB^*_\sigma})$. The fact that, conversely, for every non-empty stump σ, the set *Share*$(\overline{CB^*_\sigma})$ does not reduce to the set *Share*(CB^*_σ), is a consequence of Theorem 3.8(ii).

Theorem 3.9. *The set Share$(CB^*_{1^*})$ does not reduce to the set Share$(\overline{CB^*_{1^*}})$.*

Proof. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set *Share*$(CB^*_{1^*})$ to the set *Share*$(\overline{CB^*_{1^*}})$.

We let ζ be an element of $\mathcal{N}$ such that $\zeta^0 = \underline{0}$, and, for each n, $\zeta^{n+1} = \overline{\underline{0}}n * \langle 1\rangle * \underline{0}$. Note that ζ is an enumeration of the set $CB^*_{1^*}$.

Note that, for each α, for each n, $\alpha_{(\zeta^n)}$ belongs to the set *Share*$(CB^*_{1^*})$, and, therefore, $\gamma|(\alpha_{(\zeta^n)})$ belongs to the set *Share*$(\overline{CB^*_{1^*}})$. Using the Second Axiom of Continuous Choice, we determine δ in $\mathcal{N}$ such that, for each n, δ^n is a function from $\mathcal{N}$ to $\mathcal{N}$, such that, for each α, for each n, $\gamma|(\alpha_{(\zeta^n)})$ admits $\delta^n|\alpha$ and $\delta^n|\alpha$ belongs to $\overline{CB^*_{1^*}}$. It follows that, for each α, for each n, if α admits ζ^n, then α coincides with $\alpha_{(\zeta^n)}$ and $\gamma|\alpha$ admits $\delta^n|\alpha$.

Our intention is to show that the set $\overline{CB^*_{1^*}}$ coincides with the set $CB^*_{1^*}$. As we know that the set $CB^*_{1^*}$ is not a closed subset of $\mathcal{N}$, we then will have obtained the desired contradiction.

We first prove the following *important statement*:

> *For all a such that length$(a) > 0$ and $a(\langle\,\rangle) = a(0) = 0$, for all k, there exist b, n such that $n > k$ and $a \sqsubseteq b$ and $\overline{0}k \sqsubseteq \delta^n|b$ and, for each $i < k$, b forbids ζ^{i+1}.*

Let a, k be given and assume: *length*$(a) > 0$ and $a(0) = 0$. Find p such that, for all i, if $i > p$, then $\overline{(\zeta^{i+1})}1 >$ *length*(a) and, therefore, a admits ζ^{i+1}, and consider the infinite sequence $a * \underline{0}$. For each $i \leq k$, we consider the finite sequence $\overline{\big(\delta^{p+i}|(a * \underline{0})\big)}k$. Note that *either* there exists $i \leq k$ such that $\overline{\big(\delta^{p+i}|(a * \underline{0})\big)}k$ coincides with $\overline{0}k$, *or* we find i, j, m such that $i < j < k$ and $m < k$ such that $\overline{0}m * \langle 1 \rangle$ is an initial part of both $\overline{\big(\delta^{p+i}|(a * \underline{0})\big)}k$ and $\overline{\big(\delta^{p+j}|(a * \underline{0})\big)}k$. We show that the second one of these two alternatives can not occur.

Assume $i < j < k$ and $m < k$ and $\overline{0}m * \langle 1 \rangle$ is an initial part of both $\overline{\big(\delta^{p+i}|(a * \underline{0})\big)}k$ and $\overline{\big(\delta^{p+j}|(a * \underline{0})\big)}k$. We determine q such that $\overline{0}m * \langle 1 \rangle$ is an initial part of both $\delta^{p+i}|(a * \overline{0}q)$ and $\delta^{p+j}|(a * \overline{0}q)$. We then determine a function ε from $\mathcal{N}$ to $\mathcal{N}$ such that, for every β,

(1) $\varepsilon|\beta$ passes through $a * \overline{0}q$, and
(2) for each n, if $n \leq q$, then $(\varepsilon|\beta)\big(\overline{(\zeta^{p+i})}(n)\big) = 0$, and, if $n > q$, then for each n, $(\varepsilon|\beta)\big(\overline{(\zeta^{p+i})}(n)\big) = \beta^0(n - q - 1)$, and
(3) for each n, if $n \leq q$, then $(\varepsilon|\beta)\big(\overline{(\zeta^{p+j})}(n)\big) = 0$, and, if $n > q$, then for each n, $(\varepsilon|\beta)\big(\overline{(\zeta^{p+j})}(n)\big) = \beta^1(n - q - 1)$, and
(4) for every l, if $l \neq p+i$ and $l \neq p+j$, then there exists r such that $(\varepsilon|\beta)\big(\overline{(\zeta^l)}r\big) \neq 0$.

Observe that, for every β, $\varepsilon|\beta$ admits ζ^{p+i} if and only if $\beta^0 = \underline{0}$ and $\varepsilon|\beta$ admits ζ^{p+j} if and only if $\beta^1 = \underline{0}$, and, for every l, if $l \neq p + i$ and $l \neq p + j$, then ζ^l is not admitted by $\varepsilon|\beta$.

Note that, for each β, if $\beta^0 = \underline{0}$, then $\varepsilon|\beta$ admits ζ^{p+i}, and, therefore, $\gamma|(\varepsilon|\beta)$ admit $\delta^k|(\varepsilon|\beta)$, and, as $\varepsilon|\beta$ passes through $a * \overline{0}q$, $\delta^k|(\varepsilon|\beta)$ passes through $\overline{0}m * \langle 1 \rangle$, and $\gamma|(\varepsilon|\beta)$ admits $\overline{0}m * \langle 1 \rangle * \underline{0}$.

Also note that, for similar reasons, for each β, if $\beta^1 = \underline{0}$, then $\gamma|(\varepsilon|\beta)$ admits $\overline{0}m * \langle 1 \rangle * \underline{0}$.

Conversely, for each β, if $\gamma|(\varepsilon|\beta)$ admits $\overline{0}m * \langle 1 \rangle * \underline{0}$, then $\gamma|(\varepsilon|\beta)$ belongs to *Share*$(\overline{CB^*_{1^*}})$, and $\varepsilon|\beta$ belongs to *Share*$(CB^*_{1^*})$, so there exists l such that $\varepsilon|\beta$ admits ζ^l, and: either $\varepsilon|\beta$ admits ζ^{p+i} or $\varepsilon|\beta$ admits ζ^{p+j}, and: either $\beta^0 = \underline{0}$ or $\beta^1 = \underline{0}$.

We thus see: for every β, β belongs to $D^2(A_1)$ if and and only if $\gamma|(\varepsilon|\beta)$ admits $\overline{0}m * \langle 1 \rangle * \underline{0}$, that is: the set $D^2(A_1)$ reduces to the set A_1.

It follows from Lemma 2.1(v) and (vii) that the set $D^2(A_1)$ does not reduce to the set A_1.

Thus we are left with the first of the two alternatives mentioned above: there exists $i \leq k$ such that $\overline{\big(\delta^{p+i}|(a * \underline{0})\big)}k$ coincides with $\overline{0}k$. We now determine $i \leq k$ and q such that the finite sequence $\overline{0}k$ is an initial part of the finite sequence $\delta^{p+i}|(a * \overline{0}q)$ and we define: $c := a * \overline{0}q$. Finally, we determine b such that $c \sqsubseteq b$, and, for every $l < k$, b forbids ζ^{l+1}.

It will be clear that b satisfies the requirements.

Using the just-proven *important statement* repeatedly we construct infinite sequences $a_0, a_1, a_2, \ldots$ and $n_0, n_1, n_2, \ldots$ of natural numbers such that *length*$(a_0) > 0$ and $a(0) = 0$, and, for each k, $a_k \sqsubseteq a_{k+1}$ and $n_k < n_{k+1}$ and $\overline{0}k \sqsubseteq (\delta^{n_k}|a_k)$, and, for each $l < k$, a_k forbids ζ^{k+1}.

Note that, for each α, for each k, if α passes through a_k, then δ^{n_k} passes through $\underline{\overline{0}}k$.

We let η be a function from $\overline{CB^*_{1^*}}$ to $\overline{CB^*_{1^*}}$ such that, for each λ in $\overline{CB^*_{1^*}}$, for each k, $k = \mu n[\lambda(n) = 1]$ if and only if $n_k = \mu n[(\eta|\lambda)(n) = 1]$.
Note that $\lambda|\zeta^0 = \zeta^0$and, for each k, $\lambda|(\zeta^{k+1}) = \zeta^{n_k+1}$.

We now are ready to prove that the set $\overline{CB^*_{1^*}}$ is a subset of the set $CB^*_{1^*}$.

Let λ be an element of $\overline{CB^*_{1^*}}$.
We let α be an element of $\mathcal{N}$ satisfying the following conditions:

(1) α admits $\eta|\lambda$, and
(2) for each β in $\overline{CB^*_{1^*}}$, if $\beta \# \eta|\lambda$, then there exists r such that $\alpha(\overline{\beta}r) \neq 0$, so α forbids β, and
(3) for each k, if λ passes through $\underline{\overline{0}}k$, then α passes through a_k.

We now claim that $\gamma|\alpha$ belongs to *Share*$(\overline{CB^*_{1^*}})$. In order to prove this claim, we have to construct ν in $\overline{CB^*_{1^*}}$ such that $\gamma|\alpha$ admits ν. We define ν as follows:

> For each k, if $\overline{\lambda}k = \underline{\overline{0}}k$, then $\overline{\nu}(n_k) = \underline{\overline{0}}(n_k)$, and, if k is the least i such that $\lambda(i) \neq 0$, then $\nu = \delta^{n_k}|\alpha$.

Note that ν belongs to $\overline{CB^*_{1^*}}$. Also note that $\gamma|\alpha$ admits ν: for each k, if $\overline{\lambda}k = \underline{\overline{0}}k$, then α passes through a_k and $\underline{\overline{0}}k \sqsubseteq (\delta^{n_k}|a_k)$, so $\delta^{n_k}|\alpha$ passes through $\underline{\overline{0}}k$, and $\gamma|\alpha$ admits $\underline{\overline{0}}k = \overline{\nu}k$; on the other hand, if $\overline{\lambda}k \neq \underline{\overline{0}}k$ and k_0 is the least k such that $\lambda(k) \neq 0$, then $\lambda = \zeta^{k_0+1}$ and $\eta|\lambda = \zeta^{n_{k_0}+1}$ and α admits $\zeta^{n_{k_0}+1}$, so α coincides with $\alpha_{(\zeta^{n_{k_0}+1})}$ and $\gamma|\alpha$ admits $\nu = \delta^{n_{k_0+1}}|\alpha$, so $\gamma|\alpha$ admits $\overline{\nu}k$.
We may conclude that $\gamma|\alpha$ belongs to *Share*$(\overline{CB^*_{1^*}})$. It follows that α belongs to *Share*$(CB^*_{1^*})$. Thus we may determine n such that λ coincides with ζ^n.

We now have seen that, for all λ in $\overline{CB^*_{1^*}}$ one may determine n such that λ coincides with ζ^n, and, therefore, one may decide: *either* $\lambda = \underline{0}$ *or* $\lambda \# \underline{0}$. As the set $\overline{CB^*_{1^*}}$ is a spread one may apply Brouwer's Continuity Principle and determine m such that *either*, for every λ in $\overline{CB^*_{1^*}}$, if λ passes through $\underline{\overline{0}}m$, then $\lambda = \underline{0}$, *or*, for every λ in $\overline{CB^*_{1^*}}$, if λ passes through $\underline{\overline{0}}m$, then $\lambda \# \underline{0}$. Both alternatives are false. □

3.7. **Final observations on the two hierarchies.** We define a binary operation $\oplus$ on the set of stumps, by induction, as follows:

(1) For all stumps σ, $\sigma \oplus \emptyset = \oplus(\emptyset, \sigma) := \sigma$.
(2) For all non-empty stumps σ, τ, $\sigma \oplus \tau$ is the non-empty stump ρ satisfying: for all m, $\rho^{2m} = (\sigma^m) \oplus \tau$ and $\rho^{2m+1} = \sigma \oplus (\tau^m)$.

One might call $\sigma \oplus \tau$ the *natural sum* or the *Hessenberg sum* of the stumps σ, τ.

Note that, for all non-empty stumps σ, τ, for all n, $(\sigma^n) \oplus \tau < \sigma \oplus \tau$ and $\sigma \oplus (\tau^n) < \sigma \oplus \tau$, and, for each stump ρ, if, for all n, $(\sigma^n) \oplus \tau < \rho$ and $\sigma \oplus (\tau^n) < \rho$, then $\sigma \oplus \tau \leq \rho$.

Also note that the operation $\oplus$ is commutative and associative: for all stumps σ, τ, $\sigma \oplus \tau \equiv \tau \oplus \sigma$ and, for all stumps σ, τ, ρ, $(\sigma \oplus \tau) \oplus \rho \equiv \sigma \oplus (\tau \oplus \rho)$.

Lemma 3.10.

(i) *There exists a continuous function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for all stumps σ, τ, γ maps the spread $\overline{CB^*_{\sigma\oplus\tau}}$ onto the spread $\overline{CB^*_\sigma \times CB^*_\tau}$ and the enumerable set $CB^*_\sigma \times CB^*_\tau$ onto the enumerable set $CB_{\sigma\oplus\tau}$.*
(ii) *For all subsets X, Y of $\mathcal{N}$, if there exists a continuous function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set X onto the set Y, then the set Share(Y) reduces to the set Share(X).*
(iii) *There exists a continuous function from $\mathcal{N}$ to $\mathcal{N}$, reducing, for all subsets X, Y of $\mathcal{N}$, the set $C\big(Share(X), Share(Y)\big)$ to the set Share$(X \times Y)$.*

(iv) *For all stumps σ, τ,*
the set $C\big(Share(CB^*_\sigma), Share(CB^*_\tau)\big)$ reduces to the set $Share(CB^*_{\sigma\oplus\tau})$ and
the set $C\big(Share(\overline{CB^*_\sigma}), Share(\overline{CB^*_\tau})\big)$ reduces to the set $Share(\overline{CB^*_{\sigma\oplus\tau}})$.

Proof. (i) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ satisfying: for each α, for each m, γ maps the infinite sequence $\underline{\overline{0}}(2m) * \langle 1 \rangle * \alpha$ onto the infinite sequence $\langle \underline{\overline{0}}m * \langle 1 \rangle * \alpha, \underline{0}\rangle$ and γ maps the infinite sequence $\underline{\overline{0}}(2m+1) * \langle 1 \rangle * \alpha$ onto the infinite sequence $\langle \underline{0}, \underline{\overline{0}}m * \langle 1 \rangle * \alpha\rangle$.

It is not difficult to see that γ satisfies the requirements.

(ii) Let X, Y be subsets of $\mathcal{N}$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α, α belongs to Y if and only if there exists ε in X such that $\gamma|\varepsilon$ coincides with α. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, for each s, $(\delta|\beta)(s) = 0$ if and only if, for each t, if $t \sqsubseteq (\gamma|s)$, then $\beta(t) = 0$. Note that, for each β, there exists α in Y such that β admits α if and only if there exists ε in X such that β admits $\gamma|\varepsilon$ if and only if there exists ε in X such that $\delta|\beta$ admits ε. Clearly, the function δ reduces the set $Share(Y)$ to the set $Share(X)$.

(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if both $\alpha^0(s_I) = 0$ and $\alpha^1(s_{II}) = 0$. Note that, for all subsets X, Y of $\mathcal{N}$, for each α, α belongs to $C\big(Share(X), Share(Y)\big)$ if and only if α^0 belongs to $Share(X)$ and α^1 belongs to $Share(Y)$ if and only if there exists δ such that α^0 admits δ_I and δ_I belongs to X and α^1 admits δ_{II} and δ_{II} belongs to Y if and only if there exists δ such that δ belongs to $X \times Y$ and $\gamma|\alpha$ admits δ. We thus see that γ reduces the set $C\big(Share(X), Share(Y)\big)$ to the set $Share(X \times Y)$.

(iv) Note that, for all stumps σ, τ, the set $C\big(Share(CB^*_\sigma), Share(CB^*_\tau)\big)$ reduces to the set $Share(CB^*_\sigma \times CB^*_\tau)$, according to (iii), and that the set $Share(CB^*_\sigma \times CB^*_\tau)$ reduces to the set $Share(CB^*_{\sigma\oplus\tau})$, by (i) and (ii).

For analogous reasons, the set $C\big(Share(\overline{CB^*_\sigma}), Share(\overline{CB^*_\tau})\big)$
reduces to the set $Share([\overline{CB^*_{\sigma\oplus\tau}})$. □

A stump σ is called a *critical point for the natural sum* $\oplus$ if and only if for all stumps τ_0, τ_1, if both $\tau_0 < \sigma$ and $\tau_1 < \sigma$, then also $\tau_0 \oplus \tau_1 < \sigma$.

Recall that $J : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection, a so-called pairing function on $\mathbb{N}$. Its inverse functions are called K, L: for all p, q, there exists n such that $K(n) = p$ and $L(n) = q$, and, for every n, $J\big(K(n), L(n)\big) = n$. Note that, for every p, there exist infinitely many n such that $K(n) = p$.

Let $\sigma_0, \sigma_1, \sigma_2, \dots$ be an infinite sequence of stumps.

For each n, we define a stump $\oplus_{i=0}^{i=n}\sigma_i$, as follows, by induction.

(1) $\oplus_{i=0}^{i=0}\sigma_i := \sigma_0$, and
(2) $\oplus_{i=0}^{i=n+1}\sigma_i := (\oplus_{i=0}^{i=n}\sigma_i) \oplus \sigma_{n+1}$.

Lemma 3.11.

(i) *For every non-empty stump σ, σ is a critical point for the natural sum $\oplus$ if and only if, for all m, n, there exists p such that $\sigma^m \oplus \sigma^n \leq \sigma^p$.*
(ii) *For every stump σ, there exists a stump ρ such that $\sigma \leq \rho$ and ρ is a critical point for the natural sum.*
(iii) *For every stump σ, if σ is a critical point for the natural sum, then the two infinite sequences*

$$Share(CB_{\sigma^0}), Share(CB_{\sigma^1}), Share(CB_{\sigma^2}), \dots$$

and

$$Share(\overline{CB_{\sigma^0}}), Share(\overline{CB_{\sigma^1}}), Share(\overline{CB_{\sigma^2}}), \dots$$

are, each of them, conjunctively closed.

Proof. (i) The proof is left to the reader.

(ii) Let σ be a non-empty stump. We let ρ be the non-empty stump such that, for each n, $\rho^n = \oplus_{i=0}^{i=n}\sigma^{K(i)}$. The proof that ρ satisfies the requirements is left to the reader.

(iii) This is a consequence of (i) and Lemma 3.10(iv). $\square$

Theorem 3.12.
For every stump σ there exists a stump τ such that $\sigma < \tau$ and the sets Share(CB^*_τ) and Share$(\overline{CB^*_\tau})$ are both disjunctively and conjunctively productive.

Proof. Let σ be a stump. We let τ be the nonempty stump such that $\tau^0 = \mathbb{H}(\sigma)$, and, for each n, $\tau^{n+1} = \mathbb{H}(\tau^n \oplus \tau^n)$. Note that, for each n, τ^n is hereditarily increasing, and, therefore, by Theorem 3.8, the sets *Share*$(CB^*_{\tau^n})$ and *Share*$(\overline{CB^*_{\tau^n}})$ are disjunctively productive. Also observe that $\sigma \leq \tau^0 < \tau$ and that, for each m, n, if $m \leq n$, then $\tau^m \oplus \tau^n \leq \tau^{n+1}$, and therefore, by Lemma 3.11(i), τ is a critical point for the natural sum. Finally note that, for each n, $\tau^n \leq \tau^{n+1}$ and, therefore, by Theorem 3.8, the set *Share*$(CB^*_{\tau^n})$ reduces to the set *Share*$(CB^*_{\tau^{n+1}})$ and the set *Share*$(\overline{CB^*_{\tau^n}})$ reduces to the set *Share*$(\overline{CB^*_{\tau^{n+1}}})$. It now follows from Theorem 3.6 and Lemma 3.11(iii), and from Theorem 3.4 that the sets *Share*(CB^*_τ) and *Share*$(\overline{CB^*_\tau})$ are conjunctively and disjunctively productive. $\square$

4. An intermezzo on *Share*$(\mathcal{C})$

From a classical point of view, Theorem 3.12 is very surprising.

In the first place, the classical mathematician does not see any difference between a Cantor-Bendixson-set CB^*_σ and its closure $\overline{CB^*_\sigma}$.

Recall that a subset X of $\mathcal{N}$ is called a *fan* or a *finitary spread* if and only if there exists β in $\mathcal{C}$ such that,

1. for every s, s contains an element of X if and only if $\beta(s) = 0$, and
2. for every α, α belongs to X if and only if, for each n, $\beta(\overline{\alpha}n) = 0$, and
3. for each s, there are only finitely many numbers n such that $\beta(s * \langle n\rangle) = 0$.

Note that, for each stump σ, the set $\overline{CB^*_\sigma}$ is a fan.

For this reason, the classical mathematician would say, every set *Share*$(\overline{CB^*_\sigma})$ is a closed subset of $\mathcal{N}$.

For, suppose X is a fan, and let β in $\mathcal{C}$ satisfy the condition: for every s, s contains an element of X if and only if $\beta(s) = 0$. Find γ in $\mathcal{N}$ such that, for each n, for each s, if *length*$(s) = n$, and $\beta(s) = 0$, then $s \leq \gamma(n)$. Then, by the (constructively invalid) Infinity Lemma of D. König: for every α, α belongs to *Share*(X) if and only if $\forall n \exists s \leq \gamma(n)[\beta(s) = 0]$. Clearly then, *Share*$(X)$ is a closed subset of $\mathcal{N}$. The classical mathematician will further conclude that, for each σ, for each τ, the set *Share*$(\overline{CB^*_\sigma})$ reduces to the set *Share*$(\overline{CB^*_\tau})$.

In this Section, we want to make some intuitionistic observations on the set *Share*$(\mathcal{C})$. Note that Cantor space $\mathcal{C}$ is a fan.

Let X be a subset of $\mathcal{N}$ and let B be a subset of $\mathbb{N}$. We define: *the set B is a bar in the set X* if and only if, for each α in X, there exists n such that $\overline{\alpha}n$ belongs to B.

We now define an element bar_{01} of $\mathcal{N}$ satisfying the following conditions:

1. for each n, *length*$\big(bar_{01}(n)\big) = n+1$, and
2. $bar_{01}(0) = \langle 0\rangle = \langle\langle\ \rangle\rangle$, and
3. for each n, $\big(bar_{01}(n+1)\big)(0) = \big(\big(bar_{01}(n)\big)(0)\big) * \langle 0\rangle$ and $\big(bar_{01}(n+1)\big)(1) = \big(\big(bar_{01}(n)\big)(0)\big) * \langle 1\rangle$, and for each j, if $0 < j < n$, then $\big(bar_{01}(n+1)\big)(j+1) = \big(bar_{01}(n)\big)(j)$.

Note that, for each n, the finite set $\{\big(bar_{01}(n)\big)(0), \big(bar_{01}(n)\big)(1), \ldots, \big(bar_{01}(n)\big)(n)\}$ is a bar in $\mathcal{C}$, and, for each i, for each j, if $i < j \leq n$, then $\big(bar_{01}(n)\big)(i) \perp \big(bar_{01}(n)\big)(j)$.

Theorem 4.1.

(i) *For each subset X of $\mathcal{N}$, if X is a fan, then there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{C}$ onto X.*

(ii) *For each subset X of $\mathcal{N}$, if X is a fan, then the set $Share(X)$ reduces to the set $Share(\mathcal{C})$.*

(iii) *For each stump σ, the set $Share(\overline{CB^*_\sigma})$ reduces to the set $Share(\mathcal{C})$, and the set $Share(\mathcal{C})$ does not reduce to the set $Share(\overline{CB^*_\sigma})$.*

(iv) *For every infinite sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, the set X_n reduces to the set $Share(\mathcal{C})$, then the set $\bigcap_{n\in\mathbb{N}} X_n$ also reduces to the set $Share(\mathcal{C})$.*

(v) *For all subsets X, Y of $\mathcal{N}$, if both X and Y reduce to the set $Share(\mathcal{C})$, then also $X \cup Y$ reduces to the set $Share(\mathcal{C})$.*

Proof. (i) Suppose X is a subset of $\mathcal{N}$ and a fan, and let β in $\mathcal{C}$ satisfy the condition: for every s, s contains an element of X if and only if $\beta(s) = 0$. We determine γ in $\mathcal{N}$ such that, for each s, $\gamma(s)$ is the code of a finite sequence enumerating the different numbers i such that $|beta(s * \langle i\rangle) = 0$, that is, for each i, $\beta(s * \langle i\rangle) = 0$ if and only if there exists $j < length\big(\gamma(s)\big)$ such that $i = \big(\gamma(s)\big)(j)$, and for all j, for all k, if $j < k < length(s)$, then $\big(\gamma(s)\big)(j) \neq \big(\gamma(s)\big)(k)$. We then define δ in $\mathcal{N}$ such that for each s, if $\beta(s) = 0$, then $\delta(s) \in \{0,1\}^*$ and

(1) $\delta(\langle\,\rangle) = \langle\,\rangle$ and

(2) for each s, if $\beta(s) = 0$, then for each j, if $j < length\big(\gamma(s)\big)$, then $\delta(s * \langle\big(\gamma(s)\big)(j)\rangle = \delta(s) * bar_{01}(length\big(\gamma(s)\big)(j)$.

Note that, for each ζ, if, for each n, $\beta(\overline{\zeta}n) = 0$, then there exists exactly one α in $\mathcal{C}$ such that, for each n, $\overline{\zeta}n \sqsubseteq \delta(\overline{\alpha}n)$.

We now let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α in $\mathcal{C}$, for each s, if α passes through $\delta(s)$, then $\varepsilon|\alpha$ passes through s.

One may verify that ε is a function mapping the set $\mathcal{C}$ strongly one-to-one onto the set F_β.

(ii) Use (i) and Theorem 3.10(ii).

(iii) Let σ be a stump. Find hereditarily increasing stumps τ, ρ such that $\sigma \leq \tau < \rho$. According to Theorem 3.8(iv), the set $Share(\overline{CB^*_\rho})$ does not reduce to the set $Share(\overline{CB^*_\tau})$. As, according to (ii), the set $Share(\overline{CB^*_\rho})$ reduces to the set $Share(\mathcal{C})$, the set $Share(\mathcal{C})$ does not reduce to the set $Share(\overline{CB^*_\tau})$. As, by Theorem 2.8(i), the set $Share(\overline{CB^*_\sigma})$ reduces to the set $Share(\overline{CB^*_\tau})$, the set $Share(\mathcal{C})$ also does not reduce to $Share(\overline{CB^*_\sigma})$.

(iv) Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$, each of them reducing to the set $Share(\mathcal{C})$. Using the Second Axiom of Countable Choice, find γ in $\mathcal{N}$ such that, for each n, γ^n is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set X_n to the set $Share(\mathcal{C})$. We let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\delta|\alpha)(s) = 0$ if and only if s belongs to $\{0,1\}^*$ and, for each n, if $n \leq length(s)$, then, for each i, if $i \leq length(s^n)$, then $(\gamma^n|\alpha)\big(\overline{(s^n)}i\big) = 0$. Note that, for each α, for each β in $\mathcal{C}$, $\delta|\alpha$ admits β if and only if, for each n, $\gamma^n|\alpha$ admits β^n. It follows that, for each α, α belongs to $\bigcap_{n\in\mathbb{N}} X_n$ if and only if for each n there exists β in $\mathcal{C}$ such that $\gamma^n|\alpha$ admits β if and only if there exists β in $\mathcal{C}$ such that, for each n, $\gamma^n|\alpha$ admits β^n if and only there exists β in $\mathcal{C}$ such that $\delta|\alpha$ admits β if and only if $\delta|\alpha$ belongs to $Share(\mathcal{C})$.

(v) Let X, Y be subsets of $\mathcal{N}$, both of them reducing to the set $Share(\mathcal{C})$. Let γ, δ be functions from $\mathcal{N}$ to $\mathcal{N}$ reducing the sets X, Y, respectively, to the set $Share(\mathcal{C})$. Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\varepsilon|\alpha)(\langle\,\rangle) = 0$ and, for each s in $\{0,1\}^*$, $(\varepsilon|\alpha)(\langle 0\rangle * s) = (\gamma|\alpha)(s)$ and $(\varepsilon|\alpha)(\langle 1\rangle * s) = (\delta|\alpha)(s)$. Note that, for each α, for each β in $\mathcal{C}$, $\varepsilon|\alpha$ admits $\langle 0\rangle * \beta$ if and only if $\gamma|\alpha$ admits β and $\varepsilon|\alpha$ admits $\langle 1\rangle * \beta$ if and only if $\delta|\alpha$ admits β. It follows that, for each α, α belongs to $X \cup Y$ if and only if either $\gamma|\alpha$ or $\delta|\alpha$ admits an element of $\mathcal{C}$ if and only if $\varepsilon|\alpha$ admits an element of $\mathcal{C}$ if and only if $\varepsilon|\alpha$ belongs to $Share(\mathcal{C})$. □

Theorem 4.2.

(i) *For each stump σ, for every subset B of $\mathbb{N}$, if B is a bar in CB^*_σ, then there exists a finite subset B' of B such that B' is a bar in $\overline{CB^*_\sigma}$.*

(ii) *For each stump σ, for every subset B of $\mathbb{N}$,*
if $\forall\beta \in CB^*_\sigma \neg\neg\exists n[\overline{\beta}n \in B]$, then $\neg\neg\forall\beta \in CB^*_\sigma \exists n[\overline{\beta}n \in B]$.

(iii) *For every stump σ, the set E_1 does not reduce to the set $Share(CB^*_\sigma)$ or to the set $Share(\overline{CB^*_\sigma})$.*

Proof. (i) One proves this by induction on the set of stumps. Note that, for every subset B of $\mathbb{N}$, if B is a bar in $CB^*_{\underline{0}} = \{\underline{0}\}$, then there exists n such that $\{\underline{0}n\}$ is a subset of B and a bar in $CB^*_{\underline{0}}$. Now assume that σ is a non-empty stump and the statement has been verified for every immediate substump σ^n of σ. Let B be a subset of $\mathbb{N}$ and a bar in CB^*_σ. Find n such that $\overline{0}n$ belongs to B. Note that, for each $j < n$, the set of all s such that $\overline{0}j * \langle 1\rangle * s$ has an initial part in B is a bar in $CB^*_{\sigma^j}$. For each $j < n$, we now find, using the induction hypothesis, a finite subset B_j of the set B such that every element of CB_σ passing through $\overline{0}j * \langle 1\rangle$ has an initial part in B_j. Let C be the set $\{\overline{0}n\} \cup \bigcup_{j<n} B_j$. Note that C is a finite subset of B and a bar in CB^*_σ.

