ANALYTIC, CO-ANALYTIC AND PROJECTIVE SETS
FROM BROUWER’S INTUITIONISTIC PERSPECTIVE

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Abstract. We study projective subsets of Baire space $\mathcal{N}$ from Brouwer’s intuitionistic point of view, using his Thesis on Bars and his continuity axioms. Note that, also intuitionistically, $\mathcal{N}$ is homeomorphic to $\mathcal{N} \times \mathcal{N}$. A subset of $\mathcal{N}$ is projective if it results from a closed or an open subset of $\mathcal{N} \times \mathcal{N} = \mathcal{N}$ by a finite number of applications of the two operations of projection and universal projection or: co-projection. We first study the analytic subsets of $\mathcal{N}$; these are the projections of the closed subsets of $\mathcal{N}$. We consider a number of examples and discover a fine structure in the class of the analytic subsets of $\mathcal{N}$ that fail to be positively Borel. We then turn to the strictly analytic subsets of $\mathcal{N}$. A subset of $\mathcal{N}$ is called strictly analytic if it coincides with the range of a continuous function from $\mathcal{N}$ to $\mathcal{N}$. We explain the importance of the difference between strictly analytic and analytic subsets of $\mathcal{N}$. Using Brouwer’s Thesis on Bars we prove separation and boundedness theorems for strictly analytic subsets of $\mathcal{N}$. We then examine co-analytic subsets of $\mathcal{N}$; these are the co-projections of the open subsets of $\mathcal{N} \times \mathcal{N}$. We give some examples showing that the classical duality between analytic and co-analytic sets either disappears or takes some very subtle forms. We show different ways to prove that some co-analytic sets are not analytic and that some analytic sets are not co-analytic. We consider the set of the codes of the closed and located subsets of $\mathcal{N}$ that are almost-countable as an example of a set that is a projection of a co-analytic subset of $\mathcal{N}$.

I sat upon the shore
Fishing, with the arid plain behind me
Shall I at least set my lands in order?
London Bridge is falling down falling down falling down

T.S.Eliot, The Waste Land

1. Introduction

This is the third in a series of papers on intuitionistic descriptive set theory. We think it worthwhile to study the questions asked by Baire, Borel, Lebesgue, and the mathematicians who joined their enterprise, from Brouwer’s intuitionistic point of view.

In [45] we proved an intuitionistic version of the Borel hierarchy theorem. In [46], we explored the remarkable fine structure of the intuitionistic Borel hierarchy, and, in particular, the fine structure of the class $\Sigma^0_2$, consisting of the countable unions of closed subsets of $\mathcal{N}$. The earlier paper [41] already contains some important results concerning analytic and co-analytic subsets of $\mathcal{N}$.

This introductory section is divided into three parts. In the first part, we briefly present the basic assumptions of intuitionistic analysis. In the second part, we survey some of the results of the earlier papers. In the third part, we give a sketch of the further contents of the paper and the main results.

1.1. The basic assumptions of intuitionistic analysis.
The logical constants are used in their intuitionistic sense. A statement \( P \lor Q \) is considered proven only if one either has a proof of \( P \) or a proof of \( Q \). A statement \( \exists x \in V [P(x)] \) is considered proven only if one is able to produce an element \( x \) of \( V \) with a proof of the fact that \( x \) has the property \( P \).

In the following we list the axioms of intuitionistic analysis as we are going to use them, without arguing their plausibility. L.E.J. Brouwer was the first to use them, see 2 3 4 5 6, and the question how to state and defend them has been further discussed by others, see 12 13 16 23 31 30 32 45.

Some of the axioms in our list imply other ones. The reason that we do not present the axioms in their strongest form only is that, the stronger the statement, the more one may have doubts.

We let \( \mathbb{N} \) denote the set of the natural numbers and \( \mathcal{N} \) the set of all infinite sequences of natural numbers, calling it Baire space.

We use \( m, n, \ldots \) as variables over the set \( \mathbb{N} \) and \( \alpha, \beta, \ldots \) as variables over the set \( \mathcal{N} \).

An element of \( \mathcal{N} \) is thought of as a function from \( \mathbb{N} \) to \( \mathbb{N} \), and, given \( \alpha, n \), we denote the result of applying \( \alpha \) to \( n \) by \( \alpha(n) \).

For all \( \alpha, \beta \) we define: \( \alpha \# \beta \), \( \alpha \) is apart from \( \beta \), if and only if, for some \( n \), \( \alpha(n) \neq \beta(n) \), and \( \alpha = \beta \), \( \alpha \) coincides with \( \beta \), if and only if, for all \( n \), \( \alpha(n) = \beta(n) \).

For each \( m \) in \( \mathbb{N} \) we let \( \overline{m} \) be the element of \( \mathcal{N} \) satisfying: for all \( n \), \( \overline{m}(n) = m \).

**Axiom 1** (First Axiom of Countable Choice). For every binary relation \( R \) on \( \mathbb{N} \), if for every \( m \) there exists \( n \) such that \( mRn \), then there exists \( \alpha \) such that, for every \( m \), \( mR\alpha(m) \).

We let \( J \) be a one-to-one function from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \). \( J \) is called a pairing function on \( \mathbb{N} \).

Let \( R \) be a binary relation on \( \mathbb{N} \). \( R \) is called a decidable relation on \( \mathbb{N} \) if and only if there exists \( \beta \) in \( \mathcal{N} \) such that, for all \( m, n, mRn \) if and only if \( \beta(J(m, n)) \neq 0 \).

**Axiom 2** (Minimal Axiom of Countable Choice). For every decidable binary relation \( R \) on \( \mathbb{N} \), if for every \( m \) there exists \( n \) such that \( mRn \), then there exists \( \alpha \) such that, for every \( m \), \( mR\alpha(m) \).

Axiom 2 is a weak version of Axiom 1. Note that, in the case of a decidable relation on \( \mathbb{N} \), one may define the promised \( \alpha \) by: let, for each \( m \), \( \alpha(m) \) be the least number \( n \) such that \( mRn \). The axiom guarantees that the set \( \mathcal{N} \) of functions from \( \mathbb{N} \) to \( \mathbb{N} \) is closed under this operation, the operation of minimalization or searching for the least number with a given decidable property.

For every \( \alpha \), for every \( m \), we define an element \( \alpha^m \) of \( \mathcal{N} \) by: for all \( n \), \( \alpha^m(n) = \alpha(J(m, n)) \). \( \alpha^m \) is called the \( m \)-th subsequence of \( \alpha \).

**Axiom 3** (Second Axiom of Countable Choice). For every binary relation \( R \subseteq \mathbb{N} \times \mathcal{N} \), if for every \( m \) there exists \( \alpha \) such that \( mR\alpha \), then there exists \( \alpha \) such that, for every \( m \), \( mR\alpha^m \).

\( \mathbb{N}^* \) is the set of all finite sequences of natural numbers. We let \( \langle \ldots \rangle \) be a one-to-one function from \( \mathbb{N}^* \) onto \( \mathbb{N} \). We call the number \( \langle a(0), a(1), \ldots, a(n-1) \rangle \) the code of the finite sequence \( \langle a(0), a(1), \ldots, a(n-1) \rangle \). For each \( \alpha \), for each \( n \), we define: \( \overline{\alpha^m} = \langle a(0), a(1), \ldots, a(n-1) \rangle \).

We do not mention the Axioms of Dependent Choice as we are not going to apply them in the arguments of this paper.

**Axiom 4** (Brouwer’s Continuity Principle). For every binary relation \( R \subseteq \mathcal{N} \times \mathbb{N} \), if for every \( \alpha \) there exists \( n \) such that \( \alpha Rn \), then, for every \( \alpha \) there exist \( m, n \) such that, for every \( \beta \), if \( \beta m = \overline{\alpha^m} \), then \( \beta Rn \).
Let \( X \) be a subset of \( N \). \( X \) is called a spread or a located and closed subset of \( N \) if and only if there exists \( \beta \) such that (i) for every \( s, \beta(s) = 0 \) if and only if, for some \( \alpha \) in \( X, s = \overline{\alpha n} \), and (ii) for every \( \alpha \), \( \beta \) belongs to \( X \) if and only if, for all \( n, \beta(\overline{\alpha n}) = 0 \).

It is a well-known fact that Brouwer’s Continuity Principle generalizes to spreads:

**Axiom 5** (Brouwer’s Continuity Principle, generalized version). For every subset \( X \) of \( N \) that is a spread, for every binary relation \( R \subseteq X \times N \), if for every \( \alpha \) in \( X \) there exists \( n \) such that \( \alpha Rn \), then, for every \( \alpha \) in \( X \) there exist \( m, n \) such that, for every \( \beta \) in \( X \), if \( \overline{\beta m} = \overline{\alpha n} \), then \( \beta Rn \).

Let \( \gamma \) be an element of \( N \). We define: \( \gamma \) codes a (continuous) function from \( N \) to \( N \), or: \( \gamma \) is a (continuous) function from \( N \) to \( N \) if and only if, for each \( \alpha \), there exists \( n \) such that \( \gamma(\overline{\alpha n}) \neq 0 \).

For every \( \gamma \) coding a function from \( N \) to \( N \), for every \( \alpha \), we let \( \gamma(\alpha) \) be the natural number \( p \) satisfying: there exists \( n \) such that \( \gamma(\overline{\alpha n}) = p + 1 \), and, for each \( i < n \), \( \gamma^i(\overline{\alpha n}) = 0 \).

**Axiom 6** (First Axiom of Continuous Choice). For every binary relation \( R \subseteq N \times N \), if for every \( \alpha \) there exists \( n \) such that \( \alpha Rn \), then there exists \( \gamma \) coding a function from \( N \) to \( N \) such that, for every \( \alpha \), \( \alpha R(\gamma(\alpha)) \).

Let \( \gamma \) be an element of \( N \). We define: \( \gamma \) codes \( \alpha \) or: \( \gamma \) is a (continuous) function from \( N \) to \( N \) if and only if, for each \( n \), \( \gamma^\alpha \) codes a function from \( N \) to \( N \).

For every \( \gamma \) coding a function from \( N \) to \( N \), for every \( \alpha \), we let \( \gamma|\alpha \) be the element \( \beta \) of \( N \) satisfying: for each \( n, \beta(n) = \gamma^\alpha(n) \).

**Axiom 7** (Second Axiom of Continuous Choice). For every binary relation \( R \subseteq N \times N \), if for every \( \alpha \) there exists \( \beta \) such that \( \alpha R\beta \), then there exists \( \gamma \) coding a function from \( N \) to \( N \) such that, for every \( \alpha \), \( \alpha R(\gamma|\alpha) \).

Like Brouwer’s Continuity Principle, Axioms 5, 6, and 7 of Continuous Choice, Axioms 8 and 9 generalize to spreads. We refrain from writing down these generalized versions, as their formulation is straightforward.

We let \( * \) denote the binary function on \( N \) corresponding to concatenation of finite sequences, so, for all \( s, t \) in \( N \), \( s * t \) is the code number of the finite sequence one obtains by putting the finite sequence coded by \( t \) behind the finite sequence coded by \( s \).

We assume that the code number of the empty sequence is \( 0 \), so: \( \langle \rangle \) \( = 0 \).

We also assume that the pairing function on \( N \) and the function coding the finite sequences of natural numbers by natural numbers are connected as follows: for all \( n \), for all \( s, J(n, s) = \langle n \rangle * s \).

Note that \( J \) maps \( N \times N \) onto the set \( N \setminus \{0\} \).

**Axiom 8** (On the existence of the set of stumps).

There exists a subset of \( N \), called the set \( \text{Stp} \) of stumps, satisfying the following conditions:

(i) For every \( \beta \), if \( \beta(0) \neq 0 \), then \( \beta \) belongs to \( \text{Stp} \).

(ii) For every \( \beta \), if \( \beta(0) = 0 \), and, for each \( n, \beta^n \) belongs to \( \text{Stp} \), then \( \beta \) itself belongs to \( \text{Stp} \).

(iii) For every subset \( Q \) of \( \text{Stp} \), if

- For every \( \beta \), if \( \beta(0) \neq 0 \), then \( \beta \) belongs to \( Q \), and
- For every \( \beta \), if \( \beta(0) = 0 \), and, for each \( n, \beta^n \) belongs to \( Q \), then \( \beta \) itself belongs to \( Q \),

then \( \text{Stp} \) coincides with \( Q \).

1There is a small difference between the set \( \text{Stp} \) as it is introduced here and the sets called \( \text{Stp} \) in [13, 16], respectively.
We shall follow the convention of using \( \sigma, \tau, \ldots \) as variables over the set \( \text{Stp} \).

If \( \sigma \) is a stump satisfying \( \sigma(0) \neq 0 \), we sometimes say: \( \sigma \) is/codes the empty stump.

Note that, for each \( \alpha \), if \( \sigma(0) = 0 \), then, for each \( n, \sigma^n \) is a stump: the \( n \)-th immediate substump of \( \sigma \).

We define relations \(<, \leq \) on the set \( \text{Stp} \), as follows, by simultaneous transfinite induction.

For all stumps \( \sigma, \tau: \sigma \leq \tau \) if and only if either \( \sigma \) is the empty stump, or \( \sigma \) is non-empty and, for all \( n, \sigma^n < \tau \) and: \( \sigma < \tau \) if and only if \( \tau \) is not the empty stump and there exists \( n \) such that \( \sigma \leq \tau^n \).

Let \( Q \) be a subset of \( \text{Stp} \). \( Q \) is an inductive subset of \( \text{Stp} \) if and only if every empty stump belongs to \( Q \), and for every non-empty stump \( \tau \), \( \tau \) belongs to \( Q \) if every immediate substump \( \tau^n \) of \( \tau \) belongs to \( Q \). \( Q \) is a hereditary subset of \( \text{Stp} \) if and only if, for every stump \( \tau \), if every stump \( \sigma < \tau \) belongs to \( Q \), then \( \tau \) itself belongs to \( Q \).

The proof of the next theorem is an exercise in transfinite induction.

**Theorem 1.1.**

(i) Every inductive subset \( Q \) of \( \text{Stp} \) coincides with \( \text{Stp} \).

(ii) For all \( \sigma, \tau, \rho \) in \( \text{Stp} \),

- \( \sigma \leq \sigma \), and not: \( \sigma < \sigma \),
- if \( \sigma < \tau \), then \( \sigma \leq \tau \),
- if \( \sigma \leq \tau \) and \( \tau < \rho \), then \( \sigma < \rho \),
- if \( \sigma < \tau \) and \( \tau \leq \rho \), then \( \sigma < \rho \), and
- if \( \sigma \leq \tau \) and \( \tau \leq \rho \), then \( \sigma \leq \rho \).

(iii) Every hereditary subset \( Q \) of \( \text{Stp} \) coincides with \( \text{Stp} \).

Let \( s \) belong to \( \mathbb{N} \). We denote the length of the finite sequence of natural numbers coded by \( s \) by \( \text{length}(s) \).

Let \( s \) belong to \( \mathbb{N} \) and let \( n = \text{length}(s) \). \( s \) is thought of as a function from the set \( \{0, 1, \ldots, n-1\} \) to \( \mathbb{N} \). For each \( i < \text{length}(s) \), we let \( s(i) \) be the value \( s \) assumes at \( i \). Note that \( s = \langle s(0), s(1), \ldots, s(n-1) \rangle \).

For each \( s \), for each \( i \) \leq \text{length}(s) \), we let \( \overline{s} \) be the code number of the finite sequence of length \( i \) that is an initial part of \( s \).

Let \( s \) be an element of \( \mathbb{N} \) and let \( \alpha \) belong to \( \mathcal{N} \). We define: \( \alpha \) admits \( s \) if and only if, for each \( i \), if \( i \leq \text{length}(s) \), then \( \alpha(\overline{s}) = 0 \).

Let \( B \) be a subset of \( \mathbb{N} \). \( B \) is called a bar in \( \mathcal{N} \) if and only if, for each \( \alpha \), there exists \( n \) such that \( \overline{s}n \) belongs to \( B \).

**Axiom 9 (Brouwer’s Thesis on Bars).** Let \( B \) be a subset of \( \mathbb{N} \) that is a bar in \( \mathcal{N} \). There exists a stump \( \sigma \) such that the set of all \( s \) in \( B \) that are admitted by \( \sigma \) is a bar in \( \mathcal{N} \).

**Lemma 1.2.** Let \( \sigma \) be a stump, and let \( \alpha \) belong to \( \mathcal{N} \). If, for all \( \beta \), there exists \( n \) such that \( \overline{s}n \) is admitted by \( \sigma \) and \( \alpha(\overline{s}n) \neq 0 \), then \( \alpha \) itself is a stump.

**Proof.** The proof is by induction on the set of stumps. Note that there is nothing to prove if \( \sigma(0) \neq 0 \). Now assume \( \sigma(0) = 0 \) and the statement has been seen to hold for every immediate substump \( \sigma^n \) of \( \sigma \). Assume that \( \alpha \) belongs to \( \mathcal{N} \) and that for all \( \beta \), there exists \( n \) such that \( \overline{s}n \) is admitted by \( \sigma \) and \( \alpha(\overline{s}n) \neq 0 \). If \( \alpha(0) \neq 0 \), \( \alpha \) is a stump and we are done. If \( \alpha(0) = 0 \), then, for each \( p \), for all \( \beta \), there exists \( n \) such that \( \overline{s}n \) is admitted by \( \sigma^p \) and \( \alpha(\overline{s}n) \neq 0 \), and therefore, by the induction hypothesis, \( \alpha^p \) is a stump. It follows that \( \alpha \) itself is a stump. \( \square \)

We let \( A_1^1 \) be the set of all \( \alpha \) such that, for each \( \beta \), there exists \( n \) such that \( \alpha(\overline{s}n) \neq 0 \). Note that \( A_1^1 \) is the set of all \( \alpha \) that code a function from \( \mathcal{N} \) to \( \mathbb{N} \). For this reason, \( A_1^1 \) is sometimes called **Fun.**
Also note that, for each $\alpha$, $\alpha$ belongs to $A^1_1$ if and only if the set of all $s$ that are not admitted by $\alpha$ is a bar in $\mathcal{N}$. For this reason, $A^1_1$ is sometimes called $\text{Bar}$.

**Theorem 1.3.**
(i) $\text{Stp}$ is a subset of $A^1_1$.
(ii) $A^1_1$ is a subset of $\text{Stp}$.

**Proof.** (i) One proves this by induction on the set of stumps.
(ii) Suppose that $\alpha$ belongs to $A^1_1$. Applying Axiom 9 we find a stump $\sigma$ such that, for every $\beta$, there exists $n$ such that $\overline{\sigma n}$ is admitted by $\sigma$ and $\alpha(\overline{\sigma n}) \neq 0$. Applying Lemma 12 we conclude that $\alpha$ is a stump. \(\square\)

A subset $C$ of $\mathbb{N}$ is called *monotone* if and only if, for each $s$, for each $n$, if $s$ belongs to $C$, then $s \ast \langle n \rangle$ belongs to $C$.

A subset $C$ of $\mathbb{N}$ is called *inductive* if and only if, for each $s$, if, for each $n$, $s \ast \langle n \rangle$ belongs to $C$, then $s$ itself belongs to $C$.

**Axiom 10 (Principle of Induction on Monotone Bars).** Let $B$ be a monotone subset of $\mathbb{N}$ that is a bar in $\mathcal{N}$. Let $C$ be an inductive subset of $\mathbb{N}$ that contains $B$. Then $0 = \langle \rangle$ belongs to $C$.

Cantor space $\mathcal{C}$ is the set of all $\alpha$ in $\mathcal{N}$ such that, for each $n$, $\alpha(n) = 0$ or $\alpha(n) = 1$.

**Axiom 11 (Unrestricted Fan Theorem).** Let $B$ be a subset of $\mathbb{N}$ that is a bar in Cantor space $\mathcal{C}$. There exists a finite subset $B'$ of $B$ that is a bar in $\mathcal{C}$.

For each $\beta$ in $\mathcal{N}$ we let $D_\beta$ be the set of all $n$ in $\mathbb{N}$ such that $\beta(n) \neq 0$. The set $D_\beta$ sometimes is called the set decided by $\beta$.

**Axiom 12 ((Restricted) Fan Theorem).** Let $\beta$ be an element of $\mathbb{N}$ such that $D_\beta$ is a bar in $\mathcal{C}$. There exists a finite subset $B'$ of $D_\beta$ that is a bar in $\mathcal{C}$.

Let $X$ be a subset of $\mathcal{N}$. $X$ is called a *finitary spread* or a *fan* or a *compact located and closed subset* of $\mathcal{N}$ if and only if there exists $\beta$ such that (i) for every $s$, $\beta(s) = 0$ if and only if, for some $\alpha$ in $X$, $s = \overline{\alpha n}$, and (ii) for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for all $n$, $\beta(\overline{\sigma n}) = 0$, and (iii) for every $s$ there exists $n$ such that, for every $m$, if $m > n$, then $\beta(s \ast \langle m \rangle) \neq 0$.

**Theorem 1.4 (Fan Theorem, general formulation).**
(i) (Using Axiom 11): Let $X$ be a compact located and closed subset of $\mathcal{N}$. Let $B$ be a subset of $\mathbb{N}$ that is a bar in $X$. There exists a finite subset $B'$ of $B$ that is a bar in $X$.
(ii) (Using Axiom 12): Let $X$ be a compact located and closed subset of $\mathcal{N}$. Let $\beta$ be a subset of $\mathbb{N}$ such that $D_\beta$ is a bar in $X$. There exists a finite subset $B'$ of $D_\beta$ that is a bar in $X$.

1.2. Some results from the earlier papers.

Let $s$ be an element of $\mathbb{N}$ and let $\alpha$ belong to $\mathcal{N}$. We define: $\alpha$ passes through $s$ or $s$ is an initial part of $\alpha$ if and only if, for some $n$, $\overline{\alpha n} = s$.

Let $X$ be a subset of Baire space $\mathcal{N}$. $X$ is *basic open* if and only if either $X$ is empty or there exists $s$ such that $X$ consists of all $\alpha$ in $\mathcal{N}$ passing through $s$. $X$ is open if and only if $X$ is a countable union of basic open sets, that is, there exists $\gamma$ in $\mathcal{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $n$, $\gamma(n) > 1$ and $\alpha$ passes through $\gamma(n) - 1$.

For each $\beta$ in $\mathcal{N}$, we let $G_\beta$ be the set of all $\alpha$ such that, for some $n$, $\beta(\overline{\sigma n}) \neq 0$. One may prove that a subset $X$ of $\mathcal{N}$ is open if and only if, for some $\beta$, $X$ coincides with $G_\beta$. 
We let $E_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $n$, $\alpha(n) \neq 0$.

The following notion is very important. Let $X, Y$ be subsets of $\mathcal{N}$ and let $\gamma$ be (the code of) a continuous function from $\mathcal{N}$ to $\mathcal{N}$: $\gamma$ reduces the set $X$ to the set $Y$ if and only if, for each $\alpha$, $\alpha$ belongs to $X$ if and only if $\gamma(\alpha)$ belongs to $Y$. $X$ reduces to $Y$ if and only if there exists a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $Y$.

One may prove that a subset $X$ of $\mathcal{N}$ is open if and only if $X$ reduces to $E_1$.

Let $X$ be a subset of $\mathcal{N}$. $X$ is closed if and only if there exists an open subset $Y$ of $\mathcal{N}$ such that $X$ is the set of all $\alpha$ in $\mathcal{N}$ that do not belong to $Y$.

Let $\alpha, \beta$ be elements of $\mathcal{N}$. We define: $\beta$ admits $\alpha$ if and only if, for each $n$, $\beta(\alpha(n)) = 0$.

For each $\beta$ in $\mathcal{N}$, we let $F_\beta$ be the set all $\alpha$ admitted by $\beta$. One may prove that a subset $X$ of $\mathcal{N}$ is closed if and only if, for some $\beta$, $X$ coincides with $F_\beta$. We sometimes say that $\beta$ is a code for the set $F_\beta$.

We let $A_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for each $n$, $\alpha(n) = 0$.

One may prove that a subset $X$ of $\mathcal{N}$ is closed if and only if $X$ reduces to $A_1$.

Let $X$ be a subset of $\mathcal{N}$. $X$ is positively Borel if and only if one may obtain $X$ from open subsets of $\mathcal{N}$ by the repeated application of the operations of countable union and countable intersection.

We let $1^*$ be the element of $\mathcal{N}$ satisfying: $1^*(0) = 0$ and, for each $n > 0$, $1^*(n) = 1$.

Note that $1^*$ is a stump. We define a subset of the set of stumps, the set of the non-zero stumps as follows, by induction: a stump $\sigma$ is a non-zero stump if either $\sigma = 1^*$, or, $\sigma(0) = 0$, and, for each $n$, $\sigma(n) = 0$ and $\sigma^n$ is a non-zero stump.

For each non-zero stump $\sigma$ we define subsets $E_\sigma$ and $A_\sigma$ of $\mathcal{N}$, as follows, by induction on the set of stumps. $E_{1^*} := E_1$ and $A_{1^*} = A_1$, and, for each non-zero stump different from $1^*$, for each $\alpha$, $\alpha$ belongs to $E_\sigma$ if and only if, for some $n$, $\alpha(n)$ belongs to $A_\sigma$, and $\alpha$ belongs to $A_\sigma$ if and only if, for all $n$, $\alpha^n$ belongs to $E_\sigma$.

We define a subset of the set of stumps, the hereditarily repetitive stumps, as follows, by induction: a stump $\sigma$ is hereditarily repetitive if and only if either $\sigma(0) \neq 0$ or: $\sigma(0) = 0$ and for each $m$ there exists $n > m$ such that $\sigma^n = \sigma^m$ and, for each $n$, $\sigma^n$ is hereditarily repetitive.

The following theorem is the main result of 

**Theorem 1.5 (Borel Hierarchy Theorem).**

(i) For each non-zero stump $\sigma$, every element of $A_\sigma$ is apart from every element of $E_\sigma$.

(ii) For each subset $X$ of $\mathcal{N}$, $X$ is positively Borel if and only if there exists a hereditarily repetitive non-zero stump $\sigma$ such that $X$ reduces to $E_\sigma$.

(iii) For all non-zero stumps $\sigma, \tau$, if $\sigma < \tau$, then both $A_\sigma$ and $E_\sigma$ reduce to both $A_\tau$ and $E_\tau$.

(iv) For every hereditarily repetitive non-zero stump $\sigma$, $A_\sigma$ positively fails to reduce to $E_\sigma$, that is, for every continuous function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$, if $\gamma$ maps $A_\sigma$ into $E_\sigma$, then there exists $\alpha$ in $E_\sigma$ such that $\gamma(\alpha)$ belongs to $E_\sigma$, and $E_\sigma$ positively fails to reduce to $A_\sigma$, that is, for every continuous function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$, if $\gamma$ maps $E_\sigma$ into $A_\sigma$, then there exists $\alpha$ in $A_\sigma$ such that $\gamma(\alpha)$ belongs to $A_\sigma$.

For all subsets $X, Y$ of $\mathcal{N}$, we let $D(X, Y)$ be the set of all $\alpha$ such that either $\alpha^0$ belongs to $X$ or $\alpha^1$ belongs to $Y$. The set $D(X, Y)$ is called the disjunction of the sets $X$ and $Y$. One may prove that a subset $Z$ of $\mathcal{N}$ reduces to the set $D(X, Y)$ if and only if there exists subsets $Z_0, Z_1$ of $Z$ such that $Z = Z_0 \cup Z_1$ and $Z_0$ reduces to $X$ and $Z_1$ reduces to $Y$. 

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For each $\alpha$ in $\mathcal{N}$ we let $S(\alpha)$ be the element of $\mathcal{N}$ satisfying: for each $n$, $(S(\alpha))^n = \alpha$ and $(S(\alpha))(0) = 0$.

Note that, if $\sigma$ is a (non-zero) (hereditarily repetitive) stump, then so is $S(\sigma)$. $S(\sigma)$ is called: the \textit{successor} of $\sigma$.

Let $\sigma$ be a non-empty stump. $\sigma$ is called \textit{weakly comparative} if and only if, for all $m, n$ there exists $p$ such that $\sigma^m \leq \sigma^n$ and $\sigma^n \leq \sigma^p$. This notion has been introduced in [15], page 39.

The following result is Theorem 8.8 of [45].

**Theorem 1.6** (The persisting difficulty of disjunction). For each hereditarily repetitive and weakly comparative non-zero stump $\sigma$, the set $D(A_1, A_\sigma)$ does not reduce to the set $A_S(\sigma)$.

Recall that $0$ is the element of $\mathcal{N}$ satisfying: for each $n$, $0(n) = 0$.

We define a function associating to every stump a subset $CB_\sigma$ of $\mathcal{N}$, called the \textit{Cantor-Bendixson set associated to} $\sigma$, as follows, by induction.

1. For each stump $\sigma$, if $\sigma(0) \neq 0$, then $CB_\sigma = \emptyset$.
2. For each non-empty stump $\sigma$, $CB_\sigma = \{0\} \cup \bigcup_{n \in \mathbb{N}} \mathfrak{m}_n * (1) * CB_{\sigma^n}$.

We want to mention some of the results proven for these sets in [46].

**Theorem 1.7** (The Cantor-Bendixson hierarchy, part 1).

(i) For each non-empty stump $\sigma$, the set $CB_\sigma$ is an enumerable subset of $\mathcal{N}$.

(ii) For all hereditarily iterative stumps $\sigma, \tau$, if $\sigma \leq \tau$, then the set $CB_\sigma$ reduces to the set $CB_\tau$.

(iii) For all hereditarily iterative stumps $\sigma, \tau$, if $\sigma < \tau$, then the set $CB_\sigma$ does not reduce to the set $CB_\tau$.

Following in Cantor’s footsteps, we define, for every subset $X$ of $\mathcal{N}$, a function associating to each stump $\sigma$ a subset $\text{Der}(\sigma, X)$ of $\mathcal{N}$, called the $\sigma$-th \textit{Cantor-Bendixson derivative set} of $X$, as follows, by induction.

1. For each stump $\sigma$, if $\sigma(0) \neq 0$, then $\text{Der}(\sigma, X) = X$.
2. For each non-empty stump $\sigma$, $\text{Der}(\sigma, X)$ is the set of all $\alpha$ such that, for each $n$, there exists $\beta$ in $\bigcap_{\gamma \in \mathbb{N}} \text{Der}(\sigma^n, X)$ such that $\beta$ passes through $\mathfrak{m}_n$ and $\beta \neq \alpha$.

**Theorem 1.8** (The Cantor-Bendixson hierarchy, part 2).

(i) For all stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $\text{Der}(\tau, CB_\sigma) = \emptyset$.

(ii) For all stumps $\sigma, \tau$, if $\tau < \sigma$, then $0$ belongs to $\text{Der}(\tau, CB_\sigma)$.

We also define, for every subset $X$ of $\mathcal{N}$, a function associating to each stump $\sigma$ a subset $\mathbb{P}(\sigma, X)$ of $\mathcal{N}$, called the $\sigma$-th \textit{parsesive extension} of $X$, as follows, by induction.

1. For each stump $\sigma$, if $\sigma(0) \neq 0$, then $\mathbb{P}(\sigma, X) = X$.
2. For each non-empty stump $\sigma$, $\mathbb{P}(\sigma, X)$ is the set of all $\alpha$ such that, for some $\beta$ in $X$, if $\alpha \neq \beta$, then there exists $n$ such that $\alpha$ belongs to $\mathbb{P}(\sigma^n, X)$.

**Theorem 1.9** (The Cantor-Bendixson hierarchy, part 3).

(i) For every inhabited subset $X$ of $\mathcal{N}$, for all hereditarily iterative stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $\mathbb{P}(\sigma, X)$ is a subset of $\mathbb{P}(\tau, X)$.

(ii) For all hereditarily iterative stumps $\sigma$, the set $\mathbb{P}(\sigma, CB_\sigma)$ is a subset of the set $\mathbb{P}(\sigma, CB_\sigma)$.

(iii) For all hereditarily iterative stumps $\sigma, \tau$, if the set $\mathbb{P}(\sigma, CB_\sigma)$ is a subset of the set $\mathbb{P}(\tau, CB_\sigma)$, then $\sigma \leq S(\tau)$.

For each $n$, for all subsets $X_0, X_1, \ldots, X_n$ of $\mathcal{N}$ we let $P_{i=0}^n(X_i)$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $i \leq n$, $\alpha^i$ belongs to $X_i$, and we let $C_{i=0}^n(X_i)$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for all $i \leq n$, $\alpha^i$ belongs to $X_i$. 

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We define a function associating to every stump $\sigma$ a subset $DCB_{\sigma}$ of $\mathcal{N}$, called the disjunctive Cantor-Bendixson set associated to $\sigma$, as follows, by induction.

1. For each stump $\sigma$, if $\sigma(0) \neq 0$, then $DCB_{\sigma} = \emptyset$.
2. For each non-empty stump $\sigma$, $DCB_{\sigma} = \{0\} \cup \bigcup_{i \in \mathbb{N}} \, D_{i=0} \ast (1) \ast D_{i=0}^{n}(DCB_{\sigma})$.

We also define a function associating to every stump $\sigma$ a subset $CCB_{\sigma}$ of $\mathcal{N}$, called the conjunctive Cantor-Bendixson set associated to $\sigma$, as follows, by induction.

1. For each stump $\sigma$, if $\sigma(0) \neq 0$, then $CCB_{\sigma} = D^{2}(\{0\})$.
2. For each non-empty stump $\sigma$, $CCB_{\sigma} = \{0\} \cup \bigcup_{i \in \mathbb{N}} \, D_{i=0} \ast (1) \ast C_{i=0}^{n}(CCB_{\sigma})$.

For each $n$, for each subset $X$ of $\mathcal{N}$, we let $D_{n}(X)$ be the set of all $\alpha$ such that, for some $i < n$, $\alpha^{i}$ belongs to $X$, and we let $C_{n}(X)$ be the set of all $\alpha$ such that, for all $i < n$, $\alpha^{i}$ belongs to $X$.

Let $X$ be a subset of $\mathcal{N}$. $X$ is called disjunctively productive if and only if, for each $n$, the set $D_{n+1}(X)$ does not reduce to the set $D_{n}(X)$, and $X$ is called conjunctively productive if and only if, for each $n$, the set $C_{n+1}(X)$ does not reduce to the set $C_{n}(X)$.

The following theorem is a main result of [46].

**Theorem 1.10** (Two more hierarchies within $\Sigma_{0}^{2}$).

(i) For each stump $\sigma$, the sets $DCB_{\sigma}$ and $CCB_{\sigma}$ are countable unions of closed sets and thus belong to the class $\Sigma_{0}^{2}$.

(ii) For all hereditarily repetitive stumps $\sigma$, $\tau$, if $\sigma \leq \tau$, then the set $DCB_{\sigma}$ reduces to the set $DCB_{\tau}$, and the set $CCB_{\sigma}$ reduces to the set $CCB_{\tau}$.

(iii) For all hereditarily repetitive stumps $\sigma$, $\tau$, if $\sigma < \tau$, then the set $DCB_{\sigma}$ does not reduce to the set $DCB_{\tau}$, and the set $CCB_{\sigma}$ does not reduce to the set $CCB_{\tau}$.

(iv) For each hereditarily repetitive stump $\sigma$, the set $DCB_{\sigma}$ is disjunctively productive, and the set $CCB_{\sigma}$ is conjunctively productive.

The paper [41] contains some results that are very important in connection with the subject of this paper.

First, the set $\text{Fin}$, consisting of all $\alpha$ in Cantor space $\mathcal{C}$ such that there exists $n$ such that for all $m$, if $m > n$, then $\alpha(m) = 0$ is introduced. This set belongs to the class $\Sigma_{0}^{1}$ and it is contrasted with the set $\text{Almost}^{+}\text{Fin}$ consisting of all $\alpha$ in $\mathcal{C}$ such that, for every strictly increasing $\gamma$, there exists $n$ such that $\alpha(\gamma(n)) = 0$. It is not difficult to see that $\text{Almost}^{+}\text{Fin}$ is a co-projection of an open subset of $\mathcal{N}$ and thus a co-analytic subset of $\mathcal{N}$. The permissible extensions of the set $\text{Fin}$, that is, the sets $\mathcal{P}(\sigma, \text{Fin})$, where $\sigma$ is a stump, are called, in this paper, the Borel approximations of the set $\text{Almost}^{+}\text{Fin}$.

Note that, for each stump $\sigma$, the set $\mathcal{P}(\sigma, \text{Fin})$ is a positively Borel subset of $\mathcal{N}$.

For each stump $\sigma$, the set $\mathcal{P}(\sigma, \text{Fin})$ is a subset of $\mathcal{N}$ containing $\text{Fin}$ and contains $\text{Fin}$ with the property: if $\alpha \neq \beta$, then $\alpha$ belongs to $X$.

The following results are proven in [41].

**Theorem 1.11** (A simple co-analytic subset of $\mathcal{N}$ that is not positively Borel).

(i) For every stump $\sigma$, the set $\mathcal{P}(\sigma, \text{Fin})$ is a subset of the set $\text{Almost}^{+}\text{Fin}$.

(ii) For all stumps $\sigma, \tau$, if $\sigma < \tau$, then $\mathcal{P}(\sigma, \text{Fin})$ is a proper subset of $\mathcal{P}(\tau, \text{Fin})$.

(iii) For every stump $\sigma$, the set $(\mathcal{P}(\sigma, \text{Fin}))^{+}$ coincides with the set $\mathcal{P}(S(\sigma), \text{Fin})$, and the set $\mathcal{P}(\sigma, \text{Fin})$ is a proper subset of $(\mathcal{P}(\sigma, \text{Fin}))^{+}$.

(iv) For every stump $\sigma$, the set $(\text{Almost}^{+}\text{Fin})^{+}$ coincides with the set $\text{Almost}^{+}\text{Fin}$.

(v) For every positively Borel set $X$ that is a subset of $\text{Almost}^{+}\text{Fin}$ and contains $\text{Fin}$ there exists a positively Borel set $Y$ that is a subset of $\text{Almost}^{+}\text{Fin}$ and contains $X$ such that $Y$ does not reduce to $X$, (and thus does not coincide with $X$).

The second set that is subjected to close inspection in [41] is the set $\text{MonPath}_{01}$ consisting of all $\alpha$ with the property that there exists $\gamma$ such that, for all $n$, $\gamma(n) \leq
\(\gamma(n + 1) \leq 1\) and \(\alpha(\gamma n) = 0\). \(\text{MonPath}_{01}\) is the projection of a closed subset of \(\mathcal{N}\) and, therefore, an analytic subset of \(\mathcal{N}\), but \(\text{MonPath}_{01}\) is not positively Borel. (From a classical point of view, \(\text{MonPath}_{01}\) is a closed subset of \(\mathcal{N}\).)

Note that, for each \(\alpha\), for each \(s\), \(\alpha^0(s) = \alpha((0) + s)\).

Recall that \(1\) is the element of \(\mathcal{N}\) satisfying: for each \(n\), \(1(n) = 1\).

For every subset \(X\) of \(\mathcal{N}\) we let \(X^\sim\) be the set of all \(\alpha\) such that either \(\alpha\) admits \(1\) or \(\alpha^0\) belongs to \(X\).

The class \(\mathcal{B}\) of the Borel-approximations of the set \(\text{MonPath}_{01}\) is defined inductively, as follows.

The set \((\text{MonPath}_{01})^\sim\) is the initial Borel-approximation of the set \(\text{MonPath}_{01}\).

Note that, for each \(\alpha\), \(\alpha\) belongs to the closed set \((\text{MonPath}_{01})^\sim\) if and only if, for each \(n\), there exists \(s\) admitted by \(\alpha\) such that \(\text{length}(s) = n\), and, for each \(i < \text{length}(s) - 1\), \(s(i) \leq s(i + 1) \leq 1\).

For each subset \(X\) of \(\mathcal{N}\), \(X\) belongs to the class \(\mathcal{B}\) if and only if either \(X\) coincides with \((\text{MonPath}_{01})^\sim\), or there exists \(Y\) in \(\mathcal{B}\) such that \(X\) coincides with \(Y^\sim\) or there exists an infinite sequence \(Y_0, Y_1, \ldots\) of elements of \(\mathcal{B}\) such that \(X\) coincides with \(\bigcap_{n \in \mathbb{N}} Y_n\).

**Theorem 1.12** (A simple analytic subset of \(\mathcal{N}\) that is not positively Borel).

(i) Every set in the class \(\mathcal{B}\) is positively Borel.

(ii) The set \(\text{MonPath}_{01}\) is a subset of every set in the class \(\mathcal{B}\).

(iii) For every \(X\) in \(\mathcal{B}\) there exists a function from \(\mathcal{N}\) to \(X\) that is not a function from \(\mathcal{N}\) to \(\text{MonPath}_{01}\), so \(X\) does not coincide with \(\text{MonPath}_{01}\).

(iv) For each positively Borel set \(X\) containing the set \(\text{MonPath}_{01}\) there exists a set \(Y\) in the class \(\mathcal{B}\) properly contained in \(X\).

### 1.3. The main results of this paper.

Apart from this introductory Section, the paper contains Sections numbered 2 to 9.

In Section 2, we discover that the set of the positively ill-founded trees, that is, the set of all \(\alpha\) that admit every element of an infinite sequence of (codes of) finite sequences decreasing in the sense of the Kleene-Brouwer-ordering, is not a complete element of the class \(\Sigma_1^0\) of the analytic sets. We also explain that the set of codes of the located and closed subsets of \(\mathcal{N}\) that are positively uncountable is a complete element of the class \(\Sigma_1^0\), and that the same holds for the set of codes of the closed subsets of \(\mathcal{N}\) that contain an element \(\alpha\) of \(\mathcal{N}\) satisfying \(\forall m3n > m|\alpha(n) = 1|\).

In Section 3, we find a hierarchy of analytic sets. The basic idea is, to consider, for each stump \(\sigma\), two sets: first, the set \(\text{Share}(CB_{\sigma})\) consisting of all \(\beta\) that admit a member of the Cantor-Bendixon-set \(CB_{\sigma}\) and, secondly, the set \(\text{Share}(CB_{\sigma})\) consisting of all \(\beta\) that admit a member of the closure \(\overline{CB_{\sigma}}\) of the \(CB_{\sigma}\). Each set \(\text{Share}(CB_{\sigma})\) belongs to the class \(\Sigma_1^0\) and each set \(\text{Share}(CB_{\sigma})\) is an analytic set, and for all stumps \(\sigma, \tau\), if \(\sigma \leq \tau\) then the set \(\text{Share}(CB_{\sigma})\) reduces to the set \(\text{Share}(CB_{\tau})\) and the set \(\text{Share}(CB_{\tau})\) reduces to the set \(\text{Share}(CB_{\tau})\). It turns out to be useful to introduce so-called hereditarily increasing stumps. For all hereditarily increasing stumps \(\sigma, \tau\), if \(\sigma < \tau\), then the set \(\text{Share}(CB_{\sigma})\) does not reduce to the set \(\text{Share}(CB_{\sigma})\) and the set \(\text{Share}(CB_{\tau})\) does not reduce to the set \(\text{Share}(CB_{\tau})\). The fact that, for almost every stump \(\sigma\), the set \(\text{Share}(CB_{\sigma})\) is not positively Borel, is a consequence of Theorem 1.12.

In Section 4, we study the set \(\text{Share}(C)\), consisting of all \(\alpha\) in \(\mathcal{N}\) that admit an element of Cantor space \(C\). The class of the subsets of \(\mathcal{N}\) reducing to the set \(\text{Share}(C)\) is a proper extension of the class of the closed subsets of \(\mathcal{N}\) although, in classical mathematics, the two classes coincide: the set \(\text{MonPath}_{01}\) reduces to the set \(\text{Share}(C)\) but is not a closed subset of \(\mathcal{N}\) as it fails to be positively Borel, see Theorem 1.12. The class of the closed subsets of \(\mathcal{N}\) is closed under the operation of countable intersection but not under the operation of finite union. The class of all subsets of \(\mathcal{N}\) reducing to the set \(\text{Share}(C)\),
however, is closed under both operations. The set $D^2(A_1)$ is another and easier example of a set that is not closed but reduces to $\text{Share}(C)$.

The Section also contains some observations on the assumption that the set $E_1$ reduces to the set $\text{Share}(C)$.

Section 5 is devoted to the important distinction between analytic and strictly analytic subsets of $\mathcal{N}$. A subset $X$ of $\mathcal{N}$ is \textit{strictly analytic} if there exists $\gamma$ in $\mathcal{N}$ coding a continuous function from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of the function, that is, for each $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, $\gamma|\beta = \alpha$.

In Section 6, we give constructive proofs of the Lusin Separation Theorem and the Novikov Separation Theorem. These theorems are proven for strictly analytic subsets of $\mathcal{N}$. The proofs depend on Brouwer’s Thesis on Bars. It is also shown that the positively Novikov Separation Theorem. These theorems are proven for strictly analytic subsets of $\mathcal{N}$.

In Section 7, we consider co-analytic subsets of $\mathcal{N}$. It turns out that the set of codes of the \textit{well-founded trees}, that is the set of all $\alpha$ such that every infinite sequence either positively fails to be a sequence of (codes) of finite sequences decreasing in the sense of the Kleene-Brouwer ordering or has a member that is not admitted by $\alpha$, coincides with the set $A_1^1 = \text{Bar}$. The proof depends on Brouwer’s Thesis on Bars, Axiom \ref{AxiomBar}. The set $\text{Sink}(\text{Almost}^{\ast}\text{Fin})$ consisting of all $\beta$ satisfying the condition that every $\alpha$ admitted by $\beta$ belongs to the set $\text{Almost}^{\ast}\text{Fin}$ is another complete element of the class $\Pi^1_1$ of the co-analytic subsets of $\mathcal{N}$. We also study the set $E_1^1$ consisting of those $\beta$ that admit exactly one element of $\mathcal{N}$. Every co-analytic subset of $\mathcal{N}$ reduces to the set $E_1^1$, but $E_1^1$ itself is not co-analytic.

In Section 8, we prove that complete elements of the class of the analytic subsets of $\mathcal{N}$ positively fail to be co-analytic and that complete elements of the class of the co-analytic sets are not analytic. We also show that complete elements of either class positively fail to be positively Borel. Using Brouwer’s Thesis on bars, Axiom \ref{AxiomBar} we prove Souslin’s theorem saying that every subset of $\mathcal{N}$ that is both strictly analytic and co-analytic is positively Borel.

In Section 9, we study co-projections of analytic sets and projections of co-analytic sets. We study the set of all $\beta$ coding a located and closed subset of $\mathcal{N}$ that is, in a sense to be defined, \textit{almost-countable} as an example of a set that is a projection of a co-analytic set. We prove that the Second Axiom of Continuous Choice, Axiom \ref{AxiomChoice} implies the collapse of the projective hierarchy.

2. Analytic subsets of $\mathcal{N}$: some examples

2.1. The class of the analytic subsets of $\mathcal{N}$. We let $\langle \alpha, \beta \rangle$ denote the following pairing function on $\mathcal{N}$. We define, for each $\alpha$, for each $\beta$, for each $n$, $\langle \alpha, \beta \rangle(2n) = \alpha(n)$ and $\langle \alpha, \beta \rangle(2n + 1) = \beta(n)$.

Conversely, for each $\alpha$, we define elements $\alpha_I$ and $\alpha_{II}$ of $\mathcal{N}$ by: for each $n$, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n + 1)$.

Note that, for each $\alpha$, $\alpha = \langle \alpha_I, \alpha_{II} \rangle$.

For every subset $X$ of $\mathcal{N}$, we let \textit{the (existential) projection} of $X$, notation $Ex(X)$, be the set of all $\alpha$ such that, for some $\beta$, $\langle \alpha, \beta \rangle$ belongs to $X$.

A subset $X$ of $\mathcal{N}$ is called \textit{analytic} if and only if there exists a closed subset $Y$ of $\mathcal{N}$ such that $X$ coincides with $Ex(Y)$. The class of the analytic subsets of $\mathcal{N}$ is denoted by $\Sigma^1_1$. 

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One may parametrize the class $\mathbf{II}^0_1$ of the closed subsets of $\mathcal{N}$ as follows. As we defined in Section 1, for each $\beta$, $F_\beta$ is the set of all $\alpha$ in $\mathcal{N}$ such that, for each $n$, $\beta(\overline{\alpha}_n) = 0$. A subset $X$ of $\mathcal{N}$ is closed if and only there exists $\beta$ such that $X$ coincides with $F_\beta$.

One may parametrize the class $\mathbf{\Sigma}^1_1$ of the analytic subsets of $\mathcal{N}$ as follows. For each $\beta$, let $A_\beta$ be the set $\text{Ex}(F_\beta)$, that is the set of all $\alpha$ in $\mathcal{N}$ such that, for some $\gamma$, the pair $\langle \alpha, \gamma \rangle$ belongs to $F_\beta$. A subset $X$ of $\mathcal{N}$ is analytic if and only there exists $\beta$ such that $X$ coincides with $A_\beta$.

We let $US^1_1$ be the set of all $\alpha$ such that $\alpha_{II}$ belongs to $A_{\alpha_I}$.

For each subset $X$ of $\mathcal{N}$, for each $\beta$, we let $X \upharpoonright \beta$ be the set of all $\alpha$ such that $\langle \beta, \alpha \rangle$ belongs to $X$.

For each class $K$ of subsets of $\mathcal{N}$, for each subset $X$ of $\mathcal{N}$, $X$ is called a cataloguing or a universal element of the class $K$ if and only if $X$ belongs to $K$, and, for every element $Y$ of $K$, there exists $\beta$ such that $Y$ coincides with $X \upharpoonright \beta$.

We use $S^*$ to denote the successor-function from $\mathbb{N}$ to $\mathbb{N}$, and $\circ$ to denote the operation of composition on $\mathcal{N}$. So, for each $\alpha, \beta$ in $\mathcal{N}$, for each $n \in \mathbb{N}$, $\alpha \circ \beta(n) = \alpha(\beta(n))$.

For each $\alpha$, for each $i$, for each $j$, we define: $\alpha^{i,j} := (\alpha^i)^j$.

**Theorem 2.1.**

(i) The set $US^1_1$ is a universal element of the class $\mathbf{\Sigma}^1_1$.

(ii) For every sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each $n$, $X_n$ belongs to $\mathbf{\Sigma}^1_1$ then both $\bigcup_{n \in \mathbb{N}} X_n$ and $\bigcap_{n \in \mathbb{N}} X_n$ belong to $\mathbf{\Sigma}^1_1$.

(iii) For every subset $X$ of $\mathcal{N}$, if $X$ is positively Borel, then $X$ is analytic.

(iv) For every subset $X$ of $\mathcal{N}$, if $X$ belongs to $\mathbf{\Sigma}^1_1$, then $\text{Ex}(X)$ belongs to $\mathbf{\Sigma}^1_1$.

(v) For all subsets $X, Y$ of $\mathcal{N}$, if $X$ reduces to $Y$ and $Y$ is analytic, then $X$ is analytic.

**Proof.** (i) Note that, for each $\alpha, \alpha$ belongs to $US^1_1$ if and only if $\alpha_{II}$ belongs to $A_{\alpha_I}$ if and only if, for some $\gamma$, $\langle \alpha_{II}, \gamma \rangle$ belongs to $F_{\alpha_I}$ if and only if, for some $\gamma$, for every $n$, $\alpha_I(\langle \alpha_{II}, \gamma \rangle n) = 0$. We thus see that $US^1_1$ belongs to the class $\mathbf{\Sigma}^1_1$. The fact that, for each element $Y$ of $\mathbf{\Sigma}^1_1$, there exists $\beta$ such that $Y$ coincides with $US^1_1 \upharpoonright \beta$ follows from the observations preceding this theorem.

(ii) Let $Y_0, Y_1, \ldots$ be a sequence of closed subsets of $\mathcal{N}$ such that, for each $n$, $X_n = \text{Ex}(Y_n)$. Using the Second Axiom of Countable Choice, Axiom 3 find $\beta$ in $\mathcal{N}$ such that, for each $n$, $Y_n = F_\beta$ and $X_n = \text{Ex}(Y_n) = A_{\beta(n)}$. Define subsets $Z_0, Z_1$ of $\mathcal{N}$, as follows. $Z_0$ is the set of all $\alpha$ such that $\langle \alpha^0, \alpha^1 \circ S^* \rangle$ belongs to $Y_{\alpha^1(0)}$ and $Z_1$ is the set of all $\alpha$ such that, for all $n$, $\langle \alpha^0, \alpha^{1,n} \rangle$ belongs to $Y_n$. The sets $Z_0, Z_1$ are closed subsets of $\mathcal{N}$. one may determine $\gamma$ in $\mathcal{N}$ satisfying: for each $\alpha$, there exists $n$ such that $\gamma(\overline{\alpha}_n) \neq 0$ if and only if there exists $n$ such that $\beta^{\alpha^1(0)}(\langle \overline{\alpha}^0, \alpha^1 \circ S^* \rangle n) \neq 0$ and one may determine $\delta$ satisfying: for each $\alpha$, there exists $n$ such that $\delta(\overline{\alpha}_n) \neq 0$ if and only if $\exists i, j$ such that $i \leq n$ and $j \leq n$ and $\beta^j(\overline{\alpha}^0, \alpha^{1,j}) i) \neq 0$.

Note that the set $Z_0$ coincides with the set $F_{\gamma}$ and that the set $Z_1$ coincides with the set $F_{\delta}$.

Note that, for each $\alpha, \alpha$ belongs to $\bigcup_{n \in \mathbb{N}} X_n$ if and only if there exist $n, \beta$ such that $\langle \alpha, \beta \rangle$ belongs to $X_n$ if and only if there exists $\beta$ such that $\langle \alpha, \beta \circ S^* \rangle$ belongs to $X_{\beta(0)}$ if and only if $\alpha$ belongs to $\text{Ex}(Z_0)$. It follows that the set $\bigcup_{n \in \mathbb{N}} X_n$ coincides with the set $\text{Ex}(Z_0) = \text{Ex}(F_{\gamma}) = A_{\gamma}$.

Also note that, for each $\alpha, \alpha$ belongs to $\bigcap_{n \in \mathbb{N}} X_n$ if and only if, for each $n$, there exists $\beta$ such that $\langle \alpha, \beta \rangle$ belongs to $X_n$ if and only if, by the Second Axiom of Countable Choice, there exists $\beta$ such that, for each $n$, $\langle \alpha, \beta^m \rangle$ belongs to $X_n$ if and only there
exists $\beta$ such that $\alpha, \beta$ belongs to $Z_1$. It follows that the set $\bigcap_{n \in \mathbb{N}} X_n$ coincides with the set $\text{Ex}(Z_1) = \text{Ex}(F_\delta) = A_\delta$.

(iii) follows easily from (ii), by induction on the class of the positively Borel sets.

(iv) Suppose that $X$ belongs to $\Sigma^1_1$. Let $Y$ be a closed subset of $\mathcal{N}$ such that $X = \text{Ex}(Y)$. Observe that for every $\alpha$, $\alpha$ belongs to $\text{Ex}(X)$ if and only if, for some $\beta$, $(\alpha, \beta)$ belongs to $X$ if and only if, for some $\gamma$, the sequence $(\langle \alpha, \beta \rangle, \gamma)$ belongs to $Y$. Let $Z$ be the set of all $\alpha$ such that $\langle \langle \alpha^0, \alpha^1 \rangle, \alpha^{1,1} \rangle$ belongs to $Y$. Observe that $Z$ is a closed subset of $\mathcal{N}$ and $\text{Ex}(X)$ coincides with $\text{Ex}(Z)$, and, therefore, $\text{Ex}(X)$ belongs to $\Sigma^1_1$.

(v) Let $X, Y$ be subsets of $\mathcal{N}$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $Y$.

Assume that $Y$ is analytic. We have to show that also $X$ is analytic. Let $Z$ be a closed subset of $\mathcal{N}$ such that $Y$ coincides with $\text{Ex}(Z)$. Observe that for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\gamma|\alpha$ belongs to $Y$ if and only if, for some $\beta$, $(\gamma|\alpha, \beta)$ belongs to $Z$. Let $V$ be the set of all $\alpha$ such that $\langle \gamma|\alpha^0, \alpha^1 \rangle$ belongs to $Z$. Observe that $V$ is closed and $X$ coincides with $\text{Ex}(V)$, and, therefore, $X$ is analytic.  

\section{2.2. The set $E^1_1 = \text{Path}$ and the set $\text{PIF}$ of all $\alpha$ admitting each element of an infinite $<_{KB}$-decreasing sequence.}

Recall that, for every class $\mathcal{K}$ of subsets of $\mathcal{N}$, for every subset $X$ of $\mathcal{N}$, $X$ is a complete element of $\mathcal{K}$ if and only if $X$ belongs to $\mathcal{K}$, and every subset $Y$ of $\mathcal{N}$ that belongs to $\mathcal{K}$ reduces to $X$.

We let $E^1_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that for some $\beta$, for all $n$, $\alpha(\beta^n) = 0$. $E^1_1$ might be described as the set of all $\alpha$ that admit an infinite path in Baire space, and, for this reason, we sometimes call it $\text{Path}$. We shall prove, in the next theorem, that $E^1_1$ is a complete element of the class $\Sigma^1_1$, or, as we will say, a complete analytic set.

We now define the Kleene-Brouwer ordering $<_{KB}$ on $\mathbb{N}$. Sometimes, this ordering is called the Lusin-Sierpiński ordering, see [15], Section 2.G, p. 11.

For all $s, t \in \mathbb{N}$, $s <_{KB} t$ if and only if either $t$ is (or: codes a finite sequence of natural numbers that is) a strict initial part of (the finite sequence of natural numbers coded by) $s$, or there exists $i$ such that $i < \text{length}(s)$ and $i < \text{length}(t)$, and $\overline{si} = \overline{ti}$ while $s(i) < t(i)$, that is, $s \sqcup t$ and (the finite sequence coded by) $s$ comes lexicografically before (the finite sequence coded by) $t$.

$<_{KB}$ is a well-known decidable and linear ordering of $\mathbb{N}$.

We also define: for all $s, t \in \mathbb{N}$, $s \sqsubseteq t$, $s$ is an initial part of $t$, if and only if there exists $i \leq \text{length}(t)$ such that $s = \overline{ti}$, and $s \sqsubseteq t$, $s$ is a proper initial part of $t$, if and only if there exists $i < \text{length}(t)$ such that $s = \overline{ti}$.

Let $\alpha$ belong to $\mathcal{N}$. Recall that an element $s$ of $\mathbb{N}$ is said to be admitted by $\alpha$ if and only if, for every $t$, if $t \sqsubseteq s$, then $\alpha(t) = 0$.

We let $T_\alpha$, the tree determined by $\alpha$, be the set of all $s$ that are admitted by $\alpha$.

We let $\text{PIF}$ be the set of all $\alpha$ in $\mathcal{N}$ such that the tree $T_\alpha$ is positively ill-founded with respect to $<_{KB}$, that is, $\text{PIF}$ is the set of all $\alpha$ in $\mathcal{N}$ such that, for some $\beta$, for every $n$, $\beta(n)$ belongs to $T_\alpha$, that is, $\alpha$ admits $\beta(n)$, and, for every $n$, $\beta(n + 1) <_{KB} \beta(n)$.

We shall prove, in the next theorem, that $\text{PIF}$ is not a complete analytic set. In classical mathematics, the sets $E^1_1$ and $\text{PIF}$ coincide.

Let $A, B$ be subsets of $\mathcal{N}$. $A$ is a subset of $B$ if only if every element of $A$ is an element of $B$. $A$ is a proper subset of $B$ if $A$ is a subset of $B$ and, in addition, $B$ is not a subset of $A$, that is, the assumption that $B$ is a subset of $A$ leads to a contradiction. Note that, if $A$ is a proper subset of $B$, it may be true that the assumption that there is
an element of $B$ that does not belong to $A$ also leads to a contradiction. For instance, let $A$ be the real interval $[-1, 1]$ and let $B$ be the set of all $x$ in $[-1, 1]$ such that either $x$ is really-apart from 0 or $x$ really-coincides with 0. It follows from Brouwer’s Continuity Principle that $B$ is a proper subset of $A$. On the other hand, there is no number $x$ such that $x$ does not belong to $B$, as, for such a number $x$, one would have both: $x$ is not really-apart from 0, that is: $x$ really-coincides with 0 and: $x$ does not really-coincide with 0.

**Theorem 2.2.**

(i) The set $E^1_1$ is a complete analytic set.

(ii) The set $E^1_1$ is a proper subset of the set $\text{PIF}$. 

(iii) The set $D^2(A_1)$ does not reduce to the set $\text{PIF}$.

(iv) The set $\text{PIF}$ is not a complete analytic set.

**Proof.** (i) Let $X$ belong to $\Sigma^1_1$ and let $Y$ be a closed subset of $\mathcal{N}$ such that $X = E x(Y)$ and let $C$ be a decidable subset of $\mathcal{N}$ such that, for every $\beta$, $\beta$ belongs to $Y$ if and only if, for each $n$, $\beta n$ belongs to $C$. Observe that for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, for every $n$, $(\alpha, \beta)n$ belongs to $C$. Define a function $f$ from $\mathcal{N}$ to $\mathcal{N}$ such that for every $\alpha, \beta \in \mathcal{N}$, for every $n \in \mathbb{N}$, $(f|\alpha)(\beta n) = 0$ if and only if $(\alpha, \beta)n$ belongs to $C$ and observe that $f$ reduces $X$ to $E^1_1$.

(ii) First, we prove that every element of $E^1_1$ belongs to $\text{PIF}$. Let $\alpha$ belong to $E^1_1$. Find $\beta$ such that, for all $n$, $a(\beta n) = 0$, and define $\gamma$ in $\mathcal{N}$ such that, for each $n$, $\gamma(n) = \gamma(n).$ Note that, for each $n$, $\gamma(n)$ belongs to $T_o$ and $\gamma(n + 1) \not<_{KB} \gamma(n)$, so $\alpha$ belongs to $\text{PIF}$.

We now observe that there are elements of $\text{PIF}$ we cannot prove to belong to $E^1_1$.

Let $\gamma$ be an element of $\mathcal{N}$. Let $\alpha$ be an element of $\mathcal{N}$ such that, for every $s$, $a(s) = 0$ if and only if either there exists $n$ such that $s = \overline{n}$ or $s = \overline{n} n$ and there is no $i \leq n$ such that $\gamma(i) > 0$, or there exists $n$ such that $s = \overline{n}$ and there is $i \leq n$ such that $\gamma(i) > 0$ and the least such $i$ is odd, or there exists $n$ such that $s = \overline{n}$ and there is $i \leq n$ such that $\gamma(i) > 0$ and the least such $i$ is even. Note that $\alpha$ admits 0 if and only if: if there exists $i$ such that $\gamma(i) \neq 0$, then the first such $i$ is odd, and: $\alpha$ admits 1 if and only if: if there exists $i$ such that $\gamma(i) \neq 0$, then the first such $i$ is even.

We claim that $\alpha$ belongs to $\text{PIF}$. Let $\beta$ be an element of $\mathcal{N}$ such that, for each $n$, (i) if there is no $i \leq n$ such that $\gamma(i) > 0$, then $\beta(n) = \overline{n}$, and (ii) if there is $i \leq n$ such that $\gamma(i) > 0$ and the least such $i$ is even, then $\beta(n) = \overline{n}$, and (iii) if there exists $i \leq n$ such that $\gamma(i) > 0$ and the least such $i$ is odd, then $\beta(n) = \overline{n}$.

Note that, for each $n$, $\beta(n + 1) \not<_{KB} \beta(n)$ and $a(\beta(n)) = 0$, so $\alpha$ belongs to $\text{PIF}$.

Now assume that $\alpha$ belongs to $E^1_1$ and find $\beta$ in $\mathcal{N}$ such that, for each $n$, $a(\beta n) = 0$. Note that, if $\beta(0) = 0$, then, for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is odd, and if $\beta(0) = 1$, then, for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is even.

Suppose we define an infinite sequence $\gamma$ as follows.

Let $d$ be the element of $\mathcal{N}$ that is the decimal expansion of $\pi$, that is, that is, $\pi = 3 + \sum_{n \in \mathbb{N}} d(n).10^{-n-1}$. Let $\gamma$ satisfy the following requirement:
for each $i$, if, for each $j < 99$, $d(i + j) = 9$, then $\gamma(i) = 1$, and, if, for some $j < 99$, $d(i + j) \neq 9$, then $\gamma(i) = 0$.

For this $\gamma$, we have no proof of the statement: for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is odd, and we also have no proof of the statement: for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is even. If $\alpha$ results from this particular $\gamma$ in the above way, then, clearly, we have no proof of the statement: “$\alpha$ belongs to $E^1_1$”, and we must say that the statement: “$\alpha$ belongs to $E^1_1$”, is a reckless one.
One obtains a contradiction from the assumption: “PIF is a subset of $E_1$” in the following way:

Assume PIF is a subset of $E_1$. Then, for every $\gamma$, either for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is odd, or for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is even. Applying Brouwer’s Continuity Principle, Axiom 4 we determine $m$ such that either, for each $\gamma$ passing through $\overline{0}m$, for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is odd, or, for each $\gamma$ passing through $\overline{0}m$, for each $n$, if $n$ is the least $i$ such that $\gamma(i) > 0$, then $n$ is even. Considering the two sequences $\overline{0}(2m) \ast (1) \ast \overline{0}$ and $\overline{0}(2m + 1) \ast (1) \ast \overline{0}$, we see that this conclusion is false.

(iii) Assume that $\gamma$ is an element of $N$ coding a function from $N$ to $N$ reducing $D^2(A_1)$ to PIF. We claim that $\gamma$ will map every element of the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ into PIF. We then must conclude that the set $D^2(A_1)$ coincides with its closure $\overline{D^2(A_1)}$, and this is false.

Note that, for every $\alpha$, (1): $\alpha$ belongs to $D^2(A_1)$ if and only if, either, for each $n$, $\alpha^0(n) = 0$, or, for each $n$, $\alpha^1(n) = 0$, and (2): $\alpha$ belongs to the closure $\overline{D^2(A_1)}$ if and only if, for each $n$, either $\alpha^0n = \overline{0}n$ or $\alpha^1n = \overline{0}n$.

Let $\alpha$ belong to $N$. We define $\alpha_0, \alpha_1$ in $N$ as follows. For each $n$, $(\alpha_0)^0(n) = 0$ and, for each $m$, if there is no $i$ such that $m = (0, i)$, then $\alpha_0(m) = \alpha(m)$, and, for each $n$, $(\alpha_1)^0(n) = 0$ and, for each $m$, if there is no $i$ such that $m = (1, i)$, then $\alpha_1(m) = \alpha(m)$.

Note that $\alpha$ belongs to $D^2(A_1)$ if and only if either $\alpha = \alpha_0$ or $\alpha = \alpha_1$. Also note that, for each $i < 2$, if $\alpha$ belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ and $\alpha \neq \alpha_i$, then $\alpha = \alpha_{i-1}$.

Next, observe that, as both $\alpha_0$ and $\alpha_1$ belong to $D^2(A_1)$, both $|\alpha_0|$ and $|\alpha_1|$ belong to PIF. We determine $\gamma_0, \gamma_1$ in $N$ such that, for each $i < 2$, for each $n$, $\delta_i(n + 1) <_{KB} \delta_i(n)$, and for each $t$, if $t \subseteq \delta_i(n)$, then $(\gamma_i)(t) = 0$.

We now define $\epsilon$ in $N$ such that, for each $n$, $\epsilon(n) = \delta_0(n)$ if $\delta_1(n) \leq_{KB} \delta_0(n)$ and $\epsilon(n) = \delta_1(n)$ if $\delta_0(n) <_{KB} \delta_1(n)$. Note that, for each $n$, both $\delta_1(n + 1) <_{KB} \epsilon(n)$ and $\delta_0(n + 1) <_{KB} \epsilon(n)$, and, therefore, $\epsilon(n + 1) <_{KB} \epsilon(n)$.

Suppose we find $n, \ell$ in $N$ such that $t \subseteq \epsilon(n)$ and $(\gamma_i)(t) \neq 0$. Using the continuity of the function coded by $\gamma$, we find $m$ such that, for every $\beta$, if $\beta$ passes through $\overline{ \gamma \ell m}$, then $(\gamma_\beta)(t) = (\gamma_i)(t) \neq 0$. Observe that, if $\epsilon(n) = \delta_0(n)$, then $\gamma \ell m \neq \gamma \ell \overline{0}m$, and, if $\epsilon(n) = \delta_1(n)$, then $\gamma \ell m = \gamma \ell \overline{0}m$.

We define $\zeta$ in $N$ such that, for each $n$, if for each $j \leq n$, if or each $t \subseteq \epsilon(j)$, $(\gamma_j)(t) = 0$, then $\zeta(n) = \epsilon(n)$, and if there exists $j \leq n$, such that, for some $t \subseteq \epsilon(\ell)$, $(\gamma_j)(t) \neq 0$, and $j_0$ is the least such $j$, then either $\epsilon(j_0) = \delta_0(j_0)$ and $\zeta(n) = \delta_1(n)$ or $(\gamma_j)(t) \neq \delta_0(j_0)$ and $\zeta(n) = \delta_0(n)$.

Note that, for each $n$, $\zeta(n + 1) <_{KB} \zeta(n)$.

Now assume that $\alpha$ belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$. We claim that, for each $n$, for each $t$, if $t \subseteq \zeta(n)$, then $(\gamma_\alpha)(t) = 0$. We prove this claim as follows.

Let $n$ be a natural number. We have two cases to consider.

Case (1). For each $j \leq n$, for each $t$, if $t \subseteq \epsilon(j)$, then $(\gamma_\alpha)(t) = 0$. As, in this case, $\zeta(n) = \epsilon(n)$, we conclude that, for each $t$, if $t \subseteq \zeta(n)$, then $(\gamma_\alpha)(t) = 0$.

Case (2). There exists $j \leq n$ such that, for some $t$, $t \subseteq \epsilon(j)$ and $(\gamma_\alpha)(t) \neq 0$ and $j_0$ is the least such $j$. Note that if $\epsilon(j_0) = \delta_0(j_0)$, then $\alpha \neq \alpha_0$, and, therefore, as $\alpha$ belongs to $\overline{D^2(A_1)}$, $\alpha = \alpha_1$, and thus, for each $k$, for each $t$, if $t \subseteq \delta_1(k)$, then $(\gamma_\alpha)(t) = 0$, and, in particular, as $\zeta(n) = \delta_1(n)$, for each $t$, if $t \subseteq \zeta(n)$, then $(\gamma_\alpha)(t) = 0$.

Similarly, if $\epsilon(j_0) \neq \delta_0(j_0)$, then $\alpha \neq \alpha_1$, and, therefore, as $\alpha$ belongs to $\overline{D^2(A_1)}$, $\alpha = \alpha_0$, and thus, for each $k$, for each $t$, if $t \subseteq \delta_0(k)$, then $(\gamma_\alpha)(t) = 0$, and, in particular, as $\zeta(n) = \delta_0(n)$, for each $t$, if $t \subseteq \zeta(n)$, then $(\gamma_\alpha)(t) = 0$.

This ends the proof of our claim.
We must conclude that, for each $\alpha$, if $\alpha$ belongs to $D^2(A_1)$, then $\gamma|\alpha$ belongs to PIF. As we are assuming that $\gamma$ reduces the set $D^2(A_1)$ to the set PIF, we find that the sets $D^2(A_1)$ and $D^2(A_1)$ coincide. As the set $D^2(A_1)$ is a spread containing $\Psi$, we apply Brouwer’s Continuity Principle and find $m$ such that either for every $\alpha$ in $D^2(A_1)$, if $\overline{\alpha}m = \overline{\alpha}n$, then $\alpha^0 = \overline{\alpha}$, or for every $\alpha$ in $D^2(A_1)$, if $\overline{\alpha}m = \overline{\alpha}n$, then $\alpha^1 = \overline{\alpha}$. Both alternatives are false.

(iii) As the set $D^2(A_1)$ is an analytic set, the conclusion follows from (iii).  

2.3. The set of all $\beta$ coding a positively uncountable located and closed subset of $N$.

We intend to give another example of a complete analytic subset of $N$.

Let $X$ be a subset of $N$. We say that $X$ is a perfect spread if and only if $X$ is a closed subset of $N$ and there exist $\beta$ in $N$ satisfying the following conditions:

(i) for each $s$, $\beta(s) = 0$ if and only if $s$ contains an element of $X$, and

(ii) for each $s$, if $\beta(s) = 0$, then there exist $t$, $u$ such that $s \subseteq t$ and $t \subseteq u$ and $t \perp u$ and $\beta(t) = \beta(u) = 0$.

Note that Cantor space $C$, the set of all $\alpha$ in $N$ such that, for each $n$, $\alpha(n) = 0$ or $\alpha(n) = 1$, is a perfect spread.

Let $\gamma$ be an element of $N$. $\gamma$ codes or: is a (continuous) function from Cantor space $C$ to the set $\mathbb{N}$ of the natural numbers if and only if, for each $\alpha$ in $C$, there exists $n$ such that $\gamma(\overline{\alpha}n) \neq 0$.

Let $\gamma$ be a function from $C$ to $\mathbb{N}$ and let $\alpha$ belong to $C$. We let $\gamma(\alpha)$, the value that the function $\gamma$ assumes at the argument $\alpha$, be the natural number $p$ such that, for some $i$, $\gamma(\overline{\alpha}i) = p + 1$, while, for each $j$, if $j < i$, then $\gamma(\overline{\alpha}j) = 0$.

The set of all functions from $C$ to $\mathbb{N}$ will sometimes be denoted by $\mathbb{N}^C$.

Let $\gamma$ be an element of $N$. $\gamma$ codes or: is a (continuous) function from Cantor space $C$ to Baire space $N$ if and only if, for each $n$, $\gamma^n$ is a function from $C$ to $\mathbb{N}$.

Let $\gamma$ be a function from $C$ to $N$ and let $\alpha$ belong to $C$. We let $\gamma|\alpha$, the value that the function $\gamma$ assumes at the argument $\alpha$, be the infinite sequence $\beta$ in $N$ such that, for each $n$, $\beta(n) = \gamma^n(\alpha)$.

We let $\text{Bin}$ be the set of all natural numbers coding a finite binary sequence, that is, for all $s$, $s$ belongs to $\text{Bin}$ if and only if, for all $i$, if $i < \text{length}(s)$, then $s(i) = 0$ or $s(i) = 1$.

Let $\gamma$ be a function from $C$ to $N$ and let $s$ belong to $\text{Bin}$. We let $\gamma|s$ be the largest number $t$ such that $\text{length}(t) \leq \text{length}(s)$ and, for each $j$, if $j < \text{length}(t)$, then there exists $p$ such that $p \leq \text{length}(s)$ and $\gamma^j(\overline{\alpha}p) = t(j) + 1$ and, for every $k$, if $k < p$, then $\gamma^k(\overline{\alpha}k) = 0$. Observe that $\text{length}(\gamma|s) \leq \text{length}(s)$.

Also observe that, for every $n$, for every $\alpha$ in $C$, $\gamma(\overline{\alpha}n) \subseteq (\overline{\gamma|\alpha}n)$, that is, for all $s$ in $\text{Bin}$ for all $t$, if $\gamma|s = t$, then $\gamma$ maps every infinite sequence passing through $s$ onto an infinite sequence passing through $t$.

Note that, for all $\alpha$ in $C$, for all $\beta$ in $N$, $\gamma|\alpha = \beta$ if and only if, for each $n$, there exists $m$ such that $\overline{\alpha}m \subseteq \overline{\beta}m$.

The set of all functions from $C$ to $N$ will sometimes be denoted by $\mathbb{N}^C$.

Let $\gamma$ be a function from $C$ to $N$. $\gamma$ is called a strongly injective function from $C$ to $N$ if and only if, for all $\alpha$ in $C$, for all $\beta$ in $C$, if $\alpha \neq \beta$, then $\gamma|\alpha \neq \gamma|\beta$.

**Lemma 2.3.**

(i) For every subset $X$ of $N$, if $X$ is a perfect spread, then there exists a strongly injective function from $C$ to $X$.

(ii) For every strongly injective function $\gamma$ from $C$ to $N$, the set $\text{Range}(\gamma) = \{\gamma|\alpha \mid \alpha \in C\}$ is a perfect spread.
(iii) For all $\alpha$, the set $\{\alpha, 0\}$ is a closed subset of $\mathcal{N}$ if and only if either $\alpha = 0$ or $\alpha \neq 0$. The assumption that, for all $\alpha$, the set $\{\alpha, 0\}$ is a closed subset of $\mathcal{N}$, leads to a contradiction.

(iv) The assumption that, for every function $\gamma$ from $\mathcal{C}$ to $\mathcal{N}$, $\text{Range}(\gamma)$ is a closed subset of $\mathcal{N}$, leads to a contradiction.

(v) The set $\mathbb{N}^\mathcal{C}$ is an open subset of $\mathcal{N}$ and a complete element of the class $\Sigma^0_1$.

(vi) The set $\mathcal{N}^\mathcal{C}$ is a countable intersection of open subsets of $\mathcal{N}$ and a complete element of the class $\Pi^0_2$.

(vii) The set of all strongly injective functions from $\mathcal{C}$ to $\mathcal{N}$ is a countable intersection of open subsets of $\mathcal{N}$ and a complete element of the class $\Pi^0_2$.

**Proof.** (i) Let $X$ be a subset of $\mathcal{N}$ and a perfect spread. Find $\beta$ in $\mathcal{N}$ such that, for each $s$, $\beta(s) = 0$ if and only if $s$ contains an element of $X$. We now define a mapping $F$ from the set $\text{Bin}$ of all natural numbers coding a finite binary sequence to the set $\mathbb{N}$, as follows:

(i) $F(\langle \rangle) = \langle \rangle$.

(ii) Suppose $s$ belongs to $\text{Bin}$ and $F(s)$ has been defined already and $\beta(F(s)) = 0$.

We then search for the least number $u$ such that $s \subseteq u(0)$ and $s \subseteq u(1)$ and $\beta(u(0)) = \beta(u(1)) = 0$ and $u(0) \perp u(1)$ and we define: $F(s \ast \langle 1 \rangle) := u(0)$ and $F(s \ast \langle 0 \rangle) := u(1)$.

Note that, for every $s$ in $\text{Bin}$, for every $i$ in $\{0, 1\}$, $F(s)$ is a proper initial part of $F(s \ast \langle i \rangle)$.

We now let $\gamma$ be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for every $\alpha$ in $\mathcal{C}$, for every $n$, $\gamma|\alpha$ passes through $F(\pi_n)$. It will be clear that $\gamma$ is a strongly injective function from $\mathcal{C}$ to $X$.

(ii) Suppose that $\gamma$ is a strongly injective function from $\mathcal{C}$ to $\mathcal{N}$. Note that, for every $\alpha$ in $\mathcal{C}$, for every $n$, there exists $m$ such that $\text{length}(\gamma|\pi(m)) \geq n$. It follows from the (Restricted) Fan Theorem, Axiom [12], that, for each $n$, there exists $m$ such that, for every $\alpha$ in $\mathcal{C}$, $\text{length}(\gamma|\pi(m)) \geq n$. Using the Minimal Axiom of Countable Choice, we find $\delta$ in $\mathcal{N}$ such that, for each $n$, for every $s$ in $\text{Bin}$, if $\text{length}(s) \geq \delta(n)$, then $\text{length}(\gamma|s) \geq n$.

Note that, for every $t$ in $\mathbb{N}$, there exists $\alpha$ in $\mathcal{C}$ such that $\gamma|\alpha$ passes through $t$ if and only if there exists $s$ in $\text{Bin}$ such that $\text{length}(s) = \delta(\text{length}(t))$ and $t \in \gamma|s$.

As, for each $n$, the set of all $s$ in $\text{Bin}$ such that $\text{length}(s) = n$ is a finite set, one may decide, for every $t$ in $\mathbb{N}$, whether or not there exists $\alpha$ in $\mathcal{C}$ such that $\gamma|\alpha$ passes through $t$.

We define $\beta$ in $\mathcal{N}$ such that, for every $t$, $\beta(t) = 0$ if and only if there exists $\alpha$ in $\mathcal{C}$ such that $\gamma|\alpha$ passes through $t$.

We claim that $\text{Range}(\gamma)$ coincides with the closed set $F_3$. It suffices to prove that $F_3$ is a subset of $\text{Range}(\gamma)$.

Let $\alpha$ belong to $\mathcal{N}$. Note that for every $\alpha$ in $\mathcal{C}$, for every $\beta$ in $\mathcal{C}$, if $\pi_n \neq \beta n$, then there exists $p$ such that $\gamma|((\pi p) \perp \gamma)(\beta p))$. Using the (Restricted) Fan Theorem, Axiom [12], we find $p$ such that, for every $\alpha$ in $\mathcal{C}$, for every $\beta$ in $\mathcal{C}$, if $\pi_n \neq \beta n$, then $\gamma|((\pi p) \perp \gamma)(\beta p))$. Note that, for all $s$ in $\text{Bin}$, $\text{length}(\gamma|s) \leq \text{length}(s)$, and conclude that, for all $s$, $t$ in $\text{Bin}$, if $\text{length}(s) = \text{length}(t) = p$ and $\pi_n \neq \tau n$, then $\gamma|s \perp \gamma|t$. Using the Minimal Axiom of Countable Choice, Axiom [2], we find $\eta$ in $\mathcal{N}$ such that, for every $n$, for every $s$, $t$ in $\text{Bin}$, if $\text{length}(s) = \text{length}(t) = \eta(n)$ and $\pi n \neq \tau n$, then $\gamma|s \perp \gamma|t$.

It follows that, for every $n$, for every $\alpha$ in $\mathcal{C}$, for every $\beta$ in $\mathcal{C}$, if $\pi n \neq \beta n$, then $\gamma|((\pi(n)) \rightarrow (\beta \gamma)(\eta(n)))$, and, therefore, as $\gamma|((\pi \gamma)(\eta(n))) \subseteq (\gamma|\alpha)(\eta(n))$ and $\gamma|((\beta \gamma)(\eta(n))) \subseteq (\gamma|\beta)(\eta(n))$, also $(\gamma|\alpha)(\eta(n)) \neq (\gamma|\beta)(\eta(n))$.

Now suppose that $\alpha$ belongs to $F_3$. Note that, for each $n$, $\beta(\pi n) = 0$, and, therefore, there exists $\alpha$ in $\mathcal{C}$ such that $\gamma|\alpha$ passes through $\pi n$, and, therefore, there exists $s$ in
Bin such that \( \pi n \subseteq \gamma | s \). We define \( \zeta \) in \( N \) such that, for each \( n \), \( \zeta(n) \) is the least \( s \) such that \( s \) belongs to Bin and and \( \pi n \subseteq \gamma | s \).

Note that, for every \( n \), for every \( p \), if \( p > \eta(n) \), then there is exactly one \( s \) in Bin such that \( \text{length}(s) = n \) and \( \gamma | s \subseteq \pi p \). It follows that, for every \( n \), for every \( p \), if \( p > \eta(n) \), then \( (\zeta(p))(n) = (\zeta(p + 1))(n) \). Let \( \alpha \) be the element of \( C \) such that, for every \( n \), \( \alpha(n) = \zeta(\eta(n + 1))(n) \). Observe that \( \alpha \) belongs to \( C \) and \( \gamma | \alpha = \varepsilon \), so \( \varepsilon \) belongs to \( \text{Range}(\gamma) \).

(iii) Let \( \alpha \) belong to \( N \). Let \( \beta \) be an element of \( N \) such that, for every \( s \), \( \beta(s) = 0 \) if and only if either \( \alpha \) passes through \( s \) or \( \emptyset \) passes through \( s \). Note that \( \beta \) coincides with the closure \( \{\alpha, \emptyset\} \) of the set \( \{\alpha, \emptyset\} \). Also note that \( \beta \) is a spread.

Observe that, if either \( \alpha \neq \emptyset \) or \( \alpha = \emptyset \), then \( \beta \) coincides with \( \{\alpha, \emptyset\} \).

Now assume that \( \beta \) coincides with the set \( \{\alpha, \emptyset\} \). Then, for every \( \gamma \) in the spread \( \beta \), either \( \gamma = \alpha \) or \( \gamma = \emptyset \). Applying Bronner’s Continuity Principle, we find \( n \) such that, for every \( \gamma \) in \( \beta \), if \( \gamma = \alpha \), if \( \gamma = \emptyset \), and, therefore, \( \text{Range}(\gamma) \) coincides with the set \( \{\alpha, \emptyset\} \). Applying (iii), we find that it is not true that, for every function \( \gamma \) from \( C \) to \( N \), \( \text{Range}(\gamma) \) is a closed subset of \( N \).

(iv) Note that, for every \( \alpha \) in \( N \), there exists a continuous function from Cantor space \( C \) to \( N \) such that, for every \( \beta \) in \( C \), if \( \beta(0) = 0 \), then \( \gamma | \beta = \alpha \) and, if \( \beta(0) = 1 \), then \( \gamma | \beta = \emptyset \), and, therefore, \( \text{Range}(\gamma) \) coincides with the set \( \{\alpha, \emptyset\} \). Applying (iii), we find that it is not true that, for every function \( \gamma \) from \( C \) to \( N \), \( \text{Range}(\gamma) \) is a closed subset of \( N \).

(v) Note that, for every \( \gamma \) in \( N \), \( \gamma \) is a function from Cantor space \( C \) to \( N \) if and only if for every \( \alpha \) in \( C \), there exists \( n \) such that \( \gamma(\pi n) \neq 0 \). Using the (Restricted) Fan Theorem, Axiom 12, we conclude that, for every \( \gamma \) in \( N \), \( \gamma \) is a function from Cantor space \( C \) to \( N \) if and only if there exists \( b, n \) such that (1) for every \( i \), if \( i < \text{length}(b) \), then \( \gamma(b(i)) \neq 0 \) and (2) for every \( s \) in Bin, if \( \text{length}(s) = n \), then there exists \( i < \text{length}(b) \) such that \( b(i) \subseteq s \). Thus we see that the set \( N^C \), consisting all functions \( \gamma \) from \( C \) to \( N \), is an open subset of \( N \).

In order to prove that the set \( N^C \) is a complete element of the class \( \Sigma^0_1 \), we construct a function \( \delta \) from \( N \) to \( N \) such that, for every \( \alpha \), for every \( s \) in Bin, \( (\delta | \alpha)(s) = \alpha | \text{length}(s) \). Clearly, \( \gamma \) reduces the set \( E_1 = \{\alpha \in N | \pi n | \alpha(n) \neq 0 \} \) to the set \( N^C \). As the set \( E_1 \) is a complete element of the class \( \Sigma^0_1 \), also the set \( N^C \) is a complete element of the class \( \Sigma^0_2 \).

(vi) For every \( \gamma \) in \( N \), \( \gamma \) belongs to the set \( N^C \) if and only if, for each \( n \), \( \gamma^n \) belongs to the set \( N^C \). It follows that the set \( N^C \) is a countable intersection of open subsets of \( N \) and belongs to the class \( \Pi^0_2 \). Let \( \delta \) be a function from \( N \) to \( N \) reducing the set \( E_1 \) to the set \( N^C \). Let \( \varepsilon \) be a function from \( N \) to \( N \) such that, for each \( n \), \( (\varepsilon | \alpha)(n) = \delta | (\alpha^n) \). Clearly, \( \varepsilon \) reduces the set \( A_2 \) to the set \( N^C \), and, therefore, the set \( N^C \) is a complete element of the class \( \Pi^0_2 \).

(vii) In our proof of (iii), we saw that, for every \( \gamma \) in the set \( N^C \), \( \gamma \) is strongly injective if and only if, for each \( n \), there exists \( p \) such that \( p > n \) and, for all \( s \) in Bin, for all \( t \) in Bin, if \( \text{length}(s) = \text{length}(t) = p \) and \( \pi n \neq \pi t \), then \( \gamma | s \notin \gamma | t \). As, for each \( p \), the set of all \( s \) in Bin such that \( \text{length}(s) = p \) is a finite set, and the set \( N^C \) itself has been shown to belong to the class \( \Pi^0_2 \), we conclude that the set of all \( \gamma \) in \( N^C \) that are strongly injective also belongs to the class \( \Pi^0_2 \).

Let \( A_2 \) be the set of all \( \alpha \) in \( N \) with the property that, for each \( m \), there exists \( n \) such that \( \alpha^n(m) \neq 0 \). The set \( A_2 \) is a complete element of the class \( \Pi^0_2 \), and, therefore, a subset \( X \) of \( N \) that belongs to the class \( \Pi^0_2 \) is a complete element of the class \( \Pi^0_2 \) if and only if the set \( A_2 \) reduces to the set \( X \).
Let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ satisfying the condition that, for each $\alpha$, for each $m$, for each $s$ in $\text{Bin}$, $(\delta(\alpha))^{m}(s) = s(m) + 1$ if $\text{length}(s) > m$ and there exists $p < \text{length}(s)$ such that $\alpha^{n}(p) \neq 0$, and $(\delta(\alpha))^{m}(s) = 0$ if one of these conditions fails to hold.

Note that, for every $\alpha$ in $\mathcal{N}$, if $\alpha$ belongs to the set $A_2$, then $\delta(\alpha)$ will be a function from $\mathcal{C}$ to $\mathcal{N}$ such that, for every $\beta$ in $\mathcal{C}$, $(\delta(\alpha))\beta$ coincides with $\beta$, and, conversely, if $\delta(\alpha)$ is a strongly injective function from $\mathcal{C}$ to $\mathcal{N}$, then for every $\beta$ in $\mathcal{C}$, $(\delta(\alpha))\beta$ coincides with $\beta$, and $\alpha$ belongs to $A_2$. It follows that $\delta$ reduces the set $A_2$ to the set of all strongly injective functions from $\mathcal{C}$ to $\mathcal{N}$. $\square$

Let $X$ be a subset of $\mathcal{N}$.

We call the set $X$ strongly (positively) uncountable if and only if, for each $\alpha$, there exists $\beta$ in $X$ such that, for all $n$, $\beta \not\equiv \alpha^n$.

We call the set $X$ weakly (positively) uncountable if and only if $X$ is inhabited and, for each $\alpha$, if, for each $n$, $\alpha^n$ belongs to $X$, then there exists $\beta$ in $X$ such that, for all $n$, $\beta \not\equiv \alpha^n$.

For each $\beta$ in $\mathcal{N}$ we have introduced $F_\beta$, the closed set parametrized by $\beta$ as the set of all $\alpha$ in $\mathcal{N}$ such that, for every $n$, $\beta(\overline{\alpha^n}) = 0$.

We now introduce, for each $\beta$ in $\mathcal{N}$, the set $A_\beta$, the analytic set parametrized by $\beta$ as the set of all $\alpha$ such that, for some $\gamma$, for all $n$, $\beta(\overline{\alpha^n}, \overline{\gamma^n}) = 0$. Note that $A_\beta$ coincides with $E_{x}(F_{\beta})$, the existential projection of the closed set $F_{\beta}$.

We let $\text{UNC}$ the set of all $\beta$ in $\mathcal{N}$ such that (i) $\beta$ is a spread-law admitting at least one element of $\mathcal{N}$, that is: $\beta(\langle \rangle) = 0$ and, for each $s$, $\beta(s) = 0$ if and only, for some $n$, $\beta(s + \langle n \rangle) = 0$, and (ii) the set $F_{\beta}$, that is, the set consisting of all $\alpha$ in $\mathcal{N}$ satisfying, for all $n$, $\beta(\overline{\alpha^n}) = 0$, is weakly positively uncountable.

We let $\text{UNC}'$ the set of all $\beta$ in $\mathcal{N}$ such that the (closed) set $F_{\beta}$, that is, the set consisting of all $\alpha$ in $\mathcal{N}$ satisfying, for all $n$, $\beta(\overline{\alpha^n}) = 0$ is strongly positively uncountable.

We let $\text{UNC}''$ the set of all $\beta$ in $\mathcal{N}$ such that the (analytic) set $A_\beta$, that is, the set consisting of all $\alpha$ in $\mathcal{N}$ with the property that, for some $\gamma$ in $\mathcal{N}$, for all $n$, $\beta(\overline{\alpha^n}, \overline{\gamma^n}) = 0$, is strongly positively uncountable.

The classical result corresponding to the fourth item of the following theorem is due to W. Hurewicz, see [15], Theorem 27.5. The proof of this theorem in [15] is not constructive.

Theorem 2.4.

(i) For every subset $X$ of $\mathcal{N}$, if $X$ is an inhabited spread, and $X$ is weakly positively uncountable, then $X$ is strongly positively uncountable.

(ii) For every subset $X$ of $\mathcal{N}$, if $X$ is strongly positively uncountable, then there exists a strongly injective function from $\mathcal{C}$ to $X$.

(iii) For every subset $X$ of $\mathcal{N}$, if $X$ has a subset that is a perfect spread, then $X$ is strongly positively uncountable.

(iv) The sets $\text{UNC}$, $\text{UNC}'$ and $\text{UNC}''$ are complete analytic sets.

Proof. (i) Let $X$ be a subset of $\mathcal{N}$ that is an inhabited spread and let $\gamma$ be (the code of) a continuous function from $\mathcal{N}$ to $\mathcal{N}$ with the property that, for all $\alpha$, for all $n$, if $\overline{\alpha^n}$ does contain an element of $X$, then $\overline{\gamma(\alpha)}n = \overline{\alpha^n}$, and, if $\overline{\alpha^n}$ does not contain an element of $X$, then $\gamma(\alpha)(n)$ is the least $k$ such that $\overline{\gamma(\alpha)(n)}k$ does contain an element of $X$. Note that $\gamma$ retracts $\mathcal{N}$ onto $X$, that is, for every $\alpha$ in $\mathcal{N}$, $\gamma(\alpha)$ belongs to $X$, and, for every $\alpha$ in $X$, $\gamma(\alpha) = \alpha$. Suppose that $X$ is weakly positively uncountable. Now let $\alpha$ belong to $\mathcal{N}$ and consider the sequence $\beta$ satisfying, for each $n$, $\beta^n = \gamma(\alpha^n)$. Note that, for each $n$, $\gamma(\alpha^n)$ belongs to $X$ and find $\delta$ in $X$ such that, for each $n$, $\delta \not\equiv \beta^n$. Note
that, for each \( n \), either \( \delta \neq \alpha^n \) or \( \gamma(\langle \alpha^n \rangle) = \beta^n \neq \alpha^n \). In the latter case, there exists \( m \) such that \( \alpha^{-m} \) does not contain an element of \( X \), and therefore, as \( \delta \) does contain an element of \( X \), also \( \delta \neq \alpha^n \). Thus we see that, given any \( \alpha \), we may determine \( \delta \) in \( X \) such that, for each \( n \), \( \delta \neq \alpha^n \), that is, \( X \) is strongly positively uncountable.

(ii) Let \( X \) be a strongly positively uncountable subset of \( \mathcal{N} \). Using the Second Axiom of Continuous Choice we find \( \gamma \) in \( \mathcal{N} \) such that \( \gamma \) is the code of a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) with the property: for all \( \alpha \) in \( \mathcal{N} \), \( \gamma(\alpha) \) belongs to \( X \) and, for all \( n \), \( \gamma(\alpha) \neq \alpha^n \). We now claim the following:

For all \( s \) in \( \mathbb{N} \) there exist \( t \), \( u \) in \( \mathbb{N} \) such that \( s \subseteq t \) and \( s \subseteq u \) and 
\[
(\gamma | s) \not\subseteq (\gamma | t).
\]

We prove this claim as follows. Let \( \delta \) be the infinite sequence \( s \ast 0 \) and let \( \varepsilon \) be an infinite sequence passing through \( s \) such that \( \varepsilon^k \) coincides with \( \gamma | \delta \), where \( k = \text{length}(s) \), and, for each \( n \), if \( n > \text{length}(s) \) and there is no \( i \) such that \( n = J(k, i) \), then \( \varepsilon(n) = 0 \). Note that \( \gamma | \varepsilon \) is apart from \( \varepsilon^k = \gamma | \delta \). Find \( n \) such that \( (\gamma | \delta)n \neq (\gamma | \varepsilon)n \) and then find \( m \) such that \( \delta m \subseteq \gamma(\delta m) \) and \( \tau m \subseteq \gamma(\tau m) \).
Now define: \( t := \delta m \) and \( u := \tau m \).