(ii) One proves this by induction on the set of stumps. Note that the statement is true if $\sigma = \underline{0}$. Now assume that σ is a non-empty stump and the statement has been verified for every immediate substump σ^n of σ.

Let B be a subset of $\mathbb{N}$ such that $\forall\beta \in CB^*_\sigma \neg\neg\exists n[\overline{\beta}n \in B]$.

Assume we find n such that $\overline{0}n$ belongs to B. Note that, for each $j < n$, for every β in $CB^*_{\sigma^j}$, $\neg\neg$(there exists n such that some initial part of $\overline{0}j * \langle 1\rangle * \overline{\beta}n$ belongs to B). Using the induction hypothesis, we conclude: $\forall j < n \neg\neg\forall\beta \in CB^*_\sigma$[If β passes through $\overline{0} * \langle 1\rangle$, then $\exists n[\overline{\beta}n \in B]$].

In intuitionistic logic, one may conclude $\neg\neg(P \wedge Q)$ from $\neg\neg P \wedge \neg\neg Q$ and $\neg\neg\forall j < n[P(j)]$ from $\forall j < n[\neg\neg P(j)]$.

We thus obtain the conclusion: $\neg\neg\forall j < n\forall\beta \in CB^*_\sigma$[If β passes through $\overline{0}j * \langle 1\rangle$, then $\exists n[\overline{\beta}n \in B]$].

And, reminding ourselves that $\overline{0}n$ belongs to B, we find: $\neg\neg\forall\beta \in CB^*_\sigma \exists n[\overline{\beta}n \in B]$.

This conclusion has been reached from the assumption: $\exists n[\overline{0}n \in B]$. As we know: $\neg\neg\exists n[\overline{0}n \in B]$, and intuitionistically, the statement $\neg\neg P \to \neg\neg Q$ is a consequence of the statement $P \to \neg\neg Q$, we may conclude: $\neg\neg\forall\alpha \in CB^*_\sigma \exists n[\overline{\beta}n \in B]$.

(iii) Suppose that σ is a stump and that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set E_1 to the set $Share(CB^*_\sigma)$ or to the set $Share(\overline{CB^*_\sigma})$. We study the infinite sequence $\gamma|\underline{0}$. Note that, for each n, we may decide: *either* there exists s such that $length(s) = n$ and s contains an element of CB^*_σ and $\gamma|\underline{0}$ admits s, that is, for all i, if $i \leq n$ then $(\gamma|\underline{0})(\overline{s}i) = 0$ *or* there is no such s. We claim that the first alternative holds.

For assume, for some n, for all s, if s contains an element of CB^*_σ and $length(s) = n$, then there exists $i \leq n$ such that $(\gamma|\underline{0})(\overline{s}i) \neq 0$. Using the continuity of the function γ, we find m such that, for every α, if α passes through $\overline{0}m$, then for all s, if s contains an element of CB^*_σ and $length(s) = n$, then there exists $i \leq n$ such that $(\gamma|\alpha)(\overline{s}i) = (\gamma|\underline{0})(\overline{s}i) \neq 0$. We must conclude that, for every α, if α passes through $\overline{0}m$, then $\gamma|\alpha$ does not belong to $Share(CB^*_\sigma)$, and α itself does not belong to E_1, so $\alpha = \underline{0}$. This is not true, as α may coincide with $\overline{0}m * \underline{1}$.

Using (i), we may conclude: $\neg\forall\beta \in CB^*_\sigma \exists n[(\gamma|\underline{0})(\overline{\beta}n) \neq 0]$.

Using (ii), we obtain the conclusion: $\neg\forall\beta \in CB^*_\sigma\neg\neg\exists n[(\gamma|\underline{0})(\overline{\beta}n) \neq 0]$, that is: $\neg\neg\exists\beta \in CB^*_\sigma\forall n[(\gamma|\underline{0})(\overline{\beta}n) = 0]$, and, therefore, $\neg\neg$ ($\gamma|\underline{0}$ belongs to *Share*(CB_σ)), and: $\neg\neg(\underline{0} \in E_1)$. As $\underline{0} \notin E_1$, we have a contradiction. □

The following statement positively contradicts the Fan Theorem, Axiom 12 This statement is called: *Kleene's Alternative (to the Fan Theorem)* in [42]. Kleene has shown that this alternative holds in *constructive recursive mathematics* where every α in $\mathcal{N}$ is supposed to be given by an algorithm in the sense of Turing.

There exists a decidable subset B of N that is a bar in $\mathcal{C}$ while every finite subset B' of B positively fails to be a bar in $\mathcal{C}$, that is: $\exists\beta[\forall\alpha \in \mathcal{C}\exists n[\beta(\overline{\alpha}n) \neq 0] \wedge \forall m\exists s \in \mathbf{Bin}[\mathit{length}(s) = n \wedge \forall i \leq n[\beta(\overline{s}i) = 0]]]$.

We also want to make some observations on the *generalized Principle of Markov*, or: *Markov's Principle*. This the following statement:

For every α, if $\neg\neg\exists n[\alpha(n) \neq 0]$, then $\exists n[\alpha(n) \neq 0]$.

Markov introduced and defended this principle for infinite sequences that are given by specifying a finite algorithm. The intuitionistic notion of an infinite sequence of natural numbers is wider; hence we are using the predicate '*generalized*'.

We do not want to introduce the generalized Principle of Markov as an axiom of intuitionistic analysis. Some of its consequences have been studied by J.R. Moschovakis, see [23].

Theorem 4.3.

(i) *If the set E_1 reduces to the set Share($\mathcal{C}$), then there exists a decidable subset of $\{0,1\}^*$ that is an infinite tree without an infinite path, that is:*
$\exists\beta[\forall\alpha \in \mathcal{C}\neg\neg\exists n[\beta(\overline{\alpha}n) \neq 0] \wedge \forall m\exists s \in \mathbf{Bin}[\mathit{length}(s) = n \wedge \forall i \leq n[\beta(\overline{s}i) = 0]]]$.

(ii) *If Kleene's Alternative holds, then there exists a strongly one-to-one surjective mapping from $\mathcal{N}$ onto $\mathcal{C}$ and the set E^1_1 reduces to the set Share($\mathcal{C}$).*

(iii) *The generalized Principle of Markov implies: Kleene's Alternative is equivalent to the statement: the set E^1_1 reduces to the set Share($\mathcal{C}$), and also to the statement: the set E_1 reduces to the set Share($\mathcal{C}$).*

(iv) *The generalized Principle of Markov, (together with the (Restricted) Fan Theorem), implies: the set E_1 does not reduce to the set Share($\mathcal{C}$).*

Proof. (i) Suppose: γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set E_1 to the set *Share*($\mathcal{C}$). Consider the infinite sequence $\gamma|\underline{0}$ and observe, as in the proof of Theorem 3.2(iii): for each n, there are finitely elements s of $\{0,1\}^*$ such that $\mathit{length}(s) = n$, and we may decide if there exists s in $\{0,1\}^*$ such that $\mathit{length}(s) = n$ and, for all i, if $i \leq n$, then $(\gamma|\underline{0})(\overline{s}i) = 0$, or not. Suppose we find n such that there is no such s. We then calculate m such that, for all α, if α passes through $\overline{\underline{0}}m$, then, for all s in $\{0,1\}^*$, if $\mathit{length}(s) = n$, then there exists $i \leq n$ such that $(\gamma|\alpha)(\overline{s}i) = (\gamma|\underline{0})(\overline{s}i) \neq 0$. Now $\overline{\underline{0}}m * \underline{1}$ will belong to E_1, while $\gamma|(\overline{\underline{0}}m * \underline{1})$ does not belong to *Share*($\mathcal{C}$). Contradiction.

Let T be the set of all elements s of $\{0,1\}^*$ that are admitted by $\gamma|\underline{0}$, that is, for all $i \leq \mathit{length}(s)$, $(\gamma|\underline{0})(\overline{s}i) = 0$. We just have seen that T is an infinite subtree of $\{0,1\}^*$. On the other hand, $\underline{0}$ does not belong to E_1, and, therefore, $\gamma|\underline{0}$ does not belong to *Share*($\mathcal{C}$), that is, the infinite tree T does not have an infinite path.

Find β in $\mathcal{C}$ such that $T = D_\beta = \{s|\beta(s) = 1\}$ and note: $\forall\alpha \in \mathcal{C}\neg\neg\exists n[\beta(\overline{\alpha}n) \neq 0] \wedge \forall m\exists s \in \mathbf{Bin}[\mathit{length}(s) = n \wedge \forall i \leq n[\beta(\overline{s}i) = 0]]$.

(ii) Let B be a decidable subset of $\mathbb{N}$ that is a bar in $\mathcal{C}$ while every finite subset B' of B positively fails to be a bar in $\mathcal{C}$. Define β in $\mathcal{N}$ as follows: for each n, $\beta(n)$ is the least number s such that $s \in B$, and, for all i, if $i \leq \mathit{length}(s)$, then $\overline{s}i \notin B$, and, for all j, if $j < n$, then $\beta(n) \neq \beta(j)$. Note that β is well-defined because of the fact that, for each n, the set $\{\beta(0), \beta(1), \ldots, \beta(n-1)\}$ positively fails to be a bar in $\mathcal{C}$. Also note that, for each α in $\mathcal{C}$, there exists exactly one n such that $\beta(n)$ is an initial part of α, and that, for all m, n, if $m < n$, then $\beta(m) \perp \beta(n)$. We now let γ be an element of $\mathcal{N}$ such that $\gamma(\langle\ \rangle) = \langle\ \rangle$, and, for all s, for all n, $\gamma(s * \langle n \rangle) = \gamma(s) * \beta(n)$. Finally, we let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, for every n, $\varepsilon|\alpha$ passes through $\gamma(\overline{\alpha}n)$. The reader herself may verify that ε is a strongly one-to-one and surjective function from $\mathcal{N}$ onto $\mathcal{C}$.

The conclusion that the set E_1^1 reduces to the set $\mathit{Share}(\mathcal{C})$ now follows from Lemma 3.10(ii), where it has been shown that, for all subsets X,Y of $\mathcal{N}$, if there exists a continuous function from $\mathcal{N}$ to $\mathcal{N}$ mapping X onto Y, then the set $\mathit{Share}(Y)$ reduces to the set $\mathit{Share}(X)$. (Note that the set E_1^1 coincides with the set $\mathit{Share}(\mathcal{N})$.)

(iii) Assume the generalized principle of Markov.

By (i), if the set E_1 reduces to the set $\mathcal{C}$, then $\exists\beta[\forall\alpha \in \mathcal{C}\neg\neg\exists n[\beta(\overline{\alpha}n) \neq 0] \wedge \forall m\exists s \in \mathbf{Bin}[\mathit{length}(s) = n \wedge \forall i \leq n[\beta(\overline{s}i) = 0]]]$, so, by Markov's principle, $\exists\beta[\forall\alpha \in \mathcal{C}\exists n[\beta(\overline{\alpha}n) \neq 0] \wedge \forall m\exists s \in \mathbf{Bin}[\mathit{length}(s) = n \wedge \forall i \leq n[\beta(\overline{s}i) = 0]]]$, that is: Kleene's Alternative.

By (ii), if Kleene's Alternative holds, then the set E_1^1 reduces to the set $\mathcal{C}$, and, as the set E_1 reduces to the set E_1^1, also the set E_1 reduces to the set $\mathcal{C}$.

(iv) this is an immediate consequence of (iii). □

5. Analytic and strictly analytic subsets of $\mathcal{N}$

5.1. **Analytic and strictly analytic subsets of $\mathcal{N}$ and different kinds of closed subsets of $\mathcal{N}$.** Let X be a subset of $\mathcal{N}$. X is called *strictly analytic* if and only if there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ, that is, for every α, α belongs to X if and only if, for some β, α coincides with $\gamma|\beta$.

Note that every strictly analytic subset X of $\mathcal{N}$ is *inhabited*, that is, there exists α such that α belongs to X.

Every strictly analytic subset of $\mathcal{N}$ is an analytic and inhabited subset of $\mathcal{N}$, but, conversely, it is not true that every analytic and inhabited subset of $\mathcal{N}$ is strictly analytic, as we shall see in a moment.

The intuitionistic mathematician thus has to distinguish analytic subsets of $\mathcal{N}$ from strictly analytic subsets of $\mathcal{N}$.

The need to make the distinction between analytic and strictly analytic subsets of $\mathcal{N}$ is closely connected to the fact that, in intuitionistic mathematics, there are at least four important notions corresponding to the classical notion of a *closed subset of* $\mathcal{N}$.

Let us start with the weakest notion. A subset X of $\mathcal{N}$ is called *sequentially closed* if and only if every α such that, for each n, $\overline{\alpha}n$ contains an element of X, belongs itself to X.

Using the Second Axiom of Countable Choice, one may observe: a subset X of $\mathcal{N}$ is sequentially closed if and only if: for all α, α belongs to X if and only if there exists β such that, for each n, β^n belongs to X and $\overline{\alpha}n = \overline{\beta^n}n$, that is: α is the limit of an infinite sequence of elements of X.

Let s belong to $\mathbb{N}$ and let α belong to $\mathcal{N}$. α *passes through* s, or s *contains* α if and only if there exists n such that $\overline{\alpha}n = s$.

A subset X of $\mathcal{N}$ is called *located*, if the set of all s that contain an element of X is a *decidable* subset of $\mathbb{N}$, that is, there exists β in $\mathcal{N}$ such that, for every s, s contains an element of X if and only if $\beta(s) = 0$.

A subset X of $\mathcal{N}$ is called a *spread* if and only if X is both sequentially closed and located.

Spreads have been introduced in intuitionistic mathematics by Brouwer himself. Their importance is that Brouwer's important assumptions: the Continuity Principle and the bar theorem, generalize from Baire space to spreads.

A subset X of $\mathcal{N}$ is called *semi-located* if the set of all s that contain an element of X is an enumerable subset of $\mathbb{N}$, that is, there exists γ in $\mathcal{N}$ such that, for every s, s

contains an element of X if and only if there exists n such that $\gamma(n) = s + 1$. A subset X of $\mathcal{N}$ is called *closed-and-separable* if and only if X is both sequentially closed and semi-located.

Every spread is a closed-and-separable subset of $\mathcal{N}$, but not conversely, as we shall see in a moment. The notion of a closed-and-separable subset of $\mathcal{N}$ plays an important role in [42].

Finally, a subset X of $\mathcal{N}$ is called *closed* if and only if there exists a decidable subset C of $\mathbb{N}$ such that, for every α, α belongs to X if and only if, for each n, $\overline{\alpha}n$ belongs to C, that is, if and only if there exists β in $\mathcal{N}$ such that, for every α, α belongs to X if and only if, for all n, $\beta(\overline{\alpha}n) = 0$. This notion has been used in the definition of the positive Borel hierarchy in [32] and [45]. In this context, a subset X of $\mathcal{N}$ is called *open* if and only if there exists a decidable subset C of $\mathbb{N}$ such that, for all α, α belongs to X if and only if there exists n such that $\overline{\alpha}n$ belongs to C. It is not difficult to see that a subset X of $\mathcal{N}$ is open if and only if there exists an *enumerable* subset C of $\mathbb{N}$ such that, for all α, α belongs to X if and only if there exists n such that $\overline{\alpha}n$ belongs to C. A subset X of $\mathcal{N}$ is closed if and only there exists an open subset Y of $\mathcal{N}$ such that $X = Y^{\neg}$, that is, for all α, α belongs to X if and only if α does not belong to Y.

Note that every closed subset of $\mathcal{N}$ is a sequentially closed subset of $\mathcal{N}$.

Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$.

We let $D_{n\in\mathbb{N}}(X_n)$, the *disjunction* of the infinite sequence $X_0, X_1, X_2, \ldots$, be the set of all α in $\mathcal{N}$ such that, for some n, α^n belongs to X_n.

We let $C_{n\in\mathbb{N}}(X_n)$, the *conjunction* of the infinite sequence $X_0, X_1, X_2, \ldots$, be the set of all α in $\mathcal{N}$ such that, for all n, α^n belongs to X_n.

Theorem 5.1.

(i) *For every sequentially closed subset of X of $\mathcal{N}$, X is strictly analytic if and only if X is semi-located.*
(ii) *Every inhabited spread, that is, every inhabited and located sequentially closed subset of $\mathcal{N}$ is a strictly analytic subset of $\mathcal{N}$.*
(iii) *It is not true that every sequentially closed and semi-located subset of $\mathcal{C}$ is a located subset of $\mathcal{C}$.*
(iv) *It is not true that every closed and semi-located subset of $\mathcal{N}$ is a located subset of $\mathcal{N}$.*
(v) *It is not true that every inhabited and closed subset of $\mathcal{N}$ is semi-located.*
(vi) *It is not true that every sequentially closed subset of $\mathcal{N}$ is a closed subset of $\mathcal{N}$.*
(vii) *It is not true that, for all subsets X, Y of $\mathcal{N}$, if X, Y are spreads and $X \cap Y$ is inhabited, then $X \cap Y$ is strictly analytic.*
(viii) *For every infinite sequence $X_0, X_1, X_2, \ldots$ of subsets of $\mathcal{N}$, if, for each n, the set X_n is strictly analytic, then the sets $D_{n\in\mathbb{N}}(X_n)$ and $C_{n\in\mathbb{N}}(X_n)$ are, both of them, strictly analytic.*
(ix) *For every subset X of $\mathcal{N}$, if X is strictly analytic, then $Ex(X)$ is strictly analytic. In particular, if X is a spread, or a sequentially closed strictly analytic subset of $\mathcal{N}$, then $Ex(X)$ is strictly analytic.*

Proof. (i) Let us first assume that X is a sequentially closed strictly analytic subset of $\mathcal{N}$ and that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ. Observe that for each s, s contains a member of the range of γ if and only if there exists a such that $s \sqsubseteq \gamma|a$. Let δ be an element of $\mathcal{N}$ such that, for each n, *if* $n_I \sqsubset \gamma|n_{II}$, then $\delta(n) = n_I + 1$, and *if not*, then $\delta(n) = 0$. Note that, for each s, s contains an element of the range of γ if and only if there exists n such that $\delta(n) = s + 1$. It follows that X is semi-located and a closed-and-separable subset of $\mathcal{N}$.

Now assume that that X is an inhabited semi-located sequentially closed subset of $\mathcal{N}$ and that δ is an element of $\mathcal{N}$ such that, for each s, s contains an element of X if and only if there exists n such that $\delta(n) = s + 1$.

Note that, as X is inhabited, there exists n such that $\delta(n) = \langle\ \rangle + 1$. Observe that, for every n, for every s, if $\delta(n) = s+1$, then there exist m, p such that $\delta(m) = \delta(n) * \langle p \rangle + 1$, that is, $\delta(m) - 1$ is an immediate successor of $\delta(n)$. We now construct ε such that for each s, $\text{length}(s) = \text{length}\big(\varepsilon(s)\big)$, as follows. We define $\varepsilon(\langle\ \rangle) := \langle\ \rangle$ and, for each s, n, $\varepsilon(s*\langle n \rangle) := \delta(n) - 1$ if $\delta(n) - 1$ is an immediate successor of $\varepsilon(s)$, and $\varepsilon(a*\langle n \rangle) = \delta(p) - 1$, if $\delta(n)$ is not an immediate successor of $\varepsilon(a)$ and p is the least m such that $\delta(m) - 1$ is an immediate successor of $\varepsilon(s)$. We then let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every n, for every α, $\gamma|\alpha$ passes through $\varepsilon(\overline{\alpha}n)$. One verifies easily that the sequentially closed set X coincides with the range of γ.

(ii) This is an easy consequence of (i).

(iii) For every α, we let C_α be the set of all s in $\mathbb{N}$ such that either $\underline{0}$ passes through s, or there exists n such that $\alpha(n) \neq 0$ and $\underline{1}$ passes through s, and we let X_α be the set of all β in $\mathcal{N}$ such that, for all n, $\overline{\beta}n$ belongs to C_α. Note that, for each α, if $\alpha = \underline{0}$ then $X_\alpha = \{\underline{0}\}$, and, if there exists n such that $\alpha(n) \neq 0$, then $X_\alpha = \{\underline{0}, \underline{1}\}$. Also note that, for each α, the sequentially closed set X_α is weakly located, for, if we define γ in $\mathcal{N}$ such that, for each n, *if* either $\underline{0}$ passes through n_I, or there exists $j < n_{II}$ such that $\alpha(j) \neq 0$ and $\underline{1}$ passes through n_I, *then* $\gamma(n) = n_I + 1$, and, *if not, then* $\gamma(n) = 0$, then, for each s, s contains an element of X_α if and only if, for some n, $\gamma(n) = s + 1$.

Now assume that, for each α, X_α is a spread. Then, for each α, *either* $\langle 1 \rangle$ contains an element of X_α *or* $\langle 1 \rangle$ does not contain an element of X_α and, therefore, *either*, for some n, $\alpha(n) \neq 0$, *or*: $\alpha = \underline{0}$. We easily obtain a contradiction by the Continuity Principle.

(iv) For every α, we let C_α be the set of all s in $\mathbb{N}$ such that either $\underline{0}$ passes through s or $s(0) = 1$ and, if *length*$(s) > 1$ then $s(2)$ is the least n such that $\alpha(n) \neq 0$ and we let X_α be the set of all β in $\mathcal{N}$ such that, for all n, $\overline{\beta}n$ belongs to C_α. Note that, for each α, if $\alpha = \underline{0}$, then $X_\alpha = \{\underline{0}\}$, and, if there exists n such that $\alpha(n) \neq 0$, then $X_\alpha = \{\underline{0}\} \cup \{\langle 1, p \rangle * \alpha | \alpha \in \mathcal{N}\}$, where p is the least n such that $\alpha(n) \neq 0$. Also note that X_α is a closed subset of $\mathcal{N}$, and that X_α is weakly located, for, if we define γ in $\mathcal{N}$ such that, for each n, *if* either $\underline{0}$ passes through n_I, or there exists $p < n_{II}$ such that p is the least j such that $\alpha(j) \neq 0$ and $n_I = \langle 1 \rangle$, or *length*$(n_{II}) > 1$ and $n_{II}(0) = 1$ and $n_{II}(1)$ is is the least j such that $\alpha(j) \neq 0$, *then* $\gamma(n) = n_I + 1$, and, *if not, then* $\gamma(n) = 0$, then, for each s, s contains an element of X_α if and only if, for some n, $\gamma(n) = s + 1$.

Now assume that, for each α, X_α is a spread. Then, for each α, *either* $\langle 1 \rangle$ contains an element of X_α *or* $\langle 1 \rangle$ does not contain an element of X_α and, therefore, *either*, for some n, $\alpha(n) \neq 0$, *or*: $\alpha = \underline{0}$. We easily obtain a contradiction by the Continuity Principle.

(v) For each α, we let B_α the set of all s in $\mathbb{N}$ such that either $\underline{0}$ passes through s or $\overline{\alpha}n = \overline{\underline{0}}n$ and $\underline{1}$ passes through s. We let Y_α be the set of all β such that, for each n, $\overline{\beta}n$ belongs to B_α. Note that Y_α is a closed subset of $\mathcal{C}$. Let C_α be the set of all s that contain an element of Y_α. Observe that, for each s, s belongs to C_α if and only if either $\underline{0}$ passes through s or $\alpha = \underline{0}$ and $\underline{1}$ passes through s. Assume that, for each α, the set Y_α is semi-located. Then, for each α, there exists γ such that, for each s, s belongs to C_α if and only if there exists n such that $\gamma(n) = s + 1$. Applying the Second Axiom of Continuous Choice we find a function e from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α, for each s, s belongs to C_α if and only if there exists n such that $(e|\alpha)(n) = s + 1$. Calculate m such that $(e|\underline{0})(m) = \langle 1 \rangle + 1$. Calculate p such that, for every α, if $\overline{\alpha}p = \overline{\underline{0}}p$, then $(e|\alpha)(m) = (e|\underline{0})(m) = \langle 1 \rangle + 1$. It follows that, for every α, if α passes through $\overline{\underline{0}}p$, then $\underline{1}$ belongs to Y_α and $\alpha = \underline{0}$. Contradiction.

(vi) For each α, we let Z_α be the set of all β in $\mathcal{N}$ such that, $\beta = \underline{0}$ and $\alpha \# \underline{0}$. Note that, for each α, Z_α is sequentially closed. Assume that, for each α, Z_α is a closed subset of $\mathcal{N}$, that is, there exists γ such that, for each β, β belongs to Z_α if and only if, for each n, $\gamma(\overline{\beta}n) = 0$. Applying the Second Axiom of Continuous Choice we find a function e from $\mathcal{N}$ to $\mathcal{N}$ such that for all α, for each β, β belongs to Z_α if and only if, for each n, $(e|\alpha)(\overline{\beta}n) = 0$. It follows that, for each α, $\alpha \# \underline{0}$ if and only if, for each n, $(e|\alpha)(\overline{\underline{0}}n) = 0$. Note that, for each n, $(e|\underline{0})(\overline{\underline{0}}n) = 0$. (Let n be a natural number. Find p such that, for each α, if $\overline{\alpha}p = \overline{\underline{0}}p$, then $(e|\alpha)(\overline{\underline{0}}n) = (e|\underline{0})(\overline{\underline{0}}n)$. Consider $\alpha = \overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$ and note: $\alpha \# \underline{0}$, and therefore, $(e|\alpha)(\overline{\underline{0}}n = 0$ and $(e|\underline{0})(\overline{\underline{0}}n) = 0$.) We most conclude: $\underline{0} \# \underline{0}$: contradiction.

This argument is almost the same as the argument that E_1 positively fails to reduce to A_1, see [45], Theorem 6.2.

(vii) For each α we let V_α be the set of all β such that, for each n, *either* $\overline{\beta}n = \overline{\underline{0}}n$ *or* $\overline{\beta}n = \overline{\underline{1}}n$ and $\overline{\alpha}n = \overline{\underline{0}}n$, *or* $\overline{\beta}n = \overline{\underline{1}}(k+1) * \overline{\underline{0}}m$ where k is the least p such that $\alpha(p) \neq 0$ and $m = n - k - 1$. Note that, for each α, the set V_α is a spread. Assume that, for all α, for all β, the set $V_\alpha \cap V_\beta$ is strictly analytic. It then follows, from (i), that, for each α, there exists δ in $\mathcal{N}$ such that, for each s, s contains an element of $V_{\alpha^0} \cap V_{\alpha^1}$ if and only if, for some n, $\delta(n) = s + 1$.

Applying the Second Axiom of Continuous Choice we find a function e from $\mathcal{N}$ to $\mathcal{N}$ such that for all α, $e|\alpha$, for each s, s contains an element of $V_{\alpha^0} \cap V_{\alpha^1}$ if and only if, for some n, $(e|\alpha)(n) = s + 1$.

Note that $\langle 1 \rangle$ contains an element of $V_{\underline{0}^0} \cap V_{\underline{0}^1}$ and find p such that $(e|\underline{0})(p) = \langle 1 \rangle + 1$. Then find m such that, for each α, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $(e|\alpha)(p) = (e|\underline{0})(p)$. It follows that, for each α passing through $\overline{\underline{0}}m$, $\langle 1 \rangle$ contains an element of $V_{\alpha^0} \cap V_{\alpha^1}$. Note that, for each α, $\langle 1 \rangle$ contains an element of $V_{\alpha^0} \cap V_{\alpha^1}$ if and only if, for all k, if k is the least n such that $\alpha^0(n) \neq 0$, then k is also the least n such that $\alpha^1(n) \neq 0$. Now consider a sequence α with the property $\overline{\alpha}m = \overline{\underline{0}}m$ and $\alpha^0 = \overline{\underline{0}}m * \underline{1}$ and $\alpha^1 = \overline{\underline{0}}(m+1) * \underline{1}$. On the one hand, the set $V_{\alpha^0} \cap V_{\alpha^1}$ coincides with the set $\{\underline{0}\}$, but, on the other hand $\langle 1 \rangle$ contains an element of the set $V_{\alpha^0} \cap V_{\alpha^1}$. Contradiction.

(viii) Let $X_0, X_1, X_2, \dots$ be an infinite sequence of strictly analytic subsets of $\mathcal{N}$. Using the Second Axiom of Countable Choice, find γ in $\mathcal{N}$, such that, for each n, γ^n is a function from $\mathcal{N}$ to $\mathcal{N}$ such that X_n coincides with the range of γ^n.

Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\delta|\alpha)^{\delta(0)} = \gamma^{\delta(0)}|(\alpha^{\delta(0)})$, and, for each i, if there is no j such that $i = \langle \delta(0) \rangle * j$, then $(\delta|\alpha)(i) = \alpha(i)$. Note that $D_{n\in\mathbb{N}}(X_n)$ coincides with the range of δ.

Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each n, $(\varepsilon|\alpha)^n = \gamma^n|(\alpha^n)$, and $(\varepsilon|\alpha)(0) = \alpha(0)$. Note that $C_{n\in\mathbb{N}}(X_n)$ coincides with the range of ε.

(ix) Let X be a strictly analytic subset of $\mathcal{N}$. Find a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, α belongs to X if and only if, for some β, $\alpha = \gamma|\beta$. Note that, for each α, α belongs to $Ex(X)$ if and only if, for some β, $\alpha = (\gamma|\beta)^0$. Thus, $Ex(X)$ is seen to be strictly analytic. □

We thus see that the class of the strictly analytic subsets of $\mathcal{N}$ is a proper subclass of the class of the analytic subsets of $\mathcal{N}$, and that the class of the strictly analytic subsets of $\mathcal{N}$ is not closed under the operation of finite intersection.

Nevertheless, there are many strictly analytic sets.

Note that the sets $E_1 = \{\alpha|\exists n[\alpha(n) \neq 0]\}$ and $A_1 = \{\underline{0}\}$ are strictly analytic.

Recall, from Subsection 1.2, that 1^* is the element of $\mathcal{C}$ satisfying: $1^*(0) = 0$, for each $n > 0$, $1^*(n) = 1$.

Recall that a stump is called a *non-zero stump* if and only if either: $\sigma = 1^*$ or: $\sigma(0) = 0$ and, for each n, $\sigma(\langle n \rangle) = 0$ and σ^n is a non-zero stump.

Recall that, for each non-zero stump σ, we defined subsets E_σ and A_σ of $\mathcal{N}$ as follows, by induction.

$E_{1^*} := E_1$ and $A_{1^*} = A_1$ and, for each non-zero stump different from 1^*, for each α, α belongs to E_σ if and only if, for some n, α^n belongs to A_{σ^n}, and α belongs to A_σ if and only if, for all n, α^n belongs to E_{σ^n}.

Note that, for each non-zero stump σ, $E_\sigma := D_{n\in\mathbb{N}}(A_{\sigma^n})$ and $A_\sigma := C_{n\in\mathbb{N}}(E_{\sigma^n})$.

These sets have been introduced in [45] and are called there the *leading sets of the positive Borel hierarchy*. It follows from Theorem 5.1(viii) that they are, all of them, strictly analytic.

5.2. **The Brouwer-Kripke axiom intervenes.** The following axiom is the result of an attempt to capture Brouwer's *creating subject arguments*.