Recall that \( \text{Bin} \) be the set of all natural numbers \( s \) coding a finite binary sequence, that is, with the property: for each \( i \), \( (s(i) = 0 \text{ or } s(i) = 1) \).

We now define a mapping \( F \) from the set \( \text{Bin} \) to the set \( \mathbb{N} \), as follows:

(i) \( F(\langle \rangle) = \langle \rangle \).

(ii) Let \( s \) belong to \( \text{Bin} \) and assume \( F(s) \) has been defined already. We search for the least natural number \( u \) such that \( F(s) \subseteq u(0) \) and \( F(s) \subseteq u(1) \) and 
\[
(\gamma|u(1)) \not\subseteq (\gamma|u(0)).
\]

We let \( \varepsilon \) be a function from \( \mathcal{C} \) to \( \mathcal{N} \) such that, for each \( \alpha \) in \( \mathcal{C} \), for each \( n \), \( \delta | \alpha \) passes through \( F(\tau n) \). We let \( \varepsilon \) be a function from \( \mathcal{C} \) to \( \mathcal{N} \) such that, for each \( \alpha \) in \( \mathcal{C} \), for each \( \beta \) in \( \mathcal{C} \), if \( \alpha \neq \beta \), then, for some \( n \), \( \tau n \neq \tau m \), and, therefore, \( F(\tau n) \not\subseteq F(\tau m) \), and, therefore, \( \gamma | \langle F(\tau n) \rangle \not\subseteq \gamma | \langle F(\tau m) \rangle \) and, therefore, \( \varepsilon(\alpha) \neq \varepsilon(\beta) \). Clearly, \( \varepsilon \) is a strongly injective function from \( \mathcal{C} \) to \( X \).

(iii) Assume that \( X \) has a subset \( Y \) that is a perfect spread. Find \( \gamma \) in \( \mathcal{N} \) such that, for each \( s \), \( \gamma(s) = 0 \) if and only if \( s \) contains an element of \( Y \). Let \( \alpha \) belong to \( \mathcal{N} \). We have to find an element \( \beta \) of \( X \) such that, for each \( n \), \( \beta \neq \alpha^n \). To this end, we define an element \( \delta \) of \( \mathcal{N} \), as follows. We let \( \delta(0) \) be \( u(0) \) where \( u \) is the least natural number \( u \) such that \( u(0) \not\subseteq u(1) \) and \( \gamma(u(0)) = \gamma(u(1)) = 0 \) and \( \alpha^0 \) does not pass through \( u(0) \). For each \( n \), we let \( \delta(n + 1) \) be \( u(0) \) where \( u \) is the least natural number \( u \) such that \( \delta(n) \subseteq u(0) \) and \( \delta(n) \not\subseteq u(1) \) and \( u(0) \not\subseteq u(1) \) and \( \gamma(u(0)) = \gamma(u(1)) = 0 \) and \( \alpha^{n+1} \) does not pass through \( u(0) \). Find \( \beta \) such that, for each \( n \), \( \beta \) passes through \( \delta(n) \) and observe that, for each \( n \), \( \beta \neq \alpha^n \) and \( \gamma(\beta n) = 0 \). It follows that \( \beta \) belongs to \( Y \) and, therefore, to \( X \). Clearly, \( X \) is strongly positively uncountable.

(iv) Using Lemma 2.3, we conclude from (ii) and (iii) that the set \( \text{UNC} \) coincides with the set of all \( \beta \) such that, \( \beta(\langle \rangle) = 0 \) and \( \forall s(\beta(s) = 0 \iff \exists n(\beta(s * \langle n \rangle) = 0)) \) and \( \exists n(\gamma(n) = 0 \rightarrow \beta(n) = 0) \land \forall s(\gamma(s) = 0 \rightarrow \exists t \exists u[s \subseteq t \land s \subseteq u \land t \cup u \subseteq \gamma(t) = \gamma(u) = 0]) \). The set \( \text{UNC} \) thus is seen to be an analytic subset of \( \mathcal{N} \).

We now want to prove that the set \( E_1^1 \) reduces to the set \( \text{UNC} \). It then follows that \( \text{UNC} \), like \( E_1^1 \) itself, is a complete analytic set.

We have to define a function \( \gamma \) from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( E_1^1 \) to the set \( \text{UNC} \). Our idea is the following. Suppose \( \alpha \) belongs to \( E_1^1 \). Find \( \beta \) such that \( \alpha \) admits \( \beta \), that is, for all \( n \), \( \alpha(\beta n) = 0 \). We intend to make sure that \( \gamma | \alpha \) admits every infinite sequence \( \delta \) with the property that, for each \( n \), either \( \delta(n) = 2\beta(n) + 1 \) or \( \delta(n) = 2\beta(n) + 2 \).

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In order to achieve this goal, we let \( \gamma \) be (the code of) a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \alpha \), for each \( s \), \( (\gamma|\alpha)(s) = 0 \) if and only if, for all \( i < \text{length}(s) \), either \( s(i) = 0 \) or \( s(i) = 2t \) or \( s(i) = 2t + 1 \) where \( t \) is a natural number satisfying the following conditions: \( \text{length}(t) = i + 1 \) and for all \( j < i \), \( \alpha(t(j + 1)) = 0 \) and: either \( s(j) = 2\gamma(t(j + 1) + 1) \) or \( s(j) = 2\gamma(t(j + 1) + 1) + 2 \).

Note that, for each \( \alpha \), for each \( s \), \( (\gamma|\alpha)(s) = 0 \) if and only if \( (\gamma|\alpha)(s \ast (0)) = 0 \).

Also note that, for each \( \alpha \), for each \( s \), for each \( n \), if \( (\gamma|\alpha)(s) = 0 \) and \( n < \text{length}(s) - 1 \) and \( s(n) = 0 \), then \( s(n + 1) = 0 \).

We claim that, for each \( \alpha \), \( \alpha \) belongs to \( E_1^\gamma \) if and only if \( \gamma|\alpha \) belongs to \( \text{UNC} \).

In order to prove this claim, let us first assume that \( \alpha \) belongs to \( E_1^\gamma \). Find \( \beta \) such that \( \alpha \) admits \( \beta \). We define \( \varepsilon \) in \( \mathcal{N} \) such that, for each \( s \), \( \varepsilon(s) = 0 \) if and only if, for each \( n \), if \( n < \text{length}(s) \), then either \( s(n) = \beta(n + 1) + 1 \), or \( s(n) = \beta(n + 1) + 2 \). Note that \( F_\gamma \) is a perfect spread and a subset of the spread \( F_{\gamma|\alpha} \), and thus, by (iii), the set \( F_{\gamma|\alpha} \) is strongly positively uncountable, and, therefore, \( \gamma|\alpha \) belongs to \( \text{UNC} \).

Let us now assume that the spread \( F_{\gamma|\alpha} \) is strongly positively uncountable. Using (ii), we find \( \delta \) such that \( F_\delta \) is a perfect spread and a subset of \( F_{\gamma|\alpha} \). Note that, for each \( s \), for each \( n \), if \( (\gamma|\alpha)(s) = 0 \) and \( n < \text{length}(s) \) and \( s(n) = 0 \), then, for all \( t \), if \( (\gamma|\alpha)(t) = 0 \) and \( s \subset t \), then there exists \( k \) such that \( t = s \ast (\beta^k) \). It follows that, for each \( s \), for each \( n \), if \( \delta(s) = 0 \) and \( n < \text{length}(s) \), then \( s(n) \neq 0 \). Now find \( \beta \) such that, for each \( n \), \( \beta(n) \) is the least \( k \) such that \( \delta(s \ast (k)) = 0 \). Let \( \varepsilon \) be an element of \( \mathcal{N} \) such that, for each \( n \), either \( \beta(n) = 2\varepsilon(n) + 1 \) or \( \beta(n) = 2\varepsilon(n) + 2 \). Note that, for each \( n \), \( \alpha(\varepsilon(n)) = 0 \) and \( \text{length}(\varepsilon(n)) = n + 1 \) and \( \varepsilon(n) \leq \varepsilon(n + 1) \). Find \( \eta \) such that, for each \( n \), \( \eta \) passes through \( \beta(n) \), and note that, for each \( n \), \( \alpha(\eta n) = 0 \), so \( \alpha \) belongs to \( E_1^\gamma \).

By (ii) and (iii), the set \( \text{UNC}^\gamma \) coincides with the set of all \( \beta \) such that \( \exists \gamma \forall s [\gamma|s = 0 \rightarrow \forall t [t \subseteq s \rightarrow \beta(t) = 0]] \land \forall s [\gamma|s = 0 \rightarrow \exists u \forall [s \subseteq t \land t \subseteq u \land u \land \gamma(t) = \gamma(u) = 0]] \) and thus is seen to be an analytic subset of \( \mathcal{N} \). Note the function \( \gamma \) we just defined in order to reduce the set \( E_1^\gamma \) to the set \( \text{UNC} \), also reduces the set \( E_1^\gamma \) to the set \( \text{UNC}^\gamma \).

We may conclude that the set \( \text{UNC}^\gamma \) is a complete analytic set.

Again, by (ii) and (iii), and the Second Axiom of Countable Choice, the set \( \text{UNC}^\eta \) coincides with the set of all \( \beta \) such that \( \exists \gamma \exists \delta \forall n [\gamma|s = 0 \rightarrow (\text{length}(s) = \text{length}(\delta(s)) \land \forall i \leq \text{length}(s) [\beta(\delta(s)(i))] = 0]] \land \forall s \forall i [\delta(s) \subseteq \delta(s \ast (i))] \land \forall s [\gamma|s = 0 \rightarrow \exists u \forall [s \subseteq t \land s \subseteq u \land t \subseteq u \land \gamma(t) = \gamma(u) = 0]] \) and thus is seen to be an analytic subset of \( \mathcal{N} \).

In order to see that also \( \text{UNC}^\eta \) is a complete analytic set, we remind ourselves of the fact that every closed subset of \( \mathcal{N} \) is a spread-law and \( \mathcal{N} \). Let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for all \( \beta \), for all \( s, t \), if \( \text{length}(s) = \text{length}(t) \), then \( (\delta|\beta)(s \ast t) = 0 \) if and only if \( \beta(s) = 0 \). Note that, for every \( \beta \), for every \( \alpha \), for every \( \gamma \), for all \( n \), \( \delta(\beta)(\gamma n) = 0 \) if and only if \( \beta(\gamma n) = 0 \). It follows that, for every \( \beta \), for every \( \alpha \), \( \alpha \) belongs to the analytic \( A_{\beta|\gamma} \) if and only if \( \alpha \) belongs to the closed set \( F_\beta \), that is, \( F_\beta \) coincides with \( A_{\beta|\gamma} \).

It now easily follows that, for every \( \beta \), \( \beta \) belongs to \( \text{UNC}^\gamma \) if and only if \( \delta|\beta \) belongs to \( \text{UNC}^\gamma \), that is, \( \delta \) reduces the set \( \text{UNC}^\gamma \) to the set \( \text{UNC}^\gamma \). It follows that \( \text{UNC}^\gamma \), like \( \text{UNC}^\gamma \), is a complete analytic set.

2.4. The set of all \( \beta \) admitting an element of Inf.

Let \( X \) be a subset of \( \mathcal{N} \).

As in [30], we let \( \text{Share}(X) \) be the set of all \( \beta \) in \( \mathcal{N} \) such that the closed set \( F_\beta \) contains an element of \( X \), that is, shares a member with \( X \).

We let \( \text{Share}^\gamma(X) \) be the set of all \( \beta \) in \( \mathcal{N} \) such that \( \beta \) is a spread-law and \( \beta \) belongs to \( \text{Share}(X) \).

We let \( \text{Share}^\eta(X) \) be the set of all \( \beta \) in \( \mathcal{N} \) such that \( \beta \) is a spread-law defining a subfan of Cantor space \( \mathcal{C} \) and \( \beta \) belongs to \( \text{Share}(X) \). Note that \( \beta \) is a spread-law.
defining a subfan of Cantor space $C$ if and only if, for each $s$, $E(s) = 0$ if and only if $\beta(s(\emptyset)) = 0$ or $\beta(s(\{i\})) = 0$ and, for all $i > 1$, $\beta(s(\{i\})) \neq 0$.

We let $\text{Inf}$ be the set of all $\alpha$ in Cantor space $C$ such that, for each $m$, there exists $n$ such that $n > m$ and $\alpha(n) = 1$. Note that, for all $\alpha$ in $C$, $\alpha$ belongs to $\text{Inf}$ if and only if $\alpha$ is the characteristic function of a decidable and positively infinite subset of $\mathbb{N}$.

The next result corresponds to a well-known fact in classical descriptive set theory, see [15], page 209, Exercise 27, or [28], page 137, Exercise 4.2.3.

**Theorem 2.5.** The sets $\text{Share}(\text{Inf})$, $\text{Share}^*(\text{Inf})$ and $\text{Share}^*_0(\text{Inf})$, are, each of them, complete analytic sets.

**Proof.** The proof that the sets $\text{Share}(\text{Inf})$, $\text{Share}^*(\text{Inf})$ and $\text{Share}^*_0(\text{Inf})$ are analytic subsets of $N$ is easy and left to the reader.

We now show that the set $E_1$ reduces to the set $\text{Share}(\text{Inf})$. It then follows from Theorem 2.2 that the latter set is a complete analytic set.

We let $\delta$ be an element of $N$ such that $\delta(\{i\}) = (\gamma)$ and, for each $s$, for each $n$, $\delta(s(n)\langle i \rangle) = \delta(s)(n)\langle i \rangle \cdot (1)$.

We now let $\gamma$ be a function from $N$ to $N$ such that, for each $\alpha$, for each $s$, $(\gamma)(\alpha)(s) = 0$ if and only if there exist $t, n$ such that $s = \delta(t) \ast \overline{m}$ and $\alpha(t) = 0$. We claim that the function $\gamma$ reduces the set $E_1$ to the set $\text{Share}(\text{Inf})$, and we prove this claim as follows.

First, let $\alpha$ belong to $E_1$. Find $\beta$ such that, for all $n$, $\alpha(\overline{m}n) = 0$. Note that, for each $n$, for each $t$, if $t \subseteq \delta(\overline{m})$, then $(\gamma)(\alpha)(t) = 0$. Let $\varepsilon$ be the element of $N$ such that, for each $n$, $\varepsilon(n + 1)$ is the least $i$ such that $\varepsilon(\beta(n) + i + 1) = 1$. Note that, for each $n$, $\beta$ passes through $\delta(\overline{m}n)$. It follows that, for each $n$, $\alpha(\overline{m}n) = 0$, that is, $\alpha$ belongs to $E_1$.

This ends the proof of our claim.

Note that the function $\gamma$ also reduces the set $E_1$ to the set $\text{Share}^*(\text{Inf})$ and to the set $\text{Share}^*_0(\text{Inf})$. It follows that all these sets are complete analytic sets.

2.5. **The Suslin operation.** We want to conclude this Section with an observation concerning the famous Suslin operation.

A *Suslin system* on $N$ is a mapping $s \mapsto P_s$ that associates to very $s$ in $N$ a subset $P_s$ of $N$.

The **Suslin operation** applied to such a system produces the set

\[
A_sP_s = \bigcup_{\alpha \in N} \bigcap_{n \in N} P_s\overline{m}n.
\]

The following theorem shows that, if one applies the Suslin operation to a Suslin system consisting of analytic sets, the result is again an analytic set.

Recall that, given any $\beta$ in $N$, we let $A_\beta$ be the (analytic) subset of $N$ consisting of all $\alpha$ such that, for some $\gamma$, for every $n$, $\beta(\overline{m}n, \overline{m}n) = 0$.

**Theorem 2.6.** For each $\beta$ in $N$, the set $A_\alpha A_\beta$ is an analytic subset of $N$.

**Proof.** Note that, for each $\beta$, for each $\alpha$, $\alpha \in A_\alpha A_\beta$, if and only if $\exists \gamma \forall n[\alpha \in A_\gamma \overline{m}n]$ and only if $\exists \gamma \forall n \exists \delta [\alpha, \delta] \in F_\beta \overline{m}n$ if and only if $\exists \gamma \forall n [\langle \alpha, \gamma \rangle \in F_\beta \overline{m}n]$.

□
3. Some analytic subsets of $\mathcal{N}$ that fail to be positively Borel

3.1. Introduction. In classical descriptive set theory, one may prove, if one assumes that, for every subset $X$ of $\mathcal{N}$ in the Boolean algebra generated by the analytic subsets of $\mathcal{N}$, the usual game for two players $I, II$ in $\mathcal{N}$ with $X$ as the payoff set is determined in the sense that either player $I$ or player $II$ has a winning strategy: every analytic subset of $\mathcal{N}$ that is not a Borel subset of $\mathcal{N}$ is a complete element of the class $\Sigma^1_1$, see [15]. Theorem 26.4. In intuitionistic mathematics, however, the set $\alpha$ of all $\mathcal{N}$ of $\mathcal{N}$ with the property that, for some $\gamma$ in $\mathcal{N}$, for each $n \gamma(n) \leq \gamma(n+1) \leq 1$ and, for each $n$, $\alpha(\bar{n}) = 0$, is an example of a “simple” analytic set that is not a positively Borel subset of $\mathcal{N}$, see Theorem [12].

We now intend to fulfill a promise made in [41], Subsection 2.26, and partially fulfilled already in [32], Sections 11.21-26: we explain there exist analytic subsets of $\mathcal{N}$ that fail to be positively Borel belonging to many different degrees of reducibility.

3.2. The set $\text{Share}(A_1)$ is disjunctively productive. Let $\alpha, \beta$ belong to $\mathcal{N}$. We define: $\beta$ admits $\alpha$ if and only if $\alpha$ belongs to the closed set $F_\beta$, that is, for all $n$, $\beta(\bar{n}) = 0$. The set $F_\beta$ is the set of all $\alpha$ in $\mathcal{N}$ that are admitted by $\beta$.

Let $X$ be a subset of $\mathcal{N}$. As in the previous Section, we let $\text{Share}(X)$ be the set of all $\beta$ in $\mathcal{N}$ such that the closed set $F_\beta$ contains an element of $X$, that is, shares a member with $X$.

Note that the set $E^1_1$ coincides with the the set $\text{Share}(\mathcal{N})$.

Note that the $\text{MonPath}_{01}$ coincides with the set $\text{Share}(\mathcal{S}_{01mon})$ where $\mathcal{S}_{01mon}$ is the set of all $\gamma$ in $\mathcal{N}$ satisfying the condition: $\forall n(\gamma(n) \leq \gamma(n+1) \leq 1)$.

For every $n$, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let the (disjoint) sum of the finite sequence $(X_0, X_1, \ldots, X_{n-1})$, notation: $\bigoplus_{i<n} X_i$, be the set $\bigcup_{i<n} \langle i \rangle \star X_i$.

For every $n$, for every subset $X$ of $\mathcal{N}$, we let the $n$-fold multiple of the set $X$, notation: $n \cdot X$, be the set $\bigcup_{i<n} \langle i \rangle \star X$.

For every $n$, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let the disjunction of the finite sequence $(X_0, X_1, \ldots, X_{n-1})$, notation: $D^{\oplus}_{i<n}(X_i)$ or: $D_{i<n}(X_i)$, be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $i < n$, $\alpha^i$ belongs to $X_i$.

For every $n$, for every subset $X$ of $\mathcal{N}$, we let the $n$-fold disjunction of the set $X$, notation: $D^n(X)$, be the set of all $\alpha$ such that, for some $i < n$, $\alpha^i$ belongs to $X$.

For every $n$, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, we let the conjunction of the finite sequence $(X_0, X_1, \ldots, X_{n-1})$, notation: $C^{\cap}_{i<n}(X_i)$ or: $C_{i<n}(X_i)$, be the set of all $\alpha$ in $\mathcal{N}$ such that, for all $i < n$, $\alpha^i$ belongs to $X_i$.

For every $n$, for every subset $X$ of $\mathcal{N}$, we let the $n$-fold conjunction of the set $X$, notation: $C^n(X)$, be the set of all $\alpha$ such that, for all $i < n$, $\alpha^i$ belongs to $X$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $(\gamma|\alpha)^0 = (\gamma|\alpha)^1 = \alpha^0$ and, for each $n > 0$, $(\gamma|\alpha)^n = \alpha^{n-1}$. Observe that, for each subset $X$ of $\mathcal{N}$, for each $n > 0$, the function $\gamma$ reduces the set $D^n(X)$ to the set $D^{n+1}(X)$ and the set $C^n(X)$ to the set $C^{n+1}(X)$.

For every subset $X$ of $\mathcal{N}$ such that there exists $\alpha$ not belonging to $X$, the set $\emptyset = D^0(X)$ reduces to the set $X$ and to $D^1(X)$. For every subset $X$ of $\mathcal{N}$ such that there exists $\alpha$ in $X$, the set $\mathcal{N} = C^0(X)$ reduces to the set $X$ and to $C^1(X)$.

A subset $X$ of $\mathcal{N}$ satisfying the condition: for all $n$, $D^{n+1}(X)$ does not reduce to the set $D^n(X)$, is called disjunctively productive, see [32], Section 12.7, and [35]. Subsection 6.3.3.
A subset $X$ of $\mathcal{N}$ satisfying the condition: for all $n$, $C^{n+1}(X)$ does not reduce to the set $C^n(X)$, is called conjunctively productive, see [32], Section 12.7, and [45], Theorem 6.5(ii).

We let $A_1$ be the subset of $\mathcal{N}$ such that, for all $\alpha$, $\alpha$ belongs to $A_1$ if and only if, for all $n$, $\alpha(n) = 0$. The set $A_1$ has the infinite sequence $\langle 0 \rangle$ as its one and only member. A subset $X$ of $\mathcal{N}$ is closed if and only if $X$ reduces to $A_1$.

A subset $X$ of $\mathcal{N}$ reduces to the set $D^2(A_1)$ if and only if there exist closed subsets $Y, Z$ of $\mathcal{N}$ such that $X = Y \cup Z$.

The set $A_1$ is disjunctively productive and the set $D^2(A_1)$ is both disjunctively and conjunctively productive as has been shown in [32] and [45].

For each $\alpha$, for each $m$, for each $n$, we let $c(\alpha, m, n)$ be the number of elements of the set of all $j < m$ such that, for all $i < n$, $\alpha^j(i) = 0$.

This notion will play a role in the proof of the sixth item of next theorem, and also in the statement and proofs of Theorem 3.4 and 3.5.

The seventh item of the next lemma states a result that is easily concluded from the fifth item and the well-known and just-mentioned fact that the set $A_1$ is disjunctively productive, see [32] and [45], but it seems useful, in view of further developments, to prove it in the way it is proven here.

**Lemma 3.1.**

(i) For every subset $X$ of $\mathcal{N}$, the set $X$ reduces to the set $\text{Share}(X)$.

(ii) For every subset $X$ of $\mathcal{N}$, if $X$ is a closed subset of $\mathcal{N}$, then $\text{Share}(X)$ is an analytic subset of $\mathcal{N}$.

(iii) For every $n$, for every finite sequence $(X_0, X_1, \ldots, X_{n-1})$ of subsets of $\mathcal{N}$, the set $D_{1 < \alpha}(\text{Share}(X_i))$ coincides with the set $\text{Share}\left( \bigoplus_{i < n} X_i \right)$.

(iv) For every $n$, for every subset $X$ of $\mathcal{N}$, the set $D^n(\text{Share}(X))$ coincides with the set $\text{Share}(n \cdot X)$.

(v) The set $\text{Share}(A_1)$ reduces to the set $A_1$ and the set $A_1$ reduces to the set $\text{Share}(A_1)$.

(vi) For each $m$, for each $\gamma$, if $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^m(\text{Share}(A_1))$ to itself, then, for each $k < m$, for each $\alpha$:

- if, for each $n$, $c(\alpha, m, n) > k$, then, for each $n$, $c(\gamma|\alpha, m, n) > 0$.

(vii) The set $\text{Share}(A_1)$ is disjunctively productive.

(viii) The set $\text{Share}(\{\emptyset, 1\})$ does not reduce to the set $\{\emptyset, 1\}$.

**Proof.** (i) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $s$, $(\gamma|\alpha)(s) = 0$ if and only if $\alpha$ passes through $s$. Note that, for each $\alpha$, the set $F_{\gamma|\alpha}$ coincides with the set $\{\alpha\}$. Clearly, for each subset $X$ of $\mathcal{N}$, the function $\gamma$ reduces the set $X$ to the set $\text{Share}(X)$.

(ii) Find $\beta$ such that $X$ with $F_{\beta}$. Note that $\text{Share}(X)$ coincides with the set of all $\alpha$ such that, for some $\gamma$ in $\mathcal{N}$, for all $n$, $\beta(\gamma|n) = \alpha(\gamma|n) = 0$. Clearly, $\text{Share}(X)$ is an analytic subset of $\mathcal{N}$.

(iii) Let $n$ belong to $\mathbb{N}$ and assume $X_0, X_1, \ldots, X_{n-1}$ are subsets of $\mathcal{N}$. Note that, for each $\alpha$, $\alpha$ belongs to $D_{1 < \alpha}(\text{Share}(X_i))$ if and only if, for some $i < n$, $\alpha^i$ belongs to $\text{Share}(X_i)$ if and only if, for some $i < n$, for some $\beta$ in $X_i$, for each $n$, $\alpha^i(\beta|n) = 0$ if and only if, for some $i < n$, for some $\beta$ in $X_i$, for each $n$, $\alpha^i(i \cdot \beta|n) = 0$ if and only if $\alpha$ belongs to $\text{Share}(\bigoplus_{i < n} X_i)$.

(iv) This is an easy consequence of (iii).

(v) Clearly, the set $\text{Share}(A_1)$ is a closed subset of $\mathcal{N}$ and, therefore, the set $\text{Share}(A_1)$ reduces to the set $A_1$.

According to (i), the set $A_1$ also reduces to the set $\text{Share}(A_1)$.
(vi) Let \( m \) be a natural number and suppose \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( D^m(\text{Share}(A_1)) \) to the set \( D^m(\text{Share}(A_1)) \) itself. We claim that, for all \( \alpha \), for all \( k < m \),

if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > k \).

We establish this claim by induction and first consider the case \( k = 0 \). Suppose: for each \( n \), \( c(\alpha, m, n) > 0 \) and we find \( p \) such that \( c(\gamma|\alpha, m, p) = 0 \). Using the continuity of the function \( \gamma \), we determine \( k \) such that, for every \( \beta \) passing through \( \mathbb{T}k \), for every \( i < m \), for every \( q \leq p \), \( (\gamma|\beta)^i(\mathbb{T}q_\gamma) = (\gamma|\alpha)^i(\mathbb{T}q) \), and, therefore, \( c(\gamma|\beta, m, p) = 0 \). Find \( i < m \) such that, for all \( j \leq k \), \( \alpha^i(\mathbb{T}j) = 0 \). Observe that \((\mathbb{T}k * \mathbb{0})^i\) admits \( \mathbb{0} \). It follows that that \( \mathbb{T}k * \mathbb{0} \) belongs to the set \( D^m(\text{Share}(A_1)) \), and \( \gamma|\mathbb{T}k * \mathbb{0} \) does not, so we have a contradiction.

Now assume \( k + 1 < m \) and we have seen: for all \( \alpha \), that, if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > k \). Suppose: for each \( n \), \( c(\alpha, m, n) > k + 1 \) and we find \( p \) such that \( c(\gamma|\alpha, m, p) = k + 1 \). Using the continuity of the function \( \gamma \), we determine \( l \) such that, for every \( \beta \) passing through \( \mathbb{T}l \), for every \( i < m \), for every \( q \leq p \), \( (\gamma|\beta)^i(\mathbb{T}q_\gamma) = (\gamma|\alpha)^i(\mathbb{T}q) \), and, therefore, \( c(\gamma|\beta, m, p) = k + 1 \). Without loss of generality, one may assume that, for all \( i \leq k + 1 \), for all \( q \leq l \), \( \alpha^i(\mathbb{T}q) = 0 \), and, for all \( i \), if \( k + 1 \leq i < m \), then there exists \( q \leq p \) such that \( (\gamma|\alpha)^i(\mathbb{T}q) \neq 0 \). We now let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \beta \),

1. \( (\delta|\beta)^i = \mathbb{T}i \)
2. for every \( i \), \( (\delta|\beta)^i(\mathbb{T}q) = \alpha^i(\mathbb{T}q) = 0 \), and, for each \( n \), \( \beta|\alpha^i(\mathbb{T}q) = 0 \), and, for each \( n \), \( (\delta|\beta)^i(\mathbb{T}q) = 1 \).

Note that, for every \( \beta \), \( \delta|\beta \) belongs to \( D^m(\text{Share}(A_1)) \) if and only if there exists \( i < k + 1 \) such that \( (\delta|\beta)^i \) admits \( \mathbb{0} \) if and only if there exists \( i < k + 1 \) such that \( \beta^i = \mathbb{0} \).

Now let \( T \) be the set of all \( \beta \) in \( S \) such that, for all \( m, n \), if \( (\beta(m), \beta(n)) \neq 0 \) and \( \beta(n) \neq 0 \), then \( m = n \). The set \( T \) thus consists of all elements \( \beta \) of Cantor space \( C \) that assume the value \( 1 \) at most one time. Note that the set \( T \) is a spread.

Note that, for every \( \beta \) in \( T \), for every \( n \), the number of elements of \( \{ i \leq k + 1 | \mathbb{T}n = \mathbb{T}i \} \) is at least \( k + 1 \). It follows that, for every \( \beta \) in \( T \), for every \( n \), \( c(\beta, m, n) \geq k + 1 \). The induction hypothesis now enables us to conclude: for every \( \beta \) in \( T \), for every \( n \), \( c(\gamma|\beta|\beta, m, n) \geq k + 1 \), and, therefore, for all \( i \leq k + 1 \), \( \gamma|\beta)^i \) admits \( \mathbb{0} \), and, therefore, \( \gamma|\beta \) belongs to \( D^m(\text{Share}(A_1)) \), and, therefore, \( \beta \) belongs to \( D^m(\text{Share}(A_1)) \), and, therefore, there exists \( i \leq k + 1 \) such that \( \beta^i = \mathbb{0} \).

Applying Brouwer’s Continuity Principle, we find \( q, i \) such that, for every \( \beta \) in \( T \), if \( \beta \) passes through \( \mathbb{T}q \), \( \beta^i = \mathbb{0} \). This is false, of course, as there exists \( \beta \) in \( T \) passing through \( \mathbb{T}q \) such that, for some \( r \), \( \beta^i(r) = 1 \).

We must conclude: for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) \geq k + 1 \), then, for each \( n \), \( c(\gamma|\alpha, m, n) \geq k + 1 \).

This establishes our claim.

(vii) Let \( m \) be a natural number and let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( D^{m+1}(\text{Share}(A_1)) \) to the set \( D^{m+1}(\text{Share}(A_1)) \).

We let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \alpha \), for each \( i \), if \( i < m \), then \( (\delta|\alpha)^i = (\gamma|\alpha)^i \), and \( (\delta|\alpha)^m = \frac{1}{i} \). Note that the function \( \delta \) reduces the set \( D^{m+1}(\text{Share}(S^i_{\text{EN}}(X_1))) \) to the set \( D^{m}(\text{Share}(S^i_{\text{EN}}(X_1))) \) as well as the function \( \gamma \). The function \( \delta \), however, also reduces the set \( D^{m+1}(\text{Share}(A_1)) \) to the set \( D^{m+1}(\text{Share}(A_1)) \) itself. On the other hand, for each \( n \), \( c(\mathbb{0}, m + 1, n) = m + 1 \), and, for each \( n \), \( c(\delta|\mathbb{0}, m, n) \leq m \). This contradicts the conclusion of (v).

We must conclude that, for each \( m \), the set \( D^{m+1}(\text{Share}(A_1)) \) does not reduce to the set \( D^{m}(\text{Share}(A_1)) \), that is, the set \( \text{Share}(A_1) \) is disjunctively productive.
Proof. (i) Let the set Share be a function from $N$ to $N$ such that, for each $\alpha$, for each $n$, $(\gamma|\alpha)(\overline{\mu}1) = 0$ if and only if $\overline{\alpha}n = \overline{\mu}n$ and $(\gamma|\alpha)\overline{\mu}n = 0$ if and only if $\overline{\alpha}n = \overline{\mu}n$. Note that the function $\gamma$ reduces the set $D^\gamma(A_1)$ to the set $\text{Share}(\{0, 1\})$. It follows that the set $\text{Share}(\{0, 1\})$, like the set $D^\gamma(A_1)$ itself, is not closed and does not reduce to the closed set $\{0, 1\}$. □

3.3. Preserving the property of disjunctive productivity. Let $n$ be a natural number and let $X_0, X_1, \ldots, X_{n-1}$ be subsets of $N$. We let $S_{i<n}(X_i)$ be the set $\bigcup_{i<n} \overline{\mu}i \ast X_i$.

Lemma 3.2.

(i) For every $n$, for all subsets $X_0, X_1, \ldots, X_{n-1}$ of $N$, the set $\text{Share}(S_{i<n}(X_i))$ reduces to the set $\text{Share}\bigl(\bigoplus_{i<n} X_i\bigr)$ and

the set $\text{Share}\bigl(\bigoplus_{i<n} X_i\bigr)$ reduces to the set $\text{Share}(S_{i<n}(X_i))$.

(ii) For every $n$, for all subsets $X_0, X_1, \ldots, X_{n-1}$ of $N$, the set $\text{Share}(S_{i<n}(X_i))$ reduces to the set $D_{i<n}(\text{Share}(X_i))$ and the set $D_{i<n}(\text{Share}(X_i))$ reduces to the set $\text{Share}(S_{i<n}(X_i))$.

Proof. (i) Let $\gamma$ be a function from $N$ to $N$ such that $(\gamma|\alpha)(\overline{\mu}) = \alpha(\overline{\mu})$ and, for each $i$, $(\gamma|\alpha)(\overline{\mu}i) = 0$ if and only if, for each $j \leq i$, $\alpha(\overline{\mu}j) = 0$, and, for each $i$, for each $s$, $(\gamma|\alpha)(\overline{\mu}i * s) = \alpha(\overline{\mu}i * (1) * s)$. Clearly, $\gamma$ reduces the set $\text{Share}(S_{i<n}(X_i))$ to reduces to the set $\text{Share}\bigl(\bigoplus_{i<n} X_i\bigr)$.

Let $\delta$ be a function from $N$ to $N$ such that $(\delta|\alpha)(\overline{\mu}) = \alpha(\overline{\mu})$ and, for each $i$, $(\delta|\alpha)(\overline{\mu}i) = 0$ if and only if $i < n$, and, for each $i$, for each $s$, $(\delta|\alpha)(\overline{\mu}i * (1) * s) = \alpha(\overline{\mu}i * s)$. Clearly, $\delta$ reduces the set $\text{Share}\bigl(\bigoplus_{i<n} X_i\bigr)$ to the set $\text{Share}(S_{i<n}(X_i))$.

(ii) Use (i) and Lemma 2.1(ii).

□

Let $X_0, X_1, \ldots$ be an infinite sequence of subsets of $N$. We define two subsets of $N$, $S^0_{i\in\mathbb{N}}(X_i)$ and $S^1_{i\in\mathbb{N}}(X_i)$, as follows:

(i) For each $\alpha$, $\alpha$ belongs to $S^0_{i\in\mathbb{N}}(X_i)$ if and only if either $\alpha = \overline{\mu}$ or there exist $n$, $\beta$ such that $\alpha = \overline{\mu}n * (1) * \beta$ and $\beta$ belongs to $X_n$.

(ii) For each $\alpha$, $\alpha$ belongs to $S^1_{i\in\mathbb{N}}(X_i)$ if and only if, if $\alpha \neq \overline{\mu}$, then there exist $n$, $\beta$ such that $\alpha = \overline{\mu}n * (1) * \beta$ and $\beta$ belongs to $X_n$.

A subset $X$ of $N$ is called inhabited if and only if there exists $\alpha$ such that $\alpha$ belongs to $X$.

A subset $X$ of $N$ is called enumerable if and only if there exists $\alpha$ such that, for all $\beta$, $\beta$ belongs to $X$ if and only if, for some $n$, $\beta = \alpha^n$.

Lemma 3.3.

Let $X_0, X_1, \ldots$ be an infinite sequence of inhabited subsets of $N$.

(i) The set $S^0_{i\in\mathbb{N}}(X_i)$ is a subset of the set $S^1_{i\in\mathbb{N}}(X_i)$.

(ii) The set $S^1_{i\in\mathbb{N}}(X_i)$ is not a subset of the set $S^0_{i\in\mathbb{N}}(X_i)$.

(iii) If, for each $n$, $X_n$ is a closed subset of $N$, then $S^1_{i\in\mathbb{N}}(X_i)$ is a closed subset of $N$.

(iv) If, for each $n$, $X_n$ is a finitary spread, then $S^1_{i\in\mathbb{N}}(X_i)$ is a finitary spread.

(v) If, for each $n$, $X_n$ is an enumerable subset of $N$, then $S^1_{i\in\mathbb{N}}(X_i)$ is an enumerable subset of $N$.

Proof. Leaving the proof of the other items to the reader, we only prove (ii).

Using the Second Axiom of Countable Choice, we determine $\alpha$ such that, for each $n$, $\alpha^n$ belongs to $X_n$. We let $\gamma$ be a function from $N$ to $N$ such that, for each $\beta$, if $\beta \neq \overline{\mu}$,
and \( n \) is the the least \( i \) such that \( \beta(i) \neq 0 \), then \( \gamma|\beta = 2^n \ast \langle 1 \rangle \ast \alpha^n \). Note that \( \gamma|\beta = 0 \) and that \( \gamma \) maps \( N \) into \( S^1_{\text{even}}(X_i) \). Assume that, for every \( \beta \), \( \gamma|\beta \) belongs to \( S^0_{\text{odd}}(X_i) \), that is, for every \( \beta \), either \( \gamma|\beta = 0 \) or there exist \( n \) such that \( \gamma|\beta = 2^n \ast \langle 1 \rangle \ast \alpha^n \). Using Brouwer’s Continuity Principle, we find \( m \) such that, for every \( \beta \), if \( \beta m = n \), then \( \gamma|\beta = 0 \) or, for every \( \beta \), if \( \beta m = n \), then there exist \( n \) such that \( \gamma|\beta = 2^n \ast \langle 1 \rangle \ast \alpha^n \), and, therefore, either, for every \( \beta \), if \( \beta m = n \), then \( \beta = 0 \) or, for every \( \beta \), if \( \beta m = n \), then \( \beta \neq 0 \). Both alternatives are absurd. 

We defined the following notion just before Lemma 3.1 and intend to use it again in the proof of the next theorem.

For each \( \alpha \), for each \( m \), for each \( n \), we let \( c(\alpha, m, n) \) be the number of elements of the set of all \( j < m \) such that, for all \( i < n \), \( \alpha^i(\beta) = 0 \).

**Theorem 3.4.**

Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( N \) such that, for each \( n \), the set \( \text{Share}(X_n) \) is disjunctively productive and the set \( \text{Share}(X_n) \) reduces to the set \( \text{Share}(X_{n+1}) \).

(i) For each \( n \), the set \( \text{Share}(X_n) \) reduces to the set \( \text{Share}(S_{i<n+1}(X_i)) \).

(ii) For each \( n > 0 \), the set \( \text{Share}(S_{i<n}(X_i)) \) reduces to the set \( \text{Share}(S_{i<n+1}(X_i)) \) and, for both \( j < 2 \), the set \( \text{Share}(S_{i<n}(X_i)) \) reduces to the set \( \text{Share}(S^1_{\text{even}}(X_i)) \).

(iii) For each \( n \), for each \( p \), the set \( D^m(\text{Share}(S_{i<p}(X_i))) \) reduces to the set \( \text{Share}(S_{i<m}p(X_i)) \).

(iv) For each \( n > 0 \), the set \( D_{i<n+1}(\text{Share}(X_{n+i})) \) does not reduce to the set \( D_{i<n}(\text{Share}(X_i)) \).

(v) For both \( j < 2 \), for each \( n > 0 \), the set \( \text{Share}(S^1_{\text{even}}(X_i)) \) does not reduce to the set \( \text{Share}(S_{i<n}(X_i)) \).

(vi) For both \( j < 2 \), for each \( m \), for each \( \gamma \), if \( \gamma \) is a function from \( N \) to \( N \) reducing the set \( D^m(\text{Share}(S^1_{\text{even}}(X_i))) \) to the set \( D^m(\text{Share}(S^1_{\text{even}}(X_i))) \) itself, then, for each \( k < m \), for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma(\alpha, m, n) > k \).

(vii) For both \( j < 2 \), the set \( \text{Share}(S^1_{\text{even}}(X_i)) \) is disjunctively productive.

(viii) For both \( j < 2 \), the set \( C(D^2(A_1), \text{Share}(S^1_{\text{even}}(X_i))) \) does not reduce to the set \( \text{Share}(S^1_{\text{even}}(X_i)) \).

**Proof.** (i) Let \( n \) be a natural number. Note that the set \( \text{Share}(X_n) \) reduces to the set \( D_{i<n+1}(\text{Share}(X_i)) \) and use Lemma 3.2(ii).

(ii) Let \( n \) be a positive natural number. Let \( \gamma \) be a function from \( N \) to \( N \) such that, for each \( \alpha \), for each \( s \), if there exists \( i < n \) such that either \( s \in 2^n \ast \langle 1 \rangle \) or \( s \in 2^n \ast \langle 1 \rangle \), then \( \gamma(\alpha)(s) = \alpha(s) \), and, if not, then \( \gamma(\alpha)(s) = 1 \).

Clearly, \( \gamma \) reduces the set \( \text{Share}(S_{i<n}(X_i)) \) to the set \( \text{Share}(S_{i<n+1}(X_i)) \), and both to the set \( \text{Share}(S^0_{\text{odd}}(X_i)) \) and to the set \( \text{Share}(S^1_{\text{even}}(X_i)) \).

(iii) Let \( m, p \) be natural numbers. Note that, by Lemmas 3.1 and 3.2 the set \( D^m(\text{Share}(S_{i<p}(X_i))) \) reduces to the set \( D^m(D_{i<p}(\text{Share}(X_i))) \). We let \( Y_0, Y_1, \ldots, Y_{m-1} \) be a finite sequence of subsets of \( N \) of length \( m \cdot p \) such that, for each \( j < m \), for each \( i < p \), \( Y_{i+j} \) coincides with \( X_i \). Note that the set \( D^m(\text{Share}(S_{i<p}(X_i))) \) reduces to the set \( D_{i<m}(\text{Share}(Y_i)) \). As, for each \( n \), the set \( \text{Share}(X_n) \) reduces to the set \( \text{Share}(X_{n+1}) \), one may conclude that, for each \( i < m \), the set \( \text{Share}(Y_i) \) reduces to the set \( \text{Share}(X_i) \). It then follows that the set \( D_{i<m}(\text{Share}(Y_i)) \) reduces to the set \( D_{i<m}(\text{Share}(X_i)) \).

(iv) Let \( n \) be a positive natural number. Note that, for each \( i < n \), the set \( \text{Share}(X_i) \) reduces to the set \( \text{Share}(X_{n+1}) \). It follows that the set \( D_{i<n}(\text{Share}(X_i)) \) reduces to the set \( D^m(\text{Share}(X_{n+1})) \). Also observe that, for each \( i \leq n \), the set \( \text{Share}(X_{n+1}) \) reduces to the set \( \text{Share}(X_{n+1}) \). It follows that the set \( D^{n+1}(\text{Share}(X_{n+1})) \) reduces to
the set \( D_{i<n+1}(\text{Share}(X_{n+i})) \). As the set \( D^{n+1}(\text{Share}(X_{n-1})) \) does not reduce to the set \( D^n(\text{Share}(X_{n-1})) \), it follows that the set \( D_{i<n+1}(\text{Share}(X_{n+i})) \) does not reduce to the set \( D_{i<n}(\text{Share}(X_i)) \).

(v) One verifies easily that, for each \( n \), the set \( D_{i<n+1}(\text{Share}(X_{n+i})) \) reduces to the set \( \text{Share}(S_{i+2n+1}(X_i)) \), and therefore, by (ii), for both \( j, k \), for each \( n \), the set \( D_{k<n+1}(\text{Share}(X_{n+i})) \) reduces to the set \( \text{Share}(S^j_{i \in N}(X_i)) \). It now follows from (iii) that, for both \( j < 2 \), for each \( n \), the set \( \text{Share}(S^j_{i \in N}(X_i)) \) does not reduce to the set \( \text{Share}(S_{i<n}(X_i)) \).

(vi) Let \( j, m \) be natural numbers such that \( j < 2 \), and let \( \gamma \) be a function from \( N \) to \( N \) reducing the set \( D^m(\text{Share}(S^j_{i \in N}(X_i))) \) to itself. We have to prove:

for each \( k < m \), for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > k \).

We first consider the case \( k = 0 \). Suppose \( \alpha \) belongs to \( N \) and, for all \( n \), \( c(\alpha, m, n) > 0 \), and we find \( p \) such that \( c(\gamma|\alpha, m, p) = 0 \). Using the continuity of the function \( \gamma \), we determine \( \beta \) such that, for every \( \beta \), passing through \( \alpha \), for every \( j < m \), for every \( q \leq p \), \( (\gamma|\beta)^q(\alpha) = (\gamma|\alpha)^q(\alpha) \), and, therefore, \( c(\gamma|\beta, m, p) = 0 \). We then determine \( i_0 < m \) such that, for all \( i < 2l, c^m(i) = 0 \). It does not harm the generality of the argument to assume: \( i_0 = 0 \). We now construct a function \( \delta \) from \( N \) to \( N \) such that, for every \( \beta \),

\[
\begin{align*}
(1) \quad & \delta(\beta)k = \alpha k, \\
(2) \quad & \text{for all } i, \text{if } i \leq 2l, \text{then } (\delta|\beta)^0(i) = \alpha^0(i), \text{ and, for all } i, \text{if } i < l + 1, \text{then } \beta^{i+1}(1) = \beta^i, \text{ and } \\
(3) \quad & \text{for all } i, \text{if } i < l, \text{then } (\delta|\beta)^0(X_{i+1}) = 1, \text{ and } \beta^{i+1}(1) = \beta^i, \text{ and } \\
(4) \quad & \text{for all } i > 0, \text{for all } t > l, \text{then } (\delta|\beta)^1(t) = t.
\end{align*}
\]

Note that, for each \( \beta, \beta \) belongs to \( D_{i<n+1}(\text{Share}(X_{i+1})) \) if and only if \( (\delta|\beta)^p \) belongs to \( \text{Share}(S^j_{i \in N}(X_i)) \) if and only if \( (\delta|\beta)^p \) belongs to \( D^m(\text{Share}(S^j_{i \in N}(X_i))) \). As, for each \( \beta \), \( (\delta|\beta)^k = \alpha k \), one may conclude: for each \( \beta, \beta \) belongs to \( D_{i<n+1}(\text{Share}(X_{i+1})) \) if and only if \( (\delta|\beta)^p \) belongs to \( D^m(\text{Share}(S^j_{i \in N}(X_i))) \) if and only if \( \gamma|\beta \) belongs to \( D^m(\text{Share}(S_{i<p}(X_i))) \).

Recall that the set \( D^m(\text{Share}(S_{i<p}(X_i))) \) reduces to the set \( \text{Share}(S_{i<m}(X_i)) \) and that \( m \cdot p \leq l \). Using Lemma 2.2, we thus see that the set \( D_{i<n}(\text{Share}(X_i)) \) reduces to the set \( D_{i<n}(\text{Share}(X_i)) \), and this contradicts (iv).

We must conclude: for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) > 0 \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > 0 \).

Let us now assume that \( k + 1 < m \) and that we have proven:

for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > k \).

We intend to show:

for each \( \alpha \), if, for each \( n \), \( c(\alpha, m, n) > k + 1 \), then, for each \( n \), \( c(\gamma|\alpha, m, n) > k + 1 \).