Axiom 13 (The Brouwer-Kripke axiom)**.**
Let P be a definite mathematical proposition.
There exists β in $\mathcal{C}$ such that P if and only if, for some n, $\beta(n) = 1$.

The word *'definite'* we use in the formulation of this axiom needs some explanation. P is not allowed to be a statement about an infinite sequence we are building step by step without having fixed its future by an unequivocal description. In a formal context, one might require that the formula corresponding to the proposition should not contain a free variable over elements of Baire space.

The idea that gave rise to the axiom is that, once we are given a definite proposition P, we can start thinking about it, and the truth of P will be equivalent to our finding a proof of P at some point of time, where time is numbered by the natural numbers. So, step by step, we can build an infinite sequence β with the property: for each n, $\beta(n) \neq 0$ if and only if, at stage n, I find a proof of proposition P.

This seems to be a rather wild idea, and, certainly, it seems untenable if we allow P to be a statement about an object that itself is unfinished.

It is no surprise, therefore, that the axiom leads to a contradiction with the Second Axiom of Continuous Choice, Axiom 7, as has been first observed by J. Myhill, see [25]:

> *Suppose that for every α there exists β such that $\forall n[\alpha(n) = 0]$ if and only if $\exists n[\alpha(n) \neq 0]$. By Axiom 7, there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for all α, $\forall n[\alpha(n) = 0]$ if and only if $\exists n[(\gamma|\alpha)(n) \neq 0]$, so A_1 reduces to E_1. Contradiction.*

One might decide now, either to give up the Second Axiom of Continuous Choice, or to give up the Brouwer-Kripke axiom completely, or to restrict the applicability of the Brouwer-Kripke axiom. Myhill chose the first alternative. Johan de Iongh opted for the third one, see [10]. It is difficult to make the restriction he wanted to introduce precise, and we will not attempt to do so. We only observe that the restriction also has to be applied to the definitions of mathematical objects like subsets of the set $\mathbb{N}$ of the natural numbers or subsets of Baire space $\mathcal{N}$.

Theorem 5.2 (Some consequences of the Brouwer-Kripke axiom)**.**
(i) *Every* definite *subset of $\mathbb{N}$ is an enumerable subset of $\mathbb{N}$.*
(ii) *Every* definite *subset of $\mathcal{N}$ is a semilocated subset of $\mathcal{N}$.*
(iii) *Every* definite *analytic subset of $\mathcal{N}$ is a strictly analytic subset of $\mathcal{N}$.*

Proof. (i) Let A be a *definite* subset of $\mathbb{N}$. Given any n in $\mathbb{N}$, the Brouwer-Kripke axiom guarantees there exists β in $\mathcal{C}$ such that n belongs to A if and only if, for some n, $\beta(n) = 1$. Using the Second Axiom of Countable Choice, we find β in $\mathcal{C}$ such that for all n in $\mathbb{N}$, n belongs to A if and only if, for some m, $\beta^n(m) = 1$. Let δ be an element of $\mathcal{N}$ such that, for all p, *if* there exist n, m such that $p = \langle n, m\rangle$ and $\beta^n(m) = 1$, then $\delta(p) = n + 1$ and, *if not*, then $\delta(p) = 0$. Note that the set A coincides with the E_δ. (E_δ is the subset of $\mathbb{N}$ *enumerated by* δ, that is, the set of all n in $\mathbb{N}$ such that, for some p, $\delta(p) = n + 1$).

(ii) Let X be a *definite* subset of $\mathcal{N}$. The set of all s in $\mathbb{N}$ such that s contains an element of X will be a definite subset of $\mathbb{N}$, so we may apply (i) and conclude that this set is enumerable and that X itself is semi-located.

(iii) Let Y be an analytic subset of $\mathcal{N}$. Let X be a closed subset of $\mathcal{N}$ such that $Y = Ex(X)$. We may assume that X is a definite subset of $\mathcal{N}$ and apply (ii). X is closed and semi-located and therefore, according to Theorem 5.1(i), strictly analytic. But then, according to Theorem 5.1(ix), also the set $Y = Ex(X)$ is strictly analytic.

□

John Burgess, in [7], also studies strictly analytic subsets of $\mathcal{N}$, calling them, by a term of of Brouwer's, "*dressed spreads*". (This term is also used in [10]). Applying an unrestricted Brouwer-Kripke-axiom, in a context avoiding the Second Axiom of Continuous Choice, Burgess proves that every analytic subset of $\mathcal{N}$ is strictly analytic. The argument given here for Theorem 5.2(iii) is essentially his.

6. Separation theorems

A version of the next Theorem occurs in [32]. A related result is proven in [1].

For each α in $\mathcal{N}$, we define elements α_I and α_{II} of $\mathcal{N}$ by:

For each n, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$.

Note that, for each α, $\alpha = \langle \alpha_I, \alpha_{II} \rangle$.

For each s in $\mathbb{N}$, we define elements s_I and s_{II} of $\mathbb{N}$ by:

length(s_I) is the least k such that *length*$(s) \geq 2k$ and, for each n, if $n <$ *length*(s_I), then $s_I(n) = s(2n)$, and
length(s_{II}) is the least k such that *length*$(s) \geq 2k+1$ and, for each n, if $n <$ *length*(s_{II}), then $s_{II}(n) = s(2n+1)$.

Theorem 6.1 (Lusin's Separation Theorem)**.**
Let γ be an element of $\mathcal{N}$ such that both γ_I and γ_{II} are functions from $\mathcal{N}$ to $\mathcal{N}$ and, for every α, $\gamma_I|\alpha_I$ is apart from $\gamma_{II}|\alpha_{II}$. There exist positively Borel sets B_0, B_1, such that for every α, $\gamma_I|\alpha_I$ belongs to B_0 and $\gamma_{II}|\alpha_{II}$ belongs to B_1, and every element of B_0 is apart from every element of B_1.

Proof. We use the Principle of Induction on Monotone Bars in $\mathcal{N}$. We let P, Q be subsets of $\mathbb{N}$, such that, for every s in $\mathbb{N}$,

$P(s) :=$ $\gamma_I|s_I \perp \gamma_{II}|s_{II}$.
$Q(s) :=$ There exist positively Borel sets B_0, B_1 such that every element of B_0 is apart from every element of B_1, and, for every α, if α passes through s, then $\gamma|\alpha_I$ belongs to B_0 and $\gamma_{II}|\alpha_{II}$ belongs B_1.

Clearly, for all α in $\mathcal{N}$, there exists n such that $\overline{\alpha}n$ belongs to P and, for all s, for all n, if s belongs to P, then $s * \langle n \rangle$ belongs to P. Also, P is a subset of Q. Now assume s belong to $\mathbb{N}$ and, for every n, $s * \langle n \rangle$ belongs to Q. We want to prove that s itself belongs to Q and we distinguish two cases.

Case (1). *length*(s) is even. Note that now, for each n, $(s * \langle n \rangle)_I = s_I * \langle n \rangle$ and $(s * \langle n \rangle)_{II} = s_{II}$. Using the Second Axiom of Countable Choice, we determine an infinite sequence $(B_{0,0}, B_{0,1}), (B_{1,0}, B_{1,1}), (B_{2,0}, B_{2,1}), \ldots$ of pairs of positively Borel sets such that, for each n, for every α passing through $s * \langle n \rangle$, for every n, $\gamma_I|\alpha_I$ belongs to $B_{n,0}$ and $\gamma_{II}|\alpha_{II}$ belongs to $B_{n,1}$, and every element of $B_{n,0}$ is apart from every element of $B_{n,1}$.

Define $B_0 := \bigcup_{n \in \mathbb{N}} B_{n,0}$ and $B_1 := \bigcap_{n \in \mathbb{N}} B_{n,1}$.

Note that, for each α, if α passes through s, then $\gamma_I|\alpha_I$ belongs to B_0 and $\gamma_{II}|\alpha_{II}$ belongs to B_1, and every element of B_0 is apart from every element of B_1.

Case (2). *length*(s) is odd. Note that now, for each n, $(s*\langle n\rangle)_I = s_I$ and $(s*\langle n\rangle)_{II} = s_{II} * \langle n\rangle$. Using the Second Axiom of Countable Choice, we determine an infinite sequence $(B_{0,0}, B_{0,1}), (B_{1,0}, B_{1,1}), (B_{2,0}, B_{2,1}), \ldots$ of pairs of positively Borel sets such that, for each n, for every α passing through $s * \langle n\rangle$, for every n, $\gamma_I|\alpha_I$ belongs to $B_{n,0}$ and $\gamma_{II}|\alpha_{II}$ belongs to $B_{n,1}$, and every element of $B_{n,0}$ is apart from every element of $B_{n,1}$.

Define $B_0 := \bigcap_{n\in\mathbb{N}} B_{n,0}$ and $B_1 := \bigcup_{n\in\mathbb{N}} B_{n,1}$.

Note that, for each α, if α passes through s, then $\gamma_I|(\alpha_I)$ belongs to B_0 and $\gamma_{II}|(\alpha_{II})$ belongs to B_1, and every element of B_0 is apart from every element of B_1.

In both cases, therefore, we find that s itself belongs to Q.

The Principle of Induction on Monotone Bars in $\mathcal{N}$ allows us to conclude that the pair $\langle\ \rangle$ belongs to Q, and the conclusion of the Theorem follows. □

Lusin's Separation Theorem admits of several extensions.

Recall that, for each α in $\mathcal{N}$, for each i in $\mathbb{N}$ we defined an elements α^i of $\mathcal{N}$ by:

for each n, $\alpha^i(n) = \alpha(\langle i\rangle * n)$.

For each s in $\mathbb{N}$, for each i, we now define an elements s^i of $\mathbb{N}$ by:

length(s^i) is the least k such that *length*$(s) \geq \langle i\rangle * k$ and, for each n, if $n <$ *length*(s^i), then $s^i(n) = s(\langle i\rangle * n)$.

Let n be a positive natural number and let $X_0, X_1, \ldots, X_{n-1}$ be a finite sequence of subsets of $\mathcal{N}$ of length n. We call the finite sequence $X_0, X_1, \ldots, X_{n-1}$ *positively disjoint* if and only if, for every β in $\mathcal{N}$, if, for each $i < n$, β^i belongs to X_i, then there exist i, j such that $i < n$ and $j < n$ and β^i is apart from β^j.

Theorem 6.2 (An extension of Lusin's Separation Theorem)**.**
Let n be a positive natural number and let γ be an element of $\mathcal{N}$ such that, for each $i < n$, γ^i is a function from $\mathcal{N}$ to $\mathcal{N}$ and, for every α, there exist $i, j < n$ such that $\gamma^i|\alpha^i$ is apart from $\gamma^j|\alpha^j$. There exists a finite sequence $B_0, B_1, \ldots, B_{n-1}$ of positively Borel sets, such that for every α, for every $i < n$, $\gamma^i|\alpha^i$ belongs to B_i, and the finite sequence $B_0, B_1, \ldots, B_{n-1}$ is positively disjoint.

Proof. We use the Principle of Induction on Monotone Bars in $\mathcal{N}$. We let P, Q be subsets of $\mathbb{N}$, such that, for every s in $\mathbb{N}$,

$P(s) :=$ There exist $i, j < n$ such that $\gamma^i|s^i \perp \gamma^j|s^j$.
$Q(s) :=$ There exists a n-sequence $(B_0, B_1, \ldots, B_{n-1})$ of positively Borel sets that is positively disjoint and such that, for every α, if α passes through s, then, for each $i < n$, $\gamma^i|\alpha^i$ to B_i.

Clearly, for all α in $\mathcal{N}$, there exists n such that $\overline{\alpha}n$ belongs to P and, for all s, for all n, if s belongs to P, then $s * \langle n\rangle$ belongs to P.

The following argument shows that P is a subset of Q. Suppose that s belongs to P. Find $i, j < n$ such that $\gamma^i|s^i \perp \gamma^j|s^j$. Define a finite sequence of positively Borel sets $(B_0, B_1, \ldots, B_{n-1})$ such that B_i is the set of all α passing through $\gamma^i|s^i$, and B_j is the set of all α passing through $\gamma^j|s^j$, and, for all $k < n$, if $k \neq i$ and $k \neq j$, then $B_k = \mathcal{N}$. Note that the finite sequence $(B_0, B_1, \ldots, B_{n-1})$ is positively disjoint and for every α, if α passes through s, then, for each $i < n$, $\gamma^i|\alpha^i$ to B_i.

Now assume s belong to $\mathbb{N}$ and, for every m, $s * \langle m\rangle$ belongs to Q. We want to prove that s itself belongs to Q and we do so as follows.

First, find i, p such that *length*$(s) = \langle i\rangle * p$. Note that, for each n, $(s * \langle n\rangle)^i = s^i * \langle n\rangle$, and, for each j, if $j \neq i$, then $(s * \langle n\rangle)^j = s^j$.

We now distinguish two cases.

Case (1). $i \geq n$. Note that, in this case, the statement that s belongs to Q is equivalent to the statement that $s * \langle 0 \rangle$ belongs to Q, so we may conclude that s belongs to Q.

Case (2). $i < n$. Using the Second Axiom of Countable Choice, we determine an infinite sequence $(B_{0,0}, B_{0,1}, \ldots, B_{0,n-1})$, $(B_{1,0}, B_{1,1}, \ldots, B_{1,n-1})$, $(B_{2,0}, B_{2,1}, \ldots, B_{2,n-1}), \ldots$ of positively disjoint n-sequences of of positively Borel sets such that, for each m, for every α passing through $s * \langle m \rangle$, for every $j < n$, $\gamma^j|(\alpha^j)$ belongs to $B_{m,j}$. Define $B_i := \bigcup_{m \in \mathbb{N}} B_{m,i}$ and, for each j, if $j < n$ and $j \neq i$, then $B_j := \bigcap_{m \in \mathbb{N}} B_{m,j}$.

Note that, for each α, if α passes through s, then α^i passes through $s^i * \langle \alpha(\mathit{length}(s)) \rangle$, and $\gamma^i|\alpha^i$ belongs to $B_{\alpha(\mathit{length}(s)),i}$, and, therefore, to B_i.

Also note that, for each α, for each $j < n$, if α passes through s, then α^j passes through s^j, and, for each m, α^j passes through $(s * \langle m \rangle)^j$, as $(s * \langle m \rangle)^j$ coincides with s^j, and, therefore, $\gamma^j|(\alpha^j)$ belongs to each set $B_{m,j}$ and thus to B_j.

Finally, observe that the n-sequence $(B_0, B_1, \ldots, B_{n-1})$ is positively disjoint:

Suppose that β belongs to $\mathcal{N}$, and, for each $j < n$, β^j belongs to B_j. Find m such that β^i belongs $B_{m,i}$. Note that, for each j, if $j < n$ and $j \neq i$, then β^j belongs to $B_{m,j}$. So we may find $k, l < n$ such that β^k is apart from β^l.

In both cases, therefore, we find that s itself belongs to Q.

The Principle of Induction on Monotone Bars in $\mathcal{N}$ allows us to conclude that $\langle\ \rangle$ belongs to Q, and the conclusion of the Theorem follows. □

Let $X_0, X_1, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$. We call the infinite sequence $X_0, X_1, \ldots$ *positively disjoint* if and only if, for every β in $\mathcal{N}$, if, for each i, β^i belongs to X_i, then there exist i, j such that β^i is apart from β^j.

Theorem 6.3 (Novikov's Separation Theorem)**.**
Let γ be an element of $\mathcal{N}$ such that, for each i, γ^i is a function from $\mathcal{N}$ to $\mathcal{N}$ and, for every α, there exist i, j such that $\gamma^i|\alpha^i$ is apart from $\gamma^j|\alpha^j$. There exists an infinite sequence $B_0, B_1, \ldots$ of positively Borel sets such that, for every α, for every i, $\gamma^i|\alpha^i$ belongs to B_i, and the infinite sequence $B_0, B_1, \ldots$ is positively disjoint.

Proof. We use the Principle of Induction on Monotone Bars in $\mathcal{N}$. We let P, Q be subsets of $\mathbb{N}$, such that, for every s in $\mathbb{N}$,

$P(s) :=$ There exist i, j such that $\gamma^i|s^i \perp \gamma^j|s^j$.
$Q(s) :=$ There exist an infinite sequence of positively Borel sets $B_0, B_1, \ldots$ that is positively disjoint and such that for every α, if α passes through s, then, for each i, $\gamma^i|\alpha^i$ belongs to B_i.

Clearly, for all α in $\mathcal{N}$, there exists n such that $\overline{\alpha}n$ belongs to P and, for all s, for all n, if s belongs to P, then $s * \langle n \rangle$ belongs to P.

The following argument shows that P is a subset of Q. Suppose that s belongs to P. Find i, j, such that $\gamma^i|s^i \perp \gamma^j|s^j$. Define an infinite sequence of positively Borel sets $B_0, B_1, \ldots$ such that B_i is the set of all α passing through $\gamma^i|s^i$, and B_j is the set of all α passing through $\gamma^j|s^j$, and, for all k, if $k \neq i$ and $k \neq j$, then $B_k = \mathcal{N}$. Note that the infinite sequence $B_0, B_1, \ldots$ is positively disjoint and, for every α, if α passes through s, then, for each i, $\gamma^i|\alpha^i$ to B_i.

Now assume s belong to $\mathbb{N}$ and, for every m, $s * \langle m \rangle$ belongs to Q. We want to prove that s itself belongs to Q and we do so as follows.

First, find i, p such that $\mathit{length}(s) = \langle i \rangle * p$. Note that, for each n, $(s * \langle n \rangle)^i = s^i * \langle n \rangle$, and, for each j, if $j \neq i$, then $(s * \langle n \rangle)^j = s^j$.

Using the Second Axiom of Countable Choice, we determine an infinite sequence $B_{0,0}, B_{0,1}, \ldots$, $B_{1,0}, B_{1,1}, \ldots$, $(B_{2,0}, B_{2,1}, \ldots, \ldots$ of positively disjoint infinite sequences of of positively Borel sets such that, for each m, for every α passing through $s * \langle m \rangle$, for every $j < n$, $\gamma^j|\alpha^j$ belongs to $B_{m,j}$. Define $B_i := \bigcup_{m\in\mathbb{N}} B_{m,i}$ and, for each j, if $j \neq i$, then $B_j := \bigcap_{m\in\mathbb{N}} B_{m,j}$.

Note that, for each α, if α passes through s, then α^i passes through $s^i * \langle\alpha(length(s))\rangle$, and $\gamma^i|\alpha^i$ belongs to $B_{\alpha(length(s)),i}$, and, therefore, to B_i.

Also note that, for each α, for each $j < n$, if α passes through s, then α^j passes through s^j, and, for each m, α^j passes through $(s * \langle m\rangle)^j$, as $(s * \langle m\rangle)^j$ coincides with s^j, and, therefore, $\gamma^j|(\alpha^j)$ belongs to each set $B_{m,j}$ and thus to B_j.

Finally, observe that the infinite sequence $B_0, B_1, \ldots$ is positively disjoint:

Suppose that β belongs to $\mathcal{N}$, and, for each $j < n$, β^j belongs to B_j. Find m such that β^i belongs $B_{m,i}$. Note that, for each j, if $j \neq i$, then β^j belongs to $B_{m,j}$. So we may find k, l such that β^k is apart from β^l.

We have to admit that s itself belongs to Q.

The Principle of Induction on Monotone Bars in $\mathcal{N}$ allows us to conclude that the pair $\langle\,\rangle$ belongs to Q, and the conclusion of the Theorem follows. □

Theorem 6.4 (One half of Lusin's Regular Representation Theorem)**.**

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$, such that for all α, β, if α is apart from β, then $\gamma|\alpha$ is apart from $\gamma|\beta$.
Then the range of γ, that is, the set of all α coinciding with some $\gamma|\beta$, is positively Borel.

Proof. Assume that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ and that for all α, β, if α is apart from β, then $\gamma|\alpha$ is apart from $\gamma|\beta$.

We intend to define a sequence $H_0, H_1, \ldots$ of positively Borel sets such that the range of γ coincides with $\bigcap_{n\in\mathbb{N}} H_n$.

To this end, we first define a sequence $B_0, B_1, \ldots$ of decidable subsets of $\mathbb{N}$. Each of these will be a bar in $\mathcal{N}$.

We let B_0 be the set $\{\langle\,\rangle\}$ consisting of the empty sequence only. For each n, we let B_{n+1} be the set of all a of minimal length such that some proper initial segment of a belongs to B_n, and for some initial part b of a, $\gamma^n(b) \neq 0$. Observe that for each n, for all a in B_n, for all α passing through a, $\overline{(\gamma|\alpha)}n$ is an initial part of $\gamma|a$, and that, for each n, B_n is a decidable subset of $\mathbb{N}$ and a bar in $\mathcal{N}$.

Note that, for each n, B_n is a *thin bar in* $\mathcal{N}$, that is, for all a, b in B_n, if $a \neq b$, then $a \perp b$. It follows that, for each n, for all a, b in B_n, if $a \neq b$, then, for each α passing through a, for each β passing through b, α is apart from β, and, therefore, $\gamma|\alpha$ is apart from $\gamma|\beta$. Using Theorem 6.1 and the Second Axiom of Countable Choice we find for each n, for all a, b in B_n such that $a \neq b$, positively Borel sets $C_{a,b}$, $D_{a,b}$ such that for all α, if α passes through a, then $\gamma|\alpha$ belongs to $C_{a,b}$ and, if α passes through b, then $\gamma|\alpha$ belongs to $D_{a,b}$, and every element of $C_{a,b}$ is apart from every element of $D_{a,b}$. For each n, for each a in B_n, we let K_a be the set of all β passing through $\gamma|a$ and belonging to every set $C_{a,b} \cap D_{b,a}$, where b belongs to B_n and $b \neq a$. Remark that for all n, for all a, b in B_n, if $a \neq b$, then every element of K_a is apart from every element of K_b. For each n, for each a in B_n we let L_a be the set of all β belonging to every set K_b, where b is an initial segment of a, and for some $i \leq n$, b belongs to B_i.

Note that, for each n, for all a in B_n, for all b in B_{n+1}, if a is an initial part of B, then $L_b \subseteq L_a$. It follows that, for each n, for each k, for all a in B_n, for all b in B_{n+k}, if a is an initial part of b, then $L_b \subseteq L_a$.

Note that, for each n, for all a in B_n, for all b in B_{n+1}, if a is not an initial part of b, then $L_a \cap L_c = \emptyset$, where c the initial part of b that belongs to B_n, as c must be different from a, and, therefore, $L_a \cap L_b = \emptyset$.
It follows that, for each n, for each k, for all a in B_n, for all b in B_{n+k}, if a is not an initial part of b, then $L_a \cap L_b = \emptyset$.

We define, for each n, $H_n := \bigcup_{a \in B_n} L_a$. It will be clear that the range of γ is a subset of of every set H_n.

Now assume that β belongs to every set H_n. For each n, we determine a_n in B_n such that β belongs $L_{(a_n)}$. Observe that, for each n, a_n must be a proper initial segment of a_{n+1} and consider the sequence α that passes through every a_n. Observe that, for each n, β belongs to $K_{(a_n)}$ and therefore passes through $\gamma|(a_n)$ and also through $\overline{(\gamma|\alpha)}n$, as $\overline{(\gamma|\alpha)}n$ is an initial part of $\gamma|(a_n)$, so β coincides with $\gamma|\alpha$.

Therefore, the range of γ coincides with the set $\bigcap_{n \in \mathbb{N}} H_n$. Note that the latter set is positively Borel. $\square$

We just proved, in Theorem 6.4, one half of Lusin's classical regular Representation Theorem. The other half is the statement:

> "*For every subset X of $\mathcal{N}$, if X is Borel, then there exist a closed subset Y of $\mathcal{N}$ and a one-to-one continuous function f from Y onto X.*"

From a constructive point of view, this statement may be given several interpretations, none of them without its difficulties. We do not want to go through the various possible interpretations and only make the following observation. Suppose that we interpret "Borel" by "positively Borel" and "closed subset" by "spread" and that we define:

> A subset X of $\mathcal{N}$ is *regular in Lusin's sense* or: *has Lusin's property* if and only if there exists a subset F of $\mathcal{N}$ that is a spread and a strongly one-to-one continuous function from F onto X.

Note that every subset of $\mathcal{N}$ that is regular in Lusin's sense is also strictly analytic. We have seen, in the first and the fifth item of Theorem 5.1 that not every inhabited and closed subset of $\mathcal{N}$ is strictly analytic, and, therefore, not every inhabited and closed subset of $\mathcal{N}$ is regular in Lusin's sense.

What about positive Borel sets that are strictly analytic? One might hope that all such sets are regular in Lusin's sense, but the following theorem refutes this conjecture. We define:

> Let X be a strictly analytic subset of $\mathcal{N}$. We say that X *positively fails to be regular in Lusin's sense* if, for every subset F of $\mathcal{N}$ that is a spread, for every function γ that is a function from F to $\mathcal{N}$ mapping F onto X, there exist α, β in F such that $\alpha \# \beta$ and $\gamma|\alpha = \gamma|\beta$.

Note that the set $D^2(A_1) = \{\alpha | \alpha^0 = \underline{0} \ \vee \ \alpha^1 = \underline{0}\}$ is strictly analytic. Define a function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, if $\alpha(0) = 0$, then $(\gamma|\alpha)^0 = \underline{0}$, and for each n, if there is no m such that $n = \langle 0, m\rangle$, then $(\gamma|\alpha)(n) = \alpha(n)$ and, if $\alpha(0) \neq 0$, then $(\gamma|\alpha)^1 = \underline{0}$, and for each n, if there is no m such that $n = \langle 1, m\rangle$, then $(\gamma|\alpha)(n) = \alpha(n)$. One easily verifies that the set $D^2(A_1)$ coincides with the range of γ.

Theorem 6.5. *The set $D^2(A_1)$ positively fails to be regular in Lusin's sense.*

Proof. Suppose we find a subset F of $\mathcal{N}$ that is closed and located and an element γ in $\mathcal{N}$ such that γ is a continuous function from F onto $D^2(A_1)$. For each $i < 2$, we let P_i be the set of all α such that $\alpha^i = \underline{0}$. Note that $D^2(A_1) = P_0 \cup P_1$ and that P_0 and P_1 are spreads. Using the Second Axiom of Continuous Choice, we find δ such that, for both $i < 2$, δ^i is a continuous function from P_i to F such that, for all α in P_i, $\gamma|(\delta^i|\alpha) = \alpha$. Define ε in $\mathcal{N}$ such that $\varepsilon^0 := \delta^0|\underline{0}$ and $\varepsilon^1 := \delta^1|\underline{0}$.

We claim: for each n, there exists ζ in F such that $\overline{\zeta}n = \overline{\varepsilon^0}n$ and $(\gamma|\zeta)^1 \# \underline{0}$. We prove this claim as follows. Let n belong to $\mathbb{N}$. Find m such that, for all α in P_0, if $\overline{\alpha}m = \overline{\underline{0}}m$, then $\overline{\delta^0|\alpha}n = \overline{\varepsilon^0}n$. Determine α in P_0 such that $\overline{\alpha}m = \overline{\underline{0}}m$ and $\alpha^1(m) \neq 0$ and consider $\zeta = \delta^0|\alpha$. Note that $\overline{\zeta}n = \overline{\varepsilon^0}n$ and $(\gamma|\zeta)^1(m) \neq 0$.

Using the fact that γ maps the spread F to the set $D^2(A_1)$ and that ε belongs to F, and Brouwer's Continuity Principle, we determine n_0 and i_0 such that $i_0 < 2$ and, for all ζ in F, if $\overline{\zeta}n_0 = \overline{\varepsilon^0}n_0$, then $(\gamma|\zeta)^{i_0} = \underline{0}$. As there exists ζ in F passing through $\overline{\varepsilon^0}n_0$ such that $(\gamma|\zeta)^1 \# \underline{0}$, we must have: $i_0 = 0$.

We conclude: for all ζ in F, if $\overline{\zeta}n_0 = \overline{\varepsilon^0}n_0$, then $(\gamma|\zeta)^0 = \underline{0}$.

In a similar way, one may determine n_1 in $\mathbb{N}$ such that, for all ζ in F, if $\overline{\zeta}n_0 = \overline{\varepsilon^0}n_0$, then $(\gamma|\zeta)^1 = \underline{0}$.

Clearly, then, $\overline{\varepsilon^0}n_0 \perp \overline{\varepsilon^1}n_1$ and: $\varepsilon^0 \# \varepsilon^1$.

We thus see that the set $D^2(A_1)$ positively fails to be regular in Lusin's sense. □

As the set $D^2(A_1)$ is a fairly simple example of a positively Borel set that is also strictly analytic, Theorem 6.5 shows that it is not so easy, for a strictly analytic positively Borel set, to have Lusin's property. The set $E_2!$, to be discussed in the next Section, see Theorem 6.4, is an example of a set that is positively Borel and strictly analytic and has Lusin's property, but, like the set $D^2(A_1)$, fails to be co-analytic. It is not true, therefore, that positively Borel sets that have Lusin's property must be co-analytic.

It is a pity that we do not have a satisfying intuitionistic counterpart to the other half of Lusin's Theorem. Lusin observed that his representation theorem may help one to believe, in spite of possible qualms about generalized inductive definitions, that, after all, the collection of all positively Borel subsets of $\mathcal{N}$ is a set, see [20], pp. 38-39.

7. Co-analytic subsets of $\mathcal{N}$: some examples

7.1. The class of the co-analytic subsets of $\mathcal{N}$.

As in Section 1, we let $\langle\ ,\ \rangle$ denote the pairing function on $\mathcal{N}$ satisfying: for each α, for each β, for each n, $\langle\alpha,\beta\rangle(2n) = \alpha(n)$ and $\langle\alpha,\beta\rangle(2n+1) = \beta(n)$. For each α in $\mathcal{N}$, we defined elements α_I and α_{II} of $\mathcal{N}$ satisfying: for each n $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$. Note that, for each α, $\alpha = \langle\alpha_I, \alpha_{II}\rangle$.

For every subset X of $\mathcal{N}$, we let *the (universal) projection* of X, also called the *co-projection* of X, notation $Un(X)$, be the set of all α such that, for all β, $\langle\alpha,\beta\rangle$ belongs to X.

A subset X of $\mathcal{N}$ is called *co-analytic* if and only if there exists an open subset Y of $\mathcal{N}$ such that X coincides with $Un(Y)$. The class of the co-analytic subsets of $\mathcal{N}$ is denoted by $\mathbf{\Pi}^1_1$.

Recall that we parametrized the class $\mathbf{\Sigma}^0_1$ of the open subsets of $\mathcal{N}$ as follows. For each β, let G_β be the set of all α in $\mathcal{N}$ such that, for some n, $\beta(\overline{\alpha}n) \neq 0$. A subset X of $\mathcal{N}$ is open if and only there exists β such that X coincides with G_β.