We do so as follows. Suppose \( \alpha \) belongs to \( N \) and, for all \( n \), \( c(\alpha, m, n) > k + 1 \), and we find \( p \) such that \( c(\gamma|\alpha, m, p) = k + 1 \). Using the continuity of the function \( \gamma \), we determine \( l \) such that, for every \( \beta \), if \( \gamma(l) = \alpha(l) \), then \( c(\gamma|\beta, m, p) = c(\gamma|\alpha, m, p) \). Without loss of generality we may assume that, for all \( i < k + 1 \), for all \( q < l, \alpha^i(q) = 0 \), and that, for all \( i \) such that \( k + 1 \leq i < m \), for every \( \beta \) passing through \( \alpha(l) \), there exists \( q \leq p \) such that \( (\gamma|\beta)^q(\alpha) = 0 \). We let \( \delta \) be a function from \( N \) to \( N \) such that, for every \( \beta \),

\[
\begin{align*}
(1) \quad & \delta(\beta)l = \alpha l, \text{ and } \\
(2) \quad & \text{for all } i, \text{if } i \leq 2l, \text{then } (\delta|\beta)^0(i) = \alpha^0(i), \text{ and, for all } i, \text{if } i < l + 1, \text{then } \beta^{i+1}(1) = \beta^i, \text{ and } \\
(3) \quad & \text{for all } i, \text{if } i < l, \text{then } (\delta|\beta)^0(X_{i+1}) = 1, \text{ and } \beta^{i+1}(1) = \beta^i, \text{ and } \\
(4) \quad & \text{for all } i > 0, \text{for all } t > l, \text{then } (\delta|\beta)^1(t) = t.
\end{align*}
\]
(2) for every $i \leq k + 1$, for every $q \leq l$, $(\delta|\beta)^i(\overline{m}q) = 0$ and, for each $n$, $(\delta|\beta)^i((\overline{m}q + n + 1)) = \beta^i(n)$, and, for all $s$, if $s > l$ and $\overline{m}$ does not pass through $s$, then $(\delta|\beta)^i(s) = 1$, and

(3) for every $i$, if $k + 1 < i < m$, then, for all $n > l$, $(\delta|\beta)^i(n) = 1$.

Note that, for every $\beta$, $\delta|\beta$ belongs to $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ if and only if there exists $i \leq k + 1$ such that $(\delta|\beta)^i$ admits $0$ if and only if there exists $i \leq k + 1$ such that $\beta^i = 0$. Now let $T$ be the set of all $\beta$ in $\mathcal{C}$ such that, for all $m, n$, if $\beta(mn) \neq 0$ and $\beta(n) \neq 0$, then $m = n$. The set $T$ consists of all elements $\beta$ of Cantor space $\mathcal{C}$ that assume the value $1$ at most one time. The set $T$ is a spread.

Note that, for every $\beta$ in $T$, for every $n$, the number of elements of the set $\{i \leq k + 1\} = n$ is at least $k + 1$. It follows that, for every $\beta$ in $k$, for every $n$, $c(\delta|\beta, m, n) \geq k + 1$. The induction hypothesis now enables us to conclude: for every $\beta$ in $T$, for every $n$, $c((\delta|\beta), m, n) \geq k + 1$, and, therefore, for all $i \leq k + 1$, $(\delta|\beta)^i$ admits $0$. and, therefore, $\gamma((\delta|\beta))$ belongs to $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$, and, therefore, $\delta|\beta$ belongs to $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$, and, therefore, there exists $i \leq k + 1$ such that $\beta^i = 0$.

Applying Brouwer’s Continuity Principle, we find $q, i$ such that, for each $\beta$ in $T$, if $\beta$ passes through $\overline{m}q$, $\beta^i = 0$. This is false, of course, as there exists $\beta$ in $T$ passing through $\overline{m}q$ such that, for some $r$, $\beta^i(r) = 1$.

We must conclude: for each $\alpha$, if, for each $n$, $c(\alpha, m, n) > k + 1$, then, for each $n$, $c((\alpha|\beta, m, n)) > k + 1$.

(vii) Let $j, m$ be natural numbers such that $j < 2$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $D^{m+1}(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ to the set $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$.

We let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $i$, if $i < m$, then $(\delta|\alpha)^i = (\gamma|\alpha)^i$, and $(\delta|\alpha)^m = 1$. Note that the function $\delta$ reduces the set $D^{m+1}(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ to the set $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ as well as the function $\gamma$. The function $\delta$, however, also reduces the set $D^{m+1}(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ to the set $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ itself. On the other hand, for each $n$, $c(\overline{m}q, m + 1, n) = m + 1$, and, for each $n$, $c(\delta|\overline{m}q, m, n) \leq m$. This contradicts the conclusion of (v).

We must conclude that, for each $m$, the set $D^{m+1}(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ does not reduce to the set $D^m(\text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$, that is, the set $\text{Share}(S^j_{i\in\mathbb{N}}(X_i))$ is disjunctively productive.

(viii) Let $j < 2$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C(D^2(A_1), \text{Share}(S^j_{i\in\mathbb{N}}(X_i)))$ to the set $\text{Share}(S^j_{i\in\mathbb{N}}(X_i))$. We claim that, for all $\alpha$, if $\alpha^0$ belongs to the closure $D^2(A_1)$ of the set $D^2(A_1)$, (that is, for each $n$, either $\alpha^0|\overline{m}n = 0$ or $\alpha^0|\overline{m}n = \overline{m}n$), and $\alpha^1$ admits $0$, then $\gamma|\alpha$ admits $0$.

We prove this claim as follows. Let $\alpha$ be an element of $\mathcal{N}$ such that $\alpha^0$ belongs to $D^2(A_1)$ and $\alpha^1$ admits $0$. Suppose we find a natural number $n$ such that $(\gamma|\alpha)(\overline{m}n) \neq 0$. Using the continuity of $\gamma$, find $p$ such that, for every $\beta$, if $\beta$ passes through $\overline{m}p$, then $(\gamma|\beta)(\overline{m}n) = (\gamma|\alpha)(\overline{m}n)$. We define $q := \max(n, p)$. We know that, either, for each $i$, if $(0, 0, i) < q$, then $\alpha((0, 0, i)) = 0$ or, for each $i$, if $(0, 1, i) < q$, then $\alpha((0, 1, i)) = 0$, and without loss of generality, we may assume that the first of these two alternatives is true.

We now construct a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\beta$,

(1) $(\delta|\beta)^p = \overline{m}p$, and

(2) $(\delta|\beta)^{0,0} = \overline{m}$, and

(3) for each $i$, if $i < q$, then $(\delta|\beta)^i(\overline{m}i) = 0$ and, for all $s$, if $s > q$, then $(\delta|\beta)^i(\overline{m}*(1) \ast s) = 1$, and

(4) for each $i$, if $q \leq i < 2q$, then $(\delta|\beta)^i(\overline{m}i) = \beta^i(\overline{m}i)$ and $\overline{m}i \ast \overline{m}i \ast \overline{m}(i+1) = \beta^i$, and

(5) $(\delta|\beta)^i(\overline{m}(2q + 1)) = 1$.
Note that, for each \( \beta_i, \beta \) belongs to \( D_{<q}\{\text{Share}(X_{q+i})\} \) if and only if \( \delta|\beta \) belongs to \( \text{Share}(S_{i\in\mathbb{N}}(X_i)) \) if and only if \( \delta|\beta \) belongs to \( C(D^2(A_i), \text{Share}(S_{i\in\mathbb{N}}(X_i))) \) if and only if \( \gamma|\delta|\beta \) belongs to \( \text{Share}(S_{i\in\mathbb{N}}(X_i)) \). As, by Lemmas \ref{notitle} and \ref{notitle}, \( \text{Share}(S_{i\in\mathbb{N}}(X_i)) \) reduces to the set the set \( D_{<q}\{\text{Share}(X_i)\} \). We thus see that the set \( D_{<q}\{\text{Share}(X_{q+i})\} \) reduces to the set \( D_{<q}\{\text{Share}(X_j)\} \) and this conclusion contradicts the conclusion of (iii).

This establishes our claim that, for all \( \alpha \), if \( \alpha^0 \) belongs to \( D^2(A_1) \) and \( \alpha^1 \) admits \( \emptyset \), then \( \gamma|\alpha \) admits \( \emptyset \).

We now construct a function \( \varepsilon \) from \( N \) to \( N \) such that, for each \( \beta_i, (\varepsilon|\beta)^0 = \beta \), and \( (\varepsilon|\beta)^1 = \emptyset \).

Note that, for each \( \beta_i, \beta \) belongs to \( D^2(A_1) \), then \( (\varepsilon|\beta)^0 \) belongs to \( D^2(A_1) \) and \( (\varepsilon|\beta)^1 \) admits \( \emptyset \), and, therefore, \( \gamma|\varepsilon|\beta \) admits \( \emptyset \), so, \( \gamma|\varepsilon|\beta \) belongs to \( \text{Share}(S_{i\in\mathbb{N}}(X_i)) \), and, therefore, \( \varepsilon|\beta \) belongs to \( C(D^2(A_1), \text{Share}(S_{i\in\mathbb{N}}(X_i))) \), and, in particular, \( (\varepsilon|\beta)^0 \) belongs to \( D^3(A_1) \), so \( \beta \) belongs to \( D^3(A_1) \).

It follows that \( D^2(A_1) \) is a subset of \( D^3(A_1) \) and this is false.

We must conclude that the set \( C(D^2(A_1), \text{Share}(S_{i\in\mathbb{N}}(X_i))) \) does not reduce to the to the set \( \text{Share}(S_{i\in\mathbb{N}}(X_i)) \). \( \square \)

3.4. Preserving the property of conjunctive productivity. Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathbb{N} \). The sequence \( X_0, X_1, \ldots \) is called conjunctively closed if and only if, for each \( m \), for each \( n \), there exists \( p \) such that \( C(Y_m, Y_n) \) reduces to \( Y_p \).

**Theorem 3.5.**

Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathbb{N} \) such that \( X_1 \) is inhabited and, for each \( n \), the set \( \text{Share}(X_n) \) is disjunctively productive and the set \( \text{Share}(X_n) \) reduces to the set \( \text{Share}(X_{n+1}) \) and the infinite sequence \( \text{Share}(X_0), \text{Share}(X_1), \text{Share}(X_2), \ldots \) is conjunctively closed.

(i) For each \( m \), for each \( n \), there exists \( p \) such that

the set \( C(\text{Share}(S_{i\leq m}(X_i)), \text{Share}(S_{i\leq n}(X_i))) \) reduces to the set \( \text{Share}(S_{i\leq p}(X_i)) \).

(ii) For each \( n \), for each \( \gamma \), if \( \gamma \) is a function from \( N \) to \( N \) reducing the set \( C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i))) \) to the set \( C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i))) \) itself, then,

for each \( k < m \), for each \( \alpha \) in \( C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i))) \),

if, for each \( n \), \( c(\alpha, m, n) > k \), then, for each \( n \), \( c(\gamma(\alpha, m, n)) > k \).

(iii) For each \( n \), the set \( C(D^2(A_1), C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i))) \) does not reduce to the set \( C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i))) \).

(iv) The set \( \text{Share}(S_{i\leq n}(X_i)) \) is conjunctively productive.

**Proof.**

(i) Using Lemma \ref{notitle}, observe that the set \( C(\text{Share}(S_{i\leq m}(X_i)), \text{Share}(S_{i\leq n}(X_i))) \) reduces to the set \( C(D_{<m}\text{Share}(X_i), D_{<n}\text{Share}(X_i)) \). For each \( i < m \), for each \( j < n \), we determine a number \( p(i, j) \) in \( \mathbb{N} \) such that \( C(\text{Share}(X_i), \text{Share}(X_j)) \) reduces to \( \text{Share}(X_{p(i,j)}) \). Using the fact that, for each \( n \), the set \( \text{Share}(X_n) \) reduces to the set \( \text{Share}(X_{n+1}) \), one may determine, for each \( i < m \), for each \( j < n \), a number \( q(i,j) \) in \( \mathbb{N} \) such that the set \( C(\text{Share}(X_i), \text{Share}(X_j)) \) reduces to the set \( \text{Share}(X_{q(i,j)}) \), and, for each \( i, i' < m \), for each \( j, j' < n \), if either \( i \neq i' \) or \( j \neq j' \), then \( q(i,j) \neq q(i',j') \).

Let \( p \) be an element of \( \mathbb{N} \) such that, for each \( i < m \), for each \( j < n \), \( q(i,j) < p \). Note that the set \( D_{<n}(\text{Share}(X_j)) \) reduces to the set \( D_{<m}(D_{<n}(\text{Share}(X_{p(i,j)}))) \) and, therefore, also to the set \( D_{<p}(\text{Share}(X_i)) \) and, according to Lemma \ref{notitle}, also to the set \( \text{Share}(S_{i\leq p}(X_i)) \).

(ii) Let \( j, m \) be natural numbers such that \( j < 2 \) and let \( \gamma \) be a function from \( N \) to \( N \) reducing the set \( C^m(\text{Share}(S_{i\leq n}(X_i))) \) to itself. We have to prove:
for each $k < m$, for each $\alpha$ in $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$, if, for each $n$, $c(\alpha, m, n) > k$, then, for each $n$, $c(\gamma|\alpha, m, n) > k$.

Let us first consider the case $k = 0$. Suppose $\alpha$ belongs to $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$ and, for all $n$, $c(\alpha, m, n) > 0$, and we find $p$ such that $c(\gamma|\alpha, m, p) = 0$. Using the continuity of the function $\gamma$, we determine $k$ such that, for every $\beta$ passing through $\mathbb{N}$, for every $i < m$, for every $q \leq p$, $(\gamma|\beta)^i(q) = (\gamma|\alpha)^i(q)$, and, therefore, $c(\gamma|\beta, m, p) = 0$. Using (i), we determine $l$ such that the set $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$ reduces to the set $\text{Share}(S_{i\in\mathbb{N}}(X_i))$. We define $q := \max(n, l)$. We then determine $i_0 < m$ such that, for all $j \leq q$, $\alpha^{i_0}(i) = 0$. It does not harm the generality of the argument to assume: $i_0 = 0$. We now construct a function $\delta$ from $\mathbb{N}$ to $\mathbb{N}^*$ such that, for every $\beta$,

1. $(\delta|\beta)^l = \mathfrak{m}$, and
2. for each $i$, if $i < q$, then $(\delta|\beta)^0(\mathfrak{n}) = \alpha^0(\mathfrak{n}) = 0$ and, for all $s$, if $s > q$, then $(\delta|\beta)^0(\mathfrak{n} + 1) = 0$ and $s$.
3. for each $i$, if $i \leq q$, then $(\delta|\beta)^0(\mathfrak{n}(q+i)) = 0$, and $\mathfrak{n}^{q+i-1}(1)((\delta|\beta)^0) = \beta^i$,
4. $(\delta|\beta)^0(\mathfrak{n}(2q+1)) = 1$,
5. for all $i > 0$, $(\delta|\beta)^i = \alpha^i$.

Note that, for each $\beta$, $\beta$ belongs to $D_{i\leq q}(\text{Share}(X_{q+i}))$ if and only if $(\delta|\beta)^0$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$ and only if $(\delta|\beta)^0$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$ and only if $(\delta|\beta)^0$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$. As, for each $\beta$, $(\delta|\beta)^0 = \mathfrak{m}$, one may conclude: for each $\beta$, $\beta$ belongs to $D_{i\leq q}(\text{Share}(X_{q+i}))$ if and only if $(\delta|\beta)^0$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$ and only if $(\delta|\beta)^0$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$.

By our choice of $q$, it follows that the set $D_{i\leq q}(\text{Share}(X_{q+i}))$ reduces to the set $\text{Share}(S_{i\in\mathbb{N}}(X_i))$ and this contradicts Theorem 3.4(iii).

Let us now assume that $k + 1 < m$ and that we have proven:

for each $\alpha$ in $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$, if, for each $n$, $c(\alpha, m, n) > k$, then, for each $n$, $c(\gamma|\alpha, m, n) > k$.

We intend to show:

for each $\alpha$ in $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$, if, for each $n$, $c(\alpha, m, n) > k + 1$, then, for each $n$, $c(\gamma|\alpha, m, n) > k + 1$.

We do so as follows. Suppose $\alpha$ belongs to $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$ and, for all $n$, $c(\alpha, m, n) > k + 1$, and we find $p$ such that $c(\gamma|\alpha, m, p) = k$. Using the continuity of the function $\gamma$, we determine $l$ such that, for every $\beta$, if $\mathfrak{m} = \mathfrak{m}l$, then $c(\gamma|\beta, m, p) = c(\gamma|\alpha, m, p)$.

Without loss of generality we may assume that, for all $i \leq k + 1$, for all $r \leq l$, $\alpha^i(i) = 0$, and that, for all $i$ such that $k + 1 < i < m$, for every $\beta$ passing through $\mathfrak{m}l$, there exists $r \leq p$ such that $(\gamma|\beta)^i(r) \neq 0$. Using (i), we determine $t$ such that the set $C^{m-k-1}(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$ reduces to the set $\text{Share}(S_{i\in\mathbb{N}}(X_i))$. We define $q := \max(l, p, t)$. We let $\delta$ be a function from $\mathbb{N}$ to $\mathbb{N}^*$ such that, for every $\beta$,

1. $(\delta|\beta)^l = \mathfrak{m}$, and
2. for every $i$, if $i \leq k$, then $(\delta|\beta)^i$ admits $0$, and
3. for each $i$, if $i < q$, then $(\delta|\beta)^k+1(\mathfrak{i}) = \alpha^{k+1}(\mathfrak{i}) = 0$, and, for all $s$, if $s > q$, then $(\delta|\beta)^{k+1}(\mathfrak{i} + 1) = 1$, and
4. for each $i$, if $i \leq q$, then $(\delta|\beta)^{k+1}(\mathfrak{i}(q+i)) = 0$, and $\mathfrak{n}^{q+i}(1)((\delta|\beta)^{k+1}) = \beta^i$,
5. $(\delta|\beta)^{k+1}(\mathfrak{i}2q + 1) = 1$,
6. for every $i$, if $k + 1 < i < m$, then $(\delta|\beta)^i = \alpha^i$.

First observe that, for each $\beta$, $\beta$ belongs to $D_{i\leq q}(\text{Share}(X_{q+i}))$ if and only if $(\delta|\beta)^{k+1}$ belongs to $\text{Share}(S_{i\in\mathbb{N}}(X_i))$ and only if $\delta|\beta$ belongs to $C^m(\text{Share}(S_{i\in\mathbb{N}}(X_i)))$.
Then note that, for each $\beta$, $\delta|\beta$ passes through $\overline{\alpha}$, and therefore, if $\beta$ belongs to $D_{1\leq q}(\text{Share}(X_{q+1}))$, then, for each $i$, if $k + 1 \leq i < m$, then $(\gamma(\delta|\beta))^i$ belongs to $\text{Share}(S_{i \in \mathbb{N}}(X_i))$ and to $\text{Share}(S_{i < q}(X_i))$.

Next observe that, for each $\beta$, for each $n$, $c(\delta|\beta, m, n) > k$, and therefore, by the induction hypothesis, if $\beta$ belongs to $D_{1\leq q}(\text{Share}(X_{q+1}))$, then, for each $n$, $(\gamma(\delta|\beta))^n$ belongs to $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$, and, therefore, $\delta|\beta$ itself belongs to $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$ and $\beta$ belongs to $D_{1\leq q}(\text{Share}(X_{q+1}))$.

Summing up, we see that, for each $\beta$, $\delta|\beta$ belongs to $D_{1\leq q}(\text{Share}(X_{q+1}))$ if and only if for each $i$, if $k + 1 \leq i < m$, then $(\gamma(\delta|\beta))^i$ belongs to $\text{Share}(S_{i \in \mathbb{N}}(X_i))$ if and only if, for each $i$, if $k + 1 \leq i < m$, then $(\gamma(\delta|\beta))^i$ admits $\overline{\alpha}$, and, if $k + 1 \leq i < m$, then $(\gamma(\delta|\beta))^i$ belongs to $\text{Share}(S_{i < q}(X_i))$.

We thus see that the set $D_{1\leq q}(\text{Share}(X_{q+1}))$ reduces to the set $C(C^{k+1}(\text{Share}(A_1), \text{Share}(S_{r < p}(X_r))))$.

It now follows from our definition of the numbers $t$ and $q$ that the set $D_{1\leq q}(\text{Share}(X_{q+1}))$ reduces to the set $C(C^{q+1}(\text{Share}(A_1), \text{Share}(S_{i < q}(X_i))))$.

We now need the observation that, for every subset $Y$ of $N$, the set $C(\text{Share}(A_1), \text{Share}(Y))$ reduces to the set $\text{Share}(Y)$.

(Define a function $\eta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $s$, $(\eta|\alpha)(s) = \max(\alpha^0(s), \alpha^1(s))$. For every subset $Y$ of $\mathcal{N}$, the function $\eta$ reduces the set $C(\text{Share}(A_1), \text{Share}(Y))$ to the set $\text{Share}(Y)$.)

It follows that the set $D_{1\leq q}(\text{Share}(X_{q+1}))$ reduces to the set $\text{Share}(S_{i \in \mathbb{N}}(X_i))$, and this, together with Lemmas 2.1 and 2.2, contradicts Theorem 2.4(ii).

We must conclude: for each $\alpha$ in $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$, if, for each $n$, $c(\alpha, m, n) > k + 1$, then, for each $n$, $\delta(\alpha, m, n) > k + 1$.

(iii) Let $j, m$ be natural numbers such that $j < 2$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $C(D^2(A_1), C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i))))$ to the set $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$. Note that, for each $\alpha$, if $\alpha^0$ belongs to $D^2(A_1)$, then: $\alpha^1$ belongs to $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$ and only if $\gamma(\alpha)$ belongs to $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$. It now follows from (ii) that, for each $\alpha$, if $\alpha^0$ belongs to $D^2(A_1)$ and, for each $i < m$, $\alpha^{0,i}$ admits $\overline{\alpha}$, then, for each $i < m$, $(\gamma(\alpha))^i$ admits $\overline{\alpha}$. Using the continuity of the function $\gamma$, we may conclude that, for each $\alpha$, if $\alpha^0$ belongs to the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$, and, for each $i < m$, $\alpha^{0,i}$ admits $\overline{\alpha}$, then, for each $i < m$, $(\gamma(\alpha))^i$ admits $\overline{\alpha}$. We now let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\beta$, $(\delta(\beta))^0 = \beta$ and $(\delta(\beta))^1 = \overline{\beta}$. Note that, for each $\beta$, if $\delta$ belongs to $\overline{D^2(A_1)}$, then $(\delta(\beta))^0$ does so, and, in addition, for each $i < m$, $(\delta(\beta))^1,i$ admits $\overline{\beta}$, and therefore, $\gamma(\delta(\beta))$ belongs to $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$, and $\delta(\beta)$ belongs to $C(D^2(A_1), C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i))))$, and $(\delta(\beta))^0 = \beta$ belongs to $D^2(A_1)$. We thus see that $\overline{D^2(A_1)}$ is a subset of $D^2(A_1)$, and this is false.

We must conclude: for each $\alpha$, the set $C(D^2(A_1), C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i))))$ does not reduce to the set $C^m(\text{Share}(S_{i \in \mathbb{N}}(X_i)))$.

(iv) As the set $X_0$ is inhabited we may determine an element $\delta$ of $X_0$. We now prove that, for both $j < 2$ the set $D^2(A_1)$ reduces to the set $\text{Share}(S_{i \in \mathbb{N}}(X_i))$. To this end, we construct a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $s$, $(\gamma(\alpha))(s) = 0$ if and only if either (1) $\alpha(\delta)$ passes through $\overline{s}$, and, for each $i < \text{length}(s)$, $\alpha^0(i) = 0$, or $\overline{\beta}$ passes through $s$, and, for each $i < \text{length}(s)$, $\alpha^1(i) = 0$. One easily verifies that $\gamma$ reduces the set $D^2(A_1)$ to the set $\text{Share}(S_{i \in \mathbb{N}}(X_i))$. 

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It now easily follows from (iii) that, for both $j < 2$, for each $m$, the set $C^{n+1}(\text{Share}(S^i_{\in\mathbb{N}}(X_i)))$ does not reduce to the set $C^n(\text{Share}(S^i_{\in\mathbb{N}}(X_i)))$. □

Lemma 3.6.

Let $X_0, X_1, \ldots$ and $Y_0, Y_1, \ldots$ be infinite sequences of subsets of $\mathcal{N}$ such that, for each $n$, the set $\text{Share}(X_n)$ reduces to the set $\text{Share}(X_{n+1})$ and the set $\text{Share}(Y_n)$ reduces to the set $\text{Share}(Y_{n+1})$.

If, for each $m$, there exists $n$ such that the set $\text{Share}(X_m)$ reduces to the set $\text{Share}(Y_n)$, then, for both $j < 2$, the set $\text{Share}(S^i_{\in\mathbb{N}}(X_i))$ reduces to the set $\text{Share}(S^i_{\in\mathbb{N}}(Y_i))$.

Proof. Let $X_0, X_1, \ldots$ and $Y_0, Y_1, \ldots$ be infinite sequences of subsets of $\mathcal{N}$ satisfying the conditions of the theorem. Using the First Axiom of Countable Choice, we determine $\delta$ in $\mathcal{N}$ such that, for each $n, \delta(n) < \delta(n + 1)$ and the set $\text{Share}(X_n)$ reduces to the set $\text{Share}(Y_{\delta(n)})$ using the Second Axiom of Countable Choice, we then determine $\varepsilon$ in $\mathcal{N}$ such that, for each $n$, $\varepsilon^n$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $\text{Share}(X_n)$ to the set $\text{Share}(Y_{\delta(n)})$.

We then define a function $\zeta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, the following conditions are satisfied:

1. For each $n$, if $n$ is the least $i$ such that $\alpha(\overrightarrow{n}) \neq 0$, then, $\delta(n)$ is the least $i$ such that $\zeta(\alpha) \neq 0$, and,
2. For each $k$, if there is no $n$ such that $\delta(n) = k$, then $(\zeta | \alpha) \overrightarrow{k} \ast (1) = 1$, and,
3. For each $n$, if, for all $i \leq n$, $\alpha(\overrightarrow{i}) = 0$, then $\delta(n) \ast (1)$ and $\delta(n) \ast (1) \ast \theta$ belongs to $\text{Share}(S^i_{\in\mathbb{N}}(Y_i))$.

We first prove that the function $\zeta$ reduces the set $\text{Share}(S^i_{\in\mathbb{N}}(X_i))$ to the set $\text{Share}(S^i_{\in\mathbb{N}}(Y_i))$.

Suppose $\alpha$ belongs to $\text{Share}(S^i_{\in\mathbb{N}}(X_i))$. Find $\lambda$ in $S^0_{\in\mathbb{N}}(X_i)$ such that $\alpha$ admits $\lambda$ and distinguish two cases.

Case (a): $\lambda = \emptyset$. Note that also $\zeta | \alpha$ admits $\emptyset$ and $\emptyset$ belongs to $S^0_{\in\mathbb{N}}(Y_i)$. Therefore, $\zeta | \alpha$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(Y_i))$.

Case (b): $\lambda \neq \emptyset$. Find $n, \eta$ such that $\lambda = \overrightarrow{n} \ast (1) \ast \eta$ and $\eta$ belongs to $X_n$. Conclude that $\overrightarrow{n} \ast (1) \ast \eta$ belongs to $\text{Share}(X_n)$, and therefore, $\delta(n) \ast (1)$ and $\delta(n) \ast (1) \ast \theta$ belongs to $\text{Share}(Y_{\delta(n)})$. Find $\theta$ in $Y_{\delta(n)}$ such that $\delta(n) \ast (1) \ast \theta$ and $\delta(n) \ast (1) \ast \theta$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(Y_i))$. Therefore, $\zeta | \alpha$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(Y_i))$.

Conversely, suppose $\zeta | \alpha$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(Y_i))$. Find $\lambda$ in $S^0_{\in\mathbb{N}}(X_i)$ such that $\zeta | \alpha$ admits $\lambda$ and distinguish two cases.

Case (a): $\lambda = \emptyset$. Note that also $\alpha$ admits $\emptyset$ and $\emptyset$ belongs to $S^0_{\in\mathbb{N}}(X_i)$. Therefore, $\alpha$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(X_i))$.

Case (b): $\lambda \neq \emptyset$. Find $n, \eta$ such that $\lambda = \overrightarrow{n} \ast (1) \ast \eta$ and $\eta$ belongs to $Y_n$. Conclude that $\overrightarrow{n} \ast (1) \ast \eta$ belongs to $\text{Share}(Y_n)$, and therefore, $\delta(n) \ast (1)$ and $\delta(n) \ast (1) \ast \theta$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(Y_i))$. Find $\theta$ in $X_n$ such that $\delta(n) \ast (1) \ast \theta$ and $\delta(n) \ast (1) \ast \theta$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(X_i))$. Therefore, $\alpha$ belongs to $\text{Share}(S^0_{\in\mathbb{N}}(X_i))$.

We now prove that the function $\zeta$ reduces the set $\text{Share}(S^i_{\in\mathbb{N}}(X_i))$ to the set $\text{Share}(S^i_{\in\mathbb{N}}(Y_i))$.

Suppose $\alpha$ belongs to $\text{Share}(S^i_{\in\mathbb{N}}(X_i))$. Find $\lambda$ in $S^i_{\in\mathbb{N}}(X_i)$ such that $\alpha$ admits $\lambda$.

Note that, if $\lambda \neq \emptyset$, then there exist $n, \eta$ such that $\lambda = \overrightarrow{n} \ast (1) \ast \eta$ and $\eta$ belongs to $X_n$. We now define an element $\mu$ of $\mathcal{N}$, as follows, step by step. For each $n$, if $\overrightarrow{n} = \overrightarrow{n}$ then $\overrightarrow{n} \ast (1) = \overrightarrow{n} \ast (1)$. As soon as we discover $n$ such that $n$ is the least $i$ such that $\lambda(i) \neq 0$, then, observing that $\overrightarrow{n} \ast (1)$ belongs to $\text{Share}(X_n)$, we conclude that
\[ \mathfrak{P}(\delta(n)^{+}(1)) \langle \zeta | \alpha \rangle = \varepsilon_0^{n}(\mathfrak{P}^{n^{+}(1)} \alpha) \] belongs to \( Share(Y_n) \) and we determine \( \kappa \) in \( Y_{\delta(n)} \) such that \( \mathfrak{P}(\delta(n)^{+}(1)) \langle \zeta | \alpha \rangle \) admits \( \kappa \), and we define: \( \mu := \mathfrak{P}(\delta(n)^{+}(1)) \star 1 \star \kappa \). Clearly, \( \mu \) belongs to \( S^1_{\mathbb{N}}(Y_n) \) and is admitted by \( \zeta | \alpha \); therefore, \( \zeta | \alpha \) belongs to \( Share(S^1_{\mathbb{N}}(Y_n)) \).

Now assume \( \zeta | \alpha \) belongs to \( Share(S^1_{\mathbb{N}}(X_n)) \). Find \( \lambda \) in \( S^1_{\mathbb{N}}(X_n) \) such that \( \zeta | \alpha \) admits \( \lambda \). We define an element \( \mu \) of \( \mathcal{N} \), as follows, step by step. For each \( n \), if \( \lambda_n = \mathfrak{P}^n_n \), then, for each \( k \), if \( n < \delta(k) \), then \( \mathfrak{P}^k_n = \mathfrak{P}^k_k \). As soon as we discover \( n \) such that \( n \) is the least \( i \) such that \( \lambda(i) \neq 0 \), we determine \( k \) such that \( n = \delta(k) \) and, observing that \( \mathfrak{P}^{n^{+}(1)} \langle \zeta | \alpha \rangle \) belongs to \( Share(Y_n) \) and that \( \mathfrak{P}^{n^{+}(1)} \langle \zeta | \alpha \rangle = \varepsilon_k^{n}(\mathfrak{P}^{k^{+}(1)} \alpha) \), we conclude that \( \mathfrak{P}^{k^{+}(1)} \alpha \) belongs to \( Share(X_k) \) and we determine \( \kappa \) in \( X_k \) such that \( \mathfrak{P}^{(\delta(n)^{+}(1))} \langle \zeta | \alpha \rangle \) admits \( \kappa \), and we define: \( \mu := \mathfrak{P}^{k^{+}(1)} \star \kappa \). Clearly, \( \mu \) belongs to \( S^1_{\mathbb{N}}(X_n) \) and is admitted by \( \alpha \); therefore, \( \alpha \) belongs to \( Share(S^1_{\mathbb{N}}(X_n)) \). \( \square \)

3.5. **Building two hierarchies using hereditarily increasing stumps.** Theorem 3.2 and Lemma 3.6 enable us to build two hierarchies, the first one consisting of sets that belong to the class \( \Sigma^2_5 \), and the second one consisting of analytic sets that fail to be positively Borel.

Slightly adapting a definition given in [46], Section 3, and mentioned in the introductory Section of this paper, just before Theorem 1.7, we introduce, for every stump \( \sigma \), a subset \( CB^*_{\sigma} \) of \( \mathcal{N} \) by means of the following inductive definition:

(i) \( CB^*_0 := \{ \langle \rangle \} \).

(ii) For every nonempty stump \( \sigma \), \( CB^*_\sigma := S^0_{\mathbb{N}}(CB^*_\sigma) = \{ \langle \rangle \} \cup \bigcup_{n \in \mathbb{N}} \mathfrak{P}^n \star (1) \star CB^*_\sigma \).

The sets \( CB^*_\sigma \) are called **special Cantor-Bendixon-sets**. Note that every special Cantor-Bendixon-set is a Cantor-Bendixon-set in the sense of the earlier definition. It follows from [48], Theorem 3.3 that, for each stump \( \sigma \), if \( CB^*_\sigma \) is an infinite set, then \( CB^*_\sigma \) is a proper subset of its sequential closure \( CB_{\sigma}^\ast \). Note that, for every stump \( \sigma \), if \( \sigma \) is non-empty, then \( CB^*_\sigma \) is infinite and, therefore, not a closed subset of \( \mathcal{N} \).

We find it useful to introduce a special kind of stumps, the so-called **hereditarily increasing stumps**.

First, we define, for all \( \alpha, \beta \) in \( \mathcal{C} \): \( \alpha \) is a subset of \( \beta \), notation: \( \alpha \subseteq \beta \), if and only if, for each \( s \), if \( \alpha(s) = 1 \), then \( \beta(s) = 1 \). We now introduce the set \( His \) of the hereditarily increasing stumps by the following inductive definition.

(i) \( \langle \rangle \) is an element of \( His \).

(ii) For every non-empty stump \( \sigma \), \( \sigma \) belongs to \( His \) if and only if, for each \( n \), \( \sigma^n \) belongs to \( His \), and, for each \( n \), \( \sigma^n \) is a subset of \( \sigma^{n+1} \).

For all \( \alpha, \beta \) in Cantor space \( \mathcal{C} \), we let \( \alpha \cup \beta \), the union of \( \alpha \) and \( \beta \), be the element of Cantor space satisfying: for all \( n \), \( \alpha \cup \beta(n) = 1 \) if and only if \( \alpha(n) = \beta(n) = 1 \).

For all \( s, t \) in \( \mathbb{N} \), we define: \( s \leq^* t \) if and only if \( length(s) = length(t) \), and for all \( i < length(s) \), \( s(i) \leq t(i) \).

For every stump \( \sigma \) we let \( H(\sigma) \) be the element of Cantor space \( \mathcal{C} \) satisfying: for each \( t \), \( (H(\sigma))(1) = 1 \) if and only if there exists \( s \) such that \( s \leq^* t \) and \( \sigma(s) = 1 \).

**Lemma 3.7.**

(i) For every stump \( \sigma \), \( \sigma \) is a subset of \( H(\sigma) \).

(ii) For all stumps \( \sigma, \tau \), \( \sigma \) is a hereditarily increasing stump if and only if for each \( s \), for each \( t \), if \( \sigma(s) = 1 \) and \( s \leq^* t \), then \( \sigma(t) = 1 \).

(iii) For all stumps \( \sigma, \tau \), the infinite sequence \( \sigma \cup \tau \) is again a stump, and, if both \( \sigma \) and \( \tau \) are hereditarily increasing, then also \( \sigma \cup \tau \) is hereditarily increasing.
(iv) For every stump \( \sigma \), \( \mathbb{H}(\sigma) \) is a hereditarily increasing stump.

(v) For every stump \( \sigma \), for every hereditarily increasing stump \( \tau \), if \( \sigma \) is a subset of \( \tau \), then also \( \mathbb{H}(\sigma) \) is a subset of \( \tau \).

(vi) For every non-empty stump \( \sigma \), the set \( \overline{CB^*_v} \) coincides with the set \( S_{n \in \mathbb{N}}(\overline{CB^*_v}) \).

**Proof.**

(i) This is obvious.

(ii) We first prove: for every hereditarily increasing stump \( \sigma \), for each \( s \), for each \( t \), is \( \sigma(s) = 1 \) and \( s \leq^* t \), then \( \sigma(t) = 1 \). The proof is by induction on the set of stumps.

First, note that the statement is trivially true if \( \sigma = \emptyset \).

Now assume that that \( \sigma \) is a non-empty stump and that the statement holds for every immediate sub stump \( \sigma^n \) of \( \sigma \). Let \( s, t \) be elements of \( \mathbb{N} \) such that \( \sigma(s) = 1 \) and \( s \leq^* t \). One may assume: length of \( s \) > 0. Find \( u, v \) such that \( s = \langle s(0) \rangle * u \) and \( t = \langle t(0) \rangle * v \). As \( \sigma^{s(0)} \) is a subset of \( \sigma^{t(0)} \) and \( \sigma^{s(0)}(u) = 1 \), also \( \sigma^{t(0)}(v) = 1 \). As \( u \leq^* v \), also \( \sigma^{t(0)}(v) = 1 \) and \( \sigma(t) = 1 \).

We now prove: for every stump \( \sigma \), if \( \sigma \) satisfies the condition: for each \( s \), for each \( t \), if \( \sigma(s) = 1 \) and \( s \leq^* t \), then \( \sigma(t) = 1 \), then \( \sigma \) is hereditarily increasing. We use again induction on the set of stumps.

First, note that the statement is trivially true if \( \sigma = \emptyset \).

Now assume that \( \sigma \) is a non-empty stump and: for each \( s \), for each \( t \), if \( \sigma(s) = 1 \) and \( s \leq^* t \), then \( \sigma(t) = 1 \). Note that, for each \( n \), for each \( t \), if \( \sigma((n) * t) = 1 \), then \( \sigma((n + 1) * t) = 1 \), so \( \sigma^n \) is a subset of \( \sigma^{n + 1} \). Also, for each \( n \), for each \( s \), for each \( t \), if \( s \leq^* t \), then, if \( \sigma((n) * s) = 1 \), then \( \sigma((n) * t) = 1 \), that is: if \( \sigma^n(s) = 1 \), then \( \sigma^n(t) = 1 \). It follows from the induction hypothesis that, for each \( n \), \( \sigma^n \) is hereditarily increasing. Clearly, also \( \sigma \) itself is hereditarily increasing.

(iii) We have to prove that, for all stumps \( \sigma \),

for all stumps \( \tau \), the infinite sequence \( \sigma \cup \tau \) is again a stump,

and we do so by induction on the set of stumps.

Note that, for each stump \( \tau \), \( \emptyset \cup \tau = \tau \) is again a stump. Now assume \( \sigma \) is a non-empty stump, and for all \( n \), for every stump \( \tau \), \( \sigma^n \cup \tau \) is a stump. Note that \( \sigma \cup \emptyset = \sigma \) is a stump.

Now assume \( \tau \) is a non-empty stump. Note that, by the induction hypothesis, for each \( n \), \( (\sigma \cup \tau)^n = \sigma^n \cup \tau^n \) is a stump. It follows that \( \sigma \cup \tau \) itself is a stump.

The observation that, if both \( \sigma \) and \( \tau \) are hereditarily increasing, also \( \sigma \cup \tau \) is hereditarily increasing, is an easy consequence of (ii).

(iv) We have to prove: for every stump \( \sigma \), the infinite sequence \( \mathbb{H}(\sigma) \) is again a stump, and we do so by induction on the set of stumps.

Note that \( \mathbb{H}(\emptyset) = \emptyset \) is a stump indeed.

Now assume \( \sigma \) is a non-empty stump. Note that \( (\mathbb{H}(\sigma))(\emptyset) = 1 \), and, for each \( n \), \( \mathbb{H}(\sigma))^n = \bigcup_{i \leq n} \mathbb{H}(\sigma^i) \) and conclude, using (iii) and the induction hypothesis, that \( (\mathbb{H}(\sigma))^n \) is a stump. It follows that \( \mathbb{H}(\sigma) \) itself is a stump.

(v) This easily follows from (ii).

(vi) The proof is left to the reader. \( \square \)

**Theorem 3.8.**

(i) For all hereditarily increasing stumps \( \sigma, \tau \), if \( \sigma \leq \tau \), then the set \( \text{Share}(CB^*_v) \) reduces to the set \( \text{Share}(CB^*_v) \), and the set \( \text{Share}(\overline{CB^*_v}) \) reduces to the set \( \text{Share}(\overline{CB^*_v}) \).

(ii) For each stump \( \sigma \), the set \( \text{Share}(CB^*_v) \) belongs to the class \( \Sigma^0_2 \) and, if \( \sigma \) is non-empty, then the set \( \text{Share}(\overline{CB^*_v}) \) is an analytic set that fails to be positively Borel.

(iii) For each hereditarily increasing stump \( \sigma \), the sets \( \text{Share}(CB^*_v) \) and \( \text{Share}(\overline{CB^*_v}) \) are disjunctively productive.

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(iv) For all hereditarily increasing stumps \( \sigma, \tau \), if \( \sigma < \tau \), then the set \( \text{Share}(CB^*_\gamma) \) does not reduce to the set \( \text{Share}(CB^*_\tau) \) and the set \( \text{Share}(CB^*_\gamma) \) does not reduce to the set \( \text{Share}(CB^*_\sigma) \).

Proof. (i) One proves this by induction on the set of stumps, using Lemma 2.6.

(ii) Using Lemma 2.3(iv), one may prove, by induction on the set of stumps, that, for every stump \( \sigma, \tau \) and \( \gamma \), the set \( \text{Share}(X) \) belongs to the class \( \Sigma^0_2 \); suppose that \( \beta \) is an element of \( \mathcal{N} \) such that, for every \( \alpha \), \( \alpha \) belongs to \( X \) if and only if, for some \( n \), \( \alpha = \beta^n \); then, for every \( \alpha \), \( \alpha \) belongs to \( X \) if and only if, for some \( n \), for all \( m \), \( \alpha(\overline{\gamma}m) = 0 \) and thus the set \( \text{Share}(X) \) is seen to be a member of the class \( \Sigma^0_2 \).

As we know from Theorem 3.4(iii), a result proven in [11], the set \( \text{MonPath}_{01} \), consisting of all \( \alpha \) such that, for some \( \gamma \), for each \( n \), \( \gamma(n) = \gamma(n+1) \leq 1 \) and, for each \( n \), \( \alpha(\overline{\gamma}n) = 0 \), is not positively Borel. Now consider \( \mathcal{CB}^*_\gamma \), and note that, for each \( \gamma \), \( \gamma \) belongs to \( \mathcal{CB}^*_\gamma = \{0\} \cup \overline{\{n * (1)^* \mid n \in \mathbb{N}\}} \) if and only if, for all \( n \), \( m \), if \( \gamma(n) = 0 \) and \( \gamma(n) \neq 0 \), then \( \gamma(n) = 1 \) and \( m = n \). It is not difficult to see that the sets \( \text{Share}(\mathcal{CB}^*_\gamma) \) and \( \text{MonPath}_{01} \) reduce to each other. Note that, for each non-empty stump \( \sigma \), the set \( \text{Share}(\mathcal{CB}^*_\gamma) \) reduces to the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) and, therefore, the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) is not positively Borel.

(iii) One proves this by induction on the set of hereditarily increasing stumps, using Theorem 3.4.

(iv) It follows from (iii) and Theorem 3.4(iv) that, for each non-empty hereditarily increasing stump \( \sigma \), for each \( n \), the set \( \text{Share}(\mathcal{CB}^*_\gamma) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) and, therefore, by Theorem 3.4(i), also not to the set \( \text{Share}(\mathcal{CB}^*_\alpha) \). Now assume \( \sigma, \tau \) are hereditarily increasing stumps such that \( \sigma < \tau \). Find \( n \) such that \( \sigma \leq \tau^n \). Using (i), one easily concludes that the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\tau) \). The proof that, for all hereditarily increasing stumps \( \sigma, \tau, \) the set \( \text{Share}(\mathcal{CB}^*_\tau) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) is similar. \( \square \)

Let \( s \) belong to \( \mathbb{N} \) and \( \alpha \) to \( \mathcal{N} \). We define: \( b \) forbids \( \alpha \) if and only if there exist \( m, n \) such that \( \overline{\sigma}m < \text{length}(b) \) and \( b(\overline{\sigma}m) \neq 0 \). We define: \( b \) admits \( \alpha \) if and only if \( b \) does not forbid \( \alpha \).

For every \( \delta \) in \( \mathcal{N} \), we let \( \alpha_\delta \) be the element \( \beta \) of \( \mathcal{N} \) satisfying: for each \( n \), \( \beta(\overline{\delta}n) = 0 \) and, for each \( s \), if \( \delta \) does not pass through \( s \), then \( \beta(s) = \alpha(s) \).

3.6. An isolated but intriguing question. The following theorem reports a difficult but somewhat solitary fact. We conjecture but were unable to prove the more general result that, for every non-empty hereditarily increasing stump \( \sigma \), the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\sigma) \). The fact that, conversely, for every non-empty stump \( \sigma \), the set \( \text{Share}(\mathcal{CB}^*_\sigma) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\sigma) \), is a consequence of Theorem 3.8(ii).

Theorem 3.9. The set \( \text{Share}(\mathcal{CB}^*_\gamma) \) does not reduce to the set \( \text{Share}(\mathcal{CB}^*_\gamma) \).

Proof. Let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( \text{Share}(\mathcal{CB}^*_\gamma) \) to the set \( \text{Share}(\mathcal{CB}^*_\gamma) \).

We let \( \zeta \) be an element of \( \mathcal{N} \) such that \( \zeta^0 = 0 \) and, for each \( n \), \( \zeta^{n+1} = \overline{\{n * (1)^* \mid n \in \mathbb{N}\}} \). Note that \( \zeta \) is an enumeration of the set \( \mathcal{CB}^*_\gamma \).

Note that, for each \( \alpha \), for each \( n \), \( \alpha(\overline{\zeta^n}) \) belongs to the set \( \text{Share}(\mathcal{CB}^*_\gamma) \), and, therefore, \( \gamma(\overline{\alpha(\overline{\zeta^n})}) \) belongs to the set \( \text{Share}(\mathcal{CB}^*_\gamma) \). Using the Second Axiom of Countable Choice, we determine \( \delta \) in \( \mathcal{N} \) such that, for each \( n \), \( \delta^n \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for each \( \alpha \), for each \( n \), \( \gamma(\overline{\alpha(\overline{\zeta^n})}) \) admits \( \delta^n | \alpha \) and \( \delta^n | \alpha \) belongs to \( \mathcal{CB}^*_\gamma \). It follows that, for each \( \alpha \), for each \( n \), if \( \alpha \) admits \( \zeta^n \), then \( \alpha \) coincides with \( \alpha(\overline{\zeta^n}) \) and \( \gamma | \alpha \) admits \( \delta^n | \alpha \).
Our intention is to show that the set \( \overline{CB_1} \) coincides with the set \( CB_1 \). As we know that the set \( CB_1 \) is not a closed subset of \( \mathcal{N} \), we then will have obtained the desired contradiction.

We first prove the following important statement:

For all \( a \) such that \( \text{length}(a) > 0 \) and \( a(0) = 0 \), for all \( k \), there exist \( b, n \) such that \( n > k \) and \( a \sqsubseteq b \) and \( \mathcal{D}_k \subseteq \delta^n \) for, and, for each \( i < k \), \( b \) forbids \( \zeta^{i+1} \).

Let \( a, k \) be given and assume: \( \text{length}(a) > 0 \) and \( a(0) = 0 \). Find \( p \) such that, for all \( i \), if \( i > p \), then \( (\zeta^{i+1})_1 > \text{length}(a) \) and, therefore, \( a \) admits \( \zeta^{i+1} \), and consider the infinite sequence \( a * 0 \). For each \( i \leq k \), we consider the finite sequence \( \delta^{p+1}(a * 0) \).