One may parametrize the class $\mathbf{\Pi}^1_1$ of the co-analytic subsets of $\mathcal{N}$ as follows. For each β, let CA_β be the set $Un(G_\beta)$, that is the set of all α in $\mathcal{N}$ such that, for every γ, the pair $\langle\alpha,\gamma\rangle$ belongs to G_β. A subset X of $\mathcal{N}$ is co-analytic if and only there exists β such that X coincides with CA_β.

We let UP^1_1 be the set of all α such that α_{II} belongs to CA_{α_I}.

Recall that, for each subset X of $\mathcal{N}$, for each β, we let $X \upharpoonright \beta$ be the set of all α such that $\langle\beta,\alpha\rangle$ belongs to X.

As in Section 1, we use S^* to denote the successor-function from $\mathbb{N}$ to $\mathbb{N}$, and $\circ$ to denote the operation of composition on $\mathcal{N}$.

For every subset X of $\mathcal{N}$, we let *Perhaps*(X) be the set of all α in $\mathcal{N}$ such that there exists β in X with the property: if $\alpha \# \beta$, then α belongs to X.

For every inhabited subset X of $\mathcal{N}$, $X \subseteq$ *Perhaps*$(X) \subseteq X^{\neg\neg}$, where, for each subset Y of $\mathcal{N}$, $Y^\neg$ denotes the complement of Y, that is, the set of all α in $\mathcal{N}$ such that the assumption '$\alpha \in Y$' leads to a contradiction.

The notion *Perhaps* has been studied extensively in [35], [36], [41] en [44].

A subset X of $\mathcal{N}$ is called *perhapsive* if the set $Perhaps(X)$ coincides with the set X.

We use S^* to denote the element of $\mathcal{N}$ satisfying: for all n, $S^*(n) = n + 1$.

Theorem 7.1.

(i) *The set UP^1_1 is a universal element of the class $\mathbf{\Pi}^1_1$.*

(ii) *For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, X_n belongs to $\mathbf{\Pi}^1_1$, then $\bigcap\limits_{n\in\mathbb{N}} X_n$ belongs to $\mathbf{\Pi}^1_1$.*

(iii) *Every co-analytic subset of $\mathcal{N}$ is perhapsive.*

(iv) *The set $D^2(A_1)$ is not co-analytic.*

(v) *For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Pi}^1_1$, then $Un(X)$ belongs to $\mathbf{\Pi}^1_1$.*

(vi) *Every $\mathbf{\Pi}^0_2$-subset of $\mathcal{N}$ is co-analytic.*

(vii) *For all subsets X, Y of $\mathcal{N}$, if X reduces to Y and Y is co-analytic, then X is co-analytic.*

Proof. (i) Note that, for each α, α belongs to UP^1_1 if and only if α_{II} belongs to CA_{α_I} if and only if, for all γ, $\langle\alpha_{II}, \gamma\rangle$ belongs to G_{α_I} if and only if, for all γ, there exists n such that $\alpha_I(\overline{\langle\alpha_{II}, \gamma\rangle}n) \neq 0$. We thus see that UP^1_1 belongs to the class $\mathbf{\Pi}^1_1$. The fact that, for each element Y of $\mathbf{\Pi}^1_1$, there exists β such that Y coincides with $UP^1_1 \upharpoonright \beta$ follows from the observations preceding this theorem.

(ii) Let $Y_0, Y_1, \ldots$ be a sequence of open subsets of $\mathcal{N}$ such that, for each n, $X_n = Un(Y_n)$. Using the Second Axiom of Countable Choice, find β in $\mathcal{N}$ such that, for each n, $Y_n = G_{\beta^n}$ and $X_n = Un(Y_n) = CA_{\beta^n}$. We define a subset Z of $\mathcal{N}$ as follows. Z is the set of all α such that $\langle\alpha^0, \alpha^1 \circ S^*\rangle$ belongs to $Y_{\alpha^1(0)}$. Clearly, the set Z is an open subset of $\mathcal{N}$ and one may define γ in $\mathcal{N}$ satisfying: for each α, there exists n such that $\gamma(\overline{\alpha}n) \neq 0$ if and only if there exists n such that $\beta^{\alpha^1(0)}(\overline{\langle\alpha^0, \alpha^1 \circ S^*\rangle}n) \neq 0$.

Note that the set Z coincides wit the set G_γ.

Note that, for each α, α belongs to $\bigcap\limits_{n\in\mathbb{N}} X_n$ if and only if, for all n, for all β, $\langle\alpha, \beta\rangle$ belongs to X_n if and only if, for all β, $\langle\alpha, \beta \circ S^*\rangle$ belongs to $X_{\beta(0)}$ if and only if α belongs to $Un(Z)$. It follows that the set $\bigcap\limits_{n\in\mathbb{N}} X_n$ coincides with the set $Un(Z) = Un(G_\gamma) = CA_\gamma$.

(iii) Let X be a co-analytic subset of $\mathcal{N}$ and let Y be an open subset of $\mathcal{N}$ such that $X = Un(Y)$. Suppose that α belongs to Perhaps(X, X). Find γ in X such that, if α is apart from γ, then α belongs to X. We claim that α itself belongs to X. In order to see this, find a decidable subset C of $\mathbb{N}$ such that for every δ, δ belongs to Y if and only if, for some n, $\overline{\delta}n$ belongs to C.

Let β be an element of $\mathcal{N}$. Find n such that $\overline{\langle\gamma, \beta\rangle}n$ belongs to C, and distinguish two cases. *Either* $\overline{\langle\alpha, \beta\rangle}n = \overline{\langle\gamma, \beta\rangle}n$, and $\overline{\langle\alpha, \beta\rangle}n$ belongs to C, and, therefore, $\langle\alpha, \beta\rangle$ belongs to Y *or* $\langle\alpha, \beta\rangle \neq \overline{\langle\gamma, \beta\rangle}n$, therefore α is apart from γ, and α belongs to X and $\langle\alpha, \beta\rangle$ belongs to Y. In any case $\langle\alpha, \beta\rangle$ belongs to Y. Therefore, for every β, $\langle\alpha, \beta\rangle$ belongs to Y, and α itself belongs to X. Clearly, Perhaps(X, X) is a subset of X and X is perhapsive. We thus see that every co-analytic subset of $\mathcal{N}$ is perhapsive.

(iv) The set $D^2(A_1)$ is not perhapsive. This is a consequence of the fact that the set $Perhaps\big(D^2(A_1)\big)$ coincides with the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$ and the set $D^2(A_1)$ is not closed. It now follows from (ii) that the set $D^2(A_1)$ is not co-analytic.

(v) Suppose that X belongs to $\mathbf{\Pi}^1_1$. Let Y be an open subset of $\mathcal{N}$ such that $X = Un(Y)$. Observe that for every α, α belongs to $Un(X)$ if and only if, for all β, $\langle\alpha, \beta\rangle$ belongs to X if and only if, for all β, γ, the sequence $\big\langle\langle\alpha, \beta\rangle, \gamma\big\rangle$ belongs to Y. Let Z be the set of all α such that $\big\langle\langle\alpha^0, \alpha^{1,0}\rangle, \alpha^{1,1}\big\rangle$ belongs to Y. Observe that Z is open and $Un(X)$ coincides with $Un(Z)$, therefore $Un(X)$ belongs to $\mathbf{\Pi}^1_1$.

(vi) Use (iv) and the simple fact that every open subset of $\mathcal{N}$ is co-analytic.

(vii) Let X, Y be subsets of $\mathcal{N}$ and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to Y. Assume that Y is co-analytic. We have to show that X is co-analytic. Let Z be an open subset of $\mathcal{N}$ such that Y coincides with $Un(Z)$. Observe that for every α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y if and only if, for all β, $\langle\gamma|\alpha, \beta\rangle$ belongs to Z. Let V be the set of all α such that $\langle\gamma|\alpha^0, \alpha^1\rangle$ belongs to Z. Observe that V is an open subset of $\mathcal{N}$ and that X coincides with $Un(V)$. Clearly, X is co-analytic.

□

7.2. **The set $A^1_1 = $ Bar and the set WF of all α forbidding an element of each infinite $<_{KB}$-decreasing sequence.**

Recall that A^1_1 is the set of all α in $\mathcal{N}$ such that, for every β, there exists n such that $\alpha(\overline{\beta}n) \neq 0$.

Let α be an element of $\mathcal{N}$ and let s be an element of $\mathbb{N}$. α *forbids* s if and only if α does not admit s, that is, there exists $i \leq$ *length*(s) such that $\alpha(\overline{s}i) \neq 0$.

In Section 2, we defined the *Kleene-Brouwer ordering* on $\mathbb{N}$ as follows.

> For all s, t in $\mathbb{N}$, $s <_{KB} t$ if and only if either t is a strict initial part of s, or there exists i such that $i <$ *length*(s) and $i <$ *length*(t) and $\overline{s}i = \overline{t}i$ and $s(i) < t(i)$.

Also in Section 1, we defined, for every α in $\mathcal{N}$, a subset T_α of $\mathbb{N}$, called *the tree determined by* α: T_α is the set of all s that are *admitted by* α, that is: such that, for all i, if $i \leq$ *length*(s), then then $\alpha(\overline{s}i) = 0$.

We now let **WF** be the set of all α in $\mathcal{N}$ such that the tree T_α is *well-founded by* $<_{KB}$, that is, **WF** is the set of all α in $\mathcal{N}$ such that, for every β, if, for each n, $\beta(n)$ belongs to T_α, then there exists n such that $\beta(n) = \beta(n+1)$ or $\beta(n) \leq_{KB} \beta(n+1)$.

The following theorem should be compared with Theorem 2.2.

Theorem 7.2.
(i) *The set A^1_1 is a complete co-analytic set.*
(ii) *The set A^1_1 coincides with the set* **WF**.

Proof. (i) Let X belong to $\mathbf{\Pi}^1_1$ and let Y be an open subset of $\mathcal{N}$ such that $X = Un(Y)$ and let C be a decidable subset of $\mathbb{N}$ such that, for every β, β belongs to Y if and only if, for some n, $\overline{\beta}n$ belongs to C. Observe that for every α, α belongs to X if and only if, for every β, there exists n such that $\overline{\langle\alpha, \beta\rangle}n$ belongs to C. Define a function f from $\mathcal{N}$ to $\mathcal{N}$ such that for every α, β in $\mathcal{N}$, for every n in $\mathbb{N}$, $(f|\alpha)(\overline{\beta}n) = 0$ if and only if $\overline{\langle\alpha, \beta\rangle}n$ belongs to C and observe that f reduces X to A^1_1.

(ii) We first prove that **WF** is a subset of A^1_1.

Let α belong to **WF**. Let β belong to $\mathcal{N}$. We define γ such that $\gamma(0) = \overline{\beta}0 = \langle\ \rangle$, and, for each n, if $\overline{\beta}(n+1)$ belongs to T_α, then $\gamma(n+1) = \overline{\beta}(n+1)$, and, if $\overline{\beta}(n+1)$ does not belong to T_α, then $\gamma(n+1) = \gamma(n)$. Note that, if $\alpha(\langle\ \rangle) = 0$, then, for each n, $\gamma(n)$ belongs to T_α. Let n be the least i such that either $\alpha(\langle\ \rangle) = 0$ or $\gamma(i) = \gamma(i+1)$ or $\gamma(i) <_{KB} \gamma(i+1)$. Note that $\overline{\beta}(n+1)$ does not belong to T_α and, therefore, either $\alpha(\langle\ \rangle) \neq 0$ or $\alpha(\overline{\beta}(n+1) \neq 0$.

We thus see that, for every β, there exists n such that $\alpha(\overline{\beta}n) \neq 0$, that is, α belongs to A^1_1.

We now prove that A^1_1 is a subset of **WF**.

Let α belong to A^1_1. Let P be the set of all s such that, for some t, $t \sqsubseteq s$ and $\alpha(t) \neq 0$. Note that P is a bar in $\mathcal{N}$ and that P is monotone, that is, for each s, if s belongs to P, then, for each n, $s * \langle n\rangle$ belongs to P. We are going to use the principle of induction on monotone bars, and we let Q be the set of all s such that, for each β, if, for each n, $s \sqsubseteq \beta(n)$ and, for each n, $\beta(n)$ belongs to T_α, then there exists n such that

$\beta(n) = \beta(n+1)$ or $\beta(n) \leq_{KB} \beta(n+1)$. Note that, for each s, if s belongs to P, then, for each t, if $s \sqsubseteq t$, then t does not belong to T_α, and, therefore, s belongs to Q.

Now assume s belongs to $\mathcal{N}$ and, for each n, $s * \langle n \rangle$ belongs to Q. Let β be an element of $\mathcal{N}$ such that, for each n, $s \sqsubseteq \beta(n)$ and, for each n, $\beta(n)$ belongs to T_α. We define the statement QED, meaning (*"quod est demonstrandum, this is what we have to prove"* rather than *"quod erat demonstrandum, this is what we had to prove"*, as follows:

$$QED := \textit{There exists } n \textit{ such that } \beta(n) = \beta(n{+}1) \textit{ or } \beta(n) \leq_{KB} \beta(n{+}1).$$

Note that, for each n, *either* $\beta(n+1) = s$ and $\beta(n+1) \sqsubseteq \beta(n)$ and QED, *or* there exists m such that $s * \langle m \rangle \sqsubseteq \beta(n+1)$.

We may define γ such that, for each n, either $\beta(n+1) = s$ or $s * \langle \gamma(n) \rangle \sqsubseteq \beta(n+1)$.

Note that, for each n, *either* $\gamma(n+1) \leq \gamma(n)$, *or* $\beta(n+1) = s$, *or* $\beta(n+2) = s$, *or* $s * \langle \gamma(n) \rangle \sqsubseteq \beta(n+1)$ and $s * \langle \gamma(n+1) \rangle \sqsubseteq \beta(n+2)$ and $\gamma(n) < \gamma(n+1)$ and, therefore, $\beta(n+1) <_{KB} \beta(n+2)$. We conclude: for each n, either $\gamma(n+1) \leq \gamma(n)$ or QED.

We now claim the following:

for each m, there exists n such that $n > m$ and $\gamma(n) < \gamma(m)$ or QED.

We prove this claim as follows. Let m be a natural number. Define δ such that $\delta(0) = \beta(m{+}1)$, and, for each n, if $s{*}\langle \gamma(m) \rangle \sqsubset \beta(m{+}n{+}2)$, then $\delta(n{+}1) = \beta(m{+}n{+}2)$, and, if not $s * \langle \gamma(m) \rangle) \sqsubset \beta(m+n+2)$, then $\delta(n+1) = \delta(n)$. Note that, for each n, $s * \langle \gamma(m) \rangle \sqsubseteq \delta(n)$. Using the assumption that $s * \langle \gamma(m) \rangle$ belongs to Q, we find p such that $\delta(p) = \delta(p+1)$ or $\delta(p) <_{KB} \delta(p+1)$. Now distinguish two cases. *Either* $\delta(p) = \beta(m{+}p{+}1)$ and $\delta(p{+}1) = \beta(m{+}p{+}2)$ and, therefore, QED, *or* $\delta(p) \neq \beta(m{+}p{+}1)$ or $\delta(p+1) \neq \beta(m+p+2)$, and, therefore, $s * \langle \gamma(m) \rangle$ is not an initial part of either $\beta(m+p+1)$ or $\beta(m+p+2)$. In the latter case, either $\beta(m+p+1)$ or $\beta(m+p+2)$ coincides with s, and, therefore, QED, or: either $\gamma(m+p)$ or $\gamma(m+p+1)$ differs from $\gamma(m)$. Note that, if either $\gamma(m) < \gamma(m+p+1)$ or $\gamma(m) < \gamma(m+p+2)$, then QED. We thus may conclude: $\gamma(m+p) < \gamma(m)$ or $\gamma(m+p+1) < \gamma(m)$ or QED.

This ends the proof of our claim.

It is not difficult to see that the statement: *"for each m, there exists n such that $n > m$ and $\gamma(n) < \gamma(m)$ or QED"* forces us to the conclusion: QED.

Clearly then, for each β, if, for each n, $s \sqsubseteq \beta(n)$ and, for each n, $\beta(n)$ belongs to T_α, then there exists n such that $\beta(n) = \beta(n+1)$ or $\beta(n) \leq_{KB} \beta(n+1)$, that is: s belongs to Q.

Applying the principle of induction on monotone bars we find: the empty sequence $\langle\,\rangle$ belongs to Q, that is: for each β, if, for each n, $\beta(n)$ belongs to T_α, then there exists n such that $\beta(n) = \beta(n+1)$ or $\beta(n) \leq_{KB} \beta(n+1)$, that is: α belongs to **WF**.

We thus see: A^1_1 is a subset of **WF**. □

The statement second statement of Theorem 7.2 is also studied in [47]. It turns out that, in basic intuitionistic mathematics, this statement is an equivalent of the principle of Open Induction on Cantor space $\mathcal{C}$.

7.3. The set of the closed sets sinking into Fin or Almost*(Fin).

We now want to treat some results that, together, are a counterpart to Theorem 2.5.

Let X, Y be a subsets of $\mathcal{N}$. We introduce the colourful expression: X *sinks into* Y, for: X *is a subset of* Y.

Let α, β be elements of $\mathcal{N}$. We define: β *admits* α if and only if, for each n, $\beta(\overline{\alpha}n) = 0$.

For each β in $\mathcal{N}$, the set F_β is the set of all α admitted by β.

Recall that a subset X of $\mathcal{N}$ belongs to the class $\mathbf{\Pi}^0_1$ of the *closed* subsets of $\mathcal{N}$ if and only if, for some β, X coincides with F_β.

Let β belong to $\mathcal{N}$. β is called a *spreadlaw* if and only if, for each s, $\beta(s) = 0$ if and only if, for some n, $\beta(s * \langle n \rangle) = 0$.

Note that a closed subset X of $\mathcal{N}$ is *located* or a *spread* if and only if there exists a spreadlaw β such that X coincides with F_β.

We let $Sink(X)$ be the set of all β in $\mathcal{N}$ such that the closed set F_β sinks into X, that is: F_β is a subset of X.

We let $Sink^*(X)$ be the set of all β in $Sink(X)$ such that β is a spread-law.

We let $Sink^*_{01}(X)$ be the set of all β in $Sink(X)$ such that β is a spread-law and F_β is a subset of Cantor space $\mathcal{C}$.

We let **Fin** be the set of all α in *Baire space* $\mathcal{N}$ such that, for some m, for each n, if $n > m$, then and $\alpha(n) = 0$.

Note that, for all α in *Cantor space* $\mathcal{C}$, α belongs to **Fin** if and only if α is the characteristic function of a *finite* subset of $\mathbb{N}$. One proves easily that the sets **Fin** and $\mathbf{Fin} \cap \mathcal{C}$, as subsets of Baire space $\mathcal{N}$, reduce to each other. As has been shown in [35], see also [41] and [46], the set **Fin** belongs to the class $\mathbf{\Sigma}^0_2$, consisting of the countable unions of closed sets, and not to the class $\mathbf{\Pi}^0_2$ consisting of the countable intersections of open sets. Somewhat surprisingly, the set **Fin** is not a complete element of the class $\mathbf{\Sigma}^0_2$, although the set **Inf** is a complete element of the set $\mathbf{\Pi}^0_2$.

We let $\mathbf{Almost^*Fin}$ be the set of all α in Cantor space $\mathcal{C}$ such that, for every strictly increasing γ in $\mathcal{N}$, there exists n such that $\alpha\big(\gamma(n)\big) = 0$. Using Brouwer's Continuity Principle, one may prove that the set **Fin** is a proper subset of the set $\mathbf{Almost^*Fin}$. It has been shown in [41] that the co-analytic set $\mathbf{Almost^*Fin}$ is not positively Borel.

An important observation is that the set $\mathbf{Almost^*Fin}$ is the set of all α in $\mathcal{C}$ that are apart from every element of **Inf**.

The following notions were introduced just before Theorem 6.1 and will be used again in the proof of the second item of the next theorem.

For each α in $\mathcal{N}$, we defined elements α_I and α_{II} of $\mathcal{N}$ by:

For each n, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n+1)$.

For each s in $\mathbb{N}$, we defined elements s_I and s_{II} of $\mathbb{N}$ by:

length(s_I) is the least k such that *length*$(s) \geq 2k$ and, for each n, if $n <$ *length*(s_I), then $s_I(n) = s(2n)$, and
length(s_{II}) is the least k such that *length*$(s) \geq 2k+1$ and, for each n, if $n <$ *length*(s_{II}), then $s_{II}(n) = s(2n+1)$.

For the second item of the next theorem, it is important that one has in mind the set **Fin**, as we just defined it, a subset of Baire space, and not the set $\mathbf{Fin} \cap \mathcal{C}$.

Theorem 7.3.
(i) *The set* $Sink^*_{01}(\mathbf{Fin})$ *and the set* **Fin** *reduce to each other.*
(ii) *The set* A^1_1 *reduces to the set* $Sink^*(\mathbf{Fin})$ *but the set* $Sink^*(\mathbf{Fin})$ *is not a co-analytic subset of* $\mathcal{N}$.
(iii) *The sets* **Fin**, $Sink^*_{01}(\mathbf{Fin})$ *and* $Sink^*(\mathbf{Fin})$ *are not co-analytic.*
(iv) *The set* A^1_1 *reduces to each one of the sets* $Sink(\mathbf{Almost^*Fin})$, $Sink^*(\mathbf{Almost^*Fin})$ *and* $Sink^*_{01}(\mathbf{Almost^*Fin})$.
(v) *The sets* $Sink^*(\mathbf{Almost^*Fin})$ *and* $Sink^*_{01}(\mathbf{Almost^*Fin})$ *are co-analytic subsets of* $\mathcal{N}$.

Proof. (i) Assume that β is a spreadlaw and that F_β is a subset of Cantor space $\mathcal{C}$.

Note that β belongs to $Sink(\mathbf{Fin})$ if and only if, for each α in F_β, there exists n such that, for every m, if $m > n$, then $\alpha(m) = 0$.

Note that F_β is a spread.

Applying the First Axiom of Continuous Choice, we find a function γ from the fan F_β to $\mathbb{N}$ such that, for each α in F_β, $\alpha = \overline{\alpha}(\gamma(\alpha)) * \underline{0}$. Applying the (Restricted) Fan Theorem, Axiom 12, we conclude that there exists n such that for all α in F_β, $\gamma(\alpha) \leq n$. It follows that, for all α in F_β, for all m, if $m > n$, then $\alpha(m) = 0$. We thus see that β belongs to $Sink(\mathbf{Fin})$ if and only if there exists n such that, for all s, for all i, if $length(s) \geq n$ then $\beta(s * \langle 1 \rangle) \neq 0$.

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, for each n, *if* there exists s in **Bin** such that $\beta(s * \langle 1 \rangle) = 0$ and $length(s) = n$, then $(\gamma|\beta)(n) = 1$, and, *if not*, then $(\gamma|\beta)(n) = 0$. Note that γ reduces the set $Sink^*_{01}(\mathbf{Fin})$ to the set **Fin**.

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\gamma|\alpha)(\langle\ \rangle) = 0$, and for each s, for each n, $(\gamma|\alpha((s * \langle n \rangle) = 0$ if and only if $(\gamma|\alpha)(s) = 0$ and either: $n = 0$, or: $n = 1$ and $\alpha(n) \neq 0$. Note that γ reduces the set **Fin** to the set $Sink^*_{01}(\mathbf{Fin})$.

(In the above argument, one may avoid the use of the First Axiom of Continuous Choice. One then has to use, apart from Brouwer's Continuity Principle, Axiom 4, the unrestricted Fan Theorem, Axiom 11, rather than the Restricted Fan Theorem, Axiom 12.)

(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(\langle\ \rangle) = 0$ and, for each s, for each i, for each j, $(\gamma|\alpha)(s * \langle i, j \rangle) = 0$ if and only if $(\gamma|\alpha)(s) = 0$ and *either* $i = j = 0$ or, for all t, if $t \sqsubseteq s_I * \langle i \rangle$, then and $\alpha(t) = 0$ and $j = 1$ Note that, for each α, $\gamma|\alpha$ is a spreadlaw, as, for each s, $(\gamma|\alpha((s) = 0$ if and only if $(\gamma|\alpha)(s * \langle 0 \rangle) = 0$.
We claim that the function γ reduces the set A^1_1 reduces to the set $Sink^*(\mathbf{Fin})$, and we prove this claim as follows.
Let α belong to A^1_1 and β to $\mathcal{N}$. Assume that β belongs to $F_{\gamma|\alpha}$, that is, for each n, $(\gamma|\alpha)(\overline{\beta}n) = 0$. Consider β_I and find n such that $\alpha(\overline{\beta_I}n) \neq 0$. It follows that, for each $j > n$, $\beta(2j+1) = \beta(2j+2) = 0$, so β belongs to **Fin**.
We thus see that, for each α, if α belongs to A^1_1, then $\gamma|\alpha$ belongs to $Sink^*(\mathbf{Fin})$.
Now assume that $\gamma|\alpha$ belongs to $Sink^*(\mathbf{Fin})$. Then, for each β in the spread $F_{\gamma|\alpha}$, there exists m such that, for all n, if $n > m$, then $\beta(n) = 0$. Now let β belong to $\mathcal{N}$. Define an element β' of $\mathcal{N}$, as follows. For each n, if, for all $i \leq n+1$, $\alpha(\overline{\beta}i) = 0$, then $\beta'(2n) = \beta(n)$ and $\beta'(2n+1) = 1$, and, if there exists $i \leq n+1$ such that $\alpha(\overline{\beta}i) \neq 0$, then $\beta'(2n) = \beta'(2n+1) = 0$. Note that, for each n, $(\gamma|\alpha)(\overline{\beta'}n) = 0$, that is, β' belongs to $F_{\gamma|\alpha}$ and, therefore, to **Fin**. Find m such that for all n, if $n > m$, then $\beta'(n) = 0$. Find k such that $m+1 = 2k$ or $m+2 = 2k+1$ and note: $\beta(2k+1) = 0$ and, therefore, there exists $i \leq k$ such that $\alpha(\overline{\beta}k) \neq 0$.
Clearly then, for each α, if $\gamma|\alpha$ belongs to $Sink^*(\mathbf{Fin})$, then α belongs to A^1_1.

(iii) We first observe that the set **Fin** is not co-analytic. In [41], Theorem 3.21(v), it is shown that **Almost*****Fin** is the best co-analytic approximation of the set $\mathbf{Fin} \cap \mathcal{C}$, that is, **Almost*****Fin** is co-analytic and, for every co-analytic subset Z of $\mathcal{C}$, if $\mathbf{Fin} \subseteq Z$, then **Almost*****Fin** $\subseteq Z$. As **Fin** does not coincide with **Almost*****Fin**, one may conclude that **Fin** is not co-analytic.

Perhaps, the reader finds the just given reference too sophisticated. One may also observe: the set **Fin** is not perhapsive, and, therefore, by Theorem 7.1(ii), the set **Fin** is not co-analytic. Here is the proof that the set **Fin** is not perhapsive:

> Consider the set T consisting of all α in $\mathcal{C}$ such that, for all m, n, if $\alpha(m) = \alpha(n) = 1$, then $m = n$. T is the set of all α in $\mathcal{C}$ that assume the value 1 at most one time. Note that T is a spread. Also note that, for each α in T, if $\alpha \# \underline{0}$, then there exists n such that $\alpha = \overline{\underline{0}}n * \langle 1 \rangle * \underline{0}$, and thus, α belongs to **Fin**. We conclude: T is a subset of *Perhaps*(**Fin**). Suppose that T is a subset of **Fin**. Using Brouwer's Continuity Principle, we find m, n such that either, for all α in T, if α

passes through $\overline{\underline{0}}m$, then $\alpha = \underline{0}$ or, for all α in T, if α passes through $\overline{\underline{0}}m$, then $\alpha = \overline{\underline{0}}n * \langle 1 \rangle * \underline{0}$. Both conclusions are wrong.

It now follows from (i) that the set $Sink^*_{01}(\mathbf{Fin})$ is not co-analytic.

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each β, for each s, if s belongs to **Bin**, then $(\gamma|\beta)(s) = \beta(s)$, and, if s does not belong to **Bin**, then $(\gamma|\beta)(s) = 1$. One easily verifies that the function γ reduces the set $Sink^*_{01}(\mathbf{Fin})$ reduces to the set $Sink^*(\mathbf{Fin})$.

It follows that also the set $Sink^*(\mathbf{Fin})$ is not co-analytic.

(iv) We now show that the set A^1_1 reduces to the set $Sink(\mathbf{Almost^*Fin})$. It then follows from Theorem 7.1 that the latter set is a complete co-analytic set.

As in the proof of Theorem 2.5, we let δ be an element of $\mathcal{N}$ such that $\delta(\langle\ \rangle) = \langle\ \rangle$ and, for each s, for each n, $\delta(s * \langle n \rangle) = \delta(s) * \overline{\underline{0}}n * \langle 1 \rangle$.

We let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\gamma|\alpha)(s) = 0$ if and only if there exist t, n such that $s = \delta(t) * \overline{\underline{0}}n$ and $\alpha(t) = 0$. Note that, for each α, for each s, if $(\gamma|\alpha)(s) = 0$, then $(\gamma|\alpha)(s * \langle 0 \rangle) = 0$. It follows that, for each α, the set $F_{\gamma|\alpha}$ is a spread, and a subset of $\mathcal{C}$.

We claim that the function γ reduces the set A^1_1 to the set $Sink(\mathbf{Almost^*Fin})$, and we prove this claim as follows.

First, let α belong to A^1_1. Let β belong to $F_{\gamma|\alpha}$. We have to prove that β belongs to $\mathbf{Almost^*Fin}$. Let ε be a strictly increasing element of $\mathcal{N}$. We have to prove that there exists n such that $\beta\big(\varepsilon(n)\big) = 0$. To this end, we define an element β' of $\mathcal{N}$ as follows. For each i, if there exists n such that $i = \varepsilon(n)$, then $\beta'(i) = 1$, and, if not, then $\beta'(i) = \beta(i)$. Note that β' belongs to **Inf** and find ζ in $\mathcal{N}$ such that, for each n, β' passes through $\delta(\overline{\zeta}n)$. Now, recalling the fact that α belongs to A^1_1, find n such that $\alpha(\overline{\zeta}n) \neq 0$ and note that $(\gamma|\alpha)\big(\delta(\overline{\zeta}n)\big) \neq 0$. Find m such that $\delta(\overline{\zeta}n) = \overline{\beta'}m$. As $(\gamma|\alpha)(\overline{\beta'}m) \neq 0$ and $(\gamma|\alpha)(\overline{\beta}m) = 0$, we conclude: $\overline{\beta'}m \neq \overline{\beta}m$, and: for some $i < m$, $\beta'(i) \neq \beta(i)$, and: there exists j such that $\beta\big(\varepsilon(j)\big) \neq \beta'\big(\varepsilon(j)\big)$, and, therefore: $\beta\big(\varepsilon(j)\big) = 0$.

We thus see that $\gamma|\alpha$ belongs to $Sink(\mathbf{Almost^*Fin})$.