Note that either there exists \( i \leq k \) such that \( \delta^{p+1}(a * 0) \) coincides with \( \mathcal{D}_k \), or we find \( i, j, m \) such that \( i < j < k \) and \( m < k \) such that \( \mathcal{D}_m * \langle 1 \rangle \) is an initial part of both \( \delta^{p+1}(a * 0) \) and \( \delta^{p+1}(a * 0) \). We show that the second one of these two alternatives can not occur.

Assume \( i < j < k \) and \( m < k \) and \( \mathcal{D}_m * \langle 1 \rangle \) is an initial part of both \( \delta^{p+1}(a * 0) \) and \( \delta^{p+1}(a * 0) \). We determine \( q \) such that \( \mathcal{D}_m * \langle 1 \rangle \) is an initial part of both \( \delta^{p+1}(a * 0) \) and \( \delta^{p+1}(a * 0) \). We then determine a function \( \epsilon \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \beta \),

\[
\begin{align*}
(1) & \quad \epsilon(\beta) \text{ passes through } a * 0, \\
(2) & \quad \text{for each } n, \text{ if } n \leq q, \text{ then } (\epsilon(\beta))(\overline{\zeta^{p+1}}(n)) = 0, \text{ and, if } n > q, \text{ then for each } n, \\
& \quad (\epsilon(\beta))(\overline{\zeta^{p+1}}(n)) = \beta^0(n - q - 1), \text{ and} \\
(3) & \quad \text{for each } n, \text{ if } n \leq q, \text{ then } (\epsilon(\beta))(\overline{\zeta^{p+1}}(n)) = 0, \text{ and, if } n > q, \text{ then for each } n,
\end{align*}
\]

\[
\begin{align*}
(4) & \quad (\epsilon(\beta))(\overline{\zeta^{p+1}}(n)) = \beta^3(n - q - 1), \text{ and}
\end{align*}
\]

Observe that, for every \( \beta \), \( \epsilon(\beta) \) admits \( \zeta^{p+1} \) if and only if \( \beta^0 = 0 \) and \( \epsilon(\beta) \) admits \( \zeta^{p+1} \) if and only if \( \beta^1 = 0 \), and, for every \( l \), if \( l \neq p + i \) and \( l \neq p + j \), then \( \epsilon(\beta) \) is not admitted by \( \epsilon(\beta) \).

Note that, for each \( \beta \), if \( \beta^0 = 0 \), then \( \epsilon(\beta) \) admits \( \zeta^{p+1} \), and, therefore, \( \gamma(\epsilon(\beta)) \) admits \( \delta^k(\epsilon(\beta)) \), and, as \( \epsilon(\beta) \) passes through \( a * \mathcal{D}_q \), \( \delta^k(\epsilon(\beta)) \) passes through \( \mathcal{D}_m * \langle 1 \rangle \), and \( \gamma(\epsilon(\beta)) \) admits \( \mathcal{D}_m * \langle 1 \rangle * 0 \).

Also note that, for similar reasons, for each \( \beta \), if \( \beta^1 = 0 \), then \( \gamma(\epsilon(\beta)) \) admits \( \mathcal{D}_m * \langle 1 \rangle * 0 \).

Conversely, for each \( \beta \), if \( \gamma(\epsilon(\beta)) \) admits \( \mathcal{D}_m * \langle 1 \rangle * 0 \), then \( \gamma(\epsilon(\beta)) \) belongs to \( \text{Share}(\overline{CB_1}) \), and \( \epsilon(\beta) \) belongs to \( \text{Share}(\overline{CB_1}) \), so there exists \( l \) such that \( \epsilon(\beta) \) admits \( \zeta^l \), and: either \( \epsilon(\beta) \) admits \( \zeta^{p+1} \) or \( \epsilon(\beta) \) admits \( \zeta^{p+1} \), and: either \( \beta^0 = 0 \) or \( \beta^1 = 0 \).

We thus see: for every \( \beta, \beta \) belongs to \( D^2(A_1) \) if and only if \( \gamma(\epsilon(\beta)) \) admits \( \mathcal{D}_m * \langle 1 \rangle * 0 \), that is: the set \( D^2(A_1) \) reduces to the set \( A_1 \).

It follows from Lemma 2.1(v) and (vii) that the set \( D^2(A_1) \) does not reduce to the set \( A_1 \).

Thus we are left with the first of the two alternatives mentioned above: there exists \( i \leq k \) such that \( \delta^{p+1}(a * 0) \) coincides with \( \mathcal{D}_k \). We now determine \( i \leq k \) and \( q \) such that the finite sequence \( \mathcal{D}_k \) is an initial part of the finite sequence \( \delta^{p+1}(a * 0) \) and we define: \( c := a * \mathcal{D}_q \). Finally, we determine \( b \) such that \( c \sqsubseteq b \), and, for every \( l < k \), \( b \) forbids \( \zeta^{i+1} \).

It will be clear that \( b \) satisfies the requirements.

Using the just-proven important statement repeatedly we construct infinite sequences \( a_0, a_1, a_2, \ldots \) and \( n_0, n_1, n_2, \ldots \) of natural numbers such that \( \text{length}(a_0) > 0 \) and \( a(0) = 0 \), and, for each \( k, a_k \sqsubseteq a_{k+1} \) and \( n_k < n_{k+1} \) and \( \mathcal{D}_k \subseteq (\delta^{n_k}(a_k)) \), and, for each \( l < k \), \( a_k \) forbids \( \zeta^{i+1} \).
Note that, for each $\alpha$, for each $k$, if $\alpha$ passes through $a_k$, then $\delta^n \kappa$ passes through $\overline{\mathbb{U}} k$.

We let $\eta$ be a function from $CB^*_1$ to $CB^*_1$, such that, for each $\lambda$ in $CB^*_1$, for each $k, k = \mu n(\lambda(n) = 1)$ if and only if $n_k = \mu n(\eta(\lambda)(n) = 1)$.

Note that $\lambda|\zeta^n = \zeta^n$ and, for each $k, \lambda|((\zeta^n)_k + 1) = \zeta^n k + 1$.

We now are ready to prove that the set $CB^*_1$, is a subset of the set $CB^*_1$.

Let $\lambda$ be an element of $CB^*_1$.

We let $\alpha$ be an element of $\mathcal{N}$ satisfying the following conditions:

1. $\alpha$ admits $\eta \lambda$, and
2. for each $\beta$ in $CB^*_1$, if $\beta \neq \eta \lambda$, then there exists $r$ such that $\alpha(\overline{\beta} r) \neq 0$, so $\alpha$ forbids $\beta$, and
3. for each $k$, if $\lambda$ passes through $\overline{\mathbb{U}} k$, then $\alpha$ passes through $a_k$.

We now claim that $\gamma|\alpha$ belongs to $\text{Share}(CB^*_1)$. In order to prove this claim, we have to construct $\nu$ in $CB^*_1$ such that $\gamma|\alpha$ admits $\nu$. We define $\nu$ as follows:

For each $\lambda$, if $\overline{\mathbb{U}} k = \overline{\mathbb{U}} k$, then $\overline{\mathbb{M}} n_k = \overline{\mathbb{M}} (n_k)$, and, if $k$ is the least $i$ such that $\lambda(i) \neq 0$, then $\nu = \delta^n \kappa|\alpha$.

Note that $\nu$ belongs to $CB^*_1$. Also note that $\gamma|\alpha$ admits $\nu$: for each $\lambda$, if $\mathbb{U} k = \overline{\mathbb{U}} k$, then $\alpha$ passes through $a_k$ and $\overline{\mathbb{U}} k \subseteq (\delta^n\kappa|\alpha), \delta^n \alpha$ passes through $\overline{\mathbb{U}} k$, and $\gamma|\alpha$ admits $\overline{\mathbb{U}} k = \overline{\mathbb{U}} k$; on the other hand, if $\overline{\mathbb{U}} k \neq \overline{\mathbb{U}} k$ and $k_0$ is the least $k$ such that $\lambda(k) \neq 0$, then $\lambda = \zeta^{k_0 + 1}$ and $\eta \lambda = \zeta^{k_0 + 1}$ and $\alpha$ admits $\zeta^{k_0 + 1}$, so $\alpha$ coincides with $\alpha(\zeta^{k_0 + 1})$ and $\gamma|\alpha$ admits $\nu = \delta^{k_0 + 1}\kappa|\alpha$, so $\gamma|\alpha$ admits $\overline{\mathbb{U}} k$.

We may conclude that $\gamma|\alpha$ belongs to $\text{Share}(CB^*_1)$. It follows that $\alpha$ belongs to $\text{Share}(CB^*_1)$. Thus we may determine $n$ such that $\lambda$ coincides with $\zeta^n$.

We now have seen that, for all $\lambda$ in $CB^*_1$, one may determine $n$ such that $\lambda$ coincides with $\zeta^n$, and, therefore, one may decide: either $\lambda = \underline{0}$ or $\lambda \neq \underline{0}$. As the set $CB^*_1$, is a spread one may apply Brouwer’s Continuity Principle and determine $m$ such that either, for every $\lambda$ in $CB^*_1$, if $\lambda$ passes through $\overline{\mathbb{U}} m$, then $\lambda = \underline{0}$, or, for every $\lambda$ in $CB^*_1$, if $\lambda$ passes through $\overline{\mathbb{U}} m$, then $\lambda \neq \underline{0}$. Both alternatives are false. \hfill $\Box$

3.7. Final observations on the two hierarchies. We define a binary operation $\oplus$ on the set of stumps, by induction, as follows:

1. For all stumps $\sigma, \sigma \oplus \emptyset = \oplus(\emptyset, \sigma) := \sigma$.
2. For all non-empty stumps $\sigma, \tau$, $\sigma \oplus \tau$ is the non-empty stump $\rho$ satisfying: for all $m, \rho^m = (\sigma^m) \oplus \tau$ and $\rho^{2m+1} = \sigma \oplus (\tau^m)$.

One might call $\sigma \oplus \tau$ the natural sum or the Hessenberg sum of the stumps $\sigma, \tau$.

Note that, for all non-empty stumps $\sigma, \tau$, for all $n, (\sigma^n) \oplus \tau < \sigma \oplus \tau$ and $\sigma \oplus (\tau^n) < \sigma \oplus \tau$, and, for each stump $\rho$, if, for all $n, (\sigma^n) \oplus \tau < \rho$ and $\sigma \oplus (\tau^n) < \rho$, then $\sigma \oplus \tau < \rho$.

Also note that the operation $\oplus$ is commutative and associative: for all stumps $\sigma, \tau, \sigma \oplus \tau \equiv \tau \oplus \sigma$ and, for all stumps $\sigma, \tau, \rho, (\sigma \oplus \tau) \oplus \rho \equiv \sigma \oplus (\tau \oplus \rho)$.

Lemma 3.10.

(i) There exists a continuous function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for all stumps $\sigma, \tau$, $\gamma$ maps the spread $CB^*_\sigma \oplus \tau$ onto the spread $CB^*_\sigma \times CB^*_\tau$ and the enumerable set $CB^*_\sigma \times CB^*_\tau$ onto the enumerable set $CB^*_\sigma \oplus \tau$.

(ii) For all subsets $X, Y$ of $\mathcal{N}$, if there exists a continuous function from $\mathcal{N}$ to $\mathcal{N}$ mapping the set $X$ onto the set $Y$, then the set $\text{Share}(Y)$ reduces to the set $\text{Share}(X)$.

(iii) There exists a continuous function from $\mathcal{N}$ to $\mathcal{N}$, reducing, for all subsets $X, Y$ of $\mathcal{N}$, the set $C(\text{Share}(X), \text{Share}(Y))$ to the set $\text{Share}(X \times Y)$.
(iv) For all stumps $\sigma, \tau$, the set $C(\text{Share}(CB^{*}_{\sigma, \tau}), \text{Share}(CB^{*}_{\sigma, \tau}))$ reduces to the set $\text{Share}(CB^{*}_{\sigma \oplus \tau})$ and the set $C(\text{Share}(CB^{*}_{\sigma}), \text{Share}(CB^{*}_{\sigma}))$ reduces to the set $\text{Share}(CB^{*}_{\sigma \oplus \tau})$.

Proof. (i) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ satisfying: for each $\delta, \epsilon$, $\gamma$ maps the infinite sequence $\overline{m}(2m) \ast (1) \ast \alpha$ onto the infinite sequence $\overline{m} \ast (1) \ast \alpha, 0$ and $\gamma$ maps the infinite sequence $\overline{m}(2m+1) \ast (1) \ast \alpha$ onto the infinite sequence $0, \overline{m} \ast (1) \ast \alpha$.

It is not difficult to see that $\gamma$ satisfies the requirements.

(ii) Let $X, Y$ be subsets of $\mathcal{N}$ and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all $\alpha, \alpha$ belongs to $Y$ if and only if there exists $\epsilon$ in $X$ such that $\gamma|\epsilon$ coincides with $\alpha$. Let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\delta, \epsilon$, $\delta(s) = 0$ if and only if, for each $t$, if $t \subseteq (\gamma|s)$, then $\beta(t) = 0$. Note that, for each $\beta$, there exists $\alpha$ in $Y$ such that $\beta$ admits $\alpha$ if and only if there exists $\epsilon$ in $X$ such that $\beta$ admits $\gamma|\epsilon$ if and only if there exists $\epsilon$ in $X$ such that $\beta$ admits $\gamma|\epsilon$. Clearly, the function $\delta$ reduces the set $\text{Share}(Y)$ to the set $\text{Share}(X)$.

(iii) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha, \alpha$ belongs to $Y$ if and only if both $\alpha^{0}(s_{i}) = 0$ and $\alpha^{1}(s_{i}) = 0$. Note that, for all subsets $X, Y$ of $\mathcal{N}$, for each $\alpha, \alpha$ belongs to $C(\text{Share}(X), \text{Share}(Y))$ if and only if $\alpha^{0}$ belongs to $\text{Share}(X)$ and $\alpha^{1}$ belongs to $\text{Share}(Y)$ if and only if there exists $\delta$ such that $\alpha^{0}$ admits $\delta_{t}$ and $\delta_{t}$ belongs to $X$ and $\alpha^{1}$ admits $\delta_{t}$ and $\delta_{t}$ belongs to $Y$ if and only if there exists $\delta$ such that $\delta$ belongs to $X \times Y$ and $\gamma|\alpha$ admits $\delta$. We thus see that $\gamma$ reduces the set $C(\text{Share}(X), \text{Share}(Y))$ to the set $\text{Share}(X \times Y)$.

(iv) Note that, for all stumps $\sigma, \tau$, the set $C(\text{Share}(CB^{*}_{\sigma}), \text{Share}(CB^{*}_{\tau}))$ reduces to the set $\text{Share}(CB^{*}_{\sigma} \times CB^{*}_{\tau})$, according to (iii), and that the set $\text{Share}(CB^{*}_{\sigma} = CB^{*}_{\tau})$ reduces to the set $\text{Share}(CB^{*}_{\sigma \oplus \tau})$, by (i) and (ii).

For analogous reasons, the set $C(\text{Share}((CB^{*}_{\sigma})\overline{m}), \text{Share}(CB^{*}_{\tau}))$ reduces to the set $\text{Share}(CB^{*}_{\sigma \oplus \tau})$.

A stump $\sigma$ is called a critical point for the natural sum $\oplus$ if and only if for all stumps $\sigma, \tau$, if both $\tau_{0} < \sigma$ and $\tau_{1} < \sigma$, then also $\tau_{0} \oplus \tau_{1} < \sigma$.

Recall that $J : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is a bijection, a so-called pairing function on $\mathbb{N}$. Its inverse functions are called $K, L$: for all $p, q$, there exists $n$ such that $K(n) = p$ and $L(n) = q$, and, for every $n, J(K(n), L(n)) = n$. Note that, for every $p$, there exist infinitely many $n$ such that $K(n) = p$.

Let $\sigma_{0}, \sigma_{1}, \sigma_{2}, \ldots$ be an infinite sequence of stumps.

For each $n$, we define a stump $\oplus_{i=0}^{n} \sigma_{i}$, as follows, by induction.

1. $\oplus_{i=0}^{0} \sigma_{i} := \sigma_{0}$, and
2. $\oplus_{i=0}^{n+1} \sigma_{i} := (\oplus_{i=0}^{n} \sigma_{i}) \oplus \sigma_{n+1}$.

Lemma 3.11.

(i) For every non-empty stump $\sigma$, $\sigma$ is a critical point for the natural sum $\oplus$ if and only if for all $m, n$, there exists $p$ such that $\sigma^{m} \oplus \sigma^{n} \leq \sigma^{p}$.

(ii) For every stump $\sigma$, there exists a stump $\rho$ such that $\sigma \leq \rho$ and $\rho$ is a critical point for the natural sum.

(iii) For every stump $\sigma$, if $\sigma$ is a critical point for the natural sum, then the two infinite sequences $\text{Share}(CB_{\sigma^{m}}), \text{Share}(CB_{\sigma^{n}}), \text{Share}(CB_{\sigma^2}), \ldots$ and $\text{Share}(CB_{\sigma^m}), \text{Share}(CB_{\sigma^{n}}), \text{Share}(CB_{\sigma^2}), \ldots$ are, each of them, conjunctively closed.
Proof. (i) The proof is left to the reader.

(ii) Let \( \sigma \) be a non-empty stump. We let \( \rho \) be the non-empty stump such that, for each \( n \), \( \rho^n = \bigoplus_{i=0}^{n-1} \sigma^{K(i)} \). The proof that \( \rho \) satisfies the requirements is left to the reader.

(iii) This is a consequence of (i) and Lemma 3.10(iv). \( \square \)

**Theorem 3.12.**

For every stump \( \sigma \) there exists a stump \( \tau \) such that \( \sigma < \tau \) and the sets \( \text{Share}(\text{CB}^*_\tau) \) and \( \text{Share}(\overline{\text{CB}^*_\tau}) \) are both disjunctively and conjunctively productive.

**Proof.** Let \( \sigma \) be a stump. We let \( \tau \) be the nonempty stump such that \( \tau^0 = \mathbb{H}(\sigma) \), and, for each \( n \), \( \tau^n+1 = \mathbb{H}(\tau^n \oplus \tau^n) \). Note that, for each \( n \), \( \tau^n \) is hereditarily increasing, and, therefore, by Theorem 3.8, the sets \( \text{Share}(\text{CB}^*_\tau) \) and \( \text{Share}(\overline{\text{CB}^*_\tau}) \) are disjunctively productive. Also observe that \( \sigma \leq \tau^0 < \tau \) and that, for each \( m, n \), if \( m \leq n \), then \( \tau^m \oplus \tau^n \leq \tau^{n+1} \), and therefore, by Lemma 3.11, \( \tau \) is a critical point for the natural sum. Finally note that, for each \( n \), \( \tau^n \leq \tau^{n+1} \) and, therefore, by Theorem 3.8, the set \( \text{Share}(\text{CB}^*_\tau) \) reduces to the set \( \text{Share}(\text{CB}^*_\tau^+1) \) and the set \( \text{Share}(\overline{\text{CB}^*_\tau}) \) reduces to the set \( \text{Share}(\overline{\text{CB}^*_\tau^+1}) \). It now follows from Theorem 3.6 and Lemma 3.11(iii), and from Theorem 3.4 that the sets \( \text{Share}(\text{CB}^*_\tau) \) and \( \text{Share}(\overline{\text{CB}^*_\tau}) \) are conjunctively and disjunctively productive. \( \square \)

4. An intermezzo on \( \text{Share}(\mathcal{C}) \)

From a classical point of view, Theorem [3.12] is very surprising.

In the first place, the classical mathematician does not see any difference between a Cantor-Bendixson-set \( \text{CB}^*_\gamma \) and its closure \( \overline{\text{CB}^*_\gamma} \).

Recall that a subset \( X \) of \( \mathcal{N} \) is called a fan or a finitary spread if and only if there exists \( \beta \) in \( \mathcal{C} \) such that,

1. for every \( s, s \) contains an element of \( X \) if and only if \( \beta(s) = 0 \), and
2. for every \( \alpha, \alpha \) belongs to \( X \) if and only if, for each \( n \), \( \beta(\langle n \rangle) = 0 \), and
3. for each \( s \), there are only finitely many numbers \( n \) such that \( \beta(s \langle n \rangle) = 0 \).

Note that, for each stump \( \sigma \), the set \( \overline{\text{CB}^*_\sigma} \) is a fan.

For this reason, the classical mathematician would say, every set \( \text{Share}(\overline{\text{CB}^*_\sigma}) \) is a closed subset of \( \mathcal{N} \).

For, suppose \( X \) is a fan, and let \( \beta \) in \( \mathcal{C} \) satisfy the condition: for every \( s, s \) contains an element of \( X \) if and only if \( \beta(s) = 0 \). Find \( \gamma \) in \( \mathcal{N} \) such that, for each \( n \), for each \( s \), if \( \text{length}(s) = n \), and \( \beta(s) = 0 \), then \( s \leq \gamma(n) \). Then, by the (constructively invalid) Infinity Lemma of D. König: for every \( \alpha, \alpha \) belongs to \( \text{Share}(X) \) if and only if \( \forall n \exists s \leq \gamma(n)|\beta(s) = 0 \). Clearly then, \( \text{Share}(X) \) is a closed subset of \( \mathcal{N} \). The classical mathematician will further conclude that, for each \( \sigma \), for each \( \tau \), the set \( \text{Share}(\overline{\text{CB}^*_\beta}) \) reduces to the set \( \text{Share}(\overline{\text{CB}^*_\tau}) \).

In this Section, we want to make some intuitionistic observations on the set \( \text{Share}(\mathcal{C}) \). Note that Cantor space \( \mathcal{C} \) is a fan.

Let \( X \) be a subset of \( \mathcal{N} \) and let \( B \) be a subset of \( \mathbb{N} \). We define: the set \( B \) is a bar in the set \( X \) if and only if, for each \( \alpha \) in \( X \), there exists \( n \) such that \( \langle \alpha \rangle n \) belongs to \( B \).

We now define an element \( \text{bar}_{01} \) of \( \mathcal{N} \) satisfying the following conditions:

1. for each \( n \), \( \text{length}(\text{bar}_{01}(n)) = n + 1 \), and
2. \( \text{bar}_{01}(0) = \langle 0 \rangle = \langle \langle \rangle \rangle \), and
3. for each \( n \), \( (\text{bar}_{01}(n + 1))(0) = ((\text{bar}_{01}(n))(0)) \ast 0 \) and \( (\text{bar}_{01}(n + 1))(1) = ((\text{bar}_{01}(n))(0)) \ast 1 \), and for each \( j, 0 < j < n \), then \( (\text{bar}_{01}(n + 1))(j + 1) = (\text{bar}_{01}(n))(j) \).

Note that, for each \( n \), the finite set \( \{ (\text{bar}_{01}(n))(0), (\text{bar}_{01}(n))(1), \ldots, (\text{bar}_{01}(n))(n) \} \) is a bar in \( \mathcal{C} \), and, for each \( i \), for each \( j, i < j \leq n \), then \( (\text{bar}_{01}(n))(i) \perp (\text{bar}_{01}(n))(j) \).
Theorem 4.1.

(i) For each subset $X$ of $\mathcal{N}$, if $X$ is a fan, then there exists a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $\mathcal{C}$ onto $X$.

(ii) For each subset $X$ of $\mathcal{N}$, if $X$ is a fan, then the set $\text{Share}(X)$ reduces to the set $\text{Share}(\mathcal{C})$.

(iii) For each stamp $\sigma$, the set $\text{Share}(CB^2)\setminus\sigma$ reduces to the set $\text{Share}(\mathcal{C})$, and the set $\text{Share}(\mathcal{C})$ does not reduce to the set $\text{Share}(CB^2)\setminus\sigma$.

(iv) For every infinite sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$, if, for each $n$, the set $X_n$ reduces to the set $\text{Share}(\mathcal{C})$, then the set $\bigcap_{n \in \mathbb{N}} X_n$ also reduces to the set $\text{Share}(\mathcal{C})$.

(v) For all subsets $X, Y$ of $\mathcal{N}$, if both $X$ and $Y$ reduce to the set $\text{Share}(\mathcal{C})$, then also $X \cup Y$ reduces to the set $\text{Share}(\mathcal{C})$.

Proof. (i) Suppose $X$ is a subset of $\mathcal{N}$ and a fan, and let $\beta$ in $\mathcal{C}$ satisfy the condition: for every $s$, $s$ contains an element of $X$ if and only if $\beta(s) = 0$. We determine $\gamma$ in $\mathcal{N}$ such that, for each $s$, $\gamma(s)$ is the code of a finite sequence enumerating the different numbers $i$ such that $\beta(s + (i)) = 0$, that is, for each $i$, $\beta(s + (i)) = 0$ if and only if there exists $j < \text{length}(\gamma(s))$ such that $i = (\gamma(s))(j)$, and for all $j$, for all $k$, if $j < k < \text{length}(s)$, then $(\gamma(s))(j) \neq (\gamma(s))(k)$. We then define $\delta$ in $\mathcal{N}$ such that, for each $s$, if $\beta(s) = 0$, then $\delta(s) \equiv \{0, 1\}$, and

\begin{enumerate}
  \item $\delta(\langle \rangle) = \{\}\text{ and}$
  \item for each $s$, if $\beta(s) = 0$, then for each $j$, if $j < \text{length}(\gamma(s))$, then $\delta(s) = \langle (\gamma(s))(j) = \delta(s) \supset \text{bar}_{\nu 1}(\text{length}(\gamma(s)))(j)\rangle$.
\end{enumerate}

Note that, for each $\zeta$, if, for each $n$, $\beta(\zeta(n)) = 0$, then there exists exactly one $\alpha$ in $\mathcal{C}$ such that, for each $n$, $\zeta(n) = 0$.

We now let $\varepsilon$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $\varepsilon(\alpha)$ passes through $\delta(s)$, then $\varepsilon|\alpha$ passes through $s$.

One may verify that $\varepsilon$ is a function mapping the set $\mathcal{C}$ strongly one-to-one onto the set $\mathcal{F}_\beta$.

(ii) Use (i) and Theorem 3.10(ii).

(iii) Let $\sigma$ be a stamp. Find hereditarily increasing stamps $\tau, \rho$ such that $\sigma \leq \tau < \rho$. According to Theorem 3.8(iv), the set $\text{Share}(CB^2)\setminus\sigma$ does not reduce to the set $\text{Share}(\mathcal{C})$. As, according to (ii), the set $\text{Share}(CB^2)\setminus\sigma$ reduces to the set $\text{Share}(\mathcal{C})$, the set $\text{Share}(\mathcal{C})$ does not reduce to the set $\text{Share}(CB^2)\setminus\sigma$. As, by Theorem 2.8(i), the set $\text{Share}(CB^2)\setminus\sigma$ reduces to the set $\text{Share}(\mathcal{C})$, the set $\text{Share}(\mathcal{C})$ also does not reduce to $\text{Share}(CB^2)\setminus\sigma$.

(iv) Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$, each of them reducing to the set $\text{Share}(\mathcal{C})$. Using the Second Axiom of Countable Choice, find $\gamma$ in $\mathcal{N}$ such that, for each $n$, $\gamma^n$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing the set $X_n$ to the set $\text{Share}(\mathcal{C})$.

We let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $\delta(\alpha)(s) = 0$ if and only if $s$ belongs to $\{0, 1\}$, and, for each $n$, if $n \leq \text{length}(s)$, then, for each $i$, if $i < \text{length}(s^n)$, then $(\gamma^n(\alpha))(s^n)i) = 0$. Note that, for each $\alpha$, for each $\beta$ in $\mathcal{C}$, $\delta(\alpha)$ admits $\beta$ if and only if, for each $n$, $\gamma^n(\alpha)$ admits $\beta^n$. It follows that, for each $\alpha$, $\alpha$ belongs to $\bigcap_{n \in \mathbb{N}} X_n$ if and only if for each $n$ there exists $\beta$ in $\mathcal{C}$ such that $\gamma^n(\alpha)$ admits $\beta$ if and only if there exists $\beta$ in $\mathcal{C}$ such that, for each $n$, $\gamma^n(\alpha)$ admits $\beta^n$ if and only there exists $\beta$ in $\mathcal{C}$ such that $\delta(\alpha)$ admits $\beta$ if and only if $\delta(\alpha)$ belongs to $\text{Share}(\mathcal{C})$.

(v) Let $X, Y$ be subsets of $\mathcal{N}$, both of them reducing to the set $\text{Share}(\mathcal{C})$. Let $\gamma, \delta$ be functions from $\mathcal{N}$ to $\mathcal{N}$ reducing the sets $X, Y$, respectively, to the set $\text{Share}(\mathcal{C})$. Let $\varepsilon$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $(\varepsilon(\alpha))(\langle \rangle) = 0$ and, for each $s$ in $\{0, 1\}$, $(\varepsilon(\alpha))(0) + s = (\gamma(\alpha))(s)$ and $(\varepsilon(\alpha))(1) + s = (\delta(\alpha))(s)$. Note that, for each $\alpha$, for each $\beta$ in $\mathcal{C}$, $\varepsilon(\alpha)$ admits $0$ if and only if $\gamma(\alpha)$ admits $\beta$ and $\varepsilon(\alpha)$ admits $1$ if and
only if $\delta(\alpha)$ admits $\beta$. It follows that, for each $\alpha$, $\alpha$ belongs to $X \cup Y$ if and only if either $\gamma(\alpha)$ or $\delta(\alpha)$ admits an element of $C$ if and only if $\varepsilon(\alpha)$ admits an element of $C$ if and only if $\varepsilon(\alpha)$ belongs to $\text{Share}(C)$. \hfill $\Box$

**Theorem 4.2.**

(i) For each stump $\sigma$, for every subset $B$ of $\mathbb{N}$, if $B$ is a bar in $CB_\sigma$, then there exists a finite subset $B'$ of $B$ such that $B'$ is a bar in $CB_\sigma$.

(ii) For each stump $\sigma$, for every subset $B$ of $\mathbb{N}$, if $\forall \beta \in CB_\sigma B \neg \exists n(\exists n \in B)$, then $\forall \forall \beta \in CB_\sigma \exists n(\exists n \in B)$.

(iii) For every stump $\sigma$, the set $E_1$ does not reduce to the set $\text{Share}(CB_\sigma)$ or to the set $\text{Share}(CB_\sigma \sigma)$.

**Proof.** (i) One proves this by induction on the set of stumps. Note that, for every subset $B$ of $\mathbb{N}$, if $B$ is a bar in $CB_\sigma$, then there exists $n$ such that $\{\beta n\} = \text{bar}(\sigma B)$.

Let $B$ be a subset of $\mathbb{N}$ and a bar in $CB_\sigma$. Find $n$ such that $\beta n$ belongs to $B$. Note that, for each $j < n$, the set of all $s$ such that $\beta j + (1) * s$ has an initial part in $B$ is a bar in $CB_\sigma'$. For each $j < n$, we now find, using the induction hypothesis, a finite subset $B_j$ of the set $B$ such that every element of $CB_\sigma$ passing through $\beta j + (1)$ has an initial part in $B_j$. Let $C$ be the set $\{\beta n\} \cup \bigcup_{j < n} B_j$. Note that $C$ is a finite subset of $B$ and a bar in $CB_\sigma$.

(ii) One proves this by induction on the set of stumps. Note that the statement is true if $\sigma = \emptyset$. Now assume that $\sigma$ is a non-empty stump and the statement has been verified for every immediate sub stump $\sigma''$ of $\sigma$. Let $B$ be a subset of $\mathbb{N}$ and a bar in $CB_\sigma$. Find $n$ such that $\beta n$ belongs to $B$. Note that, for each $j < n$, the set of all $s$ such that $\beta j + (1) * s$ has an initial part in $B$ is a bar in $CB_\sigma'$. For each $j < n$, we now find, using the induction hypothesis, a finite subset $B_j$ of the set $B$ such that every element of $CB_\sigma$ passing through $\beta j + (1)$ has an initial part in $B_j$. Let $C$ be the set $\{\beta n\} \cup \bigcup_{j < n} B_j$. Note that $C$ is a finite subset of $B$ and a bar in $CB_\sigma$.

(iii) Suppose that $\sigma$ is a stump and that $\gamma$ is a function from $N$ to $\hat{N}$ reducing the set $E_1$ to the set $\text{Share}(CB_\sigma)$ or to the set $\text{Share}(CB_\sigma')$. We study the infinite sequence $\gamma(\emptyset)$. Note that, for each $n$, we may decide: either there exists $s$ such that $\text{length}(s) = n$ and $s$ contains an element of $CB_\sigma$ and $\gamma(\emptyset)$ admits $s$, that is, for all $i$, if $i \leq n$ then $(\gamma(\emptyset)(\beta i)) = 0$ or there is no such $s$. We claim that the first alternative holds.

For assume, for some $n$, for all $s$, if $s$ contains an element of $CB_\sigma$ and $\text{length}(s) = n$, then there exists $i \leq n$ such that $(\gamma(\emptyset)(\beta i)) \not= 0$. Using the continuity of the function $\gamma$, we find $m$ such that, for every $\alpha$, if $\alpha$ passes through $\beta m$, then for all $s$, if $s$ contains an element of $CB_\sigma$ and $\text{length}(s) = n$, then there exists $i \leq n$ such that $(\gamma(\alpha)(\beta i)) = (\gamma(\emptyset)(\beta i)) \not= 0$. We must conclude that, for every $\alpha$, if $\alpha$ passes through $\beta m$, then $\gamma(\alpha)$ does not belong to $\text{Share}(CB_\sigma)$, and $\alpha$ itself does not belong to $E_1$, so $\alpha = \emptyset$. This is not true, as $\alpha$ may coincide with $\beta m * \frac{1}{\beta}$.

Using (i), we may conclude: $\forall \forall \beta \in CB_\sigma \exists n(\gamma(\emptyset)(\beta n)) \not= 0$.\hfill $\Box$
Using (ii), we obtain the conclusion: \( \forall \beta \in CB_\alpha' \rightarrow \exists n[\gamma(0)(3n) \neq 0] \), that is: 
\( \neg \exists \beta \in CB_\alpha \forall n[\gamma(0)(3n) = 0] \), and, therefore, \( \neg \beta(0) \) belongs to \( \text{Share}(CB_\alpha) \), and: 
\( \neg \exists n \in E_1 \). As \( \emptyset \notin E_1 \), we have a contradiction. \( \square \)

The following statement positively contradicts the Fan Theorem, Axiom 12. This statement is called: Kleene’s Alternative (to the Fan Theorem) in [22]. Kleene has shown that this alternative holds in constructive recursive mathematics where every \( \alpha \) in \( \mathcal{N} \) is supposed to be given by an algorithm in the sense of Turing.

There exists a decidable subset \( B \) of \( \mathcal{N} \) that is a bar in \( C \) while every finite subset \( B' \) of \( B \) positively fails to be a bar in \( C \), that is: 
\( \exists \beta \forall n \in \mathcal{C} \exists n[\beta(3n) \neq 0] \land \forall m \exists s \in \text{Bin}[\text{length}(s) = n \land \forall i \leq n[\beta(3i) = 0]]. \)

We also want to make some observations on the generalized Principle of Markov, or: Markov’s Principle. This the following statement:

For every \( \alpha \), if \( \neg \exists n[\alpha(n) \neq 0] \), then \( \exists n[\alpha(n) \neq 0] \).

Markov introduced and defended this principle for infinite sequences that are given by specifying a finite algorithm. The intuitionistic notion of an infinite sequence of natural numbers is wider; hence we are using the predicate ‘generalized’.

We do not want to introduce the generalized Principle of Markov as an axiom of intuitionistic analysis. Some of its consequences have been studied by J.R. Moschovakis, see [23].

**Theorem 4.3.**

(i) If the set \( E_1 \) reduces to the set \( \text{Share}(C) \), then there exists a decidable subset of \( \{0,1\}^* \) that is a function from \( \mathcal{N} \) onto \( \mathcal{N} \) and reduces to the set \( \text{Share}(C) \).

(ii) If Kleene’s Alternative holds, then there exists a strongly one-to-one surjective mapping from \( \mathcal{N} \) onto \( \mathcal{C} \) and the set \( E_1^1 \) reduces to the set \( \text{Share}(C) \).

(iii) The generalized Principle of Markov implies: Kleene’s Alternative is equivalent to the statement: the set \( E_1^1 \) reduces to the set \( \text{Share}(C) \), and also to the statement: the set \( E_1 \) reduces to the set \( \text{Share}(C) \).

(iv) The generalized Principle of Markov, (together with the (Restricted) Fan Theorem), implies: the set \( E_1 \) does not reduce to the set \( \text{Share}(C) \).

**Proof.** (i) Suppose: \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( E_1 \) to the set \( \text{Share}(C) \). Consider the infinite sequence \( \gamma(0) \) and observe, as in the proof of Theorem 3.2(iii): for each \( n \), there are finitely many elements \( s \) of \( \{0,1\}^* \) such that \( \text{length}(s) = n \), and we may decide if there exists \( s \) in \( \{0,1\}^* \) such that \( \text{length}(s) = n \) and, for all \( i \), if \( i \leq n \), then \( (\gamma(0))(3i) = 0 \), or not. Suppose we find \( n \) such that there is no such \( s \). We then calculate \( m \) such that, for all \( i \), if \( \alpha \) passes through \( \gamma(m) \), then, for all \( s \) in \( \{0,1\}^* \), if \( \text{length}(s) = n \), then there exists \( i \leq n \) such that \( (\gamma(\alpha))(3i) = 0 \). Now \( \gamma(m) \rightarrow 1 \) will belong to \( E_1 \), while \( (\gamma(m) \rightarrow 1) \) does not belong to \( \text{Share}(C) \). Contradiction.

Let \( T \) be the set of all elements \( s \) of \( \{0,1\}^* \) that are admitted by \( \gamma(0) \), that is, for all \( i \leq \text{length}(s), (\gamma(0))(3i) = 0 \). We just have seen that \( T \) is an infinite subtree of \( \{0,1\}^* \). On the other hand, \( \emptyset \) does not belong to \( E_1 \), and, therefore, \( \gamma(0) \) does not belong to \( \text{Share}(C) \), that is, the infinite tree \( T \) does not have an infinite path.

Find \( \beta \) in \( C \) such that \( T = D_\beta = \{s | \beta(s) = 1 \} \) and note: \( \forall \alpha \in C \rightarrow \exists n[\beta(3n) \neq 0] \land \forall m \exists s \in \text{Bin}[\text{length}(s) = n \land \forall i \leq n[\beta(3i) = 0]]. \)

(ii) Let \( B \) be a decidable subset of \( \mathcal{N} \) that is a bar in \( C \) while every finite subset \( B' \) of \( B \) positively fails to be a bar in \( C \). Define \( \beta \) in \( \mathcal{N} \) as follows: for each \( n, \beta(n) \) is the least number \( s \) such that \( s \in B \), and, for all \( i, j \), if \( i \leq \text{length}(s) \), then \( 3i \notin B \), and, for all \( j, j < n \), then \( \beta(n) \neq \beta(j) \). Note that \( \beta \) is well-defined because of the fact that, for each \( n \), the set \{\( \beta(0), \beta(1), \ldots, \beta(n-1) \}\) positively fails to be a bar in \( C \). Also note that, for each \( \alpha \) in \( C \), there exists exactly one \( n \) such that \( \beta(n) \) is an initial part of \( \alpha \), and
that, for all \( m, n \), if \( m < n \), then \( \beta(m) \perp \beta(n) \). We now let \( \gamma \) be an element of \( \mathcal{N} \) such that \( \gamma(\langle \cdot \rangle) = \langle \cdot \rangle \), and, for all \( s \), for all \( n \), \( \gamma(s \cdot \langle n \rangle) = \gamma(s) \cdot \beta(n) \). Finally, we let \( \varepsilon \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha \), for every \( n \), \( \varepsilon(\alpha) \) passes through \( \gamma(\langle \alpha \rangle) \). The reader herself may verify that \( \varepsilon \) is a strongly one-to-one and surjective function from \( \mathcal{N} \) onto \( \mathcal{C} \).

The conclusion that the set \( E_1 \) reduces to the set \( \text{Share}(\mathcal{C}) \) now follows from Lemma 3.10(ii), where it has been shown that, for all subsets \( X, Y \) of \( \mathcal{N} \), if there exists a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \) mapping \( X \) onto \( Y \), then the set \( \text{Share}(Y) \) reduces to the set \( \text{Share}(X) \). (Note that the set \( E_1 \) coincides with the set \( \text{Share}(\mathcal{N}) \).)

(iii) Assume the generalized principle of Markov.

By (i), if the set \( E_1 \) reduces to the set \( \mathcal{C} \), then \( \exists \beta(\forall \alpha \in \mathcal{C} \wedge \exists n[\beta(\langle \alpha \rangle) \neq 0] \wedge \forall m \exists s \in \text{Bin} \text{length}(s) = n \wedge \forall \gamma \leq n[\beta(\langle \gamma \rangle) = 0]) \) so, by Markov's principle, \( \exists \beta(\forall \alpha \in \mathcal{C} \exists n[\beta(\langle \alpha \rangle) \neq 0] \wedge \forall m \exists s \in \text{Bin} \text{length}(s) = n \wedge \forall \gamma \leq n[\beta(\langle \gamma \rangle) = 0]) \), that is: Kleene's Alternative.

By (ii), if Kleene's Alternative holds, then the set \( E_1 \) reduces to the set \( \mathcal{C} \), and, as the set \( E_1 \) reduces to the set \( E_1 \), also the set \( E_1 \) reduces to the set \( \mathcal{C} \).

(iv) this is an immediate consequence of (iii). \( \square \)

5. **Analytic and strictly analytic subsets of \( \mathcal{N} \)**

5.1. **Analytic and strictly analytic subsets of \( \mathcal{N} \) and different kinds of closed subsets of \( \mathcal{N} \)**

Let \( X \) be a subset of \( \mathcal{N} \). \( X \) is called *strictly analytic* if and only if there exists a function \( \gamma \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that \( X \) coincides with the range of \( \gamma \), that is, for every \( \alpha \), \( \alpha \) belongs to \( X \) if and only if, for some \( \beta \), \( \alpha \) coincides with \( \gamma(\beta) \).

Note that every strictly analytic subset \( X \) of \( \mathcal{N} \) is *inhabited*, that is, there exists \( \alpha \) such that \( \alpha \) belongs to \( X \).

Every strictly analytic subset \( X \) of \( \mathcal{N} \) is an analytic and inhabited subset of \( \mathcal{N} \), but, conversely, it is not true that every analytic and inhabited subset of \( \mathcal{N} \) is strictly analytic, as we shall see in a moment.

The intuitionistic mathematician thus has to distinguish analytic subsets of \( \mathcal{N} \) from strictly analytic subsets of \( \mathcal{N} \).

The need to make the distinction between analytic and strictly analytic subsets of \( \mathcal{N} \) is closely connected to the fact that, in intuitionistic mathematics, there are at least four important notions corresponding to the classical notion of a *closed subset of \( \mathcal{N} \).*

Let us start with the weakest notion. A subset \( X \) of \( \mathcal{N} \) is called *sequentially closed* if and only if every \( \alpha \) such that, for each \( n \), \( \overline{\alpha}n \) contains an element of \( X \), belongs itself to \( X \).

Using the Second Axiom of Countable Choice, one may observe: a subset \( X \) of \( \mathcal{N} \) is sequentially closed if and only if: for all \( \alpha, \beta \) belongs to \( X \) if and only if there exists \( \beta \) such that, for each \( n \), \( \beta^n \) belongs to \( X \) and \( \overline{\beta}n = \overline{s}n \), that is: \( \alpha \) is the limit of an infinite sequence of elements of \( \mathcal{N} \).

Let \( s \) belong to \( \mathbb{N} \) and let \( \alpha \) belong to \( \mathcal{N} \). \( \alpha \) passes through \( s \), or \( s \) contains \( \alpha \) if and only if there exists \( n \) such that \( \overline{\alpha}n = s \).

A subset \( X \) of \( \mathcal{N} \) is called *located*, if the set of all \( s \) that contain an element of \( X \) is a *decidable* subset of \( \mathbb{N} \), that is, there exists \( \beta \) in \( \mathcal{N} \) such that, for every \( s \), \( s \) contains an element of \( X \) if and only if \( \beta(s) = 0 \).

A subset \( X \) of \( \mathcal{N} \) is called a *spread* if and only if \( X \) is both sequentially closed and located.

Spreads have been introduced in intuitionistic mathematics by Brouwer himself. Their importance is that Brouwer’s important assumptions: the Continuity Principle and the bar theorem, generalize from Baire space to spreads.

A subset \( X \) of \( \mathcal{N} \) is called *semi-located* if the set of all \( s \) that contain an element of \( X \) is an enumerable subset of \( \mathbb{N} \), that is, there exists \( \gamma \) in \( \mathcal{N} \) such that, for every \( s, s \)
contains an element of $X$ if and only if there exists $n$ such that $\gamma(n) = s + 1$. A subset $X$ of $\mathcal{N}$ is called closed-and-separable if and only if $X$ is both sequentially closed and semi-located.

Every spread is a closed-and-separable subset of $\mathcal{N}$, but not conversely, as we shall see in a moment. The notion of a closed-and-separable subset of $\mathcal{N}$ plays an important role in [42].

Finally, a subset $X$ of $\mathcal{N}$ is called closed if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for each $n$, $\beta(n)$ belongs to $C$, that is, if and only if there exists $\beta$ in $\mathcal{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for all $n$, $\beta(n) = 0$. This notion has been used in the definition of the positive Borel hierarchy in [32] and [15]. In this context, a subset $X$ of $\mathcal{N}$ is called open if and only if there exists a decidable subset $C$ of $\mathbb{N}$ such that, for all $\alpha$, $\alpha$ belongs to $X$ if and only if there exists $n$ such that $\beta(n)$ belongs to $C$. It is not difficult to see that a subset $X$ of $\mathcal{N}$ is open if and only if there exists an enumerable subset $C$ of $\mathbb{N}$ such that, for all $\alpha$, $\alpha$ belongs to $X$ if and only if there exists $n$ such that $\beta(n)$ belongs to $C$. A subset $X$ of $\mathcal{N}$ is closed if and only there exists an open subset $Y$ of $\mathcal{N}$ such that $X = Y^\complement$, that is, for all $\alpha$, $\alpha$ belongs to $X$ if and only if $\alpha$ does not belong to $Y$.

Note that every closed subset of $\mathcal{N}$ is a sequentially closed subset of $\mathcal{N}$.

Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of subsets of $\mathcal{N}$.

We let $D_{n \in \mathbb{N}}(X_n)$, the disjunction of the infinite sequence $X_0, X_1, X_2, \ldots$, be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $n$, $\alpha^n$ belongs to $X_n$.

We let $C_{n \in \mathbb{N}}(X_n)$, the conjunction of the infinite sequence $X_0, X_1, X_2, \ldots$, be the set of all $\alpha$ in $\mathcal{N}$ such that, for all $n$, $\alpha^n$ belongs to $X_n$.

**Theorem 5.1.**

(i) For every sequentially closed subset of $X$ of $\mathcal{N}$, $X$ is strictly analytic if and only if $X$ is semi-located.

(ii) Every inhabited spread, that is, every inhabited and located sequentially closed subset of $\mathcal{N}$ is a strictly analytic subset of $\mathcal{N}$.

(iii) It is not true that every sequentially closed and semi-located subset of $\mathcal{C}$ is a located subset of $\mathcal{C}$.

(iv) It is not true that every closed and semi-located subset of $\mathcal{N}$ is a located subset of $\mathcal{N}$.

(v) It is not true that every inhabited and closed subset of $\mathcal{N}$ is semi-located.

(vii) It is not true that, for all subsets $X, Y$ of $\mathcal{N}$, if $X, Y$ are spreads and $X \cap Y$ is inhabited, then $X \cap Y$ is strictly analytic.

(viii) For every infinite sequence $X_0, X_1, X_2, \ldots$ of subsets of $\mathcal{N}$, if, for each $n$, the set $X_n$ is strictly analytic, then the sets $D_{n \in \mathbb{N}}(X_n)$ and $C_{n \in \mathbb{N}}(X_n)$ are, both of them, strictly analytic.

(ix) For every subset $X$ of $\mathcal{N}$, if $X$ is strictly analytic, then $Ex(X)$ is strictly analytic. In particular, if $X$ is a spread, or a sequentially closed strictly analytic subset of $\mathcal{N}$, then $Ex(X)$ is strictly analytic.

**Proof.** (i) Let us first assume that $X$ is a sequentially closed strictly analytic subset of $\mathcal{N}$ and that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $\gamma$. Observe that for each $s, s$ contains a member of the range of $\gamma$ if and only if there exists $a$ such that $s \subseteq \gamma(a)$. Let $\delta$ be an element of $\mathcal{N}$ such that, for each $n_1$, if $n_1 \subseteq \gamma(n_1)$, then $\delta(n_1) = n_1 + 1$, and if not, then $\delta(n_1) = 0$. Note that, for each $s, s$ contains an element of the range of $\gamma$ if and only if there exists $n$ such that $\delta(n) = s + 1$. It follows that $X$ is semi-located and a closed-and-separable subset of $\mathcal{N}$.
Now assume that that $X$ is an inhabited semi-located sequentially closed subset of $\mathcal{N}$ and that $\delta$ is an element of $\mathcal{N}$ such that, for each $s$, $s$ contains an element of $X$ if and only if there exists $n$ such that $\delta(n) = s + 1$.

Note that, as $X$ is inhabited, there exists $n$ such that $\delta(n) = \langle \rangle + 1$. Observe that, for every $n$, for every $s$, if $\delta(n) = s + 1$, then there exist $m, p$ such that $\delta(m) = \delta(n) \ast (p) + 1$, that is, $\delta(m) - 1$ is an immediate successor of $\delta(n)$. We now construct $\varepsilon$ such that for each $s$, length($s$) = length($\varepsilon(s)$), as follows. We define $\varepsilon(\langle \rangle) := \langle \rangle$ and, for each $s, n, \varepsilon(s \langle n \rangle) := \delta(n) - 1$ if $\delta(n) - 1$ is an immediate successor of $\varepsilon(s)$, and $\varepsilon(s \ast (n)) = \delta(p) - 1$, if $\delta(n)$ is not an immediate successor of $\varepsilon(s)$ and $p$ is the least $m$ such that $\delta(m) - 1$ is an immediate successor of $\varepsilon(s)$. We then let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that for every $n$, for every $\alpha, \gamma(\alpha) \ast s$ passes through $\varepsilon(\langle \alpha \rangle n)$. One verifies easily that the sequentially closed set $X$ coincides with the range of $\gamma$.

(ii) This is an easy consequence of (i).

(iii) For every $\alpha$, we let $C_\alpha$ be the set of all $s$ in $\mathbb{N}$ such that either $\emptyset$ passes through $s$, or there exists $n$ such that $\alpha(n) \neq 0$ and $\underline{1}$ passes through $s$, and we let $X_\alpha$ be the set of all $\beta$ in $\mathcal{N}$ such that, for all $n$, $\beta n$ belongs to $C_\alpha$. Note that, for each $\alpha$, if $\alpha = \emptyset \beta n = \{0\}$ then $X_\alpha = \{0\} \sqrt{x}$, and, if there exists $n$ such that $\alpha(n) \neq 0$, then $X_\alpha = \{0\} \cup \{1, p \ast \alpha \in \mathcal{N}\}$, where $p$ is the least $n$ such that $\alpha(n) \neq 0$. Also note that $X_\alpha$ is a closed subset of $\mathcal{N}$, and that $X_\alpha$ is weakly located, for, if we define $\gamma$ in $\mathcal{N}$ such that, for each $n$, if either $\emptyset$ passes through $n_{II}$, or there exists $\gamma(p)$ such that $\langle \rangle \neq 0$ and $\underline{1}$ passes through $n_{II}$, then $\gamma(n) = n_{II} + 1$, and, if not, then $\gamma(n) = 0$, then, for each $s$, $s$ contains an element of $X_\alpha$ if and only if, for some $n$, $\gamma(n) = s + 1$.

Now assume that, for each $\alpha$, $X_\alpha$ is a spread. Then, for each $\alpha$, either (1) contains an element of $X_\alpha$, or (1) does not contain an element of $X_\alpha$ and, therefore, either, for some $n, \alpha(n) \neq 0$, or: $\alpha = \emptyset$. We easily obtain a contradiction by the Continuity Principle.

(iv) For every $\alpha$, we let $C_\alpha$ be the set of all $s$ in $\mathbb{N}$ such that either $\emptyset$ passes through $s$ or $s(0) = 1$ and, if length($s$) > 1 then $s(2)$ is the least $n$ such that $\alpha(n) \neq 0$ and we let $X_\alpha$ be the set of all $\beta$ in $\mathcal{N}$ such that, for all $n$, $\beta n$ belongs to $C_\alpha$. Note that, for each $\alpha$, if $\alpha = \emptyset \beta n = \{0\}$ then $X_\alpha = \{0\} \sqrt{x}$, and, if there exists $n$ such that $\alpha(n) \neq 0$, then $X_\alpha = \{0\} \cup \{1, p \ast \alpha \in \mathcal{N}\}$, where $p$ is the least $n$ such that $\alpha(n) \neq 0$. Also note that $X_\alpha$ is a closed subset of $\mathcal{N}$, and that $X_\alpha$ is weakly located, for, if we define $\gamma$ in $\mathcal{N}$ such that, for each $n$, if either $\emptyset$ passes through $n_{II}$, or there exists $p < n_{II}$ such that $p$ is the least $j$ such that $\alpha(j) \neq 0$ and $n_{II} = 1$, or length($n_{II}$) > 1 and $n_{II}(0) = 1$ and $n_{II}(1)$ is the least $j$ such that $\alpha(j) \neq 0$, then $\gamma(n) = n_{II} + 1$, and, if not, then $\gamma(n) = 0$, then, for each $s$, $s$ contains an element of $X_\alpha$ if and only if, for some $n$, $\gamma(n) = s + 1$.

Now assume that, for each $\alpha$, $X_\alpha$ is a spread. Then, for each $\alpha$, either (1) contains an element of $X_\alpha$, or (1) does not contain an element of $X_\alpha$ and, therefore, either, for some $n, \alpha(n) \neq 0$, or: $\alpha = \emptyset$. We easily obtain a contradiction by the Continuity Principle.

(v) For each $\alpha$, we let $B_\alpha$ the set of all $s$ in $\mathbb{N}$ such that either $\emptyset$ passes through $s$ or $\beta n = \{0\}$ and $\underline{1}$ passes through $s$. We let $Y_\alpha$ be the set of all $\beta$ such that, for each $n, \beta n$ belongs to $B_\alpha$. Note that $Y_\alpha$ is a closed subset of $C$. Let $C_\alpha$ be the set of all $s$ that contain an element of $Y_\alpha$. Observe that, for each $s$, $s$ belongs to $C_\alpha$ if and only if either $\emptyset$ passes through $s$ or $\alpha = \emptyset$ and $\underline{1}$ passes through $s$. Assume that, for each $\alpha$, the set $Y_\alpha$ is semi-located. Then, for each $\alpha$, there exists $\gamma$ such that, for each $s$, $s$ belongs to $C_\alpha$ if and only if there exists $n$ such that $\gamma(n) = s + 1$. Applying the Second Axiom of Continuous Choice we find a function $e$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for all $\alpha$, for each $s, s$ belongs to $C_\alpha$ if and only if there exists $n$ such that $\gamma(n) = s + 1$. Calculate $m$ such that $\gamma(\emptyset)(m) = (1) + 1$. Calculate $p$ such that, for every $\alpha$, if $\emptyset \ast p = \emptyset \ast p$, then $\langle \rangle \ast (\langle \rangle n) = \langle \rangle \ast (\langle \rangle n) = (1) + 1$. It follows that, for every $\alpha$, if $\alpha \ast s$ passes through $\emptyset \ast p$, then $\underline{1}$ belongs to $Y_\alpha$ and $\alpha = \emptyset$. Contradiction.

(vi) For each $\alpha$, we let $Z_\alpha$ be the set of all $\beta$ in $\mathcal{N}$ such that, $\beta = \emptyset$ and $\alpha \neq \emptyset$. Note that, for each $\alpha$, $Z_\alpha$ is sequentially closed. Assume that, for each $\alpha$, $Z_\alpha$ is a closed
subset of $\mathcal{N}$, that is, there exists $\gamma$ such that, for each $\beta$, $\beta$ belongs to $Z_\alpha$ if and only if, for each $n$, $\gamma(\beta n) = 0$. Applying the Second Axiom of Continuous Choice we find a function $e$ from $\mathcal{N}$ to $\mathcal{N}$ such that for all $\alpha$, for each $\beta$, $\beta$ belongs to $Z_\alpha$ if and only if, for each $n$, $(e(\alpha)(\beta n)) = 0$. Note that, for each $n$, $(e(\alpha)(\beta n)) = 0$. (Let $n$ be a natural number. Find $p$ such that, for each $\alpha$, if $\alpha_\beta = \beta_p$, then $(e(\alpha)(\beta n)) = (e(\alpha)(\beta n))$. Consider $\alpha = \alpha_{p+1} \ast \beta$ and note: $\alpha \neq 0$, and therefore, $(e(\alpha)(\beta n)) = 0$ and $(e(\alpha)(\beta n)) = 0$.) We must conclude: $\alpha \neq 0$: contradiction.

This argument is almost the same as the argument that $E_1$ positively fails to reduce to $A_1$, see [15], Theorem 6.2.

(vii) For each $\alpha$ we let $V_\alpha$ be the set of all $\beta$ such that, for each $n$, either $\beta n = \beta m$ or $\beta n = \beta m = \beta n = \beta n = \beta n = 1$ and $m = n - k - 1$. Note that, for all $\alpha$, the set $V_\alpha$ is a spread. Assume that, for all $\alpha$, for all $\beta$, the set $V_\alpha \cap V_\beta$ is strictly analytic. It then follows, from (i), that, for each $\alpha$, there exists $\delta$ in $\mathcal{N}$ such that, for each $s$, $s$ contains an element of $V_\alpha \cap V_\beta$ if and only if, for some $n$, $\delta(n) = s + 1$.

Applying the Second Axiom of Continuous Choice we find a function $e$ from $\mathcal{N}$ to $\mathcal{N}$ such that for all $\alpha$, $e(\alpha)$, for each $s$, $s$ contains an element of $V_\alpha \cap V_\beta$ if and only if, for some $n$, $(e(\alpha)(n)) = s + 1$.

Note that (1) contains an element of $V_\alpha \cap V_\beta$ and find $p$ such that $(e(\alpha)(p)) = (1)$. Then find $m$ such that, for each $\alpha$, if $\alpha m = \beta m$, then $(e(\alpha)(p)) = (e(0)(p))$. It follows that, for each $\alpha$ passing through $\beta m$, (1) contains an element of $V_\alpha \cap V_\beta$. Note that, for each $\alpha$, (1) contains an element of $V_\alpha \cap V_\beta$ if and only if, for all $k$, if $\alpha$ is the least $n$ such that $\alpha^0(n) \neq 0$, then $k$ is also the least $n$ such that $\alpha^1(n) \neq 0$. Now consider a sequence $\alpha$ with the property $\alpha m = \beta m$ and $\alpha^0 = \beta m \ast 1$ and $\alpha^1 = \beta m + 1$. On the one hand, the set $V_\alpha \cap V_\beta$ coincides with the set $\{0\}$, but, on the other hand (1) contains an element of the set $V_\alpha \cap V_\beta$. Contradiction.

(viii) Let $X_0, X_1, X_2, \ldots$ be an infinite sequence of strictly analytic subsets of $\mathcal{N}$. Using the Second Axiom of Countable Choice, find $\gamma$ in $\mathcal{N}$, such that, for each $n$, $\gamma^0$ is a function from $\mathcal{N}$ to $\mathcal{N}$ such that $X_n$ coincides with the range of $\gamma^0$.

Let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $\delta(\alpha)^0(0) = \gamma^0(0)(\alpha^0(0))$, and, for each $i$, if there is no $j$ such that $i = \delta(0) \ast j$, then $\delta(\alpha)(i) = \alpha(i)$. Note that $D_{\alpha \in \mathbb{N}}(X_n)$ coincides with the range of $\delta$.

Let $\varepsilon$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $n$, $(\varepsilon(\alpha)^n = \gamma^0(0)(\alpha^0))$, and $(\varepsilon(\alpha)(0)) = (\alpha(0))$. Note that, $C_{\alpha \in \mathbb{N}}(X_n)$ coincides with the range of $\varepsilon$.

(ix) Let $X$ be a strictly analytic subset of $\mathcal{N}$. Find a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\beta$, $\alpha = \gamma(\beta)$. Note that, for each $\alpha$, $\alpha$ belongs to $Ex(X)$ if and only if, for some $\beta$, $\alpha = (\gamma)(\beta)^0$. Thus, $Ex(X)$ is seen to be strictly analytic.

We thus see that the class of the strictly analytic subsets of $\mathcal{N}$ is a proper subclass of the class of the analytic subsets of $\mathcal{N}$, and that the class of the strictly analytic subsets of $\mathcal{N}$ is not closed under the operation of finite intersection.

Nevertheless, there are many strictly analytic sets.

Note that the sets $E_1 = \{ \alpha | \exists n[\alpha(n) \neq 0] \}$ and $A_1 = \{ 0 \}$ are strictly analytic.

Recall, from Subsection 1.2, that $1^*$ is the element of $\mathcal{C}$ satisfying: $1^*(0) = 0$, for each $n > 0$, $1^*(n) = 1$.

Recall that a stump is called a non-zero stump if and only if either: $\sigma = 1^*$ or: $\sigma(0) = 0$ and, for each $n$, $\sigma((n)) = 0$ and $\sigma^n$ is a non-zero stump.

Recall that, for each non-zero stump $\sigma$, we defined subsets $E_\sigma$ and $A_\sigma$ of $\mathcal{N}$ as follows, by induction.
\(E_{1^*} := E_1\) and \(A_{1^*} = A_1\) and, for each non-zero stump different from \(1^*\), for each \(\alpha\), \(\alpha\) belongs to \(E_\sigma\) if and only if, for some \(n\), \(\alpha^n\) belongs to \(A_{\sigma^n}\), and \(\alpha\) belongs to \(A_\sigma\) if and only if, for all \(n\), \(\alpha^n\) belongs to \(E_{\sigma^n}\).

Note that, for each non-zero stump \(\sigma\), \(E_\sigma := D_{\sigma} (A_{\sigma^n})\) and \(A_\sigma := C_{\sigma} (E_{\sigma^n})\).

These sets have been introduced in [12] and are called there the leading sets of the positive Borel hierarchy. It follows from Theorem 5.1(viii) that they are, all of them, strictly analytic.

5.2. The Brouwer-Kripke axiom intervenes. The following axiom is the result of an attempt to capture Brouwer’s creating subject arguments.

Axiom 13 (The Brouwer-Kripke axiom).

Let \(P\) be a definite mathematical proposition.

There exists \(\beta\) in \(\mathcal{C}\) such that \(P\) if and only if, for some \(n\), \(\beta(n) = 1\).

The word ‘definite’ we use in the formulation of this axiom needs some explanation. \(P\) is not allowed to be a statement about an infinite sequence we are building step by step without having fixed its future by an unequivocal description. In a formal context, one might require that the formula corresponding to the proposition should not contain a free variable over elements of Baire space.

The idea that gave rise to the axiom is that, once we are given a definite proposition \(P\), we can start thinking about it, and the truth of \(P\) will be equivalent to our finding a proof of \(P\) at some point of time, where time is numbered by the natural numbers. So, step by step, we can build an infinite sequence \(\beta\) with the property: for each \(n\), \(\beta(n) \neq 0\) if and only if, at stage \(n\), I find a proof of proposition \(P\).

This seems to be a rather wild idea, and, certainly, it seems untenable if we allow \(P\) to be a statement about an object that itself is unfinished.

It is no surprise, therefore, that the axiom leads to a contradiction with the Second Axiom of Continuous Choice, Axiom[7] as has been first observed by J. Myhill, see [25]:

Suppose that for every \(\alpha\) there exists \(\beta\) such that \(\forall n [\alpha(n) = 0] \) if and only if \(\exists n [\alpha(n) \neq 0]\). By Axiom [7], there exists a function \(\gamma\) from \(N\) to \(N\) such that, for all \(\alpha\), \(\forall n [\alpha(n) = 0] \) if and only if \(\exists n [\gamma(\alpha)(n) \neq 0]\), so \(A_1\) reduces to \(E_1\). Contradiction.

One might decide now, either to give up the Second Axiom of Continuous Choice, or to give up the Brouwer-Kripke axiom completely, or to restrict the applicability of the Brouwer-Kripke axiom. Myhill chose the first alternative. Johan de Jongh opted for the third one, see [10]. It is difficult to make the restriction he wanted to introduce precise, and we will not attempt to do so. We only observe that the restriction also has to be applied to the definitions of mathematical objects like subsets of the set \(\mathbb{N}\) of the natural numbers or subsets of Baire space \(\mathcal{N}\).

Theorem 5.2 (Some consequences of the Brouwer-Kripke axiom).

(i) Every definite subset of \(\mathbb{N}\) is an enumerable subset of \(\mathbb{N}\).

(ii) Every definite subset of \(\mathcal{N}\) is a semilocated subset of \(\mathcal{N}\).

(iii) Every definite analytic subset of \(\mathcal{N}\) is a strictly analytic subset of \(\mathcal{N}\).

Proof. (i) Let \(A\) be a definite subset of \(\mathbb{N}\). Given any \(n\) in \(\mathbb{N}\), the Brouwer-Kripke axiom guarantees there exists \(\beta\) in \(\mathcal{C}\) such that \(n\) belongs to \(A\) if and only if, for some \(n\), \(\beta(n) = 1\). Using the Second Axiom of Countable Choice, we find \(\beta\) in \(\mathcal{C}\) such that for all \(n\) in \(\mathbb{N}\), \(n\) belongs to \(A\) if and only if, for some \(m\), \(\beta^m(m) = 1\). Let \(\delta\) be an element of \(\mathcal{N}\) such that, for all \(p\), if there exist \(n, m\) such that \(p = (n, m)\) and \(\beta^m(m) = 1\), then \(\delta(p) = n + 1\) and, if not, then \(\delta(p) = 0\). Note that the set \(A\) coincides with the \(E_\delta\). (\(E_\delta\) is the subset of \(\mathbb{N}\) enumerated by \(\delta\), that is, the set of all \(n\) in \(\mathbb{N}\) such that, for some \(p\), \(\delta(p) = n + 1\)).
(ii) Let $X$ be a definite subset of $\mathcal{N}$. The set of all $s$ in $\mathbb{N}$ such that $s$ contains an element of $X$ will be a definite subset of $\mathbb{N}$, so we may apply (i) and conclude that this set is enumerable and that $X$ itself is semi-located.

(iii) Let $Y$ be an analytic subset of $\mathcal{N}$. Let $X$ be a closed subset of $\mathcal{N}$ such that $Y = Ex(X)$. We may assume that $X$ is a definite subset of $\mathcal{N}$ and apply (ii). $X$ is closed and semi-located and therefore, according to Theorem 5.1(i), strictly analytic. But then, according to Theorem 5.1(ix), also the set $Y = Ex(X)$ is strictly analytic. □

John Burgess, in [7], also studies strictly analytic subsets of $\mathcal{N}$, calling them, by a term of of Brouwer’s, "dressed spreads". (This term is also used in [10]). Applying an unrestricted Brouwer-Kripke-axiom, in a context avoiding the Second Axiom of Continuous Choice, Burgess proves that every analytic subset of $\mathcal{N}$ is strictly analytic. The argument given here for Theorem 5.2(iii) is essentially his.

6. Separation Theorems

A version of the next Theorem occurs in [32]. A related result is proven in [1].

For each $\alpha$ in $\mathcal{N}$, we define elements $\alpha_I$ and $\alpha_{II}$ of $\mathcal{N}$ by:

For each $n$, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n + 1)$.

Note that, for each $\alpha$, $\alpha = \langle \alpha_I, \alpha_{II} \rangle$.

For each $s$ in $\mathbb{N}$, we define elements $s_I$ and $s_{II}$ of $\mathbb{N}$ by:

- $\text{length}(s_I)$ is the least $k$ such that $\text{length}(s) \geq 2k$ and, for each $n$, if $n < \text{length}(s_I)$, then $s_I(n) = s(2n)$, and
- $\text{length}(s_{II})$ is the least $k$ such that $\text{length}(s) \geq 2k + 1$ and, for each $n$, if $n < \text{length}(s_{II})$, then $s_{II}(n) = s(2n + 1)$.

**Theorem 6.1** (Lusin’s Separation Theorem).

Let $\gamma$ be an element of $\mathcal{N}$ such that both $\gamma_I$ and $\gamma_{II}$ are functions from $\mathcal{N}$ to $\mathcal{N}$ and, for every $\alpha$, $\gamma_I[\alpha_I]$ is apart from $\gamma_{II}[\alpha_{II}]$. There exist positively Borel sets $B_0, B_1$, such that for every $\alpha$, $\gamma_I[\alpha_I]$ belongs to $B_0$ and $\gamma_{II}[\alpha_{II}]$ belongs to $B_1$, and every element of $B_0$ is apart from every element of $B_1$.

**Proof.** We use the Principle of Induction on Monotone Bars in $\mathcal{N}$. We let $P, Q$ be subsets of $\mathbb{N}$, such that, for every $s$ in $\mathbb{N}$,

- $P(s) := \gamma_I|s_I \bot \gamma_{II}|s_{II}$,
- $Q(s) := \text{There exist positively Borel sets } B_0, B_1 \text{ such that every element of } B_0 \text{ is apart from every element of } B_1, \text{ and, for every } \alpha, \text{ if } \alpha \text{ passes through } s, \text{ then } \gamma_I[\alpha_I] \text{ belongs to } B_0 \text{ and } \gamma_{II}[\alpha_{II}] \text{ belongs to } B_1$.

Clearly, for all $\alpha$ in $\mathcal{N}$, there exists $n$ such that $\bar{n}n$ belongs to $P$ and, for all $s$, for all $n$, if $s$ belongs to $P$, then $s \ast \langle n \rangle$ belongs to $P$. Also, $P$ is a subset of $Q$. Now assume $s$ belong to $P$ and, for every $n$, $s \ast \langle n \rangle$ belongs to $Q$. We want to prove that $s$ itself belongs to $Q$ and we distinguish two cases.

**Case (1).** $\text{length}(s)$ is even. Note that now, for each $n$, $(s \ast \langle n \rangle)_I = s_I \ast \langle n \rangle$ and $(s \ast \langle n \rangle)_{II} = s_{II}$. Using the Second Axiom of Countable Choice, we determine an infinite sequence $(B_{0,0}, B_{0,1}), (B_{1,0}, B_{1,1}), (B_{2,0}, B_{2,1}), \ldots$ of pairs of positively Borel sets such that, for each $n$, for every $\alpha$ passing through $s \ast \langle n \rangle$, for every $n$, $\gamma_I[\alpha_I]$ belongs to $B_{0,n}$ and $\gamma_{II}[\alpha_{II}]$ belongs to $B_{n,1}$, and every element of $B_{n,0}$ is apart from every element of $B_{n,1}$.

Define $B_0 := \bigcup_{n \in \mathbb{N}} B_{n,0}$ and $B_1 := \bigcap_{n \in \mathbb{N}} B_{n,1}$.

Note that, for each $\alpha$, if $\alpha$ passes through $s$, then $\gamma_I[\alpha_I]$ belongs to $B_0$ and $\gamma_{II}[\alpha_{II}]$ belongs to $B_1$, and every element of $B_0$ is apart from every element of $B_1$. 48
Case (2). length(s) is odd. Note that now, for each $n$, $(s*(n))_{II} = s_I$ and $(s*(n))_{II} = s_I * \langle n \rangle$. Using the Second Axiom of Countable Choice, we determine an infinite sequence $(B_{0,0}, B_{0,1}, (B_{1,0}, B_{1,1}), (B_{2,0}, B_{2,1}), \ldots)$ of pairs of positively Borel sets such that, for each $n$, for every $\alpha$ passing through $s * \langle n \rangle$, for every $n$, $\gamma_I|\alpha_I$ belongs to $B_{n,0}$ and $\gamma_I|\alpha_{II}$ belongs to $B_{n,1}$, and every element of $B_{n,0}$ is apart from every element of $B_{n,1}$.

Define $B_0 := \bigcap_{n \in \mathbb{N}} B_{n,0}$ and $B_1 := \bigcup_{n \in \mathbb{N}} B_{n,1}$.

Note that, for each $\alpha$, if $\alpha$ passes through $s$, then $\gamma_I|\alpha_I$ belongs to $B_0$ and $\gamma_I|\alpha_{II}$ belongs to $B_1$, and every element of $B_0$ is apart from every element of $B_1$.

In both cases, therefore, we find that $s$ itself belongs to $Q$.

The Principle of Induction on Monotone Bars in $\mathcal{N}$ allows us to conclude that the pair $\langle \rangle$ belongs to $Q$, and the conclusion of the Theorem follows. $\square$

Lusin’s Separation Theorem admits of several extensions.

Recall that, for each $\alpha$ in $\mathcal{N}$, for each $i$ in $\mathbb{N}$ we defined an elements $\alpha^i$ of $\mathcal{N}$ by:

for each $n$, $\alpha^i(n) = \alpha((i) * n)$.

For each $s$ in $\mathbb{N}$, for each $i$, we now define an elements $s^i$ of $\mathbb{N}$ by:

length($s^i$) is the least $k$ such that length($s$) $\geq (i) * k$ and, for each $n$, if $n < length(s^i)$, then $s^i(n) = s((i) * n)$.

Let $n$ be a positive natural number and let $X_0, X_1, \ldots, X_{n-1}$ be a finite sequence of subsets of $\mathcal{N}$ of length $n$. We call the finite sequence $X_0, X_1, \ldots, X_{n-1}$ positively disjoint if and only if, for every $\beta$ in $\mathcal{N}$, if, for each $i < n$, $\beta^i$ belongs to $X_i$, then there exist $i, j$ such that $i < n$ and $j < n$ and $\beta^i$ is apart from $\beta^j$.

**Theorem 6.2** (An extension of Lusin’s Separation Theorem).

Let $n$ be a positive natural number and let $\gamma$ be an element of $\mathcal{N}$ such that, for each $i < n$, $\gamma^i$ is a function from $\mathcal{N}$ to $\mathcal{N}$ and, for every $\alpha$, there exist $i, j < n$ such that $\gamma^i|\alpha^i$ is apart from $\gamma^j|\alpha^j$. There exists a finite sequence $B_0, B_1, \ldots, B_{n-1}$ of positively Borel sets, such that for every $\alpha$, for every $i < n$, $\gamma^i|\alpha^i$ belongs to $B_i$, and the finite sequence $B_0, B_1, \ldots, B_{n-1}$ is positively disjoint.

**Proof.** We use the Principle of Induction on Monotone Bars in $\mathcal{N}$. We let $P, Q$ be subsets of $\mathbb{N}$, such that, for every $s$ in $\mathbb{N}$,

$P(s) :=$ There exist $i, j < n$ such that $\gamma^i|s^i \perp \gamma^j|s^j$.

$Q(s) :=$ There exists a $n$-sequence $(B_0, B_1, \ldots, B_{n-1})$ of positively Borel sets that is positively disjoint and such that, for every $\alpha$, if $\alpha$ passes through $s$, then, for each $i < n$, $\gamma^i|\alpha^i$ to $B_i$.

Clearly, for all $\alpha$ in $\mathcal{N}$, there exists $n$ such that $\alpha n$ belongs to $P$ and, for all $s$, for all $n$, if $s$ belongs to $P$, then $s * \langle n \rangle$ belongs to $P$.

The following argument shows that $P$ is a subset of $Q$. Suppose that $s$ belongs to $P$.

Find $i, j < n$ such that $\gamma^i|s^i \perp \gamma^j|s^j$. Define a finite sequence of positively Borel sets $(B_0, B_1, \ldots, B_{n-1})$ such that $B_i$ is the set of all $\alpha$ passing through $\gamma^i|s^i$, and $B_j$ is the set of all $\alpha$ passing through $\gamma^j|s^j$, and, for all $k < n$, if $k \neq i$ and $k \neq j$, then $B_k = \mathcal{N}$. Note that the finite sequence $(B_0, B_1, \ldots, B_{n-1})$ is positively disjoint and for every $\alpha$, if $\alpha$ passes through $s$, then, for each $i < n$, $\gamma^i|\alpha^i$ to $B_i$.

Now assume $s$ belong to $\mathcal{N}$ and, for every $m$, $s * \langle m \rangle$ belongs to $Q$. We want to prove that $s$ itself belongs to $Q$ and we do so as follows.

First, find $i, p$ such that length($s$) = $(i) * p$. Note that, for each $n$, $(s * \langle n \rangle)^i = s^i * \langle n \rangle$, and, for each $j$, if $j \neq i$, then $(s * \langle n \rangle)^j = s^j$.

We now distinguish two cases.
Case (1). \( i \geq n \). Note that, in this case, the statement that \( s \) belongs to \( Q \) is equivalent to the statement that \( s \ast \langle 0 \rangle \) belongs to \( Q \), so we may conclude that \( s \) belongs to \( Q \).

Case (2). \( i < n \). Using the Second Axiom of Countable Choice, we determine an infinite sequence \( (B_{0,0}, B_{0,1}, \ldots, B_{0,n-1}), (B_{1,0}, B_{1,1}, \ldots, B_{1,n-1}), (B_{2,0}, B_{2,1}, \ldots, B_{2,n-1}), \ldots \) of positively disjoint \( n \)-sequences of \( n \)-sequences of positively Borel sets such that, for each \( m \), for every \( \alpha \) passing through \( s \ast \langle m \rangle \), for every \( j < n \), \( \gamma^j | \alpha^i \) belongs to \( B_{m,j} \). Define \( B_i := \bigcup_{m \in \mathbb{N}} B_{m,i} \) and, for each \( j \), if \( j < n \) and \( j \neq i \), then \( B_j := \bigcap_{m \in \mathbb{N}} B_{m,j} \).

Note that, for each \( \alpha \), if \( \alpha \) passes through \( s \), then \( \alpha \) passes through \( s^i \ast \langle \alpha(\text{length}(s)) \rangle \), and \( \gamma^i | \alpha^i \) belongs to \( B_0(\text{length}(s)), i \), and, therefore, to \( B_i \).

Also note that, for each \( \alpha \), for each \( j < n \), if \( \alpha \) passes through \( s \), then \( \alpha^j \) passes through \( s^j \), and, for each \( m \), \( \alpha^j \) passes through \( (s \ast \langle m \rangle)^j \), as \( (s \ast \langle m \rangle)^j \) coincides with \( s^j \), and, therefore, \( \gamma^j | \alpha^j \) belongs to each set \( B_{m,j} \) and thus to \( B_j \).

Finally, observe that the \( n \)-sequence \( (B_0, B_1, \ldots, B_{n-1}) \) is positively disjoint:

Suppose that \( \beta \) belongs to \( \mathcal{N} \), and, for each \( j < n \), \( \beta^j \) belongs to \( B_j \). Find \( m \) such that \( \beta^j \) belongs to \( B_{m,j} \). Note that, for each \( j \), if \( j < n \) and \( j \neq i \), then \( \beta^j \) belongs to \( B_{m,j} \).

So we may find \( k, l < n \) such that \( \beta^k \) is apart from \( \beta^l \).

In both cases, therefore, we find that \( s \) itself belongs to \( Q \).

The Principle of Induction on Monotone Bars in \( \mathcal{N} \) allows us to conclude that \( \langle \ast \rangle \) belongs to \( Q \), and the conclusion of the theorem follows.

\[ \square \]

Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathcal{N} \). We call the infinite sequence \( X_0, X_1, \ldots \) positively disjoint if and only if, for every \( \beta \) in \( \mathcal{N} \), if, for each \( i, \beta^i \) belongs to \( X_i \), then there exist \( i, j \) such that \( \beta^i \) is apart from \( \beta^j \).

**Theorem 6.3** (Novikov’s Separation Theorem).

Let \( \gamma \) be an element of \( \mathcal{N} \) such that, for each \( i, \gamma^i \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) and, for every \( \alpha \), there exist \( i, j \) such that \( \gamma^i | \alpha^i \) is apart from \( \gamma^j | \alpha^j \). There exists an infinite sequence \( B_0, B_1, \ldots \) of positively Borel sets such that, for every \( \alpha \), for every \( i, \gamma^i | \alpha^i \) belongs to \( B_i \), and the infinite sequence \( B_0, B_1, \ldots \) is positively disjoint.

**Proof.** We use the Principle of Induction on Monotone Bars in \( \mathcal{N} \). We let \( P, Q \) be subsets of \( \mathbb{N} \), such that, for every \( s \) in \( \mathbb{N} \),

\[
P(s) := \text{There exist } i, j \text{ such that } \gamma^i | s^i \perp \gamma^j | s^j.
\]

\[
Q(s) := \text{There exist an infinite sequence of positively Borel sets } B_0, B_1, \ldots \text{ that is positively disjoint and such that for every } \alpha, \text{ if } \alpha \text{ passes through } s, \text{ then, for each } i, \gamma^i | \alpha^i \text{ belongs to } B_i.
\]

Clearly, for all \( \alpha \) in \( \mathcal{N} \), there exists \( n \) such that \( \overline{\alpha n} \) belongs to \( P \) and, for all \( s \), for all \( n \), if \( s \) belongs to \( P \), then \( s \ast \langle n \rangle \) belongs to \( P \).

The following argument shows that \( P \) is a subset of \( Q \). Suppose that \( s \) belongs to \( P \). Find \( i, j \), such that \( \gamma^i | s^i \perp \gamma^j | s^j \). Define an infinite sequence of positively Borel sets \( B_0, B_1, \ldots \) such that \( B_i \) is the set of all \( \alpha \) passing through \( \gamma^i | s^i \), and \( B_j \) is the set of all \( \alpha \) passing through \( \gamma^j | s^j \), and, for all \( k, \) if \( k \neq i \) and \( k \neq j \), then \( B_k = \mathcal{N} \). Note that the infinite sequence \( B_0, B_1, \ldots \) is positively disjoint and, for every \( \alpha \), if \( \alpha \) passes through \( s \), then, for each \( i, \gamma^i | \alpha^i \) belongs to \( B_i \).

Now assume \( s \) belong to \( \mathbb{N} \) and, for every \( m, s \ast \langle m \rangle \) belongs to \( Q \). We want to prove that \( s \) itself belongs to \( Q \) and we do so as follows.

First, find \( i, p \) such that \( \text{length}(s) = (i) \ast p \). Note that, for each \( n, (s \ast \langle n \rangle)^i = s^i \ast \langle n \rangle \), and, for each \( j, \) if \( j \neq i \), then \( (s \ast \langle n \rangle)^j = s^j \).

Using the Second Axiom of Countable Choice, we determine an infinite sequence \( B_{0,0}, B_{0,1}, \ldots, B_{1,0}, B_{1,1}, \ldots, B_{2,0}, B_{2,1}, \ldots \) of positively disjoint infinite sequences of positively Borel sets such that, for each \( m \), for every \( \alpha \) passing through \( s \ast \langle m \rangle \),
for every \( j < n \), \( \gamma^j|\alpha^j \) belongs to \( B_{m,j} \). Define \( B_i := \bigcup_{m \in \mathbb{N}} B_{m,i} \) and, for each \( j \), if \( j \neq i \), then \( B_j := \bigcap_{m \in \mathbb{N}} B_{m,j} \).

Note that, for each \( \alpha \), if \( \alpha \) passes through \( s \), then \( \alpha^i \) passes through \( s^i \langle \alpha(\text{length}(s)) \rangle \), and \( \gamma^j|\alpha^j \) belongs to \( B_\alpha(\text{length}(s),i), \) and, therefore, to \( B_i \).

Also note that, for each \( \alpha \), for each \( j < n \), if \( \alpha \) passes through \( s \), then \( \alpha^j \) passes through \( s^j \), and, for each \( m \), \( \alpha^j \) passes through \( (s \ast \langle m \rangle)^j \), as \( (s \ast \langle m \rangle)^j \) coincides with \( s^j \), and, therefore, \( \gamma^j|\alpha^j \) belongs to each set \( B_{m,j} \) and thus to \( B_j \).

Finally, observe that the infinite sequence \( B_0, B_1, \ldots \) is positively disjoint:

Suppose that \( \beta \) belongs to \( \mathcal{N} \), and, for each \( j < n \), \( \beta^j \) belongs to \( B_j \). Find \( m \) such that \( \beta^m \) belongs to \( B_{m,i} \). Note that, for each \( j \), if \( j \neq i \), then \( \beta^j \) belongs to \( B_{m,j} \). So we may find \( k, l \) such that \( \beta^k \) is apart from \( \beta^l \).

We have to admit that \( s \) itself belongs to \( Q \).

The Principle of Induction on Monotone Bars in \( \mathcal{N} \) allows us to conclude that the pair \( \langle \rangle \) belongs to \( Q \), and the conclusion of the Theorem follows.

\[ \square \]

**Theorem 6.4 (One half of Lusin’s Regular Representation Theorem).**

Let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \), such that for all \( \alpha, \beta \), if \( \alpha \) is apart from \( \beta \), then \( \gamma|\alpha \) is apart from \( \gamma|\beta \).

Then the range of \( \gamma \), that is, the set of all \( \alpha \) coinciding with some \( \gamma|\beta \), is positively Borel.

**Proof.** Assume that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) and that for all \( \alpha, \beta \), if \( \alpha \) is apart from \( \beta \), then \( \gamma|\alpha \) is apart from \( \gamma|\beta \).

We intend to define a sequence \( H_0, H_1, \ldots \) of positively Borel sets such that the range of \( \gamma \) coincides with \( \bigcap_{n \in \mathbb{N}} H_n \).

To this end, we first define a sequence \( B_0, B_1, \ldots \) of decidable subsets of \( \mathbb{N} \). Each of these will be a bar in \( \mathcal{N} \).

We let \( B_0 \) be the set \( \{ \langle \rangle \} \) consisting of the empty sequence only. For each \( n \), we let \( B_{n+1} \) be the set of all \( a \) of minimal length such that some proper initial segment of \( a \) belongs to \( B_n \), and for some initial part \( b \) of \( a \), \( \gamma^n(b) \neq 0 \). Observe that for each \( n \), for all \( a \) in \( B_n \), for all \( \alpha \) passing through \( a \), \( \gamma^n(\alpha)\bar{n} \) is an initial part of \( \gamma|a \), and that, for each \( n \), \( B_n \) is a decidable subset of \( \mathbb{N} \) and a bar in \( \mathcal{N} \).

Note that, for each \( n, B_n \) is a thin bar in \( \mathcal{N} \), that is, for all \( a, b \) in \( B_n \), if \( a \neq b \), then \( a \perp b \). It follows that, for each \( n \), for all \( a, b \) in \( B_n \), if \( a \neq b \), then, for each \( \alpha \) passing through \( a \), for each \( \beta \) passing through \( b \), \( \alpha \) is apart from \( \beta \), and, therefore, \( \gamma|\alpha \) is apart from \( \gamma|\beta \). Using Theorem [13] and the Second Axiom of Countable Choice we find for each \( n \) for all \( a, b \) in \( B_n \) such that \( a \neq b \), positively Borel sets \( C_{a,b}, D_{a,b} \) such that for all \( \alpha \), if \( \alpha \) passes through \( a \), then \( \gamma|\alpha \) belongs to \( C_{a,b} \) and, if \( \alpha \) passes through \( b \), then \( \gamma|\alpha \) belongs to \( D_{a,b} \) and every element of \( C_{a,b} \) is apart from every element of \( D_{a,b} \). For each \( n \), for each \( a \) in \( B_n \), we let \( K_a \) be the set of all \( \beta \) passing through \( \gamma|a \) and belonging to every set \( C_{a,b} \cap D_{b,a} \), where \( b \) belongs to \( B_n \) and \( b \neq a \). Remark that for all \( n \), for all \( a, b \) in \( B_n \), if \( a \neq b \), then every element of \( K_a \) is apart from every element of \( K_b \). For each \( n \), for each \( a \) in \( B_n \) we let \( L_a \) be the set of all \( \beta \) belonging to every set \( K_b \), where \( b \) is an initial segment of \( a \), and for some \( i \leq n, b \) belongs to \( B_i \).

Note that, for each \( n \), for all \( a \) in \( B_n \), for all \( b \) in \( B_{n+1} \), if \( a \) is an initial part of \( B \), then \( L_b \subseteq L_a \). It follows that, for each \( n \), for each \( k \), for all \( a \) in \( B_n \), for all \( b \) in \( B_{n+k} \), if \( a \) is an initial part of \( b \), then \( L_b \subseteq L_a \).

Note that, for each \( n \), for all \( a \) in \( B_n \), for all \( b \) in \( B_{n+1} \), if \( a \) is not an initial part of \( b \), then \( L_a \cap L_b = \emptyset \), where \( c \) the initial part of \( b \) that belongs to \( B_n \), as \( a \) must be different from \( c \) and, therefore, \( L_a \cap L_b = \emptyset \).

It follows that, for each \( n \), for each \( k \), for all \( a \) in \( B_n \), for all \( b \) in \( B_{n+k} \), if \( a \) is not an initial part of \( b \), then \( L_a \cap L_b = \emptyset \).
We define, for each $n$, $H_n := \bigcup_{a \in B_n} L_a$. It will be clear that the range of $\gamma$ is a subset of every set $H_n$.

Now assume that $\beta$ belongs to every set $H_n$. For each $n$, we determine $a_n$ in $B_n$ such that $\beta$ belongs $L_{(a_n)}$. Observe that, for each $n$, $a_n$ must be a proper initial segment of $a_{n+1}$ and consider the sequence $\alpha$ that passes through every $a_n$. Observe that, for each $n$, $\beta$ belongs to $K(a_n)$ and therefore passes through $\gamma|(a_n)$ and also through $\gamma(\alpha)n$, as $\gamma(\alpha)n$ is an initial part of $\gamma|(a_n)$, so $\beta$ coincides with $\gamma(\alpha)$.

Therefore, the range of $\gamma$ coincides with the set $\bigcap_{n \in \mathbb{N}} H_n$. Note that the latter set is positively Borel.

We just proved, in Theorem 6.4, one half of Lusin’s classical regular Representation Theorem. The other half is the statement:

“For every subset $X$ of $\mathcal{N}$, if $X$ is Borel, then there exist a closed subset $Y$ of $\mathcal{N}$ and a one-to-one continuous function $f$ from $Y$ onto $X$.”

From a constructive point of view, this statement may be given several interpretations, none of them without its difficulties. We do not want to go through the various possible interpretations and only make the following observation. Suppose that we interpret “Borel” by “positively Borel” and “closed subset” by “spread” and that we define:

A subset $X$ of $\mathcal{N}$ is regular in Lusin’s sense or: has Lusin’s property if and only if there exists a subset $F$ of $\mathcal{N}$ that is a spread and a strongly one-to-one continuous function from $F$ onto $X$.

Note that every subset of $\mathcal{N}$ that is regular in Lusin’s sense is also strictly analytic. We have seen, in the first and the fifth item of Theorem 5.1 that not every inhabited and closed subset of $\mathcal{N}$ is strictly analytic, and, therefore, not every inhabited and closed subset of $\mathcal{N}$ is regular in Lusin’s sense.

What about positive Borel sets that are strictly analytic? One might hope that all such sets are regular in Lusin’s sense, but the following theorem refutes this conjecture. We define:

Let $X$ be a strictly analytic subset of $\mathcal{N}$. We say that $X$ positively fails to be regular in Lusin’s sense if, for every subset $F$ of $\mathcal{N}$ that is a spread, for every function $\gamma$ that is a function from $F$ to $\mathcal{N}$ mapping $F$ onto $X$, there exist $\alpha, \beta$ in $F$ such that $\alpha \neq \beta$ and $\gamma(\alpha) = \gamma(\beta)$.

Note that the set $D^2(\mathcal{A}_1) = \{\alpha|\alpha^0 = 0 \lor \alpha^1 = 0\}$ is strictly analytic. Define a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, if $\alpha(0) = 0$, then $\gamma(\alpha)^0 = 2$ and for each $n$, if there is no $m$ such that $n = (0, m)$, then $\gamma(\alpha)(n) = \alpha(n)$ and, if $\alpha(0) \neq 0$, then $\gamma(\alpha)^1 = 0$, and for each $n$, if there is no $m$ such that $n = (1, m)$, then $\gamma(\alpha)(n) = \alpha(n)$. One easily verifies that the set $D^2(\mathcal{A}_1)$ coincides with the range of $\gamma$.

**Theorem 6.5.** The set $D^2(\mathcal{A}_1)$ positively fails to be regular in Lusin’s sense.

**Proof.** Suppose we find a subset $F$ of $\mathcal{N}$ that is closed and located and an element $\gamma$ in $\mathcal{N}$ such that $\gamma$ is a continuous function from $F$ onto $D^2(\mathcal{A}_1)$. For each $i < 2$, we let $P_i$ be the set of all $\alpha$ such that $\alpha^i = 0$. Note that $D^2(\mathcal{A}_1) = P_0 \cup P_1$ and that $P_0$ and $P_1$ are spreads. Using the Second Axiom of Continuous Choice, we find $\delta$ such that, for both $i < 2$, $\delta^i$ is a continuous function from $P_i$ to $F$ such that, for all $\alpha$ in $P_i$, $\gamma(\delta^i(\alpha)) = \alpha$.

Define $\varepsilon$ in $\mathcal{N}$ such that $\varepsilon^0 := \delta^0 \varepsilon$ and $\varepsilon^1 := \delta^1 \varepsilon$.

We claim: for each $n$, there exists $\zeta$ in $F$ such that $\zeta n = \varepsilon^0 n$ and $(\gamma|\zeta)^1 \neq 0$. We prove this claim as follows. Let $n$ belong to $N$. Find $m$ such that, for all $\alpha$ in $P_0$, if $m \alpha n = \overline{\alpha} m n$, then $\overline{\alpha} \varepsilon \alpha n = \overline{\alpha} \varepsilon n$. Determine $\alpha$ in $P_0$ such that $\overline{\alpha} m n = \overline{\alpha} n$ and $\alpha^1(m) \neq 0$ and consider $\zeta = \delta^0 \alpha$. Note that $\zeta n = \overline{\alpha} \varepsilon n$ and $(\gamma|\zeta)^1(m) \neq 0$. 

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Using the fact that $\gamma$ maps the spread $F$ to the set $D^2(A_1)$ and that $\varepsilon$ belongs to $F$, and Brouwer’s Continuity Principle, we determine $n_0$ and $i_0$ such that $i_0 < 2$ and, for all $\zeta$ in $F$, if $\zeta n_0 = \varepsilon n_0$, then $(\gamma|\zeta)^i_0 = 0$. As there exists $\zeta$ in $F$ passing through $\varepsilon n_0$ such that $(\gamma|\zeta)^1 \neq 0$, we must have: $i_0 = 0$.

We conclude: for all $\zeta$ in $F$, if $\zeta n_0 = \varepsilon n_0$, then $(\gamma|\zeta)^0 = 0$.

In a similar way, one may determine $n_1$ in $\mathbb{N}$ such that, for all $\zeta$ in $F$, if $\zeta n_0 = \varepsilon n_0$, then $(\gamma|\zeta)^1 = 0$.

Clearly, then, $\varepsilon n_0 \perp \varepsilon n_1$ and: $\varepsilon^0 \neq \varepsilon^1$.

We thus see that the set $D^2(A_1)$ positively fails to be regular in Lusin’s sense. □

As the set $D^2(A_1)$ is a fairly simple example of a positively Borel set that is also strictly analytic, Theorem 6.4 shows that it is not so easy, for a strictly analytic positively Borel set, to have Lusin’s property. The set $E_2^1$, to be discussed in the next Section, see Theorem 6.4, is an example of a set that is positively Borel and strictly analytic and has Lusin’s property, but, like the set $D^2(A_1)$, fails to be co-analytic. It is not true, therefore, that positively Borel sets that have Lusin’s property must be co-analytic.

It is a pity that we do not have a satisfying intuitionistic counterpart to the other half of Lusin’s Theorem. Lusin observed that his representation theorem may help one to believe, in spite of possible qualms about generalized inductive definitions, that, after all, the collection of all positively Borel subsets of $\mathcal{N}$ is a set, see [20], pp. 38-39.