Conversely, assume that $\gamma|\alpha$ belongs to $Sink(\mathbf{Almost^*Fin})$. We want to prove: α belongs to A^1_1. Let ζ belong to $\mathcal{N}$. We have to prove: there exists n such that $\alpha(\overline{\zeta}n) \neq 0$. To this end, we determine β in $\mathcal{C}$ such that, for each n, β passes through $\delta(\overline{\zeta}n)$. We now use the fact that $F_{\gamma|\alpha}$ is a spread and we define an element β' of $\mathcal{C}$, by induction, as follows. For each n, if $(\gamma|\alpha)\big(\overline{\beta'}n * \langle \zeta(n) \rangle\big) = 0$, then $\beta'(n) = \beta(n)$, and, if not, then $\beta(n) = 0$. Note that, for each n, $(\gamma|\alpha)(\overline{\beta'}n) = 0$, and that β belongs to **Inf**. It follows that β is apart from β'. Determine n such that $\overline{\beta'}n \neq \overline{\beta}n$ and note that $(\gamma|\alpha)(\overline{\beta}n) \neq 0$. Find m such that $\overline{\beta}n \sqsubseteq \delta(\overline{\zeta}m)$ and note: there exists $i \leq m$ such that $\alpha(\overline{\zeta}i) \neq 0$.

We thus see that α belongs to A^1_1.

It follows that the function γ reduces the set A^1_1 to the set $Sink(\mathbf{Almost^*Fin})$

It is not difficult to see that the function γ also reduces the set A^1_1 to the set $Sink^*(\mathbf{Almost^*Fin})$ and to the set $Sink^*_{01}(\mathbf{Almost^*Fin})$.

(v) First note that, for all β, β is a spreadlaw if and only if for each s, $\beta(s) = 0$ if and only if there exists n such that $\beta(s * \langle n \rangle) = 0$. The set of all β in $\mathcal{N}$ that are a spreadlaw thus is seen to belong to the class $\mathbf{\Pi}^0_2$ and, therefore, also to the class $\mathbf{\Pi}^1_1$.

Assume now that β is a spreadlaw.

Also assume that β belongs to $Sink^*(\mathbf{Almost^*Fin})$. Then, for each α, if, for each n, $\beta(\overline{\alpha}n) = 0$, then, for each γ, if for each n, $\gamma(n) < \gamma(n+1)$, then there exists m such that $\alpha\big(\gamma(m)\big) = 0$.

We claim that, now, also the following is true:

For each α, for each γ, there exists m such that either $\gamma(m) \geq \gamma(m+1)$ or $\beta(\overline{\alpha}m) \neq 0$ or $\alpha\big(\gamma(m)\big) = 0$.

We prove this claim as follows.

Let α, γ be given. We define α' and γ' as follows.

For each n, *if* $\beta(\overline{\alpha'}n * \langle\alpha(n)\rangle) = 0$, then $\alpha'(n) = \alpha(n)$, and *if not*, then $\alpha'(n) = p$, where p is the least j such that $\beta(\overline{\alpha}'n * \langle j\rangle) = 0$.

$\gamma'(0) = \gamma(0)$, and, for each n, *if* $\gamma(n+1) > \gamma'(n)$, then $gamma'(n+1) = \gamma(n+1)$, and *if not*, then $\gamma'(n+1) = \gamma'(n) + 1$.

Note that α' belongs to F_β and that γ' is strictly increasing. Find m such that $\alpha'\big(\gamma'(m)\big) = 0$, and distinguish three cases. *Either* $\gamma'(m) \neq \gamma(m)$ and: there exists n such that $\gamma(n+1) \geq \gamma(n)$, *or* $\gamma'(m) = \gamma(m)$ and $\alpha'\big(\gamma(m)\big) \neq \alpha\big(\gamma(m)\big)$ and: there exists n such that $\beta(\overline{\alpha}n) \neq 0$, *or* $\alpha\big(\gamma(m)\big) = 0$.

This ends the proof of our claim.

Conversely, it will be clear that, *if*, for each α, for each γ, there exists m such that either $\gamma(m) \geq \gamma(m+1)$ or $\beta(\overline{\alpha}m) \neq 0$ or $\alpha\big(\gamma(m)\big) = 0$, *then*, for each α, if, for each n, $\beta(\overline{\alpha}n) = 0$, then, for each γ, if for each n, $\gamma(n) < \gamma(n+1)$, then there exists m such that $\alpha\big(\gamma(m)\big) = 0$.

It follows from the above that, for each β, β belongs to **Almost*****Fin** if and only if

$$\forall s \exists n[\beta(s) = 0 \leftrightarrow \beta(s * \langle n\rangle) = 0] \ \wedge\ \forall\alpha\exists n[\alpha^1(n) \geq \alpha^1(n+1) \vee \alpha^0\big(\alpha^1(n)\big) = 0 \vee \beta(\overline{\alpha^0}n) \neq 0].$$

We thus see that the set *Sink**(**Almost*****Fin**) belongs to the class $\mathbf{\Pi}^1_1$.

It easily follows that also the set *Sink*$^*_{01}$(**Almost*****Fin**) belongs to the class $\mathbf{\Pi}^1_1$.

It follows from (iii) that both these sets are $\mathbf{\Pi}^1_1$-complete.

□

Note that the fifth item of Theorem 6.3 is reminiscent of a classical result due to Hurewicz. This result plays a key role in the sketch of the proof of the theorem of Solovay and Kaufman in [14]. This theorem states that the class consisting of the closed sets of uniqueness and the class consisting of the closed sets of extended uniqueness are $\mathbf{\Pi}^1_1$-complete.

7.4. **The set of the trees that admit exactly one path.**

We let $E^1_1!$ be the set of all α such that there exists γ such that γ *is the only infinite sequence admitted by* α, that is, for each n, $\alpha(\overline{\gamma}n) = 0$, while, for each δ, if δ is apart from γ, that is, there exists m such that $\delta(m) \neq \gamma(m)$, then, for some n, $\alpha(\overline{\delta}n) \neq 0$. It is not difficult to verify th conclusion of the first item of the next theorem: every co-analytic set reduces to $E^1_1!$. In [15], pages 125-127, one may find a fascinating argument, due to Kechris, showing that, in classical descriptive set theory, the set $E^1_1!$ itself is $\mathbf{\Pi}^1_1$ and, therefore, $\mathbf{\Pi}^1_1$-complete. The items (ii)-(v) of the next Theorem may be taken as an intuitionistic comment upon this beautiful result.

We let $D!(A_1, A_1)$ be the set of all α such that either: $\alpha^0 = \underline{0}$ and $\alpha^1 \# \underline{0}$ or: $\alpha^0 \# \underline{0}$ and $\alpha^1 = \underline{0}$.

We let $E_2!$ be the set of all α in $\mathcal{N}$ such that, for some n, $\alpha^n = \underline{0}$, while, for each m, if $m \neq n$, then there exists p such that $\alpha^m(p) \neq 0$.

The last item of the next theorem has been added because of the discussion at the end of Section 6.

The set $E_2!$ is an example of a subset of $\mathcal{N}$ that is positively Borel and has Lusin's property but still fails to be co-analytic.

Theorem 7.4.

(i) *The set A^1_1 reduces to the set $E^1_1!$.*
(ii) *Every subset of $\mathcal{N}$ that reduces to a perhapsive subset of $\mathcal{N}$ is itself a perhapsive subset of $\mathcal{N}$.*
(iii) *The set $D!(A_1, A_1)$ reduces to the set A^1_1 and is a perhapsive subset of $\mathcal{N}$.*
(iv) *The set $D^2(A_1)$ is not a perhapsive subset of $\mathcal{N}$.*

(v) *The sets A_2 and $D!(A_1, A_1)$ reduce to the set $E_2!$, and the set $E_2!$ reduces to the set $E_1^1!$.*
(vi) *The set $D^2(A_1)$ does not reduce to the set $E_1^1!$.*
(vii) *The set $E_2!$ is not a perhapsive subset of $\mathcal{N}$.*
(viii) *The sets $E_2!$ and $E_1^1!$ do not reduce to the set A_1^1.*
(ix) *The set $E_2!$ is regular in Lusin's sense.*

Proof. (i) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, $(\gamma|\alpha)(\langle\,\rangle) = 0$ and, for every α, for every s, $(\gamma|\alpha)(\langle 0\rangle * s) = 0$ if and only if s is an initial part of $\underline{0}$, and, for every n $(\gamma|\alpha)(\langle n+1\rangle * s) = \alpha(s)$. Note that, for every α, $\gamma|\alpha$ admits $\underline{0}$ and, for every β, if $\beta(0) = 0$, then, for every n, if $\beta(n) \neq 0$, then $(\gamma|\alpha)\big(\overline{\beta}(n+1)\big) \neq 0$, and for every n, for every β, $(\gamma|\alpha)(\langle n+1\rangle * \overline{\beta}n) = \alpha(\overline{\beta}n)$. It follows that, for every α, α belongs to A_1^1 if and only if $\gamma|\alpha$ belongs to $E_1^1!$.

(ii) Let X, Y be subsets of $\mathcal{N}$ and suppose that Y is a perhapsive subset of $\mathcal{N}$ and that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set X to the set Y. We have to prove that X is a perhapsive subset of $\mathcal{N}$. Assume that α, β are elements of $\mathcal{N}$ and that β belongs to X and that, if $\alpha \# \beta$, then α belongs to X. We have to prove that α belongs to X. Note that $\gamma|\beta$ belongs to Y, and that, if $\gamma|\alpha \# \gamma|\beta$, then $\alpha \# \beta$, and, therefore, α belongs to X and $\gamma|\alpha$ belongs to Y. As Y is perhapsive, it follows that $\gamma|\alpha$ belongs to Y, and, therefore, α belongs to X.

(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\gamma|\alpha)(\langle\,\rangle) = 0$ and, for each s, $(\gamma|\alpha)(\langle 0\rangle * s) = \max(\alpha^0(\mathit{length}(s)), \alpha^1(\mathit{length}(s))$ and for each n, for each s, $(\gamma|\alpha)(\langle n+1\rangle * s) = 1$ if either $\overline{\alpha^0}n = \overline{\underline{0}}n$ or $\overline{\alpha^1}n = \overline{\underline{0}}n$, and $(\gamma|\alpha)(\langle n+1\rangle * s) = 0$ otherwise.

Suppose that α belongs to $D!(A_1, A_1)$. Note that there exists n such that $\max(\alpha^0(n), \alpha^1(n)) \neq 0$ and, therefore, for each δ, if $\delta(0) = 0$, then $(\gamma|\alpha)(\overline{\delta}(n+1)) \neq 0$. Also note that, for each n, either $\overline{\alpha^0}n = \overline{\underline{0}}n$ or $\overline{\alpha^1}n = \overline{\underline{0}}n$, and, therefore, for each δ, if $\delta(0) > 0$, then $(\gamma|\alpha)(\overline{\delta}1) \neq 0$. It follows that, for each α, if α belongs to $D!(A_1, A_1)$, then $\gamma|\alpha$ belongs to A_1^1.

Now suppose that $\gamma|\alpha$ belongs to A_1^1. Note that there exists n such that $(\gamma|\alpha)(\overline{\underline{0}}(n+1)) \neq 0$, and, therefore, either $\alpha^0(n) \neq 0$, or $\alpha^1(n) \neq 0$. Also note that , for each n, for each δ, if $\delta(0) = n+1$, then there exists p such that $(\gamma|\alpha)(\overline{\delta}p) \neq 0$, and, therefore, $(\gamma|\alpha)(\overline{\delta}1) \neq 0$, and $(\gamma|\alpha)(\langle n+1\rangle) \neq 0$ and either $\overline{\alpha^0}n = \overline{\underline{0}}n$ or $\overline{\alpha^1}n = \overline{\underline{0}}n$. As also either $\alpha^0 \# \underline{0}$ or $\alpha^1 \# \underline{0}$, we conclude: α belongs to $D!(A_1, A_1)$.

(iv) The fact that the set $D^2(A_1)$ is not perhapsive has been used in Theorem 6.1(iii). Let us briefly sketch the argument. The set $\mathit{Perhaps}(D^2(A_1)$ coincides with the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$. The set $\overline{D^2(A_1)}$ is a spread containing $\underline{0}$. Suppose that $\overline{D^2(A_1)}$ coincides with $D^2(A_1)$. Using Brouwer's Continuity Principle, we determine p such that either: for every α in $\overline{D^2(A_1)}$, if $\overline{\alpha}p = \overline{\underline{0}}p$, then $\alpha^0 = \underline{0}$, or: for every α in $\overline{D^2(A_1)}$, if $\overline{\alpha}p = \overline{\underline{0}}p$, then $\alpha^1 = \underline{0}$. But, clearly, there is no such p.

(v) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\gamma|\alpha)^0 = \underline{0}$, and, for each n, $(\gamma|\alpha)^{n+1} = \alpha^n$. It is not difficult to see that the function γ reduces the set A_2 to the set $E_2!$.

Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\delta|\alpha)^0 = \alpha^0$ and $(\delta|\alpha)^1 = \alpha^1$, and, for each $n > 1$, $\alpha^n = \underline{1}$. Clearly, δ reduces the set $D!(A_1, A_1)$ to the set $E_2!$.

Let ε be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, $(\varepsilon|\alpha)(\langle\,\rangle) = 0$, and, for each n, for each β, for each n, $(\varepsilon|\alpha)(\overline{\beta}(n+1) = 0$ if and only if there exists m such that $\overline{\beta}(n+1) = \overline{\underline{m}}(n+1)$ and $\overline{\alpha^m}n = \overline{\underline{0}}n$. Clearly, ε reduces the set $E_2!$ to the set $E_1^1!$.

(vi) Now assume that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^2(A_1)$ to the set $E_1^1!$. Let us consider the sets B_0 and B_1, where, for each $i < 2$, B_i is the set of all α

such that $\alpha^i = \underline{0}$. Note that $D^2(A_1) = B_0 \cup B_1$ and that the sets B_0 and B_1 are spreads containing $\underline{0}$.

Let p belong to $\mathbb{N}$. Using Brouwer's Continuity Principle,Axiom 4, we find s_0, s_1, m_0 and m_1 such that, $length(s_0) = length(s_1) = p$ and, for each $i < 2$, for every α in B_i, if α passes through $\overline{\underline{0}}m_i$, then there exists β passing through s_i such that $\gamma|\alpha$ admits β.

We claim: $s_0 = s_1$. In order to establish this claim, let us assume: $s_0 \neq s_1$.

Find β such that $\gamma|\underline{0}$ admits β. As $s_0 \perp s_1$, either β does not pass through s_0, or β does not pass through s_1. Without loss of generality, we assume: β does not pass through s_0. Note that, for every α in B_0, if α passes through $\overline{\underline{0}}m_0$ then there exists q such that $(\gamma|\underline{0})(\overline{\beta}q) \neq 0$. Using Brouwer's Continuity Principle, Axiom 4, find m_2, q such that $m_2 \geq m_0$ and, for every α in B_0, if α passes through $\overline{\underline{0}}m_2$, then $(\gamma|\alpha)(\overline{\beta}q) \neq 0$. It follows that $(\gamma|\underline{0})(\overline{\beta}q) \neq 0$, and that $\gamma|\underline{0}$ does not admit β. Contradiction.

Using the First Axiom of Countable Choice, we determine ε, ζ in $\mathcal{N}$ such that, for each p, $length\big(\varepsilon(p)\big) = p$ and, for each α in $B_0 \cup B_1$, if α passes through $\overline{\underline{0}}\big(\zeta(p)\big)$, then there exists η passing through $\varepsilon(p)$ such that $\gamma|\alpha$ admits η.

We claim that, for each p, $\varepsilon(p)$ is an initial part of β. In order to see this, let us assume $\varepsilon(p) \neq \overline{\beta}p$. Note that, for every α in B_0, if α passes through $\overline{\underline{0}}\zeta(p)$, then there exists q such that $(\gamma|\underline{0})(\overline{\beta}q) \neq 0$. Using Brouwer's Continuity Principle, find m_2, q such that $m_2 \geq \zeta(p)$ and, for every α in B_0, if α passes through $\overline{\underline{0}}m_2$, then $(\gamma|\alpha)(\overline{\beta}q) \neq 0$. It follows that $(\gamma|\underline{0})(\overline{\beta}q) \neq 0$, and that $\gamma|\underline{0}$ does not admit β. Contradiction.

Now let T be the set of all α in $\mathcal{C}$ with the property that, for all m, n, if $\alpha(m) = \alpha(n) = 1$, then $m = n$. Note that T is a spread and a subset of the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$. We determine η in $\mathcal{N}$ such that, for each p, $\gamma|(\overline{\underline{0}}p * \langle 1 \rangle * \underline{0})$ admits η^p, and, for all q, if $\zeta(q) \leq p$, then η^p passes through $\varepsilon(p)$. Now, using the fact that, for each q, for each $p \geq \zeta(q)$, $\overline{\eta^q}p = \varepsilon(q)$, define a function δ from T to $\mathcal{N}$ such that, for each p, $\delta|(\overline{\underline{0}}p * \langle 1 \rangle * \underline{0}) = \eta^p$.

We now claim:

(1) for each α in T, $\gamma|\alpha$ admits $\delta|\alpha$, and,

(2) for each α in T, for each η, if $\eta \# \delta|\alpha$, then there exists n such that $(\gamma|\alpha)(\overline{\eta}n) \neq 0$.

Let us first prove (1). Suppose we find α in T, n in $\mathbb{N}$ such that $(\gamma|\alpha)(\overline{\delta|\alpha}n) \neq 0$. Using the continuity of the functions (coded by) γ, δ, we find m such that for every β in T, if β passes through $\overline{\alpha}p$, then $(\gamma|\beta)(\overline{\delta|\beta}n) = (\gamma|\alpha)(\overline{\delta|\alpha}n) \neq 0$. Note that there exists p such that $\overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$ passes through $\overline{\alpha}n$ and that $\gamma|(\overline{\underline{0}}p * \langle 1 \rangle * \underline{0})$ admits $\delta|(\overline{\underline{0}}p * \langle 1 \rangle * \underline{0})$. Contradiction. We may conclude that, for each α in T, $\gamma|\alpha$ admits $\delta|\alpha$.

Let us now prove (2). Let α be an element of T and let η be an element of $\mathcal{N}$ such that $\eta \# \delta|\alpha$. Find n such that $\overline{\eta}n \neq \overline{\delta|\alpha}n$. Find m such that, for every β in T, if α passes through $\overline{\alpha}m$, then $\overline{\delta|\beta}n = \overline{\delta|\alpha}n$. We now distinguish two cases.

Case (a). $\overline{\alpha}m \neq \overline{\underline{0}}m$. Find p such that $\alpha = \overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$. Note that α belongs to $D^2(A_1)$, and that $\eta \# \delta|\alpha$. Therefore, there exists q such that $(\gamma|\alpha)(\overline{\eta}q) \neq 0$.

Case (b). $\overline{\alpha}m = \overline{\underline{0}}m$. Note that $\underline{0}$ belongs to $D^2(A_1)$, and that $\eta \# \delta|\underline{0}$. Find q such that $(\delta|\underline{0})(\overline{\eta}q) \neq 0$. Now calculate $(\delta|\alpha)(\overline{\eta}q)$ and observe: either $(\delta|\alpha)(\overline{\eta}q) = (\delta|\underline{0})(\overline{\eta}q) \neq 0$, and we are done, or $\alpha \# \underline{0}$. In the latter case, however, there exists p such that $\alpha = \overline{\underline{0}}p * \langle 1 \rangle * \underline{0}$, and α belongs to $D^2(A_1)$, and, as $\eta \# \delta|\alpha$, there exists r such that $(\delta|\underline{0})(\overline{\eta}r) \neq 0$.

It now follows that. for each α, if α belongs to T, then $\gamma|\alpha$ belongs to $E_1^1!$ and, therefore, α belongs to $D^2(A_1)$. As T is a spread, we apply Brouwer's Continuity Principle and find m such that, either, for every α in T, if α passes through $\overline{\underline{0}}m$, then $\alpha^0 = \underline{0}$, or, for every α in T, if α passes through $\overline{\underline{0}}m$, then $\alpha^1 = \underline{0}$. This conclusion is false.

We may conclude that the set $D^2(A_1)$ does not reduce to the set $E_1^1!$.

(vii) We now want to prove that the set $E_2!$ is not a perhapsive subset of $\mathcal{N}$.

Let A be the set of all α in $\mathcal{N}$ satisfying the following conditions:

(1) for all p, q, if $\alpha^0(p) \neq 0$ and $\alpha^0(q) \neq 0$, then $p = q$, and
(2) for all n, if $\alpha^0(n) = 0$, then $\alpha^{n+1} = \underline{1}$, and
(3) for all n, if $\alpha^0(n) \neq 0$, then $\alpha^{n+1} = \underline{0}$, and
(4) $\alpha(\langle\ \rangle) = 0$

Let β be the special element of A satisfying $\beta^0 = \underline{0}$ and, for each n, $\beta^{n+1} = \underline{1}$. Note that β belongs to $E_2!$ and that, for each α in A, if $\alpha \# \beta$, then there exists n such that $\alpha^0(n) \neq 0$ and $\alpha^{n+1} = \underline{0}$ and, for each i, if $i \neq n$, then $\alpha^{i+1} = \underline{1}$, and, therefore, α belongs to $E_2!$. We thus see that the set A is a subset of the set *Perhaps*$(E_2!)$.

We now prove that the set A is not a subset of the set $E_2!$. Let us assume that A is a subset of $E_2!$. Note that the set A is a spread containing β, and, using Brouwer's Continuity Principle, determine n, m such that for every α in A, if $\overline{\alpha}m = \overline{\beta}m$, then $\alpha^n = \underline{0}$. Now distinguish two cases.

Case (a). $n = 0$. Note that there exist α in A, p in $\mathbb{N}$ such that $\overline{\alpha}m = \overline{\beta}m$ and $p \geq n$ and $\alpha^0(p) \neq 0$ and $\alpha^{p+1} = \underline{0}$. Contradiction.

Case (b) $n > 0$. Note that β itself passes through $\overline{\beta}m$ and that $\beta^n = \underline{1}$. Contradiction.

We thus see that the set $E_2!$ is a proper subset of the set $E_2!$ and that the set $E_2!$ is not perhapsive.

(viii) As we just saw, the set $E_2!$ is not a perhapsive subset of $\mathcal{N}$. Using (ii), we conclude that the set $E_2!$ does not reduce to the set A^1_1 as the set A^1_1 is a perhapsive subset of $\mathcal{N}$. As, according to (v), the set $E_2!$ reduces to the set $E^1_1!$, also the set $E^1_1!$ does not reduce to the set A^1_1.

(ix) Let ε be the element of $\mathcal{N}$ satisfying: for all m, n, $\varepsilon(2^m(2n+1)-1) = \overline{\underline{0}}m * \langle n+1 \rangle$.

Let γ be (the code of) a function from $\mathcal{N}$ to $\mathcal{N}$ satisfying: for each α, $(\gamma|\alpha)(\langle\ \rangle) = \alpha(\langle\ \rangle)$, and $(\gamma|\alpha)^{\alpha^0(0)} = \underline{0}$, and for each n, if $n < \alpha^0(0)$, then $(\gamma|\alpha)^n = \varepsilon\big(\alpha^0(n+1)\big) * \alpha^{n+1}$, and if $n > \alpha^0(0)$, then $(\gamma|\alpha)^n = \varepsilon\big(\alpha^0(n)\big) * \alpha^n$.

One may verify that γ is a strongly one-to-one continuous function from the set $\mathcal{N}$ onto the set $E_2!$.

It follows that the set $E_2!$ has Lusin's property. □

8. Analytic sets face to face with co-analytic sets

8.1. **Comparing A^1_1 and Stp.** Recall that, for every γ, for every s, γ *admits* s if and only if, for each i, if $i \leq$ *length*(s), then $\gamma(\overline{s}i) = 0$.

Recall that, for all s, t, s is a *proper initial segment of* t, notation: $s \sqsubset t$ if and only if there exists $u \neq 0$ such that $s * u = t$.

Let ε be an element of $\mathcal{N}$. We say that ε is *strictly monotone* or: $\sqsubset$ *−monotone* if and only if, for all s, t, if $s \sqsubset t$, then $\varepsilon(s) \sqsubset \varepsilon(t)$.

Suppose that γ, δ belong to A^1_1.

Let ε be an element of $\mathcal{N}$. We say that ε *embeds* γ *into* δ if and only if ε is strictly monotone and ε maps every number admitted by γ onto a number admitted by δ.

We say that γ *embeds into* δ, notation: $\gamma \leq^* \delta$, if and only if there exists a strictly monotone ε embedding γ into δ. We say that γ *properly embeds into* δ, notation: $\gamma <^* \delta$, if and only if, for some n, γ embeds into δ^n.

We say that γ *positively fails to embed into* δ if and only if for every ε in $\mathcal{N}$, if ε is strictly monotone, then there exists s admitted by γ such that $\varepsilon(s)$ is not admitted by δ.

Note that, if γ positively fails to embed into δ, then γ does not embed into δ.

The first item of the next theorem shows that, on the set of stumps, the notions $\leq^*, <^*$ coincide with the notions $\leq, <$ introduced in Section 1. The theorem should be compared to Theorem 1.1.

The statements of the next theorem have an elementary proof, not involving typically intuitionistic axioms. Recall from Theorem 1.3 that one may prove, by induction on the set **Stp** of stumps: **Stp** is a subset of A^1_1.

Theorem 8.1.

(i) *For all σ, τ in* **Stp**, *$\sigma \leq \tau$ if and only if $\sigma \leq^* \tau$, and: $\sigma < \tau$ if and only if $\sigma <^* \tau$.*

(ii)
- *For all γ in A^1_1, $\gamma \leq \gamma$*
- *For all γ, δ in A^1_1, if $\gamma <^* \delta$, then $\gamma \leq^* \delta$.*
- *For all γ, δ, ζ in A^1_1, if $\gamma \leq^* \delta$ and $\delta <^* \zeta$, then $\gamma <^* \zeta$*
- *For all γ, δ, ζ in A^1_1, if $\gamma <^* \delta$ and $\delta \leq^* \zeta$, then $\gamma <^* \zeta$.*
- *For all γ, δ, ζ in A^1_1, if $\gamma \leq^* \delta$ and $\delta \leq^* \zeta$, then $\gamma \leq^* \zeta$.*

(iii) *For all γ, δ in A^1_1, if $\gamma <^* \delta$, then δ positively fails to embed into γ.*

Proof. (i) The proof is an exercise in (transfinite) induction on the set of stumps and left to the reader.

(ii)The proof is straightforward and left to the reader.

(iii) Let γ, δ be elements of A^1_1 such that $\gamma <^* \delta$. Let ε be an element of $\mathcal{N}$ that is strictly monotone, that is : for every s, t, if s is a proper initial segment of t, then $\varepsilon(s)$ is a proper initial segment of $\varepsilon(t)$. We have to prove that there exists s admitted by δ such that $\varepsilon(s)$ is not admitted by γ.

First observe that δ admits $\langle\ \rangle$ and that, if γ does not admit $\langle\ \rangle$, then γ does not admit $\varepsilon(\langle\ \rangle)$, and we are done.

In the following we assume: γ admits $\langle\ \rangle$.

Find n, ζ such that ζ is strictly monotone and ζ embeds γ into δ^n. We now define, by recursion, a sequence β in $\mathcal{N}$, such that $\beta(0) = \langle\ \rangle$ and, for each p, $\beta(p+1) = \varepsilon\big(\langle n\rangle * \zeta(\beta(p))\big)$. Note that, for each n, $\beta(n)$ is a proper initial part of $\beta(n+1)$. Recalling that γ belongs to A^1_1, find $j > 0$ such that $\beta(j-1)$ is admitted by γ and $\beta(j)$ is not. Now take $s = \langle n\rangle * \zeta(\beta(j-1)$ and observe: s is admitted by δ and $\varepsilon(s)$ is not admitted by γ.

□

8.2. **A^1_1 and E^1_1 do not reduce to each other.** The next theorem is again an elementary result.

Theorem 8.2.

(i) Cantor's diagonal argument: *For every function γ from $\mathcal{N}$ to A^1_1 there exists α in A^1_1 such that, for every β, $\gamma|\beta$ is apart from α.*

(ii) The Boundedness Theorem: *For every function γ from $\mathcal{N}$ to A^1_1 there exists α in A^1_1 such that, for every β, $\gamma|\beta$ embeds into α.*

Proof. (i) Let γ be a function from $\mathcal{N}$ to A^1_1, and let α be an element of A^1_1 such that, for every β, $\alpha(\beta) = (\gamma|\beta)(\beta) + 1$. Clearly, α is apart from every $\gamma|\beta$.

(ii). Let γ be a function from $\mathcal{N}$ to A^1_1. Observe that for every β, for every δ, there exists n such that $(\gamma|\beta)(\overline{\delta}n) \neq 0$ and there exists m such that $\gamma^{\overline{\delta}n}(\overline{\beta}m) = (\gamma|\beta)(\overline{\delta}n) + 1$ and, for each $i < m$, $\gamma^{\overline{\delta}n}(\overline{\beta}i) = 0$. We define α in $\mathcal{C}$ such that for every a, n, if $n = \text{length}(a)$ and $a = \langle a(0), \ldots, a(n-1)\rangle$, then $\alpha(a) = 1$ if and only if there exist i, j such that $2i < n$ and $2j+1 < n$ and $\gamma^{\langle a(0), a(2), \ldots, a(2i)\rangle}(\langle a(1), a(3), \ldots, a(2j+1)\rangle)$ differs from 0. Clearly, α belongs to A^1_1. Now consider any β in $\mathcal{N}$. Define an element ε of $\mathcal{N}$, as follows. For every d, n, if $n = \text{length}(d)$ and $d = \langle d(0), \ldots, d(n-1)\rangle$, then $\varepsilon(d) := \langle d(0), \beta(0), \ldots, d(n-1), \beta(n-1)\rangle$. One verifies easily that ε embeds $\gamma|\beta$ into α.
□

Note that one may obtain the conclusion of the first item of Theorem 8.2 also from its second item, as follows.

Let γ be a function from $\mathcal{N}$ to A^1_1. Using Theorem 8.2(ii), find α in A^1_1 such that, for every β, $\gamma|\beta$ embeds into α. Find δ in A^1_1 such that $\delta^0 = \alpha$ and note: α strictly embeds into δ. We claim that, for every β, $\gamma|\beta$ is apart from δ. In order to see this, let β belong to $\mathcal{N}$. Find ε in $\mathcal{N}$ embedding $\gamma|\beta$ into α. Using Theorem 8.1(iii), find s such that s is admitted by δ and $\varepsilon(s)$ is not admitted by α. Note that s is admitted by δ and not by $\gamma|\beta$, and, therefore, δ is apart from $\gamma|\beta$.

Recall that, just before Theorem 2.1 and just before Theorem 7.1 we introduced subsets US^1_1 and UP^1_1 of $\mathcal{N}$ as follows.

US^1_1 is the set of all α such that, for some γ, the sequence $\langle \alpha_{II}, \gamma\rangle$ belongs to F_{α_I}.