7. Co-analytic subsets of $\mathcal{N}$: some examples

7.1. The class of the co-analytic subsets of $\mathcal{N}$.

As in Section 1, we let $(\alpha, \beta)$ denote the pairing function on $\mathcal{N}$ satisfying: for each $\alpha$, for each $\beta$, for each $n$, $\langle \alpha, \beta \rangle(2n) = \alpha(n)$ and $\langle \alpha, \beta \rangle(2n + 1) = \beta(n)$. For each $\alpha$ in $\mathcal{N}$, we defined elements $\alpha_I$ and $\alpha_{II}$ of $\mathcal{N}$ satisfying: for each $n$ $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n + 1)$. Note that, for each $\alpha$, $\alpha = \langle \alpha_I, \alpha_{II} \rangle$.

For every subset $X$ of $\mathcal{N}$, we let the (universal) projection of $X$, also called the co-projection of $X$, notation $\text{Un}(X)$, be the set of all $\alpha$ such that, for all $\beta$, $\langle \alpha, \beta \rangle$ belongs to $X$.

A subset $X$ of $\mathcal{N}$ is called co-analytic if and only if there exists an open subset $Y$ of $\mathcal{N}$ such that $X$ coincides with $\text{Un}(Y)$. The class of the co-analytic subsets of $\mathcal{N}$ is denoted by $\Pi_1^c$.

Recall that we parametrized the class $\Sigma_1^c$ of the open subsets of $\mathcal{N}$ as follows. For each $\beta$, let $G_\beta$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $n$, $\beta(\alpha n) \neq 0$. A subset $X$ of $\mathcal{N}$ is open if and only there exists $\beta$ such that $X$ coincides with $G_\beta$.

One may parametrize the class $\Pi_1^c$ of the co-analytic subsets of $\mathcal{N}$ as follows. For each $\beta$, let $CA_\beta$ be the set $\text{Un}(G_\beta)$, that is the set of all $\alpha$ in $\mathcal{N}$ such that, for every $\gamma$, the pair $\langle \alpha, \gamma \rangle$ belongs to $G_\beta$. A subset $X$ of $\mathcal{N}$ is co-analytic if and only there exists $\beta$ such that $X$ coincides with $CA_\beta$.

We let $\text{UP}_1^c$ be the set of all $\alpha$ such that $\alpha_{II}$ belongs to $CA_{\alpha_I}$.

Recall that, for each subset $X$ of $\mathcal{N}$, for each $\beta$, we let $X \upharpoonright \beta$ be the set of all $\alpha$ such that $\langle \beta, \alpha \rangle$ belongs to $X$.

As in Section 1, we use $S^+$ to denote the successor-function from $\mathbb{N}$ to $\mathbb{N}$, and $\circ$ to denote the operation of composition on $\mathcal{N}$.

For every subset $X$ of $\mathcal{N}$, we let $\text{Perhaps}(X)$ be the set of all $\alpha$ in $\mathcal{N}$ such that there exists $\beta$ in $X$ with the property: if $\alpha \neq \beta$, then $\alpha$ belongs to $X$.

For every inhabited subset $X$ of $\mathcal{N}$, $X \subseteq \text{Perhaps}(X) \subseteq X^{-c}$, where, for each subset $Y$ of $\mathcal{N}$, $Y^{-c}$ denotes the complement of $Y$, that is, the set of all $\alpha$ in $\mathcal{N}$ such that the assumption $\langle \alpha \in Y \rangle$ leads to a contradiction.
The notion *Perhaps* has been studied extensively in \([35, 36, 41]\) on \([41]\].

A subset \(X\) of \(\mathcal{N}\) is called *perhapsive* if the set *Perhaps*(\(X\)) coincides with the set \(X\).

We use \(S^*\) to denote the element of \(\mathcal{N}\) satisfying: for all \(n\), \(S^*(n) = n + 1\).

**Theorem 7.1.**

(i) The set \(UP^1_1\) is a universal element of the class \(\Pi^1_1\).

(ii) For every sequence \(X_0, X_1, \ldots\) of subsets of \(\mathcal{N}\), if, for each \(n\), \(X_n\) belongs to \(\Pi^1_1\), then \(\bigcap_{n \in \mathbb{N}} X_n\) belongs to \(\Pi^1_1\).

(iii) Every co-analytic subset of \(\mathcal{N}\) is perhapsive.

(iv) The set \(D^2(A_1)\) is not co-analytic.

(v) For every subset \(X\) of \(\mathcal{N}\), if \(X\) belongs to \(\Pi^1_1\), then \(Un(X)\) belongs to \(\Pi^1_1\).

(vi) Every \(\Pi^1_2\)-subset of \(\mathcal{N}\) is co-analytic.

(vii) For all subsets \(X, Y\) of \(\mathcal{N}\), if \(X\) reduces to \(Y\) and \(Y\) is co-analytic, then \(X\) is co-analytic.

**Proof.** (i) Note that, for each \(\alpha, \alpha\) belongs to \(UP^1_1\) if and only if \(\alpha I\) belongs to \(CA\alpha\), if and only if, for all \(\gamma\), \(\gamma\alpha\) belongs to \(G\alpha\), if and only if, for all \(\gamma\), there exists \(n\) such that \(\alpha I(\gamma, \gamma, \gamma^0)) \neq 0\). We thus see that \(UP^1_1\) belongs to the class \(\Pi^1_1\). The fact that, for each element \(Y\) of \(\Pi^1_1\), there exists \(\beta\) such that \(Y\) coincides with \(UP^1_1 \cup \beta\) follows from the observations preceding this theorem.

(ii) Let \(Y_0, Y_1, \ldots\) be a sequence of open subsets of \(\mathcal{N}\) such that, for each \(n\), \(X_n = Un(Y_n)\). Using the Second Axiom of Countable Choice, find \(\beta\) in \(\mathcal{N}\) such that, for each \(n\), \(Y_n = G\beta^n\), and \(X_n = Un(Y_n) = CA\beta^n\). We define a subset \(Z\) of \(\mathcal{N}\) as follows. \(Z\) is the set of all \(\alpha\) such that \(\alpha \alpha, \alpha \beta^0 \circ S^*\) belongs to \(Y_{\alpha^0(0)}\). Clearly, the set \(Z\) is an open subset of \(\mathcal{N}\) and one may define \(\gamma\) in \(\mathcal{N}\) satisfying: for each \(\alpha\), there exists \(n\) such that \(\gamma(\gamma\alpha) \neq 0\) if and only if there exists \(n\) such that \(\beta^0(0)(\alpha \alpha, \alpha \beta^0 \circ S^*\alpha) \neq 0\).

Note that the set \(Z\) coincides with the set \(G\gamma\).

Note that, for each \(\alpha, \alpha\) belongs to \(\bigcap_{n \in \mathbb{N}} X_n\) if and only if, for all \(n\), for all \(\beta\), \(\alpha, \beta\) belongs to \(X_n\) if and only if, for all \(\beta\), \(\alpha \beta^0 \circ S^*\) belongs to \(X_{\beta(0)}\) if and only if \(\alpha\) belongs to \(Un(Z)\). It follows that the set \(\bigcap_{n \in \mathbb{N}} X_n\) coincides with the set \(Un(Z) = Un(G\gamma) = CA\gamma\).

(iii) Let \(X\) be a co-analytic subset of \(\mathcal{N}\) and let \(Y\) be an open subset of \(\mathcal{N}\) such that \(X = Un(Y)\). Suppose that \(\alpha\) belongs to *Perhaps*(\(X, X\)). Find \(\gamma\) in \(\mathcal{N}\) such that, if \(\alpha\) is apart from \(\gamma\), then \(\alpha\) belongs to \(X\). We claim that \(\alpha\) itself belongs to \(X\). In order to see this, find a decidable subset \(C\) of \(\mathbb{N}\) such that for every \(\delta, \delta\) belongs to \(Y\) if and only if, for some \(n, \delta\) belongs to \(C\).

Let \(\beta\) be an element of \(\mathcal{N}\). Find \(n\) such that \(\beta\beta\alpha\) belongs to \(C\), and distinguish two cases. Either \(\beta\beta\alpha\beta = \beta\alpha\beta\alpha\), and \(\alpha\beta\alpha\beta\) belongs to \(C\), and, therefore, \(\alpha\beta\alpha\beta\) belongs to \(Y\) or \(\alpha\beta\alpha\beta\beta\neq \beta\alpha\beta\alpha\beta\), therefore \(\alpha\beta\alpha\beta\beta\) belongs to \(X\) and \(\alpha\beta\alpha\beta\beta\) belongs to \(Y\). In any case \(\alpha\beta\alpha\beta\beta\) belongs to \(Y\). Therefore, for every \(\beta\), \(\alpha\beta\alpha\beta\beta\) belongs to \(Y\), and \(\alpha\beta\alpha\beta\beta\) itself belongs to \(X\). Clearly, *Perhaps*(\(X, X\)) is a subset of \(X\) and \(X\) is perhapsive. We thus see that every co-analytic subset of \(\mathcal{N}\) is perhapsive.

(iv) The set \(D^2(A_1)\) is not perhapsive. This is a consequence of the fact that the set *Perhaps*(\(D^2(A_1)\)) coincides with the closure \(\overline{D^2(A_1)}\) of the set \(D^2(A_1)\) and the set \(D^2(A_1)\) is not closed. It now follows from (ii) that the set \(D^2(A_1)\) is not co-analytic.

(v) Suppose that \(X\) belongs to \(\Pi^1_1\). Let \(Y\) be an open subset of \(\mathcal{N}\) such that \(X = Un(Y)\). Observe that for every \(\alpha\), \(\alpha\) belongs to \(Un(X)\) if and only if, for all \(\beta, \alpha\beta\) belongs to \(X\) if and only if, for all \(\beta, \gamma\), the sequence \(\langle \alpha, \beta, \gamma \rangle\) belongs to \(Y\). Let \(Z\) be the set of all \(\alpha\) such that \(\langle \alpha, \alpha, 0, \alpha, \alpha^1, 1, 1 \rangle\) belongs to \(Y\). Observe that \(Z\) is open and \(Un(X)\) coincides with \(Un(Z)\), therefore \(Un(X)\) belongs to \(\Pi^1_1\).

(vi) Use (iv) and the simple fact that every open subset of \(\mathcal{N}\) is co-analytic.
(vii) Let \( X, Y \) be subsets of \( \mathcal{N} \) and let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( X \) to \( Y \). Assume that \( Y \) is co-analytic. We have to show that \( X \) is co-analytic. Let \( Z \) be an open subset of \( \mathcal{N} \) such that \( Y \) coincides with \( Un(Z) \). Observe that for every \( \alpha, \alpha' \) belongs to \( X \) if and only if \( \langle \gamma(x), \gamma(y) \rangle \) belongs to \( \alpha \times \alpha' \), for all \( \beta, \gamma \) belongs to \( \mathcal{N} \). Let \( \mathcal{V} \) be the set of all \( \alpha \) such that \( \langle \gamma(x), \gamma(y) \rangle \) belongs to \( Z \). Observe that \( \mathcal{V} \) is an open subset of \( \mathcal{N} \) and that \( X \) coincides with \( Un(V) \). Clearly, \( X \) is co-analytic.

\[ \square \]

7.2. The set \( A^1_1 = \text{Bar} \) and the set \( \text{WF} \) of all \( \alpha \) forbidding an element of each infinite \( \leq_{KB} \)-decreasing sequence.

Recall that \( A^1_1 \) is the set of all \( \alpha \) in \( \mathcal{N} \) such that, for every \( \beta \), there exists \( n \) such that \( \alpha(\bar{\beta}n) \neq 0 \).

Let \( \alpha \) be an element of \( \mathcal{N} \) and let \( s \) be an element of \( \mathbb{N} \). \( \alpha \) forbids \( s \) if and only if \( \alpha \) does not admit \( s \), that is, there exists \( i \leq \text{length}(s) \) such that \( \alpha(\bar{\alpha}i) \neq 0 \).

In Section 2, we defined the Kleene-Brouwer ordering on \( \mathbb{N} \) as follows.

For all \( s, t \) in \( \mathbb{N} \), \( s <_{KB} t \) if and only if either \( t \) is a strict initial part of \( s \), or there exists \( i \) such that \( i < \text{length}(s) \) and \( i < \text{length}(t) \) and \( \bar{s}i = \bar{t}i \) and \( s(i) < t(i) \).

Also in Section 1, we defined, for every \( \alpha \) in \( \mathcal{N} \), a subset \( T_\alpha \) of \( \mathbb{N} \), called the tree determined by \( \alpha \). \( T_\alpha \) is the set of all \( s \) that are admitted by \( \alpha \), that is: such that, for all \( i \), if \( i \leq \text{length}(s) \), then \( \alpha(\bar{s}i) = 0 \).

We now let \( \text{WF} \) be the set of all \( \alpha \) in \( \mathcal{N} \) such that the tree \( T_\alpha \) is well-founded by \( <_{KB} \), that is, \( \text{WF} \) is the set of all \( \alpha \) in \( \mathcal{N} \) such that, for every \( \beta \), if, for each \( n \), \( \beta(n) \) belongs to \( T_\alpha \), then there exists \( n \) such that \( \beta(n) = \beta(n+1) \) or \( \beta(n) \leq_{KB} \beta(n+1) \).

The following theorem should be compared with Theorem 7.2.

**Theorem 7.2.**

(i) The set \( A^1_1 \) is a complete co-analytic set.

(ii) The set \( A^1_1 \) coincides with the set \( \text{WF} \).

**Proof.**

(i) Let \( X \) belong to \( \Pi^1_1 \) and let \( Y \) be an open subset of \( \mathcal{N} \) such that \( X = Un(Y) \) and let \( C \) be a decidable subset of \( \mathcal{N} \) such that, for every \( \beta, \beta' \) belong to \( Y \) if and only if, for some \( n \), \( \bar{\beta}n \) belongs to \( C \). Observe that for every \( \alpha, \alpha' \) belongs to \( X \) if and only if, for every \( \beta \), there exists \( n \) such that \( \langle \alpha, \beta \rangle n \) belongs to \( C \). Define a function \( f \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha, \beta \) in \( \mathcal{N} \), for every \( n \) in \( \mathbb{N} \), \( f(\alpha)\bar{\beta}n = 0 \) if and only if \( \langle \alpha, \beta \rangle n \) belongs to \( C \) and observe that \( f \) reduces \( X \) to \( A^1_1 \).

(ii) We first prove that \( \text{WF} \) is a subset of \( A^1_1 \).

Let \( \alpha \) belong to \( \text{WF} \). Let \( \beta \) belong to \( \mathcal{N} \). We define \( \gamma \) such that \( \gamma(\bar{0}) = \bar{\beta}0 = \langle \rangle \rangle \), and, for each \( n \), if \( \bar{\beta}(n+1) \) belongs to \( T_\alpha \), then \( \gamma(n+1) = \bar{\beta}(n+1) \), and, if \( \bar{\beta}(n+1) \) does not belong to \( T_\alpha \), then \( \gamma(n+1) = \gamma(n) \). Note that, if \( \alpha(\langle \rangle \rangle) = 0 \), then, for each \( n \), \( \gamma(n) \) belongs to \( T_\alpha \). Let \( n \) be the least \( i \) such that either \( \alpha(\langle \rangle \rangle) = 0 \) or \( \gamma(i) = \gamma(i+1) \) or \( \gamma(i) <_KB \gamma(i+1) \). Note that \( \bar{\beta}(n+1) \) does not belong to \( T_\alpha \) and, therefore, either \( \alpha(\langle \rangle \rangle) \neq 0 \) or \( \alpha(\bar{\beta}(n+1)) \neq 0 \).

We thus see that, for every \( \beta \), there exists \( n \) such that \( \alpha(\bar{\beta}n) \neq 0 \), that is, \( \alpha \) belongs to \( A^1_1 \).

We now prove that \( A^1_1 \) is a subset of \( \text{WF} \).

Let \( \alpha \) belong to \( A^1_1 \). Let \( P \) be the set of all \( s \) such that, for some \( t, t \subseteq s \) and \( \alpha(t) \neq 0 \). Note that \( P \) is a bar in \( \mathcal{N} \) and that \( P \) is monotone, that is, for each \( s \), if \( s \) belongs to \( P \), then, for each \( n, s \prec \langle n \rangle \) belongs to \( P \). We are going to use the principle of induction on monotone bars, and we let \( Q \) be the set of all \( s \) such that, for each \( \beta \), if, for each \( n, s \subseteq \beta(n) \) and, for each \( n, \beta(n) \) belongs to \( T_\alpha \), then there exists \( n \) such that \( \beta(n) \).
\( \beta(n) = \beta(n + 1) \) or \( \beta(n) \leq KB \beta(n + 1) \). Note that, for each \( s \), if \( s \) belongs to \( P \), then, for each \( t \), if \( s \subseteq t \), then \( t \) does not belong to \( T_\alpha \), and, therefore, \( s \) belongs to \( Q \).

Now assume \( s \) belongs to \( N \) and, for each \( n \), \( s \cdot (n) \) belongs to \( Q \). Let \( \beta \) be an element of \( N \) such that, for each \( n \), \( s \subseteq \beta(n) \) and, for each \( n \), \( \beta(n) \) belongs to \( T_\alpha \). We define the statement \( QED \), meaning \( \text{"quod est demonstrandum, this is what we have to prove"} \), rather than \( \text{"quod erat demonstrandum, this is what we had to prove"} \), as follows:

\[
QED := \text{There exists } n \text{ such that } \beta(n) = \beta(n + 1) \text{ or } \beta(n) \leq KB \beta(n + 1).
\]

Note that, for each \( n \), either \( \beta(n + 1) = s \) and \( \beta(n + 1) \subseteq \beta(n) \) and \( QED \), or there exists \( m \) such that \( s \cdot (m) \subseteq \beta(n + 1) \).

We may define \( \gamma \) such that, for each \( n \), either \( \beta(n + 1) = s \) or \( s \cdot (\gamma(n)) \subseteq \beta(n + 1) \).

Note that, for each \( n \), either \( \gamma(n + 1) \leq \gamma(n) \), or \( \beta(n + 1) = s \), or \( s \cdot (\gamma(n)) \subseteq \beta(n + 1) \) and \( s \cdot (\gamma(n + 1)) \subseteq \beta(n + 2) \) and \( \gamma(n) < \gamma(n + 1) \) and, therefore, \( \beta(n + 1) < KB \beta(n + 2) \). We conclude: for each \( n \), either \( \gamma(n + 1) \leq \gamma(n) \) or \( QED \).

We now claim the following:

for each \( m \), there exists \( n \) such that \( n > m \) and \( \gamma(n) < \gamma(m) \) or \( QED \).

We prove this claim as follows. Let \( m \) be a natural number. Define \( \delta \) such that \( \delta(0) = \beta(m + 1) \) and, for each \( n \), if \( s \cdot (\gamma(n)) \subseteq \beta(m + n + 2) \), then \( \delta(n + 1) = \beta(m + n + 2) \) and, and, if not \( s \cdot (\gamma(n)) \subseteq \beta(m + n + 2) \), then \( \delta(n + 1) = \delta(n) \). Note that, for each \( n \), \( s \cdot (\gamma(n)) \subseteq \delta(n) \). Using the assumption that \( s \cdot (\gamma(m)) \) belongs to \( Q \), we find \( p \) such that \( \delta(p) = \delta(p + 1) \) or \( \delta(p) < KB \delta(p + 1) \). Now distinguish two cases. Either \( \delta(p) = \beta(m + p + 1) \) and \( \delta(p + 1) = \beta(m + p + 2) \) and, therefore, \( QED \), or \( \delta(p) \neq \beta(m + p + 1) \) or \( \delta(p + 1) \neq \beta(m + p + 2) \) and, therefore, \( s \cdot (\gamma(m)) \) is not an initial part of either \( \beta(m + p + 1) \) or \( \beta(m + p + 2) \). In the latter case, either \( \beta(m + p + 1) \) or \( \beta(m + p + 2) \) coincides with \( s \), and, therefore, \( QED \), or: either \( \gamma(m + p) \) or \( \gamma(m + p + 1) \) differs from \( \gamma(m) \). Note that, if either \( \gamma(m) < \gamma(m + p) \) or \( \gamma(m) < \gamma(m + p + 2) \), then \( QED \). We thus may conclude: \( \gamma(m + p) < \gamma(m) \) or \( \gamma(m + p + 1) < \gamma(m) \) or \( QED \).

This ends the proof of our claim.

It is not difficult to see that the statement: \( \text{“for each } m \text{, there exists } n \text{ such that } n > m \text{ and } \gamma(n) < \gamma(m) \text{ or } QED \text{”} \) forces us to the conclusion: \( QED \).

Clearly then, for each \( \beta \), if, for each \( n \), \( s \subseteq \beta(n) \) and, for each \( n \), \( \beta(n) \) belongs to \( T_\alpha \), then there exists \( n \) such that \( \beta(n) = \beta(n + 1) \) or \( \beta(n) \leq KB \beta(n + 1) \), that is: \( s \) belongs to \( Q \).

Applying the principle of induction on monotone bars we find: the empty sequence \( (\cdot) \) belongs to \( Q \), that is: for each \( \beta \), if, for each \( n \), \( \beta(n) \) belongs to \( T_\alpha \), then there exists \( n \) such that \( \beta(n) = \beta(n + 1) \) or \( \beta(n) \leq KB \beta(n + 1) \), that is: \( s \) belongs to \( Q \).

We thus see: \( A_1^1 \) is a subset of \( WF \).

\[ QED \]

The statement second statement of Theorem \( \text{[23]} \) is also studied in \( \text{[17]} \). It turns out that, in basic intuitionistic mathematics, this statement is an equivalent of the principle of Open Induction on Cantor space \( C \).

### 7.3. The set of the closed sets sinking into Fin or Almost*\(^c\)(Fin).

We now want to treat some results that, together, are a counterpart to Theorem \( \text{[23]} \).

Let \( X, Y \) be a subsets of \( N \). We introduce the colourful expression: \( X \) sinks into \( Y \), for: \( X \) is a subset of \( Y \).

Let \( \alpha, \beta \) be elements of \( N \). We define: \( \beta \) admits \( \alpha \) if and only if, for each \( n \), \( \beta(\alpha n) = 0 \).

For each \( \beta \) in \( N \), the set \( F_\beta \) is the set of all \( \alpha \) admitted by \( \beta \).

Recall that a subset \( X \) of \( N \) belongs to the class \( \Pi_1^1 \) of the closed subsets of \( N \) if and only if, for some \( \beta \), \( X \) coincides with \( F_\beta \).
Let $\beta$ belong to $\mathcal{N}$. $\beta$ is called a spreadlaw if and only if, for each $s$, $\beta(s) = 0$ if and only if, for some $n$, $\beta(s \ast (n)) = 0$.

Note that a closed subset $X$ of $\mathcal{N}$ is located or a spread if and only if there exists a spreadlaw $\beta$ such that $X$ coincides with $F_{\beta}$.

We let $Sink(X)$ be the set of all $\beta$ in $\mathcal{N}$ such that the closed set $F_{\beta}$ sinks into $X$, that is: $F_{\beta}$ is a subset of $X$.

We let $Sink^*(X)$ be the set of all $\beta$ in $Sink(X)$ such that $\beta$ is a spread-law.

We let $Sink^*_{01}(X)$ be the set of all $\beta$ in $Sink(X)$ such that $\beta$ is a spread-law and $F_{\beta}$ is a subset of Cantor space $\mathcal{C}$.

We let $\text{Fin}$ be the set of all $\alpha$ in Baire space $\mathcal{N}$ such that, for some $m$, for each $n$, if $n > m$, then and $\alpha(n) = 0$.

Note that, for all $\alpha$ in Cantor space $\mathcal{C}$, $\alpha$ belongs to $\text{Fin}$ if and only if $\alpha$ is the characteristic function of a finite subset of $\mathbb{N}$. One proves easily that the sets $\text{Fin}$ and $\text{Fin} \cap \mathcal{C}$, as subsets of Baire space $\mathcal{N}$, reduce to each other. As has been shown in [35], see also [41] and [46], the set $\text{Fin}$ belongs to the class $\Sigma^0_1$, consisting of the countable unions of closed sets, and not to the class $\Pi^0_1$ consisting of the countable intersections of open sets. Somewhat surprisingly, the set $\text{Fin}$ is not a complete element of the class $\Sigma^0_2$, although the set $\text{Inf}$ is a complete element of the class $\Pi^0_2$.

We let $\text{Almost}^*\text{Fin}$ be the set of all $\alpha$ in Cantor space $\mathcal{C}$ such that, for every strictly increasing $\gamma$ in $\mathcal{N}$, there exists $n$ such that $\alpha(\gamma(n)) = 0$. Using Brouwer’s Continuity Principle, one may prove that the set $\text{Fin}$ is a proper subset of the set $\text{Almost}^*\text{Fin}$. It has been shown in [41] that the co-analytic set $\text{Almost}^*\text{Fin}$ is not positively Borel.

An important observation is that the set $\text{Almost}^*\text{Fin}$ is the set of all $\alpha$ in $\mathcal{C}$ that are apart from every element of $\text{Inf}$.

The following notions were introduced just before Theorem 6.1 and will be used again in the proof of the second item of the next theorem.

For each $\alpha$ in $\mathcal{N}$, we defined elements $\alpha_I$ and $\alpha_{II}$ of $\mathcal{N}$ by:

For each $n$, $\alpha_I(n) = \alpha(2n)$ and $\alpha_{II}(n) = \alpha(2n + 1)$.

For each $s$ in $\mathbb{N}$, we defined elements $s_I$ and $s_{II}$ of $\mathbb{N}$ by:

$length(s_I)$ is the least $k$ such that $length(s) \geq 2k$ and, for each $n$, if $n < length(s_I)$, then $s_I(n) = s(2n)$, and

$length(s_{II})$ is the least $k$ such that $length(s) \geq 2k + 1$ and, for each $n$, if $n < length(s_{II})$, then $s_{II}(n) = s(2n + 1)$.

For the second item of the next theorem, it is important that one has in mind the set $\text{Fin}$, as we just defined it, a subset of Baire space, and not the set $\text{Fin} \cap \mathcal{C}$.

Theorem 7.3.

(i) The set $Sink^*_{01}(\text{Fin})$ and the set $\text{Fin}$ reduce to each other.

(ii) The set $A^1_I$ reduces to the set $Sink^*(\text{Fin})$ but the set $Sink^*(\text{Fin})$ is not a co-analytic subset of $\mathcal{N}$.

(iii) The sets $\text{Fin}$, $Sink^*_{01}(\text{Fin})$ and $Sink^*(\text{Fin})$ are not co-analytic.

(iv) The set $A^1_I$ reduces to each one of the sets $Sink(\text{Almost}^*\text{Fin})$, $Sink^*(\text{Almost}^*\text{Fin})$ and $Sink^*_{01}(\text{Almost}^*\text{Fin})$.

(v) The sets $Sink^*(\text{Almost}^*\text{Fin})$ and $Sink^*_{01}(\text{Almost}^*\text{Fin})$ are co-analytic subsets of $\mathcal{N}$.

Proof. (i) Assume that $\beta$ is a spreadlaw and that $F_{\beta}$ is a subset of Cantor space $\mathcal{C}$.

Note that $\beta$ belongs to $Sink(\text{Fin})$ if and only if, for each $\alpha$ in $F_{\beta}$, there exists $n$ such that, for every $m$, if $m > n$, then $\alpha(m) = 0$.

Note that $F_{\beta}$ is a spread.
Applying the First Axiom of Continuous Choice, we find a function $\gamma$ from the fan $F_\beta$ to $\mathbb{N}$ such that, for each $\alpha$ in $F_\beta$, $\alpha = \pi(\gamma(\alpha)) \ast \underline{0}$. Applying the (Restricted) Fan Theorem, Axiom 12, we conclude that there exists $n$ such that for all $\alpha$ in $F_\beta$, $\gamma(\alpha) \leq n$. It follows that, for all $\alpha$ in $F_\beta$, for all $m$, if $m > n$, then $\alpha(m) = 0$. We thus see that $\beta$ belongs to $\text{Sink}(\text{Fin})$ if and only if there exists $n$ such that, for all $s$, for all $i$, if $\text{length}(s) \geq n$ then $\beta(s \ast (1)) \neq 0$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\beta$, for each $n$, if there exists $s$ in $\text{Bin}$ such that $\beta(s \ast (1)) = 0$ and $\text{length}(s) = n$, then $\gamma(\beta)(n) = 1$, and, if not, then $\gamma(\beta)(n) = 0$. Note that $\gamma$ reduces the set $\text{Sink}_{01}^*(\text{Fin})$ to the set $\text{Fin}$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $(\gamma(\alpha))(\langle \rangle) = 0$, and for each $s$, for each $i$, for each $j$, $(\gamma(\alpha)(s \ast \langle i, j \rangle)) = 0$ if and only if $(\gamma(\alpha)(s) = 0$ and either $i = j = 0$ or, for all $t$, if $t \subseteq s \ast \langle i \rangle$, then $\alpha(t) = 0$ and $j = 1$. Note that, for each $\alpha$, $\gamma(\alpha)$ is a spreadlaw, as, for each $s$, $(\gamma(\alpha)(s) = 0$ if and only if $(\gamma(\alpha)(s \ast \langle 0 \rangle) = 0$.

We claim that the function $\gamma$ reduces the set $A_1^2$ to the set $\text{Sink}_{01}^*(\text{Fin})$, and we prove this claim as follows.

Let $\alpha$ belong to $A_1^2$ and $\beta$ to $\mathcal{N}$. Assume that $\beta$ belongs to $F_{\gamma(\alpha)}$, that is, for each $n$, $(\gamma(\alpha)(\langle \rangle n) = 0$. Consider $\beta_0$ and find $n$ such that $\alpha(\beta_0 n) \neq 0$. It follows that, for each $j > n$, $\beta(2j + 1) = \beta(2j + 2) = 0$, so $\beta$ belongs to $\text{Fin}$.

We thus see that, for each $\alpha$, if $\alpha$ belongs to $A_1^2$, then $\gamma(\alpha)$ belongs to $\text{Sink}_{01}^*(\text{Fin})$.

Now assume that $\gamma(\alpha)$ belongs to $\text{Sink}_{01}^*(\text{Fin})$. Then, for each $\beta$ in the spread $F_{\gamma(\alpha)}$, there exists $m$ such that, for all $n$, if $n > m$, then $\beta(n) = 0$. Now let $\beta$ belong to $\mathcal{N}$. Define an element $\beta'$ of $\mathcal{N}$, as follows. For each $n$, if, for all $i \leq n + 1$, $\alpha(\beta i) = 0$, then $\beta'(2n) = \beta(n)$ and $\beta'(2n + 1) = 1$, and, if there exists $i \leq n + 1$ such that $\alpha(\beta i) \neq 0$, then $\beta'(2n) = \beta'(2n + 1) = 0$. Note that, for each $n$, $(\gamma(\alpha)(\beta n) = 0$, that is, $\beta'$ belongs to $F_{\gamma(\alpha)}$, and, therefore, to $\text{Fin}$. Find $m$ such that for all $n$, if $n > m$, then $\beta'(n) = 0$. Find $k$ such that $m + 1 = 2k$ or $m + 2 = 2k + 1$ and note: $\beta(2k + 1) = 0$, and, therefore, there exists $i \leq k$ such that $\alpha(\beta i) \neq 0$.

Clearly then, for each $\alpha$, if $\gamma(\alpha)$ belongs to $\text{Sink}_{01}^*(\text{Fin})$, then $\alpha$ belongs to $A_1^1$.

(iii) We first observe that the set $\text{Fin}$ is not co-analytic. In [11], Theorem 3.21(v), it is shown that $\text{Almost}^{\ast} \text{Fin}$ is the best co-analytic approximation of the set $\text{Fin} \cap \mathcal{C}$, that is, $\text{Almost}^{\ast} \text{Fin}$ is co-analytic and, for every co-analytic subset $Z$ of $\mathcal{C}$, if $\text{Fin} \subseteq Z$, then $\text{Almost}^{\ast} \text{Fin} \subseteq Z$. As $\text{Fin}$ does not coincide with $\text{Almost}^{\ast} \text{Fin}$, one may conclude that $\text{Fin}$ is not co-analytic.

Perhaps, the reader finds the just given reference too sophisticated. One may also observe: the set $\text{Fin}$ is not perhapsive, and, therefore, by Theorem 7.3(ii), the set $\text{Fin}$ is not co-analytic. Here is the proof that the set $\text{Fin}$ is not perhapsive:

Consider the set $T$ consisting of all $\alpha$ in $\mathcal{C}$ such that, for all $m, n$, if $\alpha(m) = \alpha(n) = 1$, then $m = n$. $T$ is the set of all $\alpha$ in $\mathcal{C}$ that assume the value 1 at most one time. Note that $T$ is a spread. Also note that, for each $\alpha$ in $T$, if $\alpha \neq \underline{0}$, then there exists $n$ such that $\alpha = \langle \underline{0} \ast \langle 1 \rangle \ast \underline{0} \rangle$, and thus, $\alpha$ belongs to $\text{Fin}$. We conclude: $T$ is a subset of $\text{Perhaps}(\text{Fin})$. Suppose that $T$ is a subset of $\text{Fin}$. Using Brouwer’s Continuity Principle, we find $m, n$ such that either, for all $\alpha$ in $T$, if $\alpha$
passes through $\overline{\beta}n$, then $\alpha = 0$ or, for all $\alpha$ in $T$, if $\alpha$ passes through $\overline{\beta}n$, then $\alpha = \overline{\beta}n + \langle 1 \rangle + 0$. Both conclusions are wrong.

It now follows from (i) that the set $\text{Sink}_{\text{Fin}}(\text{Fin})$ is not co-analytic.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\beta$, for each $s$, if $s$ belongs to $\text{Bin}$, then $(\gamma|\beta)(s) = \beta(s)$, and, if $s$ does not belong to $\text{Bin}$, then $(\gamma|\beta)(s) = 1$. One easily verifies that the function $\gamma$ reduces the set $\text{Sink}_{\text{Fin}}^{\gamma}(\text{Fin})$ reduces to the set $\text{Sink}^*(\text{Fin})$.

It follows that also the set $\text{Sink}^*(\text{Fin})$ is not co-analytic.

(iv) We now show that the set $A_{1}^{1}$ reduces to the set $\text{Sink}(\text{Almost}^*\text{Fin})$. It then follows from Theorem 2.5 that the latter set is a complete co-analytic set.

As in the proof of Theorem 2.5, we let $\delta$ be an element of $\mathcal{N}$ such that $\delta(\langle \rangle) = \langle \rangle$ and, for each $s$, for each $n$, $\delta(s * \langle n \rangle) = \delta(s) + \overline{\beta}n + \langle 1 \rangle + 0$.

We let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\gamma$, for each $\alpha$, $(\gamma|\alpha)(s) = 0$ if and only if there exist $t, n$ such that $s = \delta(t) * \overline{\beta}n + \alpha(t) = 0$. Note that, for each $\alpha$, for each $s$, if $(\gamma|\alpha)(s) = 0$, then $(\gamma)(\alpha(s * 0)) = 0$. It follows that, for each $\alpha$, the set $F_{\gamma|\alpha}$ is a spread, and a subset of $\mathcal{C}$.

We claim that the function $\gamma$ reduces the set $A_{1}^{1}$ to the set $\text{Sink}(\text{Almost}^*\text{Fin})$, and we prove this claim as follows.

First, let $\alpha$ belong to $A_{1}^{1}$. Let $\beta$ belong to $F_{\gamma|\alpha}$. We have to prove that $\beta$ belongs to $\text{Almost}^*\text{Fin}$. Let $\varepsilon$ be a strictly increasing element of $\mathcal{N}$. We have to prove that there exists $n$ such that $\beta(\varepsilon(n)) = 0$. To this end, we define an element $\beta'$ of $\mathcal{N}$ as follows. For each $i$, if there exists $n$ such that $i = \varepsilon(n)$, then $\beta'(i) = 1$, and, if not, then $\beta'(i) = \beta(i)$. Note that $\beta'$ belongs to $\text{Inf}$ and find $\zeta$ in $\mathcal{N}$ such that, for each $n$, $\beta'$ passes through $\delta(\zeta n)$. Now, recalling the fact that $\alpha$ belongs to $A_{1}^{1}$, find $n$ such that $\alpha(\zeta n) \neq 0$ and note that $(\gamma|\alpha)(\delta(\zeta n)) \neq 0$. Find $m$ such that $\delta(\zeta n) = \overline{\beta}m$. As $(\gamma|\alpha)(\overline{\beta}m) \neq 0$ and $(\gamma|\alpha)(\overline{\beta}m) = 0$, we conclude: $\overline{\beta}m \neq \overline{\beta}n$, and: for some $i < m$, $\beta'(i) \neq \beta(i)$, and: there exists $j$ such that $\beta(\varepsilon(j)) \neq \beta'(\varepsilon(j))$, and, therefore: $\beta(\varepsilon(j)) = 0$.

We thus see that $\gamma|\alpha$ belongs to $\text{Sink}(\text{Almost}^*\text{Fin})$.

Conversely, assume that $\gamma|\alpha$ belongs to $\text{Sink}(\text{Almost}^*\text{Fin})$. We want to prove: $\alpha$ belongs to $A_{1}^{1}$. Let $\zeta$ belong to $\mathcal{N}$. We have to prove: there exists $n$ such that $\alpha(\zeta n) \neq 0$. To this end, we determine $\beta$ in $\mathcal{C}$ such that, for each $n$, $\beta$ passes through $\delta(\zeta n)$. We now use the fact that $F_{\gamma|\alpha}$ is a spread and we define an element $\beta'$ of $\mathcal{C}$, by induction, as follows. For each $n$, if $(\gamma|\alpha)(\overline{\beta}n * \langle \zeta(n) \rangle) = 0$, then $\beta'(n) = \beta(n)$, and, if not, then $\beta(n) = 0$. Note that, for each $n$, $(\gamma|\alpha)(\overline{\beta}n) = 0$, and that $\beta$ belongs to $\text{Inf}$. It follows that $\beta$ is apart from $\beta'$. Determine $n$ such that $\overline{\beta}n \neq \overline{\beta}m$ and note that $(\gamma|\alpha)(\overline{\beta}m) \neq 0$. Find $m$ such that $\overline{\beta}m \subseteq \delta(\zeta m)$ and note: there exists $i \leq m$ such that $\alpha(\zeta i) \neq 0$.

We thus see that $\alpha$ belongs to $A_{1}^{1}$.

It follows that the function $\gamma$ reduces the set $A_{1}^{1}$ to the set $\text{Sink}(\text{Almost}^*\text{Fin})$.

It is not difficult to see that the function $\gamma$ also reduces the set $A_{1}^{1}$ to the set $\text{Sink}^*(\text{Almost}^*\text{Fin})$ and to the set $\text{Sink}_{\text{Fin}}^{\gamma}(\text{Almost}^*\text{Fin})$.

(v) First note that, for all $\beta$, $\beta$ is a spreadlaw if and only if for each $s$, $\beta(s) = 0$ if and only if there exists $n$ such that $\beta(s * \langle n \rangle) = 0$. The set of all $\beta$ in $\mathcal{N}$ that are a spreadlaw thus is seen to belong to the class $\Pi_{1}^{0}$ and, therefore, also to the class $\Pi_{1}^{1}$.

Assume now that $\beta$ is a spreadlaw.

Also assume that $\beta$ belongs to $\text{Sink}^*(\text{Almost}^*\text{Fin})$. Then, for each $\alpha$, if, for each $n$, $\beta(\overline{\beta}n) = 0$, then, for each $\gamma$, if for each $n, \gamma(n) < \gamma(n + 1)$, then there exists $m$ such that $\alpha(\gamma(m)) = 0$.

We claim that, now, also the following is true:

For each $\alpha$, for each $\gamma$, there exists $m$ such that either $\gamma(m) \geq \gamma(m + 1)$ or $\beta(\overline{\beta}m) \neq 0$ or $\alpha(\gamma(m)) = 0$.

We prove this claim as follows.
Let $\alpha, \gamma$ be given. We define $\alpha'$ and $\gamma'$ as follows.

For each $n$, if $\beta(\alpha' n \star (\alpha(n))) = 0$, then $\alpha'(n) = \alpha(n)$, and if not, then $\alpha'(n) = p$, where $p$ is the least $j$ such that $\beta(\alpha' n \star (j)) = 0$.

$\gamma'(0) = \gamma(0)$, and, for each $n$, if $\gamma(n+1) > \gamma'(n)$, then $\gamma'(n+1) = \gamma(n+1)$, and if not, then $\gamma'(n+1) = \gamma'(n) + 1$.

Note that $\alpha'$ belongs to $F_\beta$ and that $\gamma'$ is strictly increasing. Find $m$ such that $\alpha'(\gamma'(m)) = 0$, and distinguish three cases. Either $\gamma'(m) \neq \gamma(m)$ and: there exists $n$ such that $\gamma(n+1) \geq \gamma(n)$, or $\gamma(m) = \gamma(m)$ and $\alpha'(\gamma(m)) \neq \alpha(\gamma(m))$ and: there exists $n$ such that $\beta(\alpha n) \neq 0$, or $\alpha(\gamma(m)) = 0$.

This ends the proof of our claim.

Conversely, it will be clear that, if, for each $\alpha$, for each $\gamma$, there exists $m$ such that either $\gamma(m) \geq \gamma(m+1)$ or $\beta(\alpha n) \neq 0$ or $\alpha(\gamma(m)) = 0$, then, for each $\alpha$, if, for each $n$, $\beta(\alpha n) = 0$, then, for each $\gamma$, if for each $n$, $\gamma(n) < \gamma(n+1)$, then there exists $m$ such that $\alpha(\gamma(m)) = 0$.

It follows from the above that, for each $\beta$, $\beta$ belongs to $\text{Almost}^*\text{Fin}$ if and only if
\[
\forall s \exists n [\beta(s) = 0 \leftrightarrow \beta(s \star \langle n \rangle) = 0] \land \forall \alpha \exists n [\alpha^0(n) \geq \alpha(n+1) \lor \alpha^0(\alpha(n)) = 0 \lor \beta(\alpha n) \neq 0].
\]

We thus see that the set $\text{Sink}^* (\text{Almost}^*\text{Fin})$ belongs to the class $\Pi_1^1$.

It easily follows that also the set $\text{Sink}^*_{\Pi_1^1} (\text{Almost}^*\text{Fin})$ belongs to the class $\Pi_1^1$.

It follows from (iii) that both these sets are $\Pi_1^1$-complete.

Note that the fifth item of Theorem 6.3 is reminiscent of a classical result due to Hurewicz. This result plays a key role in the sketch of the proof of the theorem of Solovay and Kaufman in [14]. This theorem states that the class consisting of the closed sets of uniqueness and the class consisting of the closed sets of extended uniqueness are $\Pi_1^1$-complete.

7.4. The set of the trees that admit exactly one path.

We let $E_1^1$ be the set of all $\alpha$ such that there exists $\gamma$ such that $\gamma$ is the only infinite sequence admitted by $\alpha$, that is, for each $n$, $\alpha(\gamma n) = 0$, while, for each $\delta$, if $\delta$ is apart from $\gamma$, that is, there exists $m$ such that $\delta(m) \neq \gamma(m)$, then, for some $n$, $\alpha(\delta n) \neq 0$. It is not difficult to verify th conclusion of the first item of the next theorem: every co-analytic set reduces to $E_1^1$. In [15], pages 125-127, one may find a fascinating argument, due to Kechris, showing that, in classical descriptive set theory, the set $E_1^1$ itself is $\Pi_1^1$ and, therefore, $\Pi_1^1$-complete. The items (ii)-(v) of the next Theorem may be taken as an intuitionistic comment upon this beautiful result.

We let $D! (A_1, A_1)$ be the set of all $\alpha$ such that either: $\alpha^0 = \underline{0}$ and $\alpha^1 \neq \underline{0}$ or: $\alpha^0 \neq \underline{0}$ and $\alpha^1 = \underline{0}$.

We let $E_2^1$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $n$, $\alpha^n = \underline{0}$, while, for each $m$, if $m \neq n$, then there exists $p$ such that $\alpha^m(p) \neq 0$.

The last item of the next theorem has been added because of the discussion at the end of Section 6.

The set $E_2^1$ is an example of a subset of $\mathcal{N}$ that is positively Borel and has Lusin’s property but still fails to be co-analytic.

Theorem 7.4.

(i) The set $A_1^1$ reduces to the set $E_1^1$.

(ii) Every subset of $\mathcal{N}$ that reduces to a perhapsive subset of $\mathcal{N}$ is itself a perhapsive subset of $\mathcal{N}$.

(iii) The set $D! (A_1, A_1)$ reduces to the set $A_1^1$ and is a perhapsive subset of $\mathcal{N}$.

(iv) The set $D^2 (A_1)$ is not a perhapsive subset of $\mathcal{N}$.
(v) The sets $A_2$ and $D!(A_1, A_1)$ reduce to the set $E_2!$, and the set $E_2!$ reduces to the set $E_1^1$.

(vi) The set $D^2(A_1)$ does not reduce to the set $E_1^1$.

(vii) The set $E_2!$ is not a perhapsive subset of $N$.

(viii) The sets $E_2!$ and $E_1^1$ do not reduce to the set $A_1^1$.

(ix) The set $E_2!$ is regular in Lusin's sense.

**Proof.** (i) Let $\gamma$ be a function from $N$ to $N$ such that, for every $\alpha$, $(\gamma|\alpha)((\cdot)) = 0$ and, for every $\alpha$, for every $s$, $(\gamma|\alpha)((0) * s) = 0$ if and only if $s$ is an initial part of $\alpha$, and, for every $n$, $(\gamma|\alpha)((n + 1) * s) = \alpha(s)$. Note that, for every $\alpha$, $\gamma|\alpha$ admits $\emptyset$ and, for every $\beta$, if $\beta(0) = 0$, then, for every $n$, if $\beta(n) \neq 0$, then $(\gamma|\alpha)(\overline{\beta}(n + 1)) = 0$, and for every $n$, for every $\beta$, $(\gamma|\alpha)((n + 1) * \overline{\beta}n) = \alpha(\overline{\beta}n)$. It follows that, for every $\alpha$, $\alpha$ belongs to $A_1^1$ if and only if $\gamma|\alpha$ belongs to $E_1^1$.

(ii) Let $X, Y$ be subsets of $N$ and suppose that $Y$ is a perhapsive subset of $N$ and that $\gamma$ is a function from $N$ to $N$ reducing the set $X$ to the set $Y$. We have to prove that $X$ is a perhapsive subset of $N$. Assume that $\alpha, \beta$ are elements of $N$ and that $\beta$ belongs to $X$ and that, if $\alpha \neq \beta$, then $\alpha$ belongs to $X$. We have to prove that $\alpha$ belongs to $X$. Note that $\gamma|\beta$ belongs to $Y$, and that, if $\gamma|\alpha \neq \gamma|\beta$, then $\alpha \neq \beta$, and, therefore, $\alpha$ belongs to $X$ and $\gamma|\alpha$ belongs to $Y$. As $Y$ is perhapsive, it follows that $\gamma|\alpha$ belongs to $Y$ and, therefore, $\alpha$ belongs to $X$.

(iii) Let $\gamma$ be a function from $N$ to $N$ such that, for each $\alpha$, $(\gamma|\alpha)((\cdot)) = 0$ and, for each $s$, $(\gamma|\alpha)((0) * s) =$ max$(a^n(\text{length}(s)), a^1(\text{length}(s)))$ and for each $n$, for each $s$, $(\gamma|\alpha)((n + 1) * s) = 1$ if either $a^0n = \overline{\alpha}n$ or $a^1n = \overline{\alpha}n$, and $(\gamma|\alpha)((n + 1) * s) = 0$ otherwise.

Suppose that $\alpha$ belongs to $D!(A_1, A_1)$. Note that there exists $n$ such that $\max(a^n(n), a^1(n)) \neq 0$ and, therefore, for each $\delta$, if $\delta(0) = 0$, then $(\gamma|\alpha)(\overline{\delta}(n + 1)) \neq 0$. Also note that, for each $n$, either $\alpha^0n = \overline{\alpha}n$ or $\alpha^1n = \overline{\alpha}n$ and, therefore, for each $\delta$, if $\delta(0) > 0$, then $(\gamma|\alpha)(\overline{\delta}1) \neq 0$. It follows that, for each $\alpha$, if $\alpha$ belongs to $D!(A_1, A_1)$, then $\gamma|\alpha$ belongs to $A_1^1$.