UP^1_1 is the set of all α such that, for all γ, the sequence $\langle \alpha_{II}, \gamma\rangle$ belongs to G_{α_I}.

We have seen that US^1_1, UP^1_1 are cataloguing elements of the classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, respectively, in Theorem 2.1(i) and in Theorem 7.1(i).

Recall that E^1_1 is the set of all α such that for some β, for all n, $\alpha(\overline{\beta}n) = 0$ and A^1_1 is the set of all α such that for every β there exists n such that $\alpha(\overline{\beta}n) \neq 0$.

We have seen that E^1_1, A^1_1 are complete elements of the classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, respectively, in Theorem 2.2(i) and in Theorem 7.2(i).

For each α, for each β, we define: α *forbids* β if and only if, for some n, $\alpha(\overline{\beta}n) \neq 0$ and α *admits* β if and only if α does not forbid β, that is, for each n, $\alpha(\overline{\beta}) = 0$.

The following theorem should be compared to [45], Theorem 5.2.

Theorem 8.3 (The Classical Start of the Projective Hierarchy)**.**

(i) *UP^1_1 is the set of all α in $\mathcal{N}$ apart from every β in US^1_1, and A^1_1 is the set of all α in $\mathcal{N}$ apart from every β in E^1_1.*

(ii) *If UP^1_1 belongs to $\mathbf{\Sigma}^1_1$, and also if US^1_1 belongs to $\mathbf{\Pi}^1_1$, there exists α not belonging to either one of UP^1_1, US^1_1.*

(iii) *For every function γ from $\mathcal{N}$ to $\mathcal{N}$ there exists α such that, for each β, α forbids β if and only if $\gamma|\alpha$ forbids β, and therefore: α belongs to A^1_1 if and only if $\gamma|\alpha$ belongs to A^1_1 and also: α belongs to E^1_1 if and only if $\gamma|\alpha$ belongs to E^1_1.*

(iv) *If A^1_1 reduces to E^1_1 and also if E^1_1 reduces to A^1_1, there exists α not belonging to either one of A^1_1, E^1_1.*

Proof. (i) We prove that A^1_1 is the set of all α in $\mathcal{N}$ apart from every β in E^1_1.

Suppose that α belongs to A^1_1 and β to E^1_1. Find γ such that, for all n, $\beta(\overline{\gamma}n) = 0$. Find m such that $\alpha(\overline{\gamma}m) \neq 0$ and conclude: $\alpha(\overline{\gamma}m) \neq \beta(\overline{\gamma}m)$, so $\alpha \mathrel{\#} \beta$. Clearly, every α in A^1_1 is apart from every β in E^1_1.

Now assume that α is apart from every β in E^1_1. Let γ be an element of $\mathcal{N}$. We let α_γ be the element of $\mathcal{N}$ satisfying: for each s, if γ passes through s, then $\alpha_\gamma(s) = 0$, and, if γ does not pass through s, then $\alpha_\gamma(s) = \alpha(s)$. Note that α_γ belongs to E^1_1 and find s such that $\alpha_\gamma(s) \neq \alpha(s)$. Note that γ passes through s and that $\alpha(s) \neq 0$, so there exists n such that $\alpha(\overline{\gamma}n) \neq 0$. Clearly, every α that is apart from every β in E^1_1 belongs to A^1_1.

It follows that A^1_1 is the set of all α in $\mathcal{N}$ apart from every β in E^1_1.

The proof that UP^1_1 is the set of all α in $\mathcal{N}$ apart from every β in US^1_1 is similar and left to the reader.

(ii) Suppose that UP^1_1 belongs to $\mathbf{\Sigma}^1_1$ and consider the *diagonal* set DP^1_1 consisting of all α such that $\langle \alpha, \alpha\rangle$ belongs to UP^1_1. Also DP^1_1 belongs to $\mathbf{\Sigma}^1_1$. Find β such that DP^1_1 coincides with $US^1_1\restriction\beta$ and observe: $\langle \beta, \beta\rangle$ belongs to UP^1_1 if and only if $\langle \beta, \beta\rangle$ belongs to US^1_1. As every element of UP^1_1 is apart from every element of US^1_1, $\langle \beta, \beta\rangle$ can not belong to either UP^1_1 or US^1_1.

(iii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. We define α in $\mathcal{C}$ as follows. For each s, $\alpha(s) = 1$ if and only if there exist initial segments t, u of s such that $\gamma^t(u) > 1$ and for every proper initial segment v of u, $\gamma^t(v) = 0$. Observe that for every β, for some n,

$\alpha(\overline{\beta}n) > 0$ if and only if, for some n, $(\gamma|\alpha)(\overline{\beta}n) > 0$, that is α forbids β if and only if $\gamma|\alpha$ forbids β, and also: α admits β if and only if $\gamma|\alpha$ admits β. It follows that α belongs to A^1_1 if and only if $\gamma|\alpha$ belongs to A^1_1 and: α belongs to E^1_1 if and only if $\gamma|\alpha$ belongs to E^1_1.

(iv) Suppose that γ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing to A^1_1 to E^1_1. Using (iii), find α such that α belongs to A^1_1 if and only if $\gamma|\alpha$ belongs to A^1_1 and: α belongs to E^1_1 if and only if $\gamma|\alpha$ belongs to E^1_1. We also know that α belongs to A^1_1 if and only if $\gamma|\alpha$ belongs to E^1_1. As every element of A^1_1 is apart from every element of E^1_1, α cannot belong to either one of A^1_1, E^1_1. The proof of the second statement is similar. □

The classical mathematician concludes from Theorem 8.3 that the sets A^1_1 and E^1_1 do not reduce to each other, and thus that there exists an analytic set that is not co-analytic and a co-analytic set that is not analytic. The intuitionistic mathematician, however, fails to see that a contradiction follows from the assumption that some α in $\mathcal{N}$ does not belong to A^1_1 and also not to E^1_1. He may take a note that, as in the case of the second level of the Borel hierarchy, see [45], Subsection 5.3, *Markov's Principle* would enable him to obtain the contradiction, as follows:

> Assume that α does not belong to E^1_1. Then $\neg\exists\beta\forall n[\alpha(\overline{\beta}n) = 0]$. Therefore, $\forall\beta\neg\neg\exists n[\alpha(\overline{\beta}n) \neq 0]$. By Markov's Principle, $\forall\beta\exists n[\alpha(\overline{\beta}n) \neq 0]$, so α belongs to A^1_1.

Nevertheless, it is true, intuitionistically as well as classically, that the set E^1_1 is not co-analytic and that the set A^1_1 is not analytic.

In fact, we already know that E^1_1 does not reduce to A^1_1: the set $D^2(A_1)$, although clearly analytic and reducing to E^1_1, does not reduce to A^1_1, see Theorem 7.1(iv). There is no (continuous) function γ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every α, α belongs to E^1_1 if and only if $\gamma|\alpha$ belongs to A^1_1.

Exploiting the the fixed-point-idea behind Theorem 8.3(iii) one may prove, in an elementary way, a stronger statement: every (continuous) function from $\mathcal{N}$ to $\mathcal{N}$ that maps the set E^1_1 into the set A^1_1 will also map some element of A^1_1 into A^1_1, or, as we will say: E^1_1 *positively fails to reduce to* A^1_1.

We do so in the next Theorem. For the sake of comparison we have added the elementary argument showing that that the set E_2 positively fails to reduce to the set A_2. This argument occurs already in [45], Subsection 5.4.

The set $E_2 = E_{2^*}$ is the set of all α such that, for some n, $\alpha^n = \underline{0}$. The set E_2 is a complete element of the class $\mathbf{\Sigma}^0_2$ consisting of the countable unions of closed sets. ($2^* = S(1^*)$ is the stump defined defined by: for all s, $2^*(s) = 0$ if $length(s) < 2$ and $2^*(s) = 1$ if $length(s) \geq 2$). The set $A_2 = A_{2^*}$ is the set of all α such that, for all n, $\alpha^n \# \underline{0}$.The set A_2 is a complete element of the class $\mathbf{\Pi}^0_2$ consisting of the countable intersections of open subsets of $\mathcal{N}$.

Theorem 8.4 (E^1_1 positively fails to reduce to A^1_1)**.**

(i) *For every function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping E_2 into A_2 there exists α such that both α and $\gamma|\alpha$ belong to A_2.*
(ii) *For every function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping E^1_1 into A^1_1 there exists α such that both α and $\gamma|\alpha$ belong to A^1_1.*

Proof. (i) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping E_2 into A_2. Define α in $\mathcal{C}$ such that, for each s, $\alpha(s) \neq 0$ if and only if there exist n, t, j, i such that $s = \langle n \rangle * t$ and $j \leq length(t)$ and $i \leq s$ and $\gamma^{\langle n \rangle * \overline{t}j}(\overline{\alpha}i) > 1$, while, for each k, if $k < i$, then $\gamma^{\langle n \rangle * \overline{t}j}(\overline{\alpha}k) = 0$. Note that, for each n, $\alpha^n \# \underline{0}$ if and only if $(\gamma|\alpha)^n \# \underline{0}$ and, therefore, α belongs to E_2 if and only if $\gamma|\alpha$ belongs to E_2 and: α belongs to A_2 if and only if $\gamma|\alpha$ belongs to A_2.

We claim that α, and therefore also $\gamma|\alpha$, belong to A_2, and we prove this claim as follows. Let n belong to $\mathbb{N}$. Let α_n be the element of $\mathcal{N}$ satisfying: for each s, if there exists t such that $s = \langle n \rangle * t$, then $\alpha_n(s) = 0$, and, if there is no t such that $s = \langle n \rangle * t$, then $\alpha_n(s) = \alpha(s)$. Note that $(\alpha_n)^n = \underline{0}$ and that α_n belongs to E_2. It follows that $\gamma|\alpha_n$ belongs to A_2. Find m such that $(\gamma|\alpha_n)^n(m) \neq 0$. Find p such that $\gamma^{\langle n \rangle * m}(\overline{\alpha_n}p) > 2$ and, for all j, if $j < p$, then $\gamma^{\langle n \rangle * m}(\overline{\alpha_n}j) = 0$. Note that, for every δ, if $\overline{\delta}p = \overline{\alpha_n}p$, then $(\gamma|\delta)^n(m) = (\gamma|\alpha_n)^n(m) \neq 0$. Now distinguish two cases. *Either:* $\overline{\alpha}p = \overline{\alpha_n}p$ and therefore $(\gamma|\alpha)^n(m) = (\gamma|\alpha_n)^n(m) \neq 0$ and: $(\gamma|\alpha)^n \# \underline{0}$ and, therefore: $\alpha^n \# \underline{0}$, *or:* $\overline{\alpha}pneq\overline{\alpha_n}p$, and $\alpha^n \# \underline{0}$, and, therefore: $(\gamma|\alpha)^n \# \underline{0}$.

Clearly, for each n, both $\alpha^n \# \underline{0}$ and $(\gamma|\alpha)^n \# \underline{0}$ and both α and $\gamma|\alpha$ belong to A_2.

(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping E^1_1 into A^1_1. Using Theorem 8.3(iii), find α such that, for each β, α forbids β if and only if $\gamma|\alpha$ forbids β, and therefore: α belongs to A^1_1 if and only if $\gamma|\alpha$ belongs to A^1_1, and: α belongs to E^1_1 if and only if α belongs to E^1_1. We claim that α, and therefore also $\gamma|\alpha$, belong to A^1_1, and we prove this claim as follows.

Let β belong to $\mathcal{N}$. Let α_β be the element of $\mathcal{N}$ satisfying: for each s, if β passes through s, then $\alpha_\beta(s) = 0$, and, if β does not pass through s, then $\alpha_\beta(s) = \alpha(s)$. Note that α_β admits β and belongs to E^1_1. It follows that $\gamma|\alpha_\beta$ belongs to A^1_1. Find n such that $(\gamma|\alpha_\beta)(\overline{\beta}n) \neq 0$. Also find p such that $\gamma^{\overline{\beta}n}(\overline{\alpha_\beta}p) > 1$, and for all $j < p$, $\gamma^{\overline{\beta}n}(\overline{\alpha_\beta}j) = 0$. Note that, for every δ, if $\overline{\delta}p = \overline{\alpha_\beta}p$, then $(\gamma|\delta)(\overline{\beta}n) = (\gamma|\alpha_\beta)(\overline{\beta}n)$. Now distinguish two cases. *Either:* $\overline{\alpha}p = \overline{\alpha_\beta}p$, and therefore $(\gamma|\alpha)(\overline{\alpha_\beta}n) = (\gamma|\alpha_\beta)(\overline{\alpha_\beta}n) \neq 0$, and $\gamma|\alpha$ forbids β, and, therefore, α forbids β, *or:* $\overline{\alpha}m \neq \overline{\alpha_\beta}m$, and α forbids β and, therefore, $\gamma|\alpha$ forbids β.

Clearly, for each β, both α and $\gamma|\alpha$ forbid β, and both α and $\gamma|\alpha$ belong to A^1_1. □

The next question is: can we also prove that the set A^1_1 is not analytic? Note that we have seen already that A^1_1 is not a *strictly* analytic subset of $\mathcal{N}$. In fact, given any (continuous) function from $\mathcal{N}$ into A^1_1, one may produce an element β of A^1_1 such that, for every α, $\beta \# \gamma|\alpha$, see Theorem 8.2. One might say: *A^1_1 positively fails to be strictly analytic.*

The next Lemma prepares for an argument that A^1_1 also fails to be analytic.

We first introduce the following notion.

Let X, Y be subsets of $\mathcal{N}$.

Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. γ *strongly reduces the set X to the set Y* if and only if, for each α, α belongs to X if and only if $\gamma|\alpha$ belongs to Y and: *α is apart from every element of X if and only if $\gamma|\alpha$ is apart from every element of Y.*

X strongly reduces to X if and only if there exists a function γ from $\mathcal{N}$ to $\mathcal{N}$ strongly reducing X to Y.

Lemma 8.5. *Every analytic subset of $\mathcal{N}$ strongly reduces to E^1_1.*

Proof. Let X be an analytic subset of $\mathcal{N}$. Find β in $\mathcal{N}$ such that, for each α, α belongs to X if and only if, for some δ, β admits $\langle \alpha, \delta \rangle$, that is, for all n, $\beta(\overline{\langle \alpha, \delta \rangle}n) = 0$. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each δ, $(\gamma|\alpha)(\overline{\delta}n) = \beta(\overline{\langle \alpha, \delta \rangle}n)$. It will be clear that, for all α, α belongs to X if and only if $\gamma|\alpha$ belongs to E^1_1. One may verify, by an argument similar to the one used in the proof of Theorem 8.3(i), that, for all α, α is apart from every element of X if and only if, for all δ, β forbids $\langle \alpha, \delta \rangle$, that is, for some n, $\beta(\overline{\langle \alpha, \delta \rangle}n) \neq 0$. It follows that, for each α, α is apart from every element of X if and only if for all δ, $\gamma|\alpha$ forbids δ if and only if $\gamma|\alpha$ is apart from every element of E^1_1. □

Theorem 8.6. *The set A^1_1 is not analytic.*

Proof. Suppose that the set A^1_1 is analytic. Using lemma 8.5, find a function γ from $\mathcal{N}$ to $\mathcal{N}$ strongly reducing A^1_1 to E^1_1. Note that, for every α, if α belongs to E^1_1, then, by Theorem 8.3(i), α is apart from every element of A^1_1, and, therefore, $\gamma|\alpha$ is apart from every element of E^1_1, and so, again by Theorem 8.3(i), $\gamma|\alpha$ belongs to A^1_1. It follows that γ maps E^1_1 into A^1_1. Using Theorem 8.4, one finds α in A^1_1 such that $\gamma|\alpha$ belongs to A^1_1. Contradiction. □

The elementary argument for Theorem 8.6 has eluded us for some time. We then used the following reasoning in order to show that A^1_1 is not analytic. A^1_1 is a *definite* subset of $\mathcal{N}$, and, therefore, by the Brouwer-Kripke axiom, Axiom 13, in Subsection 5.2, if A^1_1 is analytic, A^1_1 is strictly analytic, see Theorem 5.2(iii). However, as we mentioned before, according to Theorem 8.6, A^1_1 positively fails to be strictly analytic.

8.3. **A^1_1 and E^1_1 are not positively Borel.** Note that it follows from Theorem 8.6 that the set A^1_1 is not positively Borel, as, according to Theorem 2.1, every subset of $\mathcal{N}$ that is positively Borel is also analytic. Like the conclusion of Theorem 8.6, this conclusion is obtained in an elementary way.

We now are going to prove that the set E^1_1 is not positively Borel. We will use the Borel Hierarchy Theorem 1.5, in the proof of which Brouwer's Continuity Principle, Axiom 4, plays a key role. We did not find an elementary argument, avoiding the typically intuitionistic axioms.

Lemma 8.7. *For every complementary pair (X, Y) of positively Borel sets there exists a (continuous) function γ from $\mathcal{N}$ to $\mathcal{N}$ reducing X to E^1_1 and mapping Y into A^1_1.*

Proof. Let (X, Y) be a complementary pair of positively Borel sets. Note that X is analytic, see Theorem 2.1(ii), and that every element of X is apart from every element of Y. According to Lemma 8.5, there exists a function strongly reducing X to E^1_1, that is, reducing X to E^1_1 and mapping every α that is apart from every element in X into A^1_1. Note that γ reduces X to E^1_1 and maps Y into A^1_1. □

Theorem 8.8 (The sets E^1_1 and A^1_1 positively fail to be positively Borel)**.**

(i) *For every positively Borel subset X of $\mathcal{N}$, for every function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping E^1_1 into X there exists α in A^1_1 such that $\gamma|\alpha$ belongs to X.*
(ii) *For every non-zero stump σ, for every positively Borel subset X of $\mathcal{N}$ for every function γ from $\mathcal{N}$ to $\mathcal{N}$ mapping A^1_1 into X there exists α in E^1_1 such that $\gamma|\alpha$ belongs to X.*
(iii) *The sets E^1_1 and A^1_1 are not positively Borel.*

Proof. (i) Let X be a positively Borel set and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping E^1_1 into X. Find a non-zero stump σ and a function δ from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_σ. Using Lemma 8.7, find a function ε from $\mathcal{N}$ to $\mathcal{N}$ reducing E_σ to E^1_1 and mapping A_σ into A^1_1. Observe that, for each α, if α belongs to E_σ, then $\varepsilon|\alpha$ belongs to E^1_1, and $\gamma|(\varepsilon|\alpha)$ belongs to X, and $\delta|\big(\gamma|(\varepsilon|\alpha)\big)$ belongs to A_σ, so the composition of the functions (coded by) δ, γ and ε maps E_σ into A_σ. Using the Borel Hierarchy Theorem, we find α such that both α and $\delta|\big(\gamma|(\varepsilon|\alpha)\big)$ belong to A_σ. Remark that $\varepsilon|\alpha$ belongs to A^1_1 and $\gamma|(\varepsilon|\alpha)$ belongs to X.

(ii) Let X be a positively Borel set and let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping A^1_1 into X. Find a non-zero stump σ and a function δ from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_σ. Using Lemma 8.7, find a function ε from $\mathcal{N}$ to $\mathcal{N}$ reducing A_σ to E^1_1 and mapping E_σ into A^1_1. Observe that, for each α, if α belongs to E_σ, then $\varepsilon|\alpha$ belongs to A^1_1, and $\gamma|(\varepsilon|\alpha)$ belongs to X, and $\delta|\big(\gamma|(\varepsilon|\alpha)\big)$ belongs to A_σ, so the composition of the functions (coded by) δ, γ and ε maps E_σ into A_σ. Using the Borel Hierarchy Theorem, we find α such that both α and $\delta|\big(\gamma|(\varepsilon|\alpha)\big)$ belong to A_σ. Remark that $\varepsilon|\alpha$ belongs to A^1_1 and $\gamma|(\varepsilon|\alpha)$ belongs to X.

(iii) is an easy consequence of (i) and (ii).

□

In [32], the question how to prove that A_1^1 is not positively Borel was asked but not answered. As we observed in Section 1 of this paper, it is shown in [41] that there exist co-analytic and analytic subsets of $\mathcal{N}$ much "simpler" than E_1^1, A_1^1, respectively, that fail to be positively Borel, see Theorem 1.11 and 1.12. The fact that the set A_1^1 is not positively Borel, one of the conclusions of Theorem 8.8(iii), also follows from Theorem 1.11 and Theorem 7.2. The fact that the set E_1^1 is not positively Borel, the other conclusion of Theorem 8.8(iii), also follows from Theorem 1.12 and and Theorem 2.2.

Theorem 8.9 (One Half of Souslin's Theorem)**.**

(i) *For every stump σ, the set of all α such that α embeds into σ is a positively Borel subset of $\mathcal{N}$.*
(ii) *For every subset X of $\mathcal{N}$, if X is both strictly analytic and co-analytic, then X is positively Borel.*

Proof. (i) This fact follows, by induction on the set of stumps, from the observation that for every non-zero stump σ, for every α, α embeds into σ if and only if for each m there exists n such that α^m embeds into σ^n.

(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that X coincides with the range of γ. Let δ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing X to A_1^1. Let h be the composition of g and f. Using the Boundedness Theorem, Theorem 8.6(ii), find β in A_1^1 such that, for every α, $\delta|\alpha$ embeds into β. Using Brouwer's Thesis, see Lemma 1.2, we may assume that β is a stump. Observe that, for each α, α is positively Borel if and only if $\delta|\alpha$ embeds into β, therefore, by (i), X is positively Borel.

□

Note that, combining Theorem 8.9 and Theorem 8.8, we find a corroboration of the conclusion of Theorem 8.2: the set A_1^1 is not strictly analytic.

Theorem 8.9(ii) is of limited application as every co-analytic subset of $\mathcal{N}$ is perhapsive and "most" positively Borel sets are not. Therefore, there are not "many" positively Borel sets that are both co-analytic and strictly analytic. The converse of Theorem 8.9(ii) is far from true.

9. The second level of the projective hierarchy

Let X be a subset of $\mathcal{N}$. X is called an (*existential*) *projection of a co-analytic set* if and only if there exists a co-analytic subset Y of $\mathcal{N}$ such that $X = Ex(Y)$. The class of these sets is denoted by $\mathbf{\Sigma}_2^1$. A subset X of $\mathcal{N}$ is called a *universal projection of an analytic set* or a *co-projection of an analytic set* if and only if there exists an analytic subset Y of $\mathcal{N}$ such that $X = Un(Y)$. The class of these sets is denoted by $\mathbf{\Pi}_2^1$.

9.1. An example: the set of all β coding a located and closed subset of $\mathcal{N}$ that is almost-countable.

Let δ be an element of $\mathcal{N}$. As in [46], we let En_δ be the set $\{\delta^0, \delta^1, \ldots\} = \{\delta^n | n \in \mathbb{N}\}$ and we call this set *the subset of $\mathcal{N}$ enumerated by δ*.

In Section 1, we have seen how one defines, given any subset X of $\mathcal{N}$, a function assigning to every stump σ a subset $\mathbb{P}(\sigma, X)$ of $\mathcal{N}$, called *the σ-th perhapsive extension of the set X*, as follows, by induction.

(i) For every stump σ, if $\sigma(\langle\ \rangle) \neq 0$, then $\mathbb{P}(\sigma, X) = X$
(ii) For every stump σ, if $\sigma(\langle\ \rangle) = 0$, then $\mathbb{P}(\sigma, X)$ is the set of all α such that, for some β, if $\alpha \# \beta$, then there exists m such that α belongs to $\mathbb{P}(\sigma^m, X)$.

It is not difficult to prove and shown in [46] that, for all inhabited subsets X of $\mathcal{N}$, for all stumps σ, τ, if $\sigma \leq \tau$, then $\mathbb{P}(\sigma, X)$ is a subset of $\mathbb{P}(\tau, X)$, and that, for all inhabited subsets X and Y of $\mathcal{N}$, if X is a subset of Y, then, for all stumps σ, $\mathbb{P}(\sigma, X)$ is a subset of $\mathbb{P}(\sigma, Y)$.

Note that, for every δ, for every non-empty stump σ, α belongs to $\mathbb{P}(\sigma, En_\delta)$ if and only if, for some n, if $\alpha \# \delta^n$, then there exists m such that α belongs to $\mathbb{P}(\sigma^m, X)$.

We let $Almost^*(En_\delta)$ be the set of all α satisfying: for each γ, there exists n such that $\overline{\alpha}\big(\gamma(n)\big) = \overline{\delta^n}\big(\gamma(n)\big)$. Note that we may think of $Almost^*(En_\delta)$ as the set of all α such that every attempt to give positive evidence that α differs from every δ^n fails in finitely many steps. It is not difficult to prove and shown in [46] that, for all δ, for all ε, if En_δ coincides with En_ε, then, for every α: (for each γ, there exists n such that $\overline{\alpha}\big(\gamma(n)\big) = \overline{\delta^n}\big(\gamma(n)\big)$) *if and only* if (for each γ, there exists n such that $\overline{\alpha}\big(\gamma(n)\big) = \overline{\varepsilon^n}\big(\gamma(n)\big)$). The definition of the set $Almost^*(En_\delta)$ is thus unambiguous in the sense that it does not depend on the enumeration δ.

It is also shown in [46] that for each δ, for each stump σ, the set $\mathbb{P}(\sigma, En_\delta)$ is a subset of the set $Almost^*(En_\delta)$ and that Brouwer's Thesis implies that the set $Almost^*(En_\delta)$ id s subset of the set $\bigcup\limits_{\sigma \in \mathbf{Stp}} \mathbb{P}(\sigma, En_\delta)$. This result will be proven again in Theorem 9.2.

We let **CS** be the set of all β coding an *countable spread*, that is the set of all β such that

(i) for all s, $\beta(s) = 0$ if and only if, for some n, $\beta(s * \langle n \rangle) = 0$, and
(ii) for some δ, F_β is a subset of En_δ.

In the above definition, one might think of strengthening (ii) as follows: "for some δ, for each n, δ^n belongs to F_β, and F_β coincides with En_δ." This would not be a real strengthening. For suppose, for some δ, F_β is a subset of En_δ. As in the proof of Theorem 2.4(i), let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ retracting $\mathcal{N}$ onto F_β. Let δ' satisfy: for each n, $(\delta')^n = \gamma|(\delta^n)$. Note that for each n, $(\delta')^n$ belongs to F_β, and F_β coincides with $En_{\delta'}$.

For each stump σ we let $\mathbf{PCS}_\sigma$ be the set of all β coding a *σ-perhaps-countable spread*, that is the set of all β such that

(i) for all s, $\beta(s) = 0$ if and only if, for some n, $\beta(s * \langle n \rangle) = 0$, and
(ii) for some δ, F_β is a subset of $\mathbb{P}(\sigma, En_\delta)$.

We let **ACS** be the set of all β coding an *almost-countable spread*, that is the set of all β such that

(i) for all s, $\beta(s) = 0$ if and only if, for some n, $\beta(s * \langle n \rangle) = 0$, and
(ii) for some δ, F_β is a subset of $Almost^*(En_\delta)$.

It is easy to see that the sets $\mathbf{PCS}_\sigma$ and the set **ACS** belong to the class $\mathbf{\Sigma}^1_2$.

Note that a classical mathematician, having seen Section 2, and in particular its conclusion that the set **UNC** is analytic, see Theorem 2.4, would expect the sets $\mathbf{PCS}_\sigma$ and the set **ACS** to be in the class $\mathbf{\Pi}^1_1$.

Recall that **Fin** is the set of all α in Cantor space $\mathcal{C}$ such that there exists n such that, for all m, if $m > n$, then $\alpha(m) = 0$.

Note that the set **Fin** and also all its perhapsive extensions are *shift-invariant* subsets of $\mathcal{C}$, that is: for every stump σ, for every α in $\mathcal{C}$, α belongs to the set $\mathbb{P}(\sigma, \mathbf{Fin})$ if and only if $\langle 0 \rangle * \alpha$ belongs to $\mathbb{P}(\sigma, \mathbf{Fin})$ if and only if $\langle 1 \rangle * \alpha$ belongs to $\mathbb{P}(\sigma, \mathbf{Fin})$.

Note that there exists δ such that $\mathbf{Fin} = En_\delta$. For instance, let δ satisfy: for each s, *if* s belongs to the set **Bin** of all natural numbers coding a finite binary sequence, then $\delta^s = s * \underline{0}$, and, *if not*, then $\delta^s = \underline{0}$.

We define a function $\sigma \mapsto CB^\dagger_\sigma$ associating to every stump σ a subset of $\mathcal{N}$, as follows, by induction on the set of stumps:

(i) For each stump σ, if $\sigma(0) \neq 0$, then $CB^\dagger_\sigma = \{\underline{0}\} \cup \{\overline{\underline{0}}n * \langle 1 \rangle * \underline{0} \mid n \in \mathbb{N}\}$.

(ii) For each stump σ, if $\sigma(0) = 0$, then
$$CB^\dagger_\sigma = \{\underline{0}\} \cup \{\overline{\underline{0}}n * \langle 1 \rangle * \underline{0} \mid n \in \mathbb{N}\} \cup \bigcup_{n\in\mathbb{N}} \bigcup_{p\in\mathbb{N}} \overline{\underline{0}}n * \langle 1 \rangle * \overline{\underline{0}}p * \langle 1 \rangle * CB^\dagger_{\sigma^{p(0)}}.$$

The sets $CB^\dagger_\sigma$ are called *very special Cantor-Bendixson-sets*. We introduced the special Cantor-Bendixson-sets CB^*_σ in Subsection 3.5. It is not difficult to prove that, for each stump σ, there exists a stump τ such that $CB^\dagger_\sigma = CB^*_\tau$, that is, every very special Cantor-Bendixson-set is a special Cantor-Bendixson-set.

Let X be a located and closed subset of $\mathcal{N}$ and let Y be a subset of $\mathcal{N}$. X *embeds into* Y if and only if there exists γ coding a function from X into Y that is *strongly injective*, that is, for all α, β in X, if $\alpha \# \beta$, then $\gamma|\alpha \# \gamma|\beta$.

Theorem 9.1.

(i) *For every β in* **CS**, *for every stump σ, if $\sigma(0) \neq 0$, then F_β embeds into $CB^\dagger_\sigma \cap$* **Fin**.

(ii) *For every stump σ, for every β in* $\mathbf{PCS}_\sigma$, *F_β embeds into $\overline{CB^\dagger_\sigma} \cap \mathbb{P}(\sigma, \mathbf{Fin})$.*

(iii) *For every β in* **ACS**, *F_β embeds into* $\mathbf{Almost}^*\mathbf{Fin}$.

Proof. (i) Let σ be a stump satisfying $\sigma(0) \neq 0$, that is, σ is or codes the empty stump. Assume that β belongs to **CS**. Find δ such that F_β is a subset of En_δ. Using the Second Axiom of Continuous Choice, Axiom 6, find γ coding a continuous function from F_β to $\mathbb{N}$ such that, for every α in F_β, α coincides with $\delta^{\gamma(\alpha)}$. Note that, for each α in F_β, $\delta^{\gamma(\alpha)}$ is the only element of F_β passing through $\overline{\alpha}m$ where m is the least j such that $\gamma(\overline{\alpha}j) \neq 0$. We let ε be the code of a continuous function from F_β to $\mathcal{N}$ such that, for each α, $\varepsilon|\alpha = \overline{\underline{0}}(\gamma(\alpha)) * \langle 1 \rangle * \underline{0}$. It is clear that γ defines a strongly injective function from F_β to $CB^\dagger_\sigma \cap \mathbf{Fin}$.

(ii) We use induction on the set of stumps. Note that, if $\sigma(\langle\,\rangle) \neq 0$, we may refer to (i). Let σ be a non-empty stump and assume the statement has been proven for every immediate substump of σ. Assume that β belongs to $\mathbf{PCS}_\sigma$. Find δ such that F_β is a subset of $\mathbb{P}(\sigma, En_\delta)$. Note that, for every α in F_β, there exists n such that, if $\delta^n \# \alpha$, then there exists m such that α belongs to $\mathbb{P}(\sigma^m, En_\delta)$. Using the (generalized) First Axiom of Continuous Choice, Axiom 6, find γ coding a continuous function from F_β to $\mathbb{N}$ such that, for each α in F_β, if $\delta^{\gamma(\alpha)} \# \alpha$, then there exists m such that α belongs to $\mathbb{P}(\sigma^m, En_\delta)$.

We define s in $\mathbb{N}$ to be *wrong* if $\beta(s) = 0$ and there exists $m \leq \mathit{length}(s)$ such that m is the least i such that $\gamma(\overline{s}i) \neq 0$ and $\delta^{\gamma(\overline{s}m)-1}$ *does not pass through* s.

Now assume s is a wrong element of $\mathbb{N}$ such that, for every n, if $n < \mathit{length}(s)$, then $\overline{s}n$ is not wrong. So s is a wrong element of $\mathbb{N}$ of minimal length. Let us denote the set of all α in F_β that pass through s by $F_\beta \cap s$. Note that, for all α in $F_\beta \cap s$, $\alpha \# \delta|\alpha$ and there exists m such that α belongs to $\mathbb{P}(\sigma^m, En_\delta)$. Also note that $F_\beta \cap s$ is a spread. Applying the First Axiom of Continuous Choice, Axiom 6, find ζ coding a function from $F_\beta \cap s$ to $\mathbb{N}$ such that for all n, if $n \leq \mathit{length}(s)$, then $\zeta(\overline{s}n) = 0$, and, for all α in $F_\beta \cap s$, α belongs to $\mathbb{P}(\sigma^{\zeta(\alpha)}, En_\delta)$.

Applying the Second Axiom of Countable Choice, Axiom 3, we find ρ such that, for each s, is s is a wrong element of $\mathbb{N}$ of minimal length, then ρ^s codes a function from $F_\beta \cap s$ to $\mathbb{N}$ such that for all n, if $n \leq \mathit{length}(s)$, then $\rho^s(\overline{s}n) = 0$, and, for all α in $F_\beta \cap s$, α belongs to $\mathbb{P}(\sigma^{\rho^s(\alpha)}, En_\delta)$.

We define t in $\mathbb{N}$ to be *right* if there exists an initial part s of t such that s is a wrong element of $\mathbb{N}$ of minimal length and $\rho^s(t) \neq 0$ and for all n, if $n < \mathit{length}(t)$, then $\rho^s(\overline{t}n) = 0$.

Using the induction hypothesis and the Second Axiom of Countable Choice, Axiom 3, we find η in $\mathcal{N}$ such that, for every t, for each m, if t is right, and m is the least

i such that $\overline{t}i$ is wrong, then η^t codes a strongly one-to-one function from $F_\beta \cap t$ into $\overline{CB^\dagger_{\sigma^{\rho^{\overline{t}m}(t)-1}}} \cap \mathbb{P}(\sigma^{\rho^{\overline{t}m}(t)-1}, \mathbf{Fin})$.

Finally, we let τ be the code of a continuous function from F_β to $\mathcal{N}$ such that, for each α in F_β,

(1) for each α in F_β, for all n, if n is the least i such that $\gamma(\overline{\alpha}i) \neq 0$, then $\tau|\alpha$ passes through $\underline{\overline{0}}(\overline{\alpha}n) * \langle 1 \rangle$.

(2) for each α in F_β, for each n, for each m, if n is the least i such that $\gamma(\overline{\alpha}i) \neq 0$ and $m > n$ and there is no $i \le m$ such that $\overline{\alpha}i$ is wrong, then $\tau|\alpha$ passes through $\underline{\overline{0}}(\overline{\alpha}n) * \langle 1 \rangle * \underline{\overline{0}}(m-n)$.

(3) for each α in F_β, for each n, for each m, for each k, if n is the least i such that $\gamma(\overline{\alpha}i) \neq 0$, and $m > n$ and m is the least i such that $\overline{\alpha}i$ is wrong, and $k > m$ and k is the least i such that $\rho^{\overline{\alpha}m}(\overline{\alpha}i) \neq 0$, then $\tau|\alpha = \underline{\overline{0}}(\overline{\alpha}n) * \langle 1 \rangle * \underline{\overline{0}}(\langle \overline{\alpha}k, \rho^{\overline{\alpha}m}(\overline{\alpha}k) - 1\rangle) * \langle 1 \rangle * \eta^{\overline{\alpha}k}|\alpha$.

Note that τ codes a strongly injective function from F_β into $\overline{CB^\dagger_\sigma}$.

Note that, for all n, $\tau|(\delta^n) = \underline{\overline{0}}(\overline{\delta^n}m) * \langle 1 \rangle * \underline{0}$, where m is the least i such that $\gamma(\overline{\delta^n}i) \neq 0$.

Note that, for all α in F_β, $\tau|\alpha$ passes through $\underline{\overline{0}}(\overline{\alpha}m) * \langle 1 \rangle$ where m is the least i such that $\gamma(\overline{\alpha}i) \neq 0$. Also note that, for all α in F_β, if $\tau|\alpha \# \tau|(\delta^{\gamma(\alpha)})$, then there exist m, n, t, p such that $\tau|\alpha = \underline{\overline{0}}m * \langle 1 \rangle * \underline{\overline{0}}n * \langle 1 \rangle * \eta^t|\alpha$ and $\eta^t|\alpha$ belongs to $\mathbb{P}(\sigma^p, \mathbf{Fin})$. As the set $\mathbb{P}(\sigma^p, \mathbf{Fin})$ is shift-invariant, it follows that τ maps F_β into $\mathbb{P}(\sigma, \mathbf{Fin})$.

(iii) Assume that β belongs to **ACS**. Find δ such that F_β is a subset of $Almost^*(En_\delta)$.

First observe that, for each s such that $\beta(s) = 0$, there exists i such that $\delta^i\big(length(s)\big) = s$. For, given s such that $\beta(s) = 0$, let α be an element of F_β that passes through s. Define $\gamma = \underline{length(s)}$, that is, for each i, $\gamma(i) = length(s)$. Find i such that $\overline{\alpha}\big(\gamma(i)\big) = \overline{\delta^i}\big(\gamma(i)\big)$ and conlude: δ^i passes through s.

We now define ε in $\mathcal{N}$ such that, for each s, if $\beta(s) = 0$, then $\varepsilon(s)$ equals the least i such that δ^i passes through s. Note that, for all s, for all n, if $\beta(s * \langle n \rangle) = 0$, then $\varepsilon(s) \le \varepsilon(s * \langle n \rangle)$. We then define η in $\mathcal{N}$ such that $\eta(\langle\ \rangle) = \langle\ \rangle$ and, for all s, for all n, if $\beta(s * \langle n \rangle) = 0$, then: *either* $\varepsilon(s * \langle n \rangle) = \varepsilon(s)$ and $\eta(s * \langle n \rangle) = \eta(s) * \langle 0 \rangle$, *or* $\varepsilon(s * \langle n \rangle) \neq \varepsilon(s)$ and $\eta(s * \langle n \rangle) = \eta(s) * \langle 0 \rangle * \underline{\overline{1}}(\varepsilon(s * \langle n \rangle))$. Finally, we let γ be an element of $\mathcal{N}$ coding a function from F_β to $\mathcal{N}$ such that, for every α in F_β, for every n, $\gamma|\alpha$ passes through $\eta(\overline{\alpha}n)$.

It is easily seen that γ is a strongly injective function from F_β to Cantor space $\mathcal{C}$.

We now prove that, for all α in F_β, $\gamma|\alpha$ belongs to $\mathbf{Almost^*Fin}$.

Let α belong to F_β. Suppose that ζ is strictly increasing, that is, for all n, $\zeta(n) < \zeta(n+1)$. Note that, for all n, there exists $m > n$ such that $(\gamma|\alpha)(m) = 0$. We define ζ' as follows. $\zeta'(0) = \zeta(0)$ and, for each n, $\zeta'(n+1) = \zeta(p)$ where p is the least j such that there exists m satisfying: $\zeta'(n) < m < \zeta(j)$ and $(\gamma|\alpha)(m) = 0$. We define η in $\mathcal{N}$ as follows. For each p, $\eta(p)$ is the number of elements of the set $\{m < \zeta'(p) | (\gamma|\alpha)(m) = 0\}$. Note that η is strictly increasing and that, for each p, for each α' in F_β, if α' passes through $\overline{\alpha}\big(\eta(p)\big)$, then $\gamma|\alpha'$ passes through $\overline{\gamma|\alpha}\big(\zeta'(p)\big)$. Note that, for each p, if $(\gamma|\alpha)\big(\zeta'(p)\big) = 1$, then $\varepsilon\big(\overline{\alpha}(\eta(p))\big) \neq \varepsilon\big(\overline{\alpha}(\eta(p-1))\big)$.

We claim that there exists i such that $\varepsilon\big(\overline{\alpha}(\eta(p))\big) = \varepsilon\big(\overline{\alpha}(\eta(p-1))\big)$. In order to prove this claim we define ρ in $\mathcal{N}$ as follows. Note that, for each i, if $i < \varepsilon(\overline{\alpha}(\eta(i+1))$, then $\delta^i \# \alpha$. Therefore, for each i, *if* $i < \varepsilon(\overline{\alpha}(\eta(i+1))$, we let $\rho(i)$ be the least j such that $\overline{\delta^i}j \neq \overline{\alpha}j$, and, *if not*, then $\rho(i) = 0$. Using the fact that α belongs to $Almost^*(En_\delta)$, we find i such that $\overline{\delta^i}(\rho(i)) = \overline{\alpha}(\rho(i))$. Note that $\rho(i) = 0$ and $i \ge \varepsilon(\overline{\alpha}(\eta(i+1))$ and there exists $p \le i$ such that $\varepsilon\big(\overline{\alpha}(\eta(p))\big) = \varepsilon\big(\overline{\alpha}(\eta(p-1))\big)$.

We may conclude that there exists p such that $(\gamma|\alpha)\big(\zeta'(p)\big) = 0$ and that there exists p such that $(\gamma|\alpha)\big(\zeta(p)\big) = 0$.

We have shown that $\gamma|\alpha$ belongs to $\mathbf{Almost}^*\mathbf{Fin}$. $\square$

The set of the *hereditarily repetitive stumps* has been introduced in Section 1, just before Theorem 1.5. One may verify that, for every hereditarily repetitive stump σ, for each n, the set $\overline{CB^*_\sigma}$ embeds into the set $\overline{CB^*_\sigma} \cap \overline{\underline{0}}n$, that is, the set of all elements of $\overline{CB^*_\sigma}$ passing through $\overline{\underline{0}}n$, see [46], Theorem 3.9(i).

In the proof of the next theorem, we shall use the fact that, for all hereditarily repetitive stumps σ, τ, if $\sigma \leq \tau$, then $\overline{CB^*_\sigma}$ embeds into $\overline{CB^*_\tau}$. This fact is proven without much difficulty, see [46], Theorem 3.9(ii).

Theorem 9.2.

(i) *For each stump σ, the set $\overline{CB^*_\sigma}$ is a σ-perhaps-countable spread.*

(ii) *For each hereditarily repetitive stump σ, the set $\overline{CB^*_{S(\sigma)}}$ does not embed into $\mathbb{P}(\sigma, \mathbf{Fin})$.*

(iii) *For each hereditarily repetitive stump σ, the set $\overline{CB^*_{S(\sigma)}}$ is not a σ-perhaps-countable spread.*

Proof. (i) For each stump σ, the set CB^*_σ is easily seen to be a countable set and its closure $\overline{CB^*_\sigma}$ coincides with the set $\mathbb{P}(\sigma, CB^*_\sigma)$, see Theorem 1.9(ii). The latter fact has been established in [46], Theorem 3.22. The proof is given there for the Cantor-Bendixson-sets CB_σ but works as well for the special Cantor-Bendixson-sets CB^*_σ, as the proof is by induction on the set of stumps and only the starting clause has to be adapted for the case of the special Cantor-Bendixson-sets introduced in Subsection 3.5 of this paper. The starting statement is trivial, however, as in the case of the Cantor-Bendixson-sets themselves.

(ii) We again use induction on the set of stumps.

First assume that σ is a stump satisfying $\sigma(0) \neq 0$. Note that $CB_{S(\sigma)}$ is the set $\{\underline{0}\} \cup \{\overline{\underline{0}}n * \langle 1 \rangle * \underline{0} | n \in \mathbb{N}\}$. Suppose that γ codes a continuous function from $\overline{CB_{S(\sigma)}}$ to $\mathcal{N}$ embedding $\overline{CB_{S(\sigma)}}$ into $\mathbb{P}(\sigma, \mathbf{Fin}) = \mathbf{Fin}$. Note that, for all α in $\overline{CB_{S(\sigma)}}$, there exists n such that, for all $m > n$, $(\gamma|\alpha)(n) = 0$. Using Brouwer's Continuity Principle, Axiom 4, we find p, m, such that, for all α in $\overline{CB_{S(\sigma)}}$, if $\overline{\alpha}p = \overline{\underline{0}}p$, then, for all n, if $n > m$, then $(\gamma|\alpha)(n) = 0$. Find $q \geq p$ such that $\overline{\gamma|\underline{0}(m+1)} \sqsubseteq \gamma|\overline{\underline{0}}q$. It follows that, for every α, if $\overline{\alpha}q = \overline{\underline{0}}q$, then $\gamma|\alpha = \gamma|\underline{0}$, so γ is not strongly injective.

Now assume that σ is a hereditarily repetitive stump such that $\sigma(0) = 0$ and, for each n, $\overline{CB^*_{S(\sigma^n)}}$ does not embed into $\mathbb{P}(\sigma^n, \mathbf{Fin})$. Suppose that γ codes a continuous function from $\overline{CB_{S(\sigma)}}$ to Cantor space $\mathcal{C}$ embedding $\overline{CB_{S(\sigma)}}$ into $\mathbb{P}(\sigma, \mathbf{Fin})$. Note that, for all α in $\overline{CB_{S(\sigma)}}$, there exists m, such that, for all n, if $n > m$ and $(\gamma|\alpha)(n) \neq 0$, then there exists k such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \mathbf{Fin})$. Using Brouwer's Continuity Principle, Axiom 4, we find p, m, such that, for all α in $\overline{CB_{S(\sigma)}}$, if $\overline{\alpha}p = \overline{\underline{0}}p$, then, for all n, if $n > m$, then there exists k such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \mathbf{Fin})$. Find $q \geq p$ such that $\overline{\gamma|\underline{0}(m+1)} \sqsubseteq \gamma|\overline{\underline{0}}q$. Now consider $\alpha_0 := \overline{\underline{0}}q * \langle 1 \rangle * \underline{0}$ and $\alpha_1 := \overline{\underline{0}}(q+1) * \langle 1 \rangle * \underline{0}$. Note that $\alpha_0 \# \alpha_1$, and for both $i < 2$, $\overline{\alpha_i}q = \overline{\underline{0}}q$. As γ is strongly injective, $\gamma|\alpha_0 \# \gamma|\alpha_1$, and there exist $i < 2$ and $n > m$ such that $(\gamma|\alpha_i)(n) \neq 0$. Find $i < 2$ and $n > m$ such that $(\gamma|\alpha_i)(n) \neq 0$. Find $r > q$ such that $\overline{\gamma|\alpha_i}(n+1) \sqsubseteq \gamma|\overline{\alpha_i}r$. Note that, for every α in $\overline{CB_{S(\sigma)}}$, if α passes through $\overline{\alpha_i}r$, then there exists k such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \mathbf{Fin})$. Using Brouwer's Continuity Principle, Axiom 4, again, find $t \geq r$ and k such that, for every α in $\overline{CB_{S(\sigma)}}$, if α passes through $\overline{\alpha_i}t$, then $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \mathbf{Fin})$. Note that γ embeds $\overline{CB^*_{S(\sigma)}} \cap \overline{\alpha_i}t$ into $\mathbb{P}(\sigma^k, \mathbf{Fin})$. On the other hand, $S(\sigma^k) \leq \sigma$ and, therefore, $\overline{CB^*_{S(\sigma^k)}}$ embeds into $\overline{CB^*_\sigma}$. Also, as σ is hereditarily repetitive, $\overline{CB^*_\sigma}$ embeds into $\overline{CB^*_{S(\sigma)}} \cap \overline{\alpha_i}t$. Combining these facts, we find that $\overline{CB^*_{S(\sigma^k)}}$ embeds into $\mathbb{P}(\sigma^k, \mathbf{Fin})$, and this contradicts the induction hypothesis.

(iii) This is an immediate consequence of (ii) and Theorem 9.1(ii). □

Let δ, α belong to $\mathcal{N}$ and let σ be a stump. We define:

σ *secures that* α *belongs to* $Almost^*(En_\delta)$

if and only if

for each γ, *there exists* n *such that* $\overline{\gamma}(n+1)$ *is admitted by* σ *and* $\overline{\alpha}\big(\gamma(n)\big) = \overline{\delta^n}\big(\gamma(n)\big)$.

Note that item (iv) of the next theorem is a *boundedness result*, to be compared with Theorems 8.2(ii) and 8.9.

Theorem 9.3.

(i) *For every stump* σ, *for all* δ, *for all* α, *if* σ *secures that* α *belongs to* $Almost^*(En_\delta)$, *then* α *belongs to* $\mathbb{P}(\sigma, En_\delta)$.

(ii) *For all* δ, *for all* α, *if* α *belongs to* $Almost^*(En_\delta)$, *then there exists a stump* σ *such that* α *belongs to* $\mathbb{P}(\sigma, En_\delta)$.

(iii) *For all* δ, *the set* $Almost^*(En_\delta)$ *coincides with the set* $\bigcup_{\sigma \in \mathbf{Stp}} \mathbb{P}(\sigma, En_\delta)$.

(iv) *For every* β, *for every* δ, *if for every* s, $\beta(s) = 0$ *if and only if, for some* n, $\beta(s * \langle n \rangle) = 0$, *and* F_β *is a subset of* $Almost^*(En_\delta)$, *then there exists a stump* τ *such that* F_β *is a subset of* $\mathbb{P}(\tau, En_\delta)$.

(v) *The set* $\mathbf{ACS}$ *coincides with the set* $\bigcup_{\sigma \in \mathbf{Stp}} \mathbf{PCS}_\sigma$.

Proof. (i) We use the principle of induction on the set of stumps, Axiom 8(iii). Note that, if $\sigma(0) \neq 0$, that is, σ codes the empty stump, then the statement to be proven is true for trivial reasons. Now assume that σ is a stump satisfying $\sigma(0) = 0$ and the statement has been verified for every immediate substump σ^n of σ. Let δ, α be elements of $\mathcal{N}$ such that σ secures that α belongs to $Almost(En_\delta)$. Suppose that $\alpha \# \delta^0$. Find m such that $\overline{\alpha}m \neq \overline{\delta^0}m$. Note that, for each γ, if $\gamma(0) = m$, then there exists $n > 0$ such that $\overline{\gamma}(n+1)$ is admitted by σ and $\overline{\alpha}(\gamma(n)) = \overline{\delta^n}(\gamma(n))$. It follows that, for each γ, there exists n such that $\overline{\gamma}(n+1)$ is admitted by σ^m and $\overline{\alpha}(\gamma(n)) = \overline{\delta^{n+1}}(\gamma(n))$, that is, σ^m secures that α belongs to $Almost^*(En_{\delta'})$, where δ' satisfies: for each n, $(\delta')^n = \delta^{n+1}$. Using the induction hypothesis, we conclude that, if $\alpha \# \delta^0$, then, for some m, α belongs to $\mathbb{P}(\sigma^m, En_{\delta'})$. As $En_{\delta'}$ is a subset of En_δ, for each m, $\mathbb{P}(\sigma^m, En_{\delta'})$ is a subset of $\mathbb{P}(\sigma^m, En_\delta)$. It follows that α belongs to $\mathbb{P}(\sigma, En_\delta)$.

(ii) Note that, if α belongs to $Almost(En_\delta)$, then, according to Brouwer's Thesis, Axiom 9, there exists a stump σ that secures that α belongs to $Almost(En_\delta)$, and apply (i).

(iii) is an easy consequence of (ii).

(iv) Let β, δ satisfy the assumptions. Let ε be the code of a continuous function from $\mathcal{N}$ to $\mathcal{N}$ retracting $\mathcal{N}$ onto the set F_β that is, for all α, $\varepsilon|\alpha$belongs to F_β, and, if α belongs to F_β, then $\varepsilon|\alpha = \alpha$. Note that, for each α, there exists n, such that $\overline{\varepsilon|\alpha_I}(\alpha_{II}(n)) = \overline{\delta^n}(\alpha_{II}(n))$. Let B be the set of all s such that $length(s)$ is even and, for some $n < length(s)$, $length(\varepsilon|s_I) \geq s_{II}(n)$ and $\overline{\varepsilon|s_I}(s_{II}(n)) = \overline{\delta^n}(s_{II}(n))$. Note that B is a bar in $\mathcal{N}$, and, using Brouwer's Thesis, Axiom 9, find a stump τ such that the set of all elements in B that are admitted by τ is a bar in $\mathcal{N}$. We now prove that, for every α in F_β, there exists a stump σ securing that α belongs to $Almost(En_\delta)$ such that $\sigma \leq^* \tau$, that is: σ *embeds into* τ. (The notion $\leq^*$ has been defined in Subsection 8.1.).

Let α be an element of F_β. We let ε be the element of $\mathcal{N}$ satisfying: for each s, $length(\varepsilon(s)) = 2length(s)$ and α passes through $(\varepsilon(s))_I$ and $(\varepsilon(s))_{II} = s$. Now let σ in $\mathcal{N}$ satisfy: for each s, $\sigma(s) = \tau\big(\varepsilon(s)\big)$. One easily verifies that σ is a stump securing that α belongs to $Almost(En_\delta)$ and that ε embeds σ into τ. It follows from (i) that α

belongs to $\mathbb{P}(\sigma, En_\delta)$, and, as $\sigma \leq \tau$, also to $\mathbb{P}(\tau, En_\delta)$. We thus see that F_β is a subset of $\mathbb{P}(\tau, En_\delta)$.

(v) is an easy consequence of (iv). □

We now may draw a conclusion similar to a conclusion Cantor found by an indirect argument, see [8], Theorem C, in [9], page 220.

Corollary 9.4. *Let X be a located and closed subset of $\mathcal{N}$. If X is almost-countable, then X is reducible in Cantor's sense, that is: there exists a stump σ such that $Der(\sigma, X) = \emptyset$.*

Proof. Let X be an almost-countable located and closed subset of $\mathcal{N}$. Using Theorem 9.3(v), we find a stump σ such that X is σ-perhaps-countable. Using Theorem 9.1(ii), we find that X embeds into $\overline{CB^{\dagger}_\sigma}$. We determine a stump τ such that $\overline{CB^{\dagger}_\sigma}$ coincides with $\overline{CB_\tau}$. We now use Theorem 1.8(ii) and the easy observation that, for each subset Y of $\mathcal{N}$, for each non-empty stump σ, the set $Der(\sigma, Y)$ coincides with the set $Der(\sigma, \overline{Y})$. We conclude $Der(\tau, \overline{CB_\tau}) = \emptyset$. Now determine a strongly injective function γ from X into $\overline{CB_\tau}$. One may prove, by induction on the set of stumps, that, for each stump σ, γ embeds the set $Der(\sigma, X)$ into the set $Der(\sigma, \overline{CB_\tau})$. It follows that γ embeds the set $Der(\tau, X)$ into the empty set, and that $Der(\tau, X) = \emptyset$.

□

The converse of Corollary 9.4, that is, the statement that every reducible located and closed subset of X is almost-countable, does not seem to be true.

Theorem 9.5.

(i) *For all located and closed subsets X, Y of $\mathcal{N}$, if X is compact and X embeds into Y, then there exists a (continuous) function from Y to X mapping Y onto X.*

(ii) *For all located and closed subsets X, Y of $\mathcal{N}$, if there exists a surjective (continuous) function from Y onto X, then X embeds into Y.*

(iii) *For every non-empty compact, located and closed subset X of $\mathcal{N}$, the following conditions are equivalent:*
  - *(a) X is almost-countable.*
  - *(b) There exists a stump σ such that X embeds into $\overline{CB^*_\sigma}$.*
  - *(c) There exists a stump σ such that $\overline{CB^*_\sigma}$ maps onto X.*

Proof. (i) Let X, Y be located and closed subsets of $\mathcal{N}$ and let X be compact. Suppose γ codes a strongly injective (continuous) function from X to Y. Note that, for each n, for each α in X, there exists m such that $length\big(\gamma|(\overline{\alpha}m)\big) \geq n$. Using the Fan Theorem, Theorem 1.4, we conclude that there exists m such that, for each α in X, $length\big(\gamma|(\overline{\alpha}m)\big) \geq n$. Let β be an element of $\mathcal{N}$ such that, for each n, $\beta(n)$ equals the least m such that for each α in X, $length\big(\gamma|(\overline{\alpha}m)\big) \geq n$. Note that, for each s, there exists α in X such that $\gamma|\alpha$ passes through s if and only if there exists t such that t contains an element of X and $s \sqsubseteq \gamma|t$ and $length(t) = length\big(\beta(length(s))\big)$. Define δ in $\mathcal{C}$ such that, for each s, $\delta(s) = 0$ if and only if there exists α in X such that $\gamma|\alpha$ passes through s. Note that F_δ is a compact located and closed subset of $\mathcal{N}$ and that, for each α in X, $\gamma|\alpha$ belongs to F_δ.

Now assume ε belongs to F_δ. We intend to show there exists α in X such that $\gamma|\alpha = \varepsilon$. First, find ζ in $\mathcal{C}$ such that, for each s, $\zeta(s) = 0$ if and only if there exists α in X passing through s. We claim:

> For all s, t, if $\zeta(s) = \zeta(t) = 0$ and $s \perp t$, then *either:* for every α in $X \cap s$, $\gamma|\alpha \# \varepsilon$, *or:* for every α in $X \cap t$, $\gamma|\alpha \# \varepsilon$.

We prove this claim as follows. Note that, for every α in $X \cap s$, for every η in $X \cap t$, $\gamma|\alpha \# \gamma|\eta$, and, therefore: either $\gamma|\alpha \# \varepsilon$ or $\gamma|\eta \# \varepsilon$, and: there exists n such that

$\gamma|(\overline{\alpha}n) \perp \overline{\varepsilon}n$ or there exists n such that $\gamma|(\overline{\eta}n) \perp \overline{\varepsilon}n$. Using the Fan Theorem, 1.4, we find finite subsets B, C of $\mathbb{N}$ such that B is a bar in $X \cap s$ and C is a bar in $X \cap t$, and, for all u in B, for all v in C, ε does not pass through $\gamma|u$ or ε does not pass through $\gamma|v$. As B and C are finite, we may infer: either: for all u in B, ε does not pass through $\gamma|u$ or: for all v in C, ε does not pass through $\gamma|v$, and therefore: either: for every α in $X \cap s$, $\gamma|\alpha \# \varepsilon$, or: for every α in $X \cap t$, $\gamma|\alpha \# \varepsilon$. This ends the proof of our claim.

Using the just proven result and the fact that X is compact, we conclude: for each n, there exists s such that $length(s) = n$ and $\zeta(s) = 0$ and, for all t, if $length(t) = n$ and $\zeta(t) = 0$ and $t \neq s$, then for every α in $X \cap t$, $\gamma|\alpha \# \varepsilon$. We define λ in $\mathcal{N}$ such that, for each n, $length\big(\lambda(n)\big) = n$ and $\zeta\big(\lambda(n)\big) = 0$ and, for all t, if $\zeta(t) = 0$ and $length(t) = n$ and $t \neq \lambda(n)$, then for every α in $X \cap t$, $\gamma|\alpha \# \varepsilon$.

We claim: for each n, not: for every α in $X \cap \lambda(n)$, $\gamma|\alpha \# \varepsilon$. For suppose we find n such that for every α in $X \cap \lambda(n)$, $\gamma|\alpha \# \varepsilon$. It follows that, for every α in X, there exists n such that $(\overline{\gamma|\alpha})n \neq \overline{\varepsilon}n$ and then, again by the Fan Theorem, Theorem 1.4, there exists n such that, for all α in X, $(\overline{\gamma|\alpha})n \neq \overline{\varepsilon}n$, contradicting the assumption: $\delta(\overline{\varepsilon}n) = 0$.

It now will be clear that, for each n, $\lambda(n) \sqsubseteq \lambda(n+1)$. Let α be the element of X passing through every $\lambda(n)$. Assume $\gamma|\alpha \# \varepsilon$. Then there exists n such that $\overline{\varepsilon}n$ does not pass through $\gamma|\overline{\alpha}n$, and, therefore, there exists n such that $\overline{\varepsilon}n$ does not pass through $\gamma|\lambda(n)$. Contradiction. Therefore, $\gamma|\alpha$ coincides with ε.

We now define a function μ from $\mathbb{N}$ to $\mathbb{N}$, by induction on the length of the argument, as follows. $\mu(0) = 0$ and, for every e, for every n, for every p, for every q, $\mu^n(e * \langle q \rangle) = p + 1$ if and only if $length(\mu|e) = n$ and $\delta(e * \langle q \rangle) = 0$ and $\zeta|\big((\mu|e) * \langle p \rangle\big) = 0$ and, for every t, if $length(t) = length(e)$ and $\zeta(t) = 0$, and $t \perp (\mu|e) * \langle p \rangle$, then $\gamma|t \perp e * \langle q \rangle$. Using the Fan Theorem, one may prove that μ is the code of a continuous function from F_δ to F_ζ satisfying: for all α in F_ζ, $\mu|(\gamma|\alpha) = \alpha$.

Assume that Y is a located and closed subset of $\mathcal{N}$ containing F_δ. Let ρ be the code of a continuous function from Y to F_δ retracting Y onto F_δ. Let ν be the code of a continuous function from Y to $F_\zeta = X$ such that, for all α in Y, $\nu|\alpha = \mu|(\rho|\alpha$. Note that ν codes a surjective function from Y to X.

(ii) Let X, Y be located and closed subsets of $\mathcal{N}$ and let γ be the code of a continuous function from X to Y that maps X onto Y. Let ρ be a function from $\mathcal{N}$ to Y that retracts $\mathcal{N}$ onto Y. Note that, for every α, there exists β in X such that $\rho|\alpha = \gamma|\beta$. Applying the Second Axiom of Continuous Choice, Axiom 7, we find δ coding a continuous function from $\mathcal{N}$ to X such that, for every α, $\rho|\alpha = \gamma|(\delta|\alpha)$, and, in particular, for every α in Y, $\alpha = \rho|\alpha = \gamma|(\delta|\alpha)$.

We only have to verify that δ is strongly injective. Suppose α, β belong to Y and $\alpha \# \beta$. Find n such that $\overline{\alpha}n \neq \overline{\beta}n$. Find m such that $\overline{\alpha}n \sqsubseteq \gamma|(\delta|\overline{\alpha}m)$. Find p such that $\delta|\overline{\alpha}m \sqsubseteq (\overline{\delta|\alpha})p$. Note that $(\overline{\delta|\alpha})p$ must be different from $(\overline{\delta|\beta})p$.

(iii) The equivalence of (b) and (c) follows from (i) and (ii).

The equivalence of (a) and (b) follows from Theorem 9.1(ii) and Theorem 9.2(i).

□

G. Ronzitti, on page 63 of her Ph.D. dissertation [26] and in the last definition of her paper [27], suggested to call a located and closed subset X of $\mathcal{N}$ *countable* if X satisfies condition (c) of Theorem 9.5(iii): *there exists a stump σ such that CB^*_σ maps onto X*. Theorem 9.5 shows this suggestion makes sense if we restrict ourselves to *compact* located and closed subsets of $\mathcal{N}$. It seems that Theorem 9.5(i) does not extend to located and closed subsets X of $\mathcal{N}$ that we do not know to be compact.

9.2. **The collapse of the projective hierarchy.** One may parametrize the class $\mathbf{\Sigma}^1_2$ of the (existential) projections of the co-analytic subsets of $\mathcal{N}$ as follows. For each β, let PCA_β be the set $Ex(CA_\beta)$, that is the set of all α in $\mathcal{N}$ such that, for some γ, the pair $\langle \alpha, \gamma \rangle$ belongs to CA_β. A subset X of $\mathcal{N}$ belongs to the class $\mathbf{\Sigma}^1_2$ if and only there exists β such that X coincides with PCA_β.

One may parametrize the class $\mathbf{\Pi}^1_2$ of the co-projections of the analytic subsets of $\mathcal{N}$ as follows. For each β, let CPA_β be the set $Un(A_\beta)$, that is the set of all α in $\mathcal{N}$ such that, for all γ, the pair $\langle \alpha, \gamma \rangle$ belongs to A_β. A subset X of $\mathcal{N}$ belongs to the class $\mathbf{\Pi}^1_2$ if and only there exists β such that X coincides with CPA_β.

Note that, for each β, every element of PCA_β is apart from every element of CPA_β.

We define subsets US^1_2 and UP^1_2 of $\mathcal{N}$, as follows. US^1_2 is the set of all α such that α_{II} belongs to PCA_{α_I} and UP^1_2 is the set of all α such that α_{II} belongs to CPA_{α_I}. US^1_2, UP^1_2 are called the *cataloguing* sets of $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, respectively.

Note that every element of US^1_2 is apart from every element of UP^1_2.

We define subsets E^1_2 and A^1_2 of $\mathcal{N}$, as follows. E^1_2 is the set of all α such that for some β, for all γ there exists n, such that $\alpha(\langle \overline{\beta}n, \overline{\gamma}n \rangle) \neq 0$, and A^1_2 is the set of all α such that for all β there exists γ such that, for all n, $\alpha(\langle \overline{\beta}n, \overline{\gamma}n \rangle) = 0$. E^1_2, A^1_2 are called the *leading* sets of $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_2$, respectively.

Note that every element of E^1_2 is apart from every element of A^1_2.

A subset X of $\mathcal{N}$ is called *positively projective* if and only if it is obtained from a closed subset of $\mathcal{N}$ by finitely many applications of the operations of existential and universal projection.

In Section 6, we defined, for each s in $\mathbb{N}$, elements s_I and s_{II} of $\mathbb{N}$ by:

> $length(s_I)$ is the least k such that $length(s) \geq 2k$ and, for each n, if $n < length(s_I)$, then $s_I(n) = s(2n)$, and
> $length(s_{II})$ is the least k such that $length(s) \geq 2k+1$ and, for each n, if $n < length(s_{II})$, then $s_{II}(n) = s(2n+1)$.

For each s in $\mathbb{N}$, we now define: $s_{I,I} := (s_I)_I$ and $s_{I,II} := (s_I)_{II}$.

Recall that S^* denotes the element of $\mathcal{N}$ satisfying: for all n, $S^*(n) = n+1$.

Theorem 9.6 (Properties of the classes $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$)**.** (i) *For every subset X of $\mathcal{N}$, X belongs to $\mathbf{\Sigma}^1_2$ if and only if, for some α, X coincides with $US^1_2 \upharpoonright \alpha$ if and only if X reduces to E^1_2.*

(ii) *For every subset X of $\mathcal{N}$, X belongs to $\mathbf{\Pi}^1_2$ if and only if, for some α, X coincides with $UP^1_2 \upharpoonright \alpha$ if and only if X reduces to A^1_2.*

(iii) *Both $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ are subclasses of $\mathbf{\Sigma}^1_2$ as well as of $\mathbf{\Pi}^1_2$.*

(iv) *For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, X_n belongs to $\mathbf{\Sigma}^1_2$, then both $\bigcup\limits_{n \in \mathbb{N}} X_n$ and $\bigcap\limits_{n \in \mathbb{N}} X_n$ belong to $\mathbf{\Sigma}^1_2$.*

(v) *For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each n, X_n belongs to $\mathbf{\Pi}^1_2$, then $\bigcap\limits_{n \in \mathbb{N}} X_n$ belongs to $\mathbf{\Pi}^1_2$.*

(vi) *For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Sigma}^1_2$, then $Ex(X)$ belongs to $\mathbf{\Sigma}^1_2$.*

(vii) *For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Pi}^1_2$, then $Un(X)$ belongs to $\mathbf{\Pi}^1_2$.*

Proof. (i) The proof of the first statement is left to the reader. Let X be a subset of $\mathcal{N}$ that belongs to the class $\mathbf{\Sigma}^1_2$. We prove that X reduces to E^1_2. Find β in $\mathcal{N}$ such that X is the set $PCA_\beta = Ex(CA_\beta)$. Note that, for every α in $\mathcal{N}$, α belongs to X if and only if $\exists \delta \forall \gamma \exists n [\beta(\overline{\langle \langle \alpha, \gamma \rangle, \delta \rangle} n) \neq 0]$. Note that, for every α, α belongs to E^1_2 if and only if $\exists \delta \forall \gamma \exists n [\alpha(\langle \overline{\gamma}n, \overline{\delta}n \rangle) \neq 0]$. Let η be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\eta|\alpha)(s) \neq 0$ if and only if there exists $u \leq s$ such that $\beta(u) \neq 0$ and α passes through $u_{I,I}$ and $u_{I,II} \sqsubseteq s(1)$ and $u_{II} \sqsubseteq s(0)$. One verifies easily that η reduces X to E^1_2.

(ii) The statement is obviously true.

(iii)The proof of the first statement is left to the reader. Let X be a subset of $\mathcal{N}$ that belongs to the class $\mathbf{\Pi}^1_2$. We prove that X reduces to A^1_2. Find β in $\mathcal{N}$ such that X is the set $CPA_\beta = Un(A_\beta)$. Note that, for every α in $\mathcal{N}$, α belongs to X if and only if $\forall\delta\exists\gamma\forall n[\beta(\overline{\langle\langle\alpha,\gamma\rangle,\delta\rangle}n) = 0]$. Note that, for every α, α belongs to A^1_2 if and only if $\forall\delta\exists\gamma\forall n[\alpha(\langle\overline{\gamma}n, \overline{\delta}n\rangle) = 0]$. Let η be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each α, for each s, $(\eta|\alpha)(s) \neq 0$ if and only if there exists $u \leq s$ such that $\beta(u) \neq 0$ and α passes through $u_{I,I}$ and $u_{I,II} \sqsubseteq s(1)$ and $u_{II} \sqsubseteq s(0)$. One verifies easily that η reduces X to A^1_2.

(iv) Let $X_0, X_1, \ldots$ be an infinite sequence of elements of $\mathbf{\Sigma}^1_2$.

Using the Second Axiom of Countable Choice, Axiom 3, we find β such that, for each n, X_n coincides with $PCA_{\beta^n} = Ex(CA_{\beta^n})$. Note that, for all α, α belongs to $\bigcap\limits_{n\in\mathbb{N}} X_n$ if and only if, for each n, there exists γ such that $\langle\alpha,\gamma\rangle$ belongs to CA_{β^n}. Using the Second Axiom of Countable Choice again, we conclude that, for all α, α belongs to $\bigcap\limits_{N\in\mathbb{N}} X_n$ if and only there exists γ such that, for all n, $\langle\alpha,\gamma^n\rangle$ belongs to CA_{β^n}. As the class $\mathbf{\Pi}^1_1$ is closed under the operation of countable intersection, see Theorem 7.1, the set of all δ such that, for all n, $\langle\delta_I, (\delta_{II})^n\rangle$ belongs to CA_{β^n}, is a member of the class $\mathbf{\Pi}^1_1$ and the set $\bigcap\limits_{n\in\mathbb{N}} X_n$ is a member of the class $\mathbf{\Sigma}^1_2$.

We also observe that, for all α, α belongs to $\bigcup\limits_{n\in\mathbb{N}} X_n$ if and only if there exist n, γ such that $\langle\alpha,\gamma\rangle$ belongs to CA_{β^n} if and only if there exists γ such that $\langle\alpha, \gamma\circ S^*\rangle$ belongs to $CA_{\beta^{\gamma(0)}}$ if and only if there exists γ such that $\langle\beta^{\gamma(0)}, \langle\alpha,\gamma\circ S^*\rangle\rangle$ belongs to UP^1_1. It follows that the set $\bigcup\limits_{n\in\mathbb{N}} X_n$ is a member of the class $\mathbf{\Sigma}^1_2$.

(v) Let $X_0, X_1, \ldots$ be an infinite sequence of elements of $\mathbf{\Pi}^1_2$. Using the Second Axiom of Countable Choice, Axiom 3, we find β such that, for each n, X_n coincides with $CPA_{\beta^n} = Un(A_{\beta^n})$. We observe that, for all α, α belongs to $\bigcap\limits_{n\in\mathbb{N}} X_n$ if and only if for all n, for all γ, $\langle\alpha,\gamma\rangle$ belongs to A_{β^n} if and only if, for all γ, $\langle\alpha,\gamma\circ S^*\rangle$ belongs to $A_{\beta^{\gamma(0)}}$ if and only if, for all γ, $\langle\beta^{\gamma(0)}, \langle\alpha,\gamma\circ S^*\rangle\rangle$ belongs to US^1_1. It follows that the set $\bigcap\limits_{n\in\mathbb{N}} X_n$ is a member of the class $\mathbf{\Pi}^1_2$.

(vi) Let X be an element of the class $\mathbf{\Sigma}^1_2$. Find Y in $\mathbf{\Pi}^1_1$ such that $X = Ex(Y)$. Note that, for all α, α belongs to $Ex(X)$ if and only if there exists β such that $\langle\alpha,\beta\rangle$ belongs to Y if and only if there exist β,γ such that $\langle\langle\alpha,\beta\rangle,\gamma\rangle$ belongs to Y if and only if there exists β such that $\langle\langle\alpha,\beta_I\rangle,\beta_{II}\rangle$ belongs to Y. It follows that the set $Ex(X)$ is a member of the class $\mathbf{\Sigma}^1_2$.

(vii) Let X be an element of the class $\mathbf{\Pi}^1_2$. Find Y in $\mathbf{\Pi}^1_1$ such that $X = Un(Y)$. Note that, for all α, α belongs to $Un(X)$ if and only if, for all β, $\langle\alpha,\beta\rangle$ belongs to Y if and only if, for all β,γ, $\langle\langle\alpha,\beta\rangle,\gamma\rangle$ belongs to Y if and only if, for all β, $\langle\langle\alpha,\beta_I\rangle,\beta_{II}\rangle$ belongs to Y. It follows that the set $Un(X)$ is a member of the class $\mathbf{\Pi}^1_2$.

□

Theorem 9.7 (Breakdown of the Projective Hierarchy)**.**

(i) *The set A^1_2 reduces to the set E^1_2 and, therefore: $\mathbf{\Pi}^1_2$ is a subclass of $\mathbf{\Sigma}^1_2$.*
(ii) *For every subset X of $\mathcal{N}$, if X belongs to $\mathbf{\Sigma}^1_2$, then $Un(X)$ belongs to $\mathbf{\Sigma}^1_2$.*
(iii) *Every (positively) projective set belongs to the class $\mathbf{\Sigma}^1_2$.*

Proof. (i) Note that, for each α, α belongs to A^1_2 if and only if, for every β, there exists γ such that, for each n, $\alpha(\langle\overline{\beta}n, \overline{\gamma}n\rangle) = 0$. a Using the Second Axiom of Continuous Choice, we conclude that α belongs to A^1_2 if and only if there exists ε in A^1_1 such that $\varepsilon(0) = 0$ and for each β, $\alpha(\langle\overline{\beta}n, \overline{(\varepsilon|\beta)}n\rangle) = 0$ if and only if there exists ε such that

$\varepsilon(0) \neq 0$ and for all β there exists n such that $\varepsilon(\overline{\beta}n) \neq 0$ and for all s, $\alpha(s) \neq 0$ if and only if $s(1)$ is not an initial part of $\varepsilon|\big(s(0)\big)$.

This shows that the set A^1_2 is a member of the class $\mathbf{\Sigma}^1_2$.

(ii) Let X be a subset of $\mathcal{N}$ that is the co-projection of a set in the class $\mathbf{\Sigma}^1_2$. Find β in $\mathcal{N}$ such that X is the set $CPA_\beta = Un(CA_\beta)$. Note that, for every α in $\mathcal{N}$, α belongs to X if and only if $\forall\varepsilon\exists\delta\forall\gamma\exists n[\beta(\overline{\langle\langle\langle\alpha,\gamma\rangle,\delta\rangle,\varepsilon\rangle}n) \neq 0]$ if and only if $\exists\zeta[\zeta(\langle\ \rangle) = 0\ \wedge\ \forall\varepsilon\exists n[\zeta(\overline{\varepsilon}n) \neq 0]\ \wedge\ \forall\varepsilon\forall\gamma\exists n[\beta(\overline{\langle\langle\langle\alpha,\gamma\rangle,(\zeta|\varepsilon)\rangle,\varepsilon\rangle}n) \neq 0]$ if and only if $\exists\zeta[\zeta(\langle\ \rangle) = 0 \wedge \forall\varepsilon\exists n[\zeta(\overline{\varepsilon}n) \neq 0] \wedge \forall\varepsilon\forall\gamma\forall\delta\forall\eta\exists n[\beta(\overline{\langle\langle\langle\alpha,\gamma\rangle,\delta\rangle,\varepsilon\rangle}n) \neq 0 \vee \zeta|\big(\overline{\varepsilon}(\eta(n))\big) \neq \overline{\delta}n]]$.
It will be clear that X belongs to the class $\mathbf{\Sigma}^1_2$.

(iii) Use (ii) and Theorem 9.6. □

9.3. The previous Theorem shows that, in intuitionistic mathematics, $\mathbf{\Sigma}^1_2$ is the class of all positively projective sets. Many difficult questions remain, for instance, if $\mathbf{\Pi}^1_2$ is a proper subclass of $\mathbf{\Sigma}^1_2$ and if the class $\mathbf{\Pi}^1_2$ is closed under the operation of disjunction. We were unable to answer these questions.

Note that the existential projection of a positively Borel set is analytic. It is not true however, that the universal projection of a positively Borel set is always co-analytic, for the simple reason that some positively Borel sets, like $D^2(A_1)$, see Theorem 7.1 are not co-analytic.

From a classical point of view, Theorem 9.7 is a great surprise. The next theorem shows which conclusion the intuitionistic mathematician draws from the classical diagonal arguments establishing the projective hierarchy. Note that these conclusions are, if one reads them classically, contradictions, and, for the classical mathematician, would refute Theorem 9.7 and the Second Axiom of Continuous Choice, Axiom 7, on which it is based.

Lemma 9.8. *For every function γ from $\mathcal{N}$ to $\mathcal{N}$ there exists α such that α belongs to E^1_2 if and only if $\gamma|\alpha$ belongs to E^1_2 and also: α belongs to A^1_2 if and only if $\gamma|\alpha$ belongs to A^1_2.*

Proof. Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$. We construct α in $\mathcal{C}$ in such a way that for all β,γ, there exists n such that $\alpha(\langle\overline{\beta}n,\overline{\gamma}n\rangle) = 1$ if and only if there exists n such that $(\gamma|\alpha)(\langle\overline{\beta}n,\overline{\gamma}n\rangle) \neq 0$.

We define α by induction, as follows. Let n be a natural number and suppose we already decided about $\alpha(0),\ldots,\alpha(n-1)$. We now say: $\alpha(n) = 1$ if and only if there exists b,c,p,q,j such that $\text{length}(b) = \text{length}(c)$ and $n = \langle b,c\rangle$ and $\text{length}(p) = \text{length}(q)$ and p,q are initial parts of b,c respectively and $j < n$ and $\gamma^{\langle p,q\rangle}(\overline{\alpha}j) > 1$ and, for all $k < j$, $\gamma^{\langle p,q\rangle}(\overline{\alpha}k) = 0$.

It is easy to see that α satisfies the requirements, and that α belongs to E^1_2 if and only if $f|\alpha$ belongs to E^1_2 and: α belongs to A^1_2 if and only if $f|\alpha$ belongs to A^1_2. □

Note that the classical mathematician would conclude, from Lemma 9.8, that the sets A^1_2 and E^1_2 do not reduce to each other.

Theorem 9.9 (The Ruins of the Classical Projective Hierarchy: de Morgan's nightmares)**.**

(i) *There exists γ belonging to neither one of US^1_2, UP^1_2.*
(ii) *There exists α belonging to neither one of E^1_2, A^1_2.*

Proof. (i) We let DP^1_2 be the set of all α such that $\langle\alpha,\alpha\rangle$ belongs to UP^1_2. According to Theorem 9.7 DP^1_2 is a member of $\mathbf{\Sigma}^1_2$ and we may find β such that DP^1_2 coincides with $US^1_2\restriction\beta$, therefore, for every α, $\langle\alpha,\alpha\rangle$ belongs to UP^1_2 if and only if $\langle\beta,\alpha\rangle$ belongs to US^1_2. Define $\gamma := \langle\beta,\beta\rangle$ and observe that γ does not belong to either US^1_2 or UP^1_2, as every element of US^1_2 is apart from every element of UP^1_2.

(ii) Let γ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing A^1_2 to E^1_2 and apply Lemma 9.8, keeping in mind that every element of E^1_2 is apart from every element of A^1_2. $\square$

Let α be an element of $\mathcal{N}$ not belonging to either E^1_2 or A^1_2. Note that

(i) $\neg\exists\beta\forall\gamma\exists n[\alpha(\langle\overline{\beta}n, \overline{\gamma}n\rangle) \neq 0$ and

(ii) $\neg\forall\beta\exists\gamma\forall n[\alpha(\overline{\beta}n, \overline{\gamma}n\rangle) = 0]$ and

(iii) $\forall\beta\forall\gamma\forall n[\alpha(\overline{\beta}n, \overline{\gamma}n\rangle) = 0 \ \vee \ \alpha(\overline{\beta}n, \overline{\gamma}n\rangle) \neq 0]$.

This example shows that, in intuitionistic mathematics it is possible that statements

(i) $\neg\exists \mathsf{x}\forall \mathsf{y}\exists \mathsf{z}[\mathsf{P}(\mathsf{x}, \mathsf{y}, \mathsf{z})]$ and

(ii) $\neg\forall \mathsf{x}\exists \mathsf{y}\forall \mathsf{z}[\neg\mathsf{P}(\mathsf{x}, \mathsf{y}, \mathsf{z})]$ and

(iii) $\forall \mathsf{x}\forall \mathsf{y}\forall \mathsf{z}[\mathsf{P}(\mathsf{x}, \mathsf{y}, \mathsf{z}) \vee \neg\mathsf{P}(\mathsf{x}, \mathsf{y}, \mathsf{z})]$

are simultaneously true. The example depends on the Second Axiom of Continuous Choice, Axiom 7. Another example, depending only on Brouwer's Continuity Principle, Axiom 4, has been given in [45], Section 5.5:

(i) $\neg\exists\alpha\forall n\exists m[\alpha(n) = 0 \ \wedge \alpha(m) \neq 0]$ and

(ii) $\neg\forall\alpha\exists n\forall m[\alpha(n) \neq 0 \ \wedge \ \alpha(m) = 0]$ and

(iii) $\forall\alpha\forall n\forall m[\big(\alpha(n) = 0 \ \wedge \alpha(m) \neq 0\big) \ \vee \ \big(\alpha(n) \neq 0 \ \vee \ \alpha(m) = 0\big)]$.

It has been observed by J.R. Moschovakis that, in the context of intuitionistic arithmetic, Church's Thesis causes the collapse of the (positively) arithmetical hierarchy, just as the Second Axiom of Continuous Choice, Axiom 7 causes the collapse of the (positively) projective hierarchy, see [23].

References

[1] P. Aczel, A constructive version of the Lusin separation theorem, in: S. Lindström, E. Palmgren, K. Segerberg, V. Stoltenberg-Hansen(ed.), *Logicism, Intuitionism, and Formalism: what has become of them?*, Springer, 2009, pp. 129-152.

[2] L.E.J. Brouwer, Begründung der Mengenlehre unabhängig vom logischen Satz vom ausgeschlossenen Dritten. Erster Teil: Allgemeine Mengenlehre. Koninklijke Nederlandsche Akademie van Wetenschappen, Verhandelingen, 1e sectie 12 no. 5, 1918, also [5], pp.150-190, with footnotes, pp. 573-585.

[3] L.E.J. Brouwer, Über Definitionsbereiche von Funktionen, *Math. Annalen* 97(1927)60-75, also in [5], pp. 390-405, see also [11], pp. 446-463.

[4] L.E.J. Brouwer, Points and Spaces, *Canad. J. Math.* 6(1954)1-17, also in: [5], pp. 522-538.

[5] L.E.J. Brouwer, *Collected Works*, Vol. I: *Philosophy and foundations of mathematics*, ed. A. Heyting, North-Holland, Amsterdam, 1975.

[6] L.E.J. Brouwer, *Intuitionismus*, herausgegeben, eingeleitet und kommentiert von D. van Dalen, Wissenschaftsverlag, Mannheim etc., 1992.

[7] J. Burgess, Brouwer and Souslin on transfinite cardinals, *Zeitschr. für Math. Log. und Grundl. der Math.* 26(1980)209-214.

[8] G. Cantor, Über unendliche lineare Punktmannigfaltigkeiten. Nr. 6, *Math. Annalen* 23(1884)453-488, also in [9], pp. 210-246.

[9] G. Cantor, *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, herausgegeben von E. Zermelo, Julius Springer, Berlin, 1932, reprint: Springer Verlag, Berlin etc.,1980.

[10] W. Gielen, H. de Swart and W. Veldman, The continuum hypothesis in intuitionism, *Journ. of Symb. Logic* 46(1981)121-136.

[11] J. van Heijenoort, *From Frege to Gödel, A Source Book in Mathematical Logic, 1879-1931*, Harvard University Press, Cambridge, Mass., 1967.

[12] A. Heyting, *Intuitionism, an Introduction*, North-Holland Publishing Company, Amsterdam etc., 1956, Second, revised edition 1966, Third, revised edition 1972.

[13] W.A. Howard and G. Kreisel, Transfinite induction and bar induction of types zero and one, and the role of continuity in intuitionistic analysis, *Journ. Symb. Logic* 31(1966)325-358.

[14] A.S. Kechris and A. Louveau, *Descriptive Set Theory and the Structure of Sets of Uniqueness*, Cambridge University Press, 1987.

[15] A.S. Kechris, *Classical Descriptive Set Theory*, Springer-Verlag, Berlin, 1996.

[16] S.C. Kleene, and R.E. Vesley, *The Foundations of Intuitionistic Mathematics, Especially in relation to Recursive Functions*, North-Holland, Amsterdam, 1965.

[17] H. Lebesgue, Sur les fonctions représentables analytiquement, *Journal de Math.* 6^{e} série 1(1905)139-216.

[18] N. Lusin, Sur la classification de M. Baire, Comptes Rendus Acad. Sci., Paris, 164(1917), pp. 91-94.

[19] N. Lusin, Sur les ensembles non measurables B et l'emploi de la diagonale de Cantor, *Comptes Rendus Acad. Sci.* 181(1925)95-96.

[20] N. Lusin, *Leçons sur les ensembles analytiques et leurs applications*, Paris, 1930, second edition: Chelsea Publishing Company, New York 1972.

[21] J.R. Moschovakis, Classical and constructive hierarchies in extended intuitionistic analysis, *Journ. Symb. Logic* 68(2003)1015-1043.

[22] J.R. Moschovakis, Hierarchies in Intuitionistic Arithmetic, Gjuletchica, Bulgaria, December 2002.

[23] J.R. Moschovakis, The effect of Markov's Principle on the intuitionistic continuum, *Proceedings of Oberwolfach Proof Theory week (April 2005)*.

[24] Y.N. Moschovakis, *Descriptive Set Theory*, North-Holland, Amsterdam, 1980.

[25] J. Myhill, Formal systems of intuitionistic analysis II. The theory of species, in: A. Kino, J. Myhill, R.E. Vesley, *Intuitionism and Proof Theory, Proc. Summer Conference at Buffalo 1968*, North Holland Publishing Company, Amsterdam etc., 1970, pp. 151-162

[26] G. Ronzitti, *On the cardinality of a spread*, Tesi di Dottorato di Filosofia, Universita degli Studi di Genova, Genova, 2001.

[27] G. Ronzitti, On some difficulties concerning the definition of an intuitionistic concept of countable set, The Logica Yearbook 2003, pp. 241-25.

[28] S.M. Srivastava, *A Course on Borel Sets*, Springer-Verlag, New York etc., 1998.

[29] M. Souslin, Sur une définition des ensembles mesurables B sans nombres transfinis, Comptes Rendus Acad. Sci., Paris 164 (1917), pp. 88-91.

[30] A.S. Troelstra(ed.), *Metamathematical Investigation of Intuitionistic Arithmetic and Analysis*, Lecture Notes in Mathematics, Vol. 344, Springer-Verlag, Berlin etc., 1973.

[31] A.S. Troelstra, and D. van Dalen, *Constructivism in Mathematics, volume I*, North-Holland, Amsterdam, 1988.

[32] W. Veldman, *Investigations in Intuitionistic Hierarchy Theory*, Ph.D. Thesis, Katholieke Universiteit Nijmegen, 1981.

[33] W. Veldman, On the constructive contrapositions of two axioms of countable choice, in: A.S. Troelstra, D. van Dalen (ed.), *The L.E.J. Brouwer Centenary Symposium*, Proceedings of the conference, Noordwijkerhout 1981, North-Holland, Amsterdam, 1982, pp. 513-523.

[34] W. Veldman, A survey of intuitionistic descriptive set theory, in: P.P. Petkov (ed.), *Mathematical Logic*, Proceedings of the Heyting Conference 1988, Plenum Press, New York and London, 1990, pp. 155-174.

[35] W. Veldman, Some intuitionistic variations on the notion of a finite set of natural numbers, in: H.C.M. de Swart, L.J.M. Bergmans (ed.), *Perspectives on Negation,essays in honour of Johan J. de Iongh on the occasion of his 80th birthday*, Tilburg University Press, Tilburg, 1995, pp. 177-202.

[36] W. Veldman, On sets enclosed between a set and its double complement, in: A. Cantini e.a.(ed.), *Logic and Foundations of Mathematics*, Proceedings Xth International Congress on Logic, Methodology and Philosophy of Science, Florence 1995, Volume III, Kluwer Academic Publishers, Dordrecht, 1999, pp. 143-154.

[37] W. Veldman, Understanding and using Brouwer's continuity principle, in: U. Berger, H. Osswald, P. Schuster (ed.), *Reuniting the Antipodes, Constructive and Nonstandard Views of the Continuum*, Proceedings of a Symposium held in San Servolo/Venice, 1999, Kluwer Academic Publishers, Dordrecht, 2001, pp. 285-302.

[38] W. Veldman, The Cantor-Bendixson-hierarchy revisited by Brouwer, in: *Proceedings of the First Seminar on the Philosophy of Mathematics in Iran, Oct. 17, 2001*, Faculty of Mathematical Sciences, Shahid Beheshti University, Teheran, 2003, pp. 79-103.

[39] W. Veldman, On the persistent difficulty of disjunction, in: A. Rojszczak, J. Cachro, and G. Kurczewski(eds.), *Philosophical Dimensions of Logic and Science*, Selected Contributed Papers from the 11th International Congress on Logic, Methodology and Philosophy of Science, Kraków, 1999, Kluwer Academic Publishers, Dordrecht, 2003, pp. 77-90.

[40] W. Veldman, An intuitionistic proof of Kruskal's theorem, *Archive for Mathematical Logic* 43(2004)215-264.

[41] W. Veldman, Two simple sets that are not positively Borel, *Annals of Pure and Applied Logic* 135(2005)151-209.

[42] W. Veldman, *Brouwer's Fan Theorem as an axiom and as a contrast to Kleene's Alternative*, Report No. 0509, Department of Mathematics, Radboud University Nijmegen, July 2005.

[43] W. Veldman, Brouwer's real Thesis on Bars, in: G. Heinzmann, G. Ronzitti (eds.), *Constructivism: Mathematics, Logic, Philosophy and Linguistics, Philosophia Scientiae, Cahier spé cial 6*, 2006, pp. 21-42.

[44] W. Veldman, Perhaps the Intermediate Value Theorem, in: C.S. Calude, H. Ishihara (eds.), *Constructivity, Computability and Logic. A Collection of Papers in Honour of the 60th Birthday of Douglas Bridges. Journal of Universal Computer Science* 11(2005)2142-2158.

[45] W. Veldman, The Borel hierarchy theorem from Brouwer's intuitionistic perspective, *The Journal of Symbolic Logic* 73(2008)1-64.

[46] W. Veldman, The fine structure of the intuitionistic Borel hierarchy, *The Review of Symbolic Logic*, 2(2009)1-72.

[47] W. Veldman, The principle of Open Induction on Cantor space $\mathcal{C}$ and some of its equivalents, *in preparation.*

[48] F.A. Waaldijk, *Modern intuitionistic topology*, Ph.D. Thesis, Katholieke Universiteit Nijmegen, 1996.

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