Now suppose that $\gamma|\alpha$ belongs to $A_1^1$. Note that there exists $n$ such that $(\gamma|\alpha)(\overline{\alpha}(n + 1)) \neq 0$ and, therefore, either $a^0(n) \neq 0$, or $a^1(n) \neq 0$. Also note that, for each $n$, for each $\delta$, if $\delta(0) = n + 1$, then there exists $p$ such that $(\gamma|\alpha)(\overline{\delta}p) \neq 0$, and, therefore, $(\gamma|\alpha)(\overline{\delta}1) \neq 0$, and $(\gamma|\alpha)((n + 1)) \neq 0$ and either $a^0n = \overline{\alpha}n$ or $a^1n = \overline{\alpha}n$. As also either $a^0 \neq \emptyset$ or $a^1 \neq \emptyset$, we conclude: $\alpha$ belongs to $D!(A_1, A_1)$.

(iv) The fact that the set $D^2(A_1)$ is not perhapsive has been used in Theorem 6.1(iii). Let us briefly sketch the argument. The set Perhapsive$(D^2(A_1))$ coincides with the closure $\overline{D^2(A_1)}$ of the set $D^2(A_1)$. The set $\overline{D^2(A_1)}$ is a spreading containing $\emptyset$. Suppose that Perhapsive$(D^2(A_1))$ coincides with $D^2(A_1)$. Using Brouwer's Continuity Principle, we determine $p$ such that either: for every $\alpha$ in $D^2(A_1)$, if $\overline{\alpha}p = \overline{\alpha}p$, then $\alpha^0 = \emptyset$, or: for every $\alpha$ in $D^2(A_1)$, if $\overline{\alpha}p = \overline{\alpha}p$, then $\alpha^1 = \emptyset$. But, clearly, there is no such $p$.

(v) Let $\gamma$ be a function from $N$ to $N$ such that, for each $\alpha$, $(\gamma|\alpha)^0 = \emptyset$, and for each $n$, $(\gamma|\alpha)^{n+1} = \alpha^n$. It is not difficult to see that the function $\gamma$ reduces the set $A_2$ to the set $E_2!$.

Let $\delta$ be a function from $N$ to $N$ such that, for each $\alpha$, $(\delta|\alpha)^0 = \alpha^0$ and $(\delta|\alpha)^1 = \alpha^1$, and, for each $n > 1$, $\alpha^n = \omega$. Clearly, $\delta$ reduces the set $D!(A_1, A_1)$ to the set $E_2!$.

Let $\varepsilon$ be a function from $N$ to $N$ such that, for each $\alpha$, $(\varepsilon|\alpha)((\cdot)) = 0$, and, for each $n$, for each $\beta$, for each $\varepsilon$, $(\varepsilon|\alpha)((\beta(n + 1)) = 0$ if and only if there exists $m$ such that $\overline{\beta}(n + 1) = \overline{\varepsilon}m(n + 1)$ and $\alpha^m = \overline{\alpha}m$. Clearly, $\varepsilon$ reduces the set $E_2!$ to the set $E_1^1$.

(vi) Now assume that $\gamma$ is a function from $N$ to $N$ reducing the set $D^2(A_1)$ to the set $E_1^1$. Let us consider the sets $B_0$ and $B_1$, where, for each $i < 2$, $B_i$ is the set of all $\alpha$
such that $\alpha^t = 0$. Note that $D^2(A_1) = B_0 \cup B_1$ and that the sets $B_0$ and $B_1$ are spreads containing $0$.

Let $p$ belong to $\mathbb{N}$. Using Brouwer’s Continuity Principle, Axiom $\mathbb{A}$, we find $s_0, s_1, m_0$ and $m_1$ such that $\text{length}(s_0) = \text{length}(s_1) = p$, and, for each $i < 2$, for every $\alpha$ in $B_i$, if $\alpha$ passes through $\overline{m}_i$, then there exists $\beta$ passing through $s_i$ such that $\gamma|\alpha$ admits $\beta$.

We claim: $s_0 = s_1$. In order to establish this claim, let us assume: $s_0 \neq s_1$.

Find $\beta$ such that $\gamma\overline{0}$ admits $\beta$. As $s_0 \perp s_1$, either $\beta$ does not pass through $s_0$, or $\beta$ does not pass through $s_1$. Without loss of generality, we assume: $\beta$ does not pass through $s_0$. Note that, for every $\alpha$ in $B_0$, if $\alpha$ passes through $\overline{m}_0$ then there exists $q$ such that $(\gamma\overline{0})(\overline{\beta}q) \neq 0$. Using Brouwer’s Continuity Principle, Axiom $\mathbb{A}$, find $m_2, q$ such that $m_2 \geq m_0$ and, for every $\alpha$ in $B_0$, if $\alpha$ passes through $\overline{m}_2$, then $(\gamma|\alpha)(\overline{\beta}q) \neq 0$. It follows that $(\gamma\overline{0})(\overline{\beta}q) \neq 0$, and that $\gamma\overline{0}$ does not admit $\beta$. Contradiction.

Using the First Axiom of Countable Choice, we determine $\varepsilon, \zeta$ in $\mathbb{N}$ such that, for each $p$, $\text{length}(\varepsilon(p)) = p$ and, for each $\alpha$ in $B_0 \cup B_1$, if $\alpha$ passes through $\overline{n}(\zeta(p))$, then there exists $\eta$ passing through $\varepsilon(p)$ such that $\gamma|\alpha$ admits $\eta$.

We claim that, for each $p$, $\varepsilon(p)$ is an initial part of $\beta$. In order to see this, let us assume $\varepsilon(p) \neq \overline{\beta}p$. Note that, for every $\alpha$ in $B_0$, if $\alpha$ passes through $\overline{m}_p$, then there exists $q$ such that $(\gamma\overline{0})(\overline{\beta}q) \neq 0$. Using Brouwer’s Continuity Principle, find $m_2, q$ such that $m_2 \geq \zeta(p)$ and, for every $\alpha$ in $B_0$, if $\alpha$ passes through $\overline{m}_2$, then $(\gamma|\alpha)(\overline{\beta}q) \neq 0$. It follows that $(\gamma\overline{0})(\overline{\beta}q) \neq 0$, and that $\gamma\overline{0}$ does not admit $\beta$. Contradiction.

Now let $T$ be the set of all $\alpha$ in $\mathcal{C}$ with the property that, for all $m, n$, if $\alpha(m) = \alpha(n) = 1$, then $m = n$. Note that $T$ is a spread and a subset of the closure $D^2(A_1)$ of the set $D^2(A_1)$. We determine $\eta$ in $\mathbb{N}$ such that, for each $p$, $\gamma|\overline{\delta}(1) \ast \overline{0}$ admits $\eta^p$, and, for all $q$, if $\zeta(q) \leq p$, then $\eta^p$ passes through $\varepsilon(p)$. Now, using the fact that, for each $q$, for each $p \geq \zeta(q)$, $\overline{m}_p = \varepsilon(q)$, define a function $\delta$ from $T$ to $\mathbb{N}$ such that, for each $p, \delta|\overline{\delta}(1) \ast \overline{0} = \eta^p$.

We now claim:

(1) for each $\alpha$ in $T$, $\gamma|\alpha$ admits $\delta|\alpha$, and,

(2) for each $\alpha$ in $T$, $\eta$, for each $\eta$ if $\eta \neq \delta|\alpha$, then there exists $n$ such that $(\gamma|\alpha)(\overline{\delta}n) = 0$.

Let us first prove (1). Suppose we find $\alpha$ in $T$, $n$ in $\mathbb{N}$ such that $(\gamma|\alpha)(\overline{\delta}n) = 0$. Using the continuity of the functions (coded by) $\gamma, \delta$, we find $m$ such that for every $\beta$ in $T$, if $\beta$ passes through $\overline{m}$, then $(\gamma|\beta)(\overline{\delta}n) = (\gamma|\alpha)(\overline{\delta}n) = 0$. Note that there exists $p$ such that $m_n \ast 1 \ast 0 \ast 0$ passes through $\overline{m}$ and that $\gamma|\overline{\delta}(1) \ast \overline{0}$ admits $\delta|\overline{\delta}(1) \ast \overline{0}$. Contradiction. We may conclude that, for each $\alpha$ in $T$, $\gamma|\alpha$ admits $\delta|\alpha$.

Let us now prove (2). Let $\alpha$ be an element of $T$ and let $\eta$ be an element of $\mathbb{N}$ such that $\eta \neq \delta|\alpha$. Find $n$ such that $\overline{m}_n \neq \overline{\delta}n$. Find $m$ such that, for every $\beta$ in $T$, if $\alpha$ passes through $\overline{m}$, then $\delta|\beta n = \delta|\alpha n$. We now distinguish two cases.

Case (a). $\overline{m}_n \neq \overline{m}$. Find $p$ such that $\alpha = \overline{m}_p \ast 1 \ast 0$. Note that $\alpha$ belongs to $D^2(A_1)$, and that $\eta \neq \delta|\alpha$. Therefore, there exists $q$ such that $(\gamma|\alpha)(\overline{\delta}q) \neq 0$.

Case (b). $\overline{m}_n = \overline{m}$. Note that $\overline{0}$ belongs to $D^2(A_1)$, and that $\eta \neq \delta|\alpha$. Find $q$ such that $(\delta|\alpha)(\overline{\delta}q) \neq 0$. Now calculate $(\overline{\delta}|\alpha)(\overline{\delta}q)$ and observe: either $(\overline{\delta}|\alpha)(\overline{\delta}q) = (\delta|\overline{\delta})(\overline{\delta}q) \neq 0$, and we are done, or $\alpha \neq 0$. In the latter case, however, there exists $p$ such that $\alpha = \overline{m}_p \ast 1 \ast 0$, and $\alpha$ belongs to $D^2(A_1)$, and, as $\eta \neq \delta|\alpha$, there exists $r$ such that $(\delta|\overline{\delta})(\overline{\delta}r) \neq 0$.

It now follows that...
(1) for all \(p, q\), if \(\alpha^0(p) \neq 0\) and \(\alpha^0(q) \neq 0\), then \(p = q\), and
(2) for all \(n\), if \(\alpha^0(n) = 0\), then \(\alpha^{n+1} = 1\) and
(3) for all \(n\), if \(\alpha^0(n) \neq 0\), then \(\alpha^{n+1} = 0\), and
(4) \(\alpha(\langle \rangle) = 0\)

Let \(\beta\) be the special element of \(A\) satisfying \(\beta^0 = 0\) and, for each \(n\), \(\beta^{n+1} = 1\). Note that \(\beta\) belongs to \(E_2\!\!^+\) and that, for each \(\alpha\) in \(A\), if \(\alpha \neq \beta\), then there exists \(n\) such that \(\alpha^0(n) \neq 0\) and \(\alpha^{n+1} = 0\) and, for each \(i\), if \(i \neq n\), then \(\alpha^{i+1} = 1\), and, therefore, \(\alpha\) belongs to \(E_2\!\!^+\). We thus see that the set \(A\) is a subset of the set \(\mathcal{P}(E_2\!\!^+)\).

We now prove that the set \(A\) is not a subset of the set \(E_2\!\!^+\). Let us assume that \(A\) is a subset of \(E_2\!\!^+\). Note that the set \(A\) is a spread containing \(\beta\), and, using Brouwer’s Continuity Principle, determine \(n, m\) such that for every \(\alpha\) in \(A\), if \(\overline{m} = \overline{m}\), then \(\alpha^n = 0\). Now distinguish two cases.

Case (a). \(n = 0\). Note that there exist \(\alpha\) in \(A\), \(p\) in \(\mathbb{N}\) such that \(\overline{m} = \overline{m}\) and \(p \geq n\) and \(\alpha^0(p) \neq 0\) and \(\alpha^{p+1} = 0\). Contradiction.

Case (b) \(n > 0\). Note that \(\beta\) itself passes through \(\overline{m}\) and that \(\beta^n = 1\). Contradiction.

We thus see that the set \(E_2\!\!^+\) is a proper subset of the set \(E_2\!\!^+\) and that the set \(E_2\!\!^+\) is not perhapsive.

(viii) As we just saw, the set \(E_2\!\!^+\) is not a perhapsive subset of \(\mathcal{N}\). Using (ii), we conclude that the set \(E_2\!\!^+\) does not reduce to the set \(A\!\!^+\) as the set \(A\!\!^+\) is a perhapsive subset of \(\mathcal{N}\). As, according to (v), the set \(E_2\!\!^+\) reduces to the set \(E_2\!\!^+\)!, also the set \(E_2\!\!^+\) does not reduce to the set \(A\!\!^+\).

(ix) Let \(\varepsilon\) be the element of \(\mathcal{N}\) satisfying: for all \(m, n\), \(\varepsilon(2^m(2n+1)-1) = \overline{m} \ast (n+1)\).

Let \(\gamma\) be (the code of) a function from \(\mathcal{N}\) to \(\mathcal{N}\) satisfying: for each \(\alpha\), \((\gamma|\alpha)(\langle \rangle) = \alpha^0(\langle \rangle)\), and \((\gamma|\alpha)^{(0)} = 0\) and for each \(n\), if \(n < \alpha^0(0)\), then \((\gamma|\alpha)^n = \varepsilon(\alpha^0(n+1)) \ast \alpha^{n+1}\), and if \(n > \alpha^0(0)\), then \((\gamma|\alpha)^n = \varepsilon(\alpha^0(n)) \ast \alpha^n\).

One may verify that \(\gamma\) is a strongly one-to-one continuous function from the set \(\mathcal{N}\) onto the set \(E_2\!\!^+\).

It follows that the set \(E_2\!\!^+\) has Lusin’s property.

\[\square\]

8. Analytic sets face to face with co-analytic sets

8.1. Comparing \(A^+\) and Stp. Recall that, for every \(\gamma\), for every \(s\), \(\gamma\) admits \(s\) if and only if, for each \(i\), if \(i \leq \text{length}(s)\), then \(\gamma(\overline{s}_i) = 0\).

Recall that, for all \(s, t\), \(s\) is a proper initial segment of \(t\), notation: \(s \preceq t\) if and only if there exists \(u \neq 0\) such that \(s \ast u = t\).

Let \(\varepsilon\) be an element of \(\mathcal{N}\). We say that \(\varepsilon\) is strictly monotone or: \(\preceq\)–monotone if and only if, for all \(s, t\), if \(s \preceq t\), then \(\varepsilon(s) \preceq \varepsilon(t)\).

Suppose that \(\gamma, \delta\) belong to \(A\).

Let \(\varepsilon\) be an element of \(\mathcal{N}\). We say that \(\varepsilon\) embeds \(\gamma\) into \(\delta\) if and only if \(\varepsilon\) is strictly monotone and \(\varepsilon\) maps every number admitted by \(\gamma\) onto a number admitted by \(\delta\).

We say that \(\gamma\) embeds into \(\delta\), notation: \(\gamma \leq^* \delta\), if and only if there exists a strictly monotone \(\varepsilon\) embedding \(\gamma\) into \(\delta\). We say that \(\gamma\) properly embeds into \(\delta\), notation: \(\gamma <^* \delta\), if and only if, for some \(n\), \(\gamma\) embeds into \(\delta^n\).

We say that \(\gamma\) positively fails to embed into \(\delta\) if and only if for every \(\varepsilon\) in \(\mathcal{N}\), if \(\varepsilon\) is strictly monotone, then there exists \(s\) admitted by \(\gamma\) such that \(\varepsilon(s)\) is not admitted by \(\delta\).

Note that, if \(\gamma\) positively fails to embed into \(\delta\), then \(\gamma\) does not embed into \(\delta\).

The first item of the next theorem shows that, on the set of stumps, the notions \(\leq^*, <^*\) coincide with the notions \(\leq, <\) introduced in Section 1. The theorem should be compared to Theorem 1.1.
The statements of the next theorem have an elementary proof, not involving typically intuitionistic axioms. Recall from Theorem 8.2.1.2 that one may prove, by induction on the set \( \text{Stp} \) of stumps: \( \text{Stp} \) is a subset of \( A^1_1 \).

**Theorem 8.2.**

(i) For all \( \sigma, \tau \in \text{Stp} \), \( \sigma \leq \tau \) if and only if \( \sigma \leq^* \tau \), and: \( \sigma < \tau \) if and only if \( \sigma <^* \tau \).

(ii) The proof is an exercise in (transfinite) induction on the set of stumps and strictly monotone, that is: for every \( s, t \), \( s \geq t \) if and only if \( s \leq^* t \).

(iii) First observe that \( \gamma \) is a proper initial segment of \( \epsilon \).

Proof. (i) The proof is an exercise in (transfinite) induction on the set of stumps and left to the reader.

(ii) The proof is straightforward and left to the reader.

(iii) Let \( \gamma, \delta \) be elements of \( A^1_1 \) such that \( \gamma <^* \delta \). Let \( \epsilon \) be an element of \( N \) that is strictly monotone, that is: for every \( s, t \), if \( s \) is a proper initial segment of \( t \), then \( \epsilon(s) \) is a proper initial segment of \( \epsilon(t) \). We have to prove that there exists \( s \) admitted by \( \gamma \) such that \( \epsilon(s) \) is not admitted by \( \gamma \).

First observe that \( \delta \) admits \( (\gamma) \) and that, if \( \gamma \) does not admit \( (\gamma) \), then \( \gamma \) does not admit \( \epsilon((\gamma)) \), and we are done.

In the following we assume: \( \gamma \) admits \( (\gamma) \).

Find \( n, \zeta \) such that \( \zeta \) is strictly monotone and \( \zeta \) embeds \( \gamma \) into \( \delta^N \). We now define, by recursion, a sequence \( \beta \) in \( N \), such that \( \beta(0) = (\gamma) \) and, for each \( p \), \( \beta(p + 1) = \epsilon((\gamma) + \zeta(\beta(p))) \). Note that, for each \( n \), \( \beta(n) \) is a proper initial part of \( \beta(n + 1) \).

Recalling that \( \gamma \) belongs to \( A^1_1 \), find \( j > 0 \) such that \( \beta(j - 1) \) is admitted by \( \gamma \) and \( \beta(j) \) is not. Now take \( s = (n) \star (\zeta(\beta(j - 1)) \) and observe: \( s \) is admitted by \( \beta \) and \( \epsilon(s) \) is not admitted by \( \gamma \).

\[ \Box \]

8.2. \( A^1_1 \) and \( E^1_1 \) do not reduce to each other. The next theorem is again an elementary result.

**Theorem 8.2.**

(i) Cantor’s diagonal argument: For every function \( \gamma \) from \( N \) to \( A^1_1 \) there exists \( \alpha \) in \( A^1_1 \) such that, for every \( \beta \), \( \gamma(\beta) \) is apart from \( \alpha \).

(ii) The Boundedness Theorem: For every function \( \gamma \) from \( N \) to \( A^1_1 \) there exists \( \alpha \) in \( A^1_1 \) such that, for every \( \beta \), \( \gamma(\beta) \) embeds into \( \alpha \).

Proof. (i) Let \( \gamma \) be a function from \( N \) to \( A^1_1 \), and let \( \alpha \) be an element of \( A^1_1 \) such that, for every \( \beta \), \( \alpha(\beta) = (\gamma(\beta)) \beta + 1 \). Clearly, \( \alpha \) is apart from every \( \gamma(\beta) \).

(ii). Let \( \gamma \) be a function from \( N \) to \( A^1_1 \). Observe that for every \( \beta \), for every \( \delta \), there exists \( n \) such that \( (\gamma(\beta)(\delta) \neq 0 \) and there exists \( m \) such that \( \gamma(\delta)(\beta) \delta n) + 1 \) and, for each \( i < m \), \( \gamma(\delta)(\beta) \delta i) = 0 \). We define \( \alpha \) in \( C \) such that for every \( a, n \), if \( n = \text{length}(a) \) and \( a = (a(0), \ldots, a(n - 1)) \), then \( \alpha(a) = 1 \) if and only if there exist \( i, j \) such that \( 2i < n \) and \( 2j + 1 < n \) and \( \gamma(a(0), a(1), \ldots, a(2j + 1)) \) differs from \( 0 \). Clearly, \( \alpha \) belongs to \( A^1_1 \). Now consider any \( \beta \) in \( N \). Define an element \( \epsilon \) of \( N \), as follows. For every \( d, n \), if \( n = \text{length}(d) \) and \( d = (d(0), \ldots, d(n - 1)) \), then \( \epsilon(d) := (d(0), \beta(0), \ldots, d(n - 1), \beta(n - 1)) \). One verifies easily that \( \epsilon \) embeds \( \gamma(\beta) \) into \( \alpha \).

\[ \Box \]

Note that one may obtain the conclusion of the first item of Theorem 8.2.1.2 also from its second item, as follows.
Let $\gamma$ be a function from $\mathcal{N}$ to $A_1^\gamma$. Using Theorem 8.2(ii), find $\alpha$ in $A_1^\gamma$ such that, for every $\beta$, $\gamma|\beta$ embeds into $\alpha$. Find $\delta$ in $A_1^\delta$ such that $\delta^0 = \alpha$ and note: $\alpha$ strictly embeds into $\delta$. We claim that, for every $\beta$, $\gamma|\beta$ is apart from $\delta$. In order to see this, let $\beta$ belong to $\mathcal{N}$. Find $\varepsilon$ in $\mathcal{N}$ embedding $\gamma|\beta$ into $\alpha$. Using Theorem 8.1(iii), find $s$ such that $s$ is admitted by $\delta$ and $\varepsilon(s)$ is not admitted by $\alpha$. Note that $s$ is admitted by $\delta$ and not by $\gamma|\beta$, and, therefore, $\delta$ is apart from $\gamma|\beta$.

Recall that, just before Theorem 2.1 and just before Theorem 7.1 we introduced subsets $US_1^\alpha$ and $UP_1^\alpha$ of $\mathcal{N}$ as follows.

$US_1^\alpha$ is the set of all $\alpha$ such that, for some $\gamma$, the sequence $\langle \alpha_1, \gamma \rangle$ belongs to $F_{\alpha_1}$.

$UP_1^\alpha$ is the set of all $\alpha$ such that, for all $\gamma$, the sequence $\langle \alpha_1, \gamma \rangle$ belongs to $G_{\alpha_1}$.

We have seen that $US_1^\alpha$, $UP_1^\alpha$ are cataloguing elements of the classes $\Sigma_1^\alpha$, $\Pi_1^\alpha$, respectively, in Theorem 2.1(i) and in Theorem 7.1(i).

Recall that $E_1^\alpha$ is the set of all $\alpha$ such that for some $\beta$, for all $n$, $\alpha(\overline{\beta}n) = 0$ and $A_1^-\alpha$ is the set of all $\alpha$ such that for every $\beta$ there exists $n$ such that $\alpha(\overline{\beta}n) \neq 0$.

We have seen that $E_1^\alpha$, $A_1^-\alpha$ are complete elements of the classes $\Sigma_1^\alpha$, $\Pi_1^\alpha$, respectively, in Theorem 2.2(i) and in Theorem 7.2(i).

For each $\alpha$, for each $\beta$, we define: $\alpha$ forbids $\beta$ if and only if, for some $n$, $\alpha(\overline{\beta}n) \neq 0$ and $\alpha$ admits $\beta$ if and only if $\alpha$ does not forbid $\beta$, that is, for each $n$, $\alpha(\overline{\beta}) = 0$.

The following theorem should be compared to [45], Theorem 5.2.

**Theorem 8.3** (The Classical Start of the Projective Hierarchy).

(i) $UP_1^\alpha$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $US_1^\alpha$, and $A_1^\alpha$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $E_1^\alpha$.

(ii) If $UP_1^\alpha$ belongs to $\Sigma_1^\alpha$, and also if $US_1^\alpha$ belongs to $\Pi_1^\alpha$, there exists $\alpha$ not belonging to either one of $UP_1^\alpha$, $US_1^\alpha$.

(iii) For every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ there exists $\alpha$ such that, for each $\beta$, $\alpha$ forbids $\beta$ if and only if $\gamma|\alpha$ forbids $\beta$, and therefore: $\alpha$ belongs to $A_1^-\alpha$ if and only if $\gamma|\alpha$ belongs to $A_1^-\alpha$ and also: $\alpha$ belongs to $E_1^\alpha$, if and only if $\gamma|\alpha$ belongs to $E_1^\alpha$.

(iv) If $A_1^\alpha$ reduces to $E_1^\alpha$, and also if $E_1^\alpha$ reduces to $A_1^\alpha$, there exists $\alpha$ not belonging to either one of $A_1^\alpha$, $E_1^\alpha$.

**Proof.** (i) We prove that $A_1^\alpha$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $E_1^\alpha$.

Suppose that $\alpha$ belongs to $A_1^\alpha$ and $\beta$ to $E_1^\alpha$. Find $\gamma$ such that, for all $n$, $\beta(\overline{\gamma}n) = 0$. Find $m$ such that $\alpha(\overline{\gamma}m) \neq 0$ and conclude: $\alpha(\overline{\gamma}m) \neq \beta(\overline{\gamma}m)$, so $\alpha \neq \beta$. Clearly, every $\alpha$ in $A_1^-\alpha$ is apart from every $\beta$ in $E_1^\alpha$.

Now assume that $\alpha$ is apart from every $\beta$ in $E_1^\alpha$. Let $\gamma$ be an element of $\mathcal{N}$. We let $\alpha_\gamma$ be the element of $\mathcal{N}$ satisfying: for each $s$, if $\gamma$ passes through $s$, then $\alpha_\gamma(s) = 0$, and, if $\gamma$ does not pass through $s$, then $\alpha_\gamma(s) = \alpha(s)$. Now that $\alpha_\gamma$ belongs to $E_1^\alpha$ and find $s$ such that $\alpha_\gamma(s) \neq \alpha(s)$. Note that $\gamma$ passes through $s$ and that $\alpha(s) \neq 0$, so there exists $n$ such that $\alpha(\overline{\gamma}n) \neq 0$. Clearly, every $\alpha$ that is apart from every $\beta$ in $E_1^\alpha$ belongs to $A_1^-\alpha$.

It follows that $A_1^\alpha$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $E_1^\alpha$.

The proof that $UP_1^\alpha$ is the set of all $\alpha$ in $\mathcal{N}$ apart from every $\beta$ in $US_1^\alpha$ is similar and left to the reader.

(ii) Suppose that $UP_1^\alpha$ belongs to $\Sigma_1^\alpha$ and consider the diagonal set $DP_1^\alpha$ consisting of all $\alpha$ such that $\langle \alpha, \alpha \rangle$ belongs to $UP_1^\alpha$. Also $DP_1^\alpha$ belongs to $\Sigma_1^\alpha$. Find $\beta$ such that $DP_1^\alpha$ coincides with $US_1^\alpha|\beta$ and observe: $\langle \beta, \beta \rangle$ belongs to $UP_1^\alpha$ if and only if $\langle \beta, \beta \rangle$ belongs to $US_1^\alpha$. As every element of $UP_1^\alpha$ is apart from every element of $US_1^\alpha$, $\langle \beta, \beta \rangle$ cannot belong to either $UP_1^\alpha$ or $US_1^\alpha$.

(iii) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$. We define $\alpha$ in $C$ as follows. For each $s$, $\alpha(s) = 1$ if and only if there exist initial segments $t, u$ of $s$ such that $\gamma(t) > 1$ and for every proper initial segment $v$ of $u$, $\gamma(v) = 0$. Observe that for every $\beta$, for some $n$, $\alpha(\overline{\beta}n) = 0$.
α(3n) > 0 if and only if, for some n, (γ|α)(3n) > 0, that is α forbids β if and only if γ|α forbids β, and also: α admits β if and only if γ|α admits β. It follows that α belongs to A^1_1 if and only if γ|α belongs to A^1_1 and: α belongs to E^1_1 if and only if γ|α belongs to E^1_1.

(iv) Suppose that γ is a function from N to N reducing to A^1_1 to E^1_1. Using (iii), find α such that α belongs to A^1_1 if and only if γ|α belongs to A^1_1 and: α belongs to E^1_1 if and only if γ|α belongs to E^1_1. We also know that α belongs to A^1_1 if and only if γ|α belongs to E^1_1. As every element of A^1_1 is apart from every element of E^1_1, α cannot belong to either one of A^1_1, E^1_1. The proof of the second statement is similar. □

The classical mathematician concludes from Theorem 8.3(iii) that the sets A^1_1 and E^1_1 do not reduce to each other, and thus that there exists an analytic set that is not co-analytic and a co-analytic set that is not analytic. The intuitionistic mathematician, however, fails to see that a contradiction follows from the assumption that some α in N does not belong to A^1_1 and also not to E^1_1. He may take a note that, as in the case of the second level of the Borel hierarchy, see [45], Subsection 5.3, Markov’s Principle would enable him to obtain the contradiction, as follows:

Assume that α does not belong to E^1_1. Then ¬∃β∀n[α(βn) = 0]. Therefore, ∀β¬∃n[α(βn) ≠ 0]. By Markov’s Principle, ∀β∃n[α(βn) ≠ 0], so α belongs to A^1_1.

Nevertheless, it is true, intuitionistically as well as classically, that the set E^1_1 is not co-analytic and that the set A^1_1 is not analytic.

In fact, we already know that E^1_1 does not reduce to A^1_1: the set D^2(A_1), although clearly analytic and reducing to E^1_1, does not reduce to A^1_1, see Theorem 7.1(iv). There is no (continuous) function γ from N to N such that, for every α, α belongs to E^1_1 if and only if γ|α belongs to A^1_1.

Exploiting the the fixed-point-idea behind Theorem 8.3(iii) one may prove, in an elementary way, a stronger statement: every (continuous) function from N to N that maps the set E^1_1 into the set A^1_1 will also map some element of A^1_1 into A^1_1, or, as we will say: E^1_1 positively fails to reduce to A^1_1.

We do so in the next Theorem. For the sake of comparison we have added the elementary argument showing that the set E_2 positively fails to reduce to the set A_2. This argument occurs already in [43], Subsection 5.4.

The set E_2 = E^*_2 is the set of all α such that, for some n, α^n = 0. The set E_2 is a complete element of the class Σ^0_2 consisting of the countable unions of closed sets. (2^* = S(1^*) is the stump defined defined by: for all s, 2^*(s) = 0 if length(s) < 2 and 2^*(s) = 1 if length(s) ≥ 2). The set A_2 = A_2 is the set of all α such that, for all n, α^n ≠ 0. The set A_2 is a complete element of the class Π^0_2 consisting of the countable intersections of open subsets of N.

Theorem 8.4 (E^1_1 positively fails to reduce to A^1_1).

(i) For every function γ from N to N mapping E_2 into A_2 there exists α such that both α and γ|α belong to A_2.

(ii) For every function γ from N to N mapping E^1_1 into A^1_1 there exists α such that both α and γ|α belong to A^1_1.

Proof. (i) Let γ be a function from N to N mapping E_2 into A_2. Define α in C such that, for each s, α(s) ≠ 0 if and only if there exist n, t, j, i such that s = ⟨n⟩ * t and j ≤ length(t) and i ≤ s and γ(⟨n⟩ * t, i) > 1, while, for each k, if k < i, then γ(⟨n⟩ * t, k) = 0. Note that, for each n, α^n ≠ 0 if and only if (γ|α)^n ≠ 0 and, therefore, α belongs to E_2 if and only if γ|α belongs to E_2 and: α belongs to A_2 if and only if γ|α belongs to A_2.
We claim that $\alpha$, and therefore also $\gamma|\alpha$, belong to $A_2$, and we prove this claim as follows. Let $n$ belong to $\mathbb{N}$. Let $\alpha_n$ be the element of $\mathcal{N}$ satisfying: for each $s$, if there exists $t$ such that $s = (n) * t$, then $\alpha_n(s) = 0$, and, if there is no $t$ such that $s = (n) * t$, then $\alpha_n(s) = \alpha(s)$. Note that $(\alpha_n)^n = \mathbf{1}$ and that $\alpha_n$ belongs to $E_2$. It follows that $\gamma|\alpha_n$ belongs to $A_2$. Find $m$ such that $(\gamma|\alpha_n)^m(m) \neq 0$. Find $p$ such that $(\gamma|\alpha_n)^m(p) > 2$ and, for all $j$, if $j < p$, then $(\gamma|\alpha_n)^m(j) = 0$. Now distinguish two cases. Either: $\mathbf{p} = \mathbf{0}$ and therefore $(\gamma|\alpha_n)^m(m) = (\gamma|\alpha_n)^m(m) \neq 0$ and: $(\gamma|\alpha)^n \neq \mathbf{0}$ and, therefore: $\alpha^n \neq \mathbf{0}$.

Clearly, for each $n$, both $\alpha^n \neq \mathbf{0}$ and $(\gamma|\alpha)^n \neq \mathbf{0}$ and both $\alpha$ and $\gamma|\alpha$ belong to $A_2$.

(ii) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1$ into $A_1$. Using Theorem [8.3(iii)], find $\alpha$ such that, for each $\beta$, $\alpha$ forbids $\beta$ if and only if $\gamma|\alpha$ forbids $\beta$, and therefore: $\alpha$ belongs to $A_1$ if and only if $\gamma|\alpha$ belongs to $A_1$, and: $\alpha$ belongs to $E_1$ if and only if $\alpha$ belongs to $E_1$. We claim that $\alpha$, and therefore also $\gamma|\alpha$, belong to $A_1$, and we prove this claim as follows.

Let $\beta$ belong to $\mathcal{N}$. Let $\alpha_\beta$ be the element of $\mathcal{N}$ satisfying: for each $s$, if $\beta$ passes through $s$, then $\alpha_\beta(s) = 0$, and, if $\beta$ does not pass through $s$, then $\alpha_\beta(s) = \alpha(s)$. Note that $\alpha_\beta$ admits $\beta$ and belongs to $E_1$. It follows that $\gamma|\alpha_\beta$ belongs to $A_1$. Find $n$ such that $(\gamma|\alpha_\beta)(\beta n) \neq 0$. Also find $p$ such that $(\gamma|\alpha_\beta)(\beta n) > 1$, and for all $j < p$, $(\gamma|\alpha_\beta)(\beta n j) = 0$. Note that, for every $\delta$, if $\delta \neq \mathbf{0}$, then $(\gamma|\alpha_\beta)(\beta n) = (\gamma|\alpha_\beta)(\beta n) \neq 0$. Now distinguish two cases. Either: $\mathbf{p} = \mathbf{0}$ and therefore $(\gamma|\alpha_\beta)(\beta n) = (\gamma|\alpha_\beta)(\beta n) \neq 0$, and $\gamma|\alpha$ forbids $\beta$, and, therefore, $\alpha$ forbids $\beta$, or: $\mathbf{m} = \mathbf{0}$ and, and therefore, $\gamma|\alpha$ forbids $\beta$.

Clearly, for each $\beta$, both $\alpha$ and $\gamma|\alpha$ forbid $\beta$, and both $\alpha$ and $\gamma|\alpha$ belong to $A_1$.

The next question is: can we also prove that the set $A_1$ is not analytic? Note that we have seen already that $A_1$ is not a strictly analytic subset of $\mathcal{N}$. In fact, given any (continuous) function from $\mathcal{N}$ into $A_1$, one may produce an element $\beta$ of $A_1$ such that, for every $\alpha$, $\beta \neq \gamma|\alpha$, see Theorem [6.32]. One might say: $A_1$ positively fails to be strictly analytic.

The next Lemma prepares for an argument that $A_1$ also fails to be analytic.

We first introduce the following notion.

Let $X,Y$ be subsets of $\mathcal{N}$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$. $\gamma$ strongly reduces the set $X$ to the set $Y$ if and only if, for each $\alpha$, $\alpha$ belongs to $X$ if and only if $\gamma|\alpha$ belongs to $Y$ and: $\alpha$ is apart from every element of $X$ if and only if $\gamma|\alpha$ is apart from every element of $Y$.

$X$ strongly reduces to $Y$ if and only if there exists a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ strongly reducing $X$ to $Y$.

**Lemma 8.5.** Every analytic subset of $\mathcal{N}$ strongly reduces to $E_1$.

**Proof.** Let $X$ be an analytic subset of $\mathcal{N}$. Find $\beta$ in $\mathcal{N}$ such that, for each $\alpha$, $\alpha$ belongs to $X$ if and only if, for some $\delta$, $\beta$ admits $\langle \alpha, \delta \rangle$, that is, for all $n$, $\beta(\langle \alpha, \delta \rangle n) = 0$. Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $\delta$, $(\gamma|\alpha)(\beta n) = \beta(\langle \alpha, \delta \rangle n)$. It will be clear that, for all $\alpha$, $\alpha$ belongs to $X$ if and only if, for all $\delta$, $\beta$ forbids $\langle \alpha, \delta \rangle$, that is, for some $n$, $\beta(\langle \alpha, \delta \rangle n) \neq 0$. It follows that, for each $\alpha$, $\alpha$ is apart from every element of $X$ if and only if, for all $\delta$, $\gamma|\alpha$ forbids $\delta$ if and only if $\gamma|\alpha$ is apart from every element of $E_1$.

**Theorem 8.6.** The set $A_1$ is not analytic.
Proof. Suppose that the set $A_1$ is analytic. Using lemma 8.5 find a function $\gamma$ from $N$ to $\mathcal{N}$ strongly reducing $A_1$ to $E_1$. Note that, for every $\alpha$, if $\alpha$ belongs to $E_1$, then, by Theorem 8.4(i), $\alpha$ is apart from every element of $A_1$ and, therefore, $\gamma|\alpha$ is apart from every element of $E_1$, and so, again by Theorem 8.4(i), $\gamma|\alpha$ belongs to $A_1$. It follows that $\gamma$ maps $E_1$ into $A_1$. Using Theorem 8.4 one finds $\alpha$ in $A_1$ such that $\gamma|\alpha$ belongs to $A_1$. Contradiction. 

The elementary argument for Theorem 8.6 has eluded us for some time. We then used the following reasoning in order to show that $A_1$ is not analytic. $A_1$ is a definite subset of $\mathcal{N}$, and, therefore, by the Brouwer-Kripke axiom, Axiom 4 in Subsection 5.2, if $A_1$ is analytic, $A_1$ is strictly analytic, see Theorem 5.2(iii). However, as we mentioned before, according to Theorem 8.6, $A_1$ positively fails to be strictly analytic.

8.3. $A_1$ and $E_1$ are not positively Borel. Note that it follows from Theorem 8.6 that the set $A_1$ is not positively Borel, as, according to Theorem 2.1 every subset of $\mathcal{N}$ that is positively Borel is also analytic. Like the conclusion of Theorem 8.6, this conclusion is obtained in an elementary way.

We now are going to prove that the set $E_1$ is not positively Borel. We will use the Borel Hierarchy Theorem 1.3 in the proof of which Brouwer’s Continuity Principle, Axiom 4 plays a key role. We did not find an elementary argument, avoiding the typically intuitionistic axioms.

Lemma 8.7. For every complementary pair $(X, Y)$ of positively Borel sets there exists a (continuous) function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $E_1$ and mapping $Y$ into $A_1$.

Proof. Let $(X, Y)$ be a complementary pair of positively Borel sets. Note that $X$ is analytic, see Theorem 2.1(ii), and that every element of $X$ is apart from every element of $Y$. According to Lemma 8.5 there exists a function strongly reducing $X$ to $E_1$, that is, reducing $X$ to $E_1$ and mapping every $\alpha$ that is apart from every element in $X$ into $A_1$. Note that $\gamma$ reduces $X$ to $E_1$ and maps $Y$ into $A_1$.

Theorem 8.8 (The sets $E_1$ and $A_1$ positively fail to be positively Borel).

(i) For every positively Borel subset $X$ of $\mathcal{N}$, for every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1$ into $X$ there exists $\alpha$ in $A_1$ such that $\gamma|\alpha$ belongs to $X$.

(ii) For every non-zero stump $\sigma$, for every positively Borel subset $X$ of $\mathcal{N}$, for every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ mapping $A_1$ into $X$ there exists $\alpha$ in $E_1$ such that $\gamma|\alpha$ belongs to $X$.

(iii) The sets $E_1$ and $A_1$ are not positively Borel.

Proof. (i) Let $X$ be a positively Borel set and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $E_1$ into $X$. Find a non-zero stump $\sigma$ and a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $A_\sigma$. Using Lemma 8.7 find a function $\varepsilon$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $E_\sigma$ to $E_1$ and mapping $A_\sigma$ into $A_1$. Observe that, for each $\alpha$, if $\alpha$ belongs to $E_\sigma$, then $\varepsilon|\alpha$ belongs to $E_1$, and $\gamma|(\varepsilon|\alpha)$ belongs to $X$, and $\delta(\gamma|(\varepsilon|\alpha))$ belongs to $A_\sigma$, so the composition of the functions (coded by) $\delta$, $\gamma$, and $\varepsilon$ maps $E_\sigma$ into $A_\sigma$. Using the Borel Hierarchy Theorem, we find $\alpha$ such that both $\alpha$ and $\delta(\gamma|(\varepsilon|\alpha))$ belong to $A_\sigma$. Remark that $\varepsilon|\alpha$ belongs to $A_1$ and $\gamma|(\varepsilon|\alpha)$ belongs to $X$.

(ii) Let $X$ be a positively Borel set and let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ mapping $A_1$ into $X$. Find a non-zero stump $\sigma$ and a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $A_\sigma$. Using Lemma 8.7 find a function $\varepsilon$ from $\mathcal{N}$ to $\mathcal{N}$ reducing $A_\sigma$ to $E_1$ and mapping $E_\sigma$ into $A_1$. Observe that, for each $\alpha$, if $\alpha$ belongs to $E_\sigma$, then $\varepsilon|\alpha$ belongs to $A_1$, and $\gamma|(\varepsilon|\alpha)$ belongs to $X$, and $\delta(\gamma|(\varepsilon|\alpha))$ belongs to $A_\sigma$, so the composition of the functions (coded by) $\delta$, $\gamma$, and $\varepsilon$ maps $E_\sigma$ into $A_\sigma$. Using the Borel Hierarchy Theorem, we find $\alpha$ such that both $\alpha$ and $\delta(\gamma|(\varepsilon|\alpha))$ belong to $A_\sigma$. Remark that $\varepsilon|\alpha$ belongs to $A_1$ and $\gamma|(\varepsilon|\alpha)$ belongs to $X$. 68
(iii) is an easy consequence of (i) and (ii).

\[ \Box \]

In [22], the question how to prove that $A_1^1$ is not positively Borel was asked but not answered. As we observed in Section 1 of this paper, it is shown in [11] that there exist co-analytic and analytic subsets of $\mathcal{N}$ much “simpler” than $E_1^1$, $A_1^1$, respectively, that fail to be positively Borel, see Theorem [1.11] and [1.12]. The fact that the set $A_1^1$ is not positively Borel, one of the conclusions of Theorem [8.8] (iii), also follows from Theorem [1.11] and Theorem [7.2]. The fact that the set $E_1^1$ is not positively Borel, the other conclusion of Theorem [8.8] (iii), also follows from Theorem [1.12] and Theorem [7.2].

**Theorem 8.9** (One Half of Souslin’s Theorem).

(i) For every stump $\sigma$, the set of all $\alpha$ such that $\alpha$ embeds into $\sigma$ is a positively Borel subset of $\mathcal{N}$.

(ii) For every subset $X$ of $\mathcal{N}$, if $X$ is both strictly analytic and co-analytic, then $X$ is positively Borel.

**Proof.** (i) This fact follows, by induction on the set of stumps, from the observation that for every non-zero stump $\sigma$, for every $\alpha$, $\alpha$ embeds into $\sigma$ if and only if for each $m$ there exists $n$ such that $\alpha^n$ embeds into $\sigma^m$.

(ii) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that $X$ coincides with the range of $\gamma$. Let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $X$ to $A_1^1$. Let $h$ be the composition of $g$ and $f$. Using the Boundedness Theorem, Theorem [8.9] (ii), find $\beta$ in $A_1^1$ such that, for every $\alpha$, $\delta|\alpha$ embeds into $\beta$. Using Brouwer’s Thesis, see Lemma [1.2] we may assume that $\beta$ is a stump. Observe that, for each $\alpha$, $\beta$ is positively Borel if and only if $\delta|\alpha$ embeds into $\beta$, therefore, by (i), $X$ is positively Borel.

\[ \Box \]

Note that, combining Theorem [8.9] and Theorem [8.8], we find a corroboration of the conclusion of Theorem [8.2]: the set $A_1^1$ is not strictly analytic.

Theorem [8.9] (ii) is of limited application as every co-analytic subset of $\mathcal{N}$ is perhapsive and “most” positively Borel sets are not. Therefore, there are not “many” positively Borel sets that are both co-analytic and strictly analytic. The converse of Theorem [8.9] (ii) is far from true.

### 9. The Second Level of the Projective Hierarchy

Let $X$ be a subset of $\mathcal{N}$. $X$ is called an \textit{(existential) projection of a co-analytic set} if and only if there exists a co-analytic subset $Y$ of $\mathcal{N}$ such that $X = Ex(Y)$. The class of these sets is denoted by $\Sigma^1_2$. A subset $X$ of $\mathcal{N}$ is called a \textit{universal projection of an analytic set} or a \textit{co-projection of an analytic set} if and only if there exists an analytic subset $Y$ of $\mathcal{N}$ such that $X = Un(Y)$. The class of these sets is denoted by $\Pi^1_2$.

#### 9.1. An example: the set of all $\beta$ coding a located and closed subset of $\mathcal{N}$ that is almost-countable.

Let $\delta$ be an element of $\mathcal{N}$. As in [16], we let $En_\delta$ be the set $\{\delta^0, \delta^1, \ldots\}$ and we call this set the \textit{subset of $\mathcal{N}$ enumerated by $\delta$}.

In Section 1, we have seen how one defines, given any subset $X$ of $\mathcal{N}$, a function assigning to every stump $\sigma$ a subset $\mathbb{P}(\sigma, X)$ of $\mathcal{N}$, called the $\sigma$-th perhapsive extension of the set $X$, as follows, by induction:

(i) For every stump $\sigma$, if $\sigma(\langle \rangle) \neq 0$, then $\mathbb{P}(\sigma, X) = X$

(ii) For every stump $\sigma$, if $\sigma(\langle \rangle) = 0$, then $\mathbb{P}(\sigma, X)$ is the set of all $\alpha$ such that, for some $\beta$, if $\alpha \neq \beta$, then there exists $m$ such that $\alpha$ belongs to $\mathbb{P}(\sigma^m, X)$.
It is not difficult to prove and shown in [46] that, for all inhabited subsets \(X\) of \(\mathcal{N}\), for all stumps \(\sigma, \tau\), if \(\sigma \leq \tau\), then \(\mathcal{P}(\sigma, X)\) is a subset of \(\mathcal{P}(\tau, X)\), and that, for all inhabited subsets \(X\) and \(Y\) of \(\mathcal{N}\), if \(X\) is a subset of \(Y\), then, for all stumps \(\sigma\), \(\mathcal{P}(\sigma, X)\) is a subset of \(\mathcal{P}(\sigma, Y)\).

Note that, for every \(\delta\), for every non-empty stump \(\sigma, \alpha\) belongs to \(\mathcal{P}(\sigma, \mathcal{E}_n)\) if and only if, for some \(n\), if \(\alpha \neq \delta^n\), then there exists \(m\) such that \(\alpha\) belongs to \(\mathcal{P}(\sigma^n, X)\).

We let Almost\(^*(\mathcal{E}_n)\) be the set of all \(\alpha\) satisfying: for each \(\gamma\), there exists \(n\) such that \(\mathcal{P}(\gamma(n)) = \delta^n(\gamma(n))\). Note that we may think of Almost\(^*(\mathcal{E}_n)\) as the set of all \(\alpha\) such that every attempt to give positive evidence that \(\alpha\) differs from every \(\delta^n\) fails in finitely many steps. It is not difficult to prove and shown in [46] that, for all inhabited subsets \(X\) of all stumps \(\alpha\), the sense that it does not depend on the enumeration \(\delta\) that.

It is also shown in [46] that for each \(\delta\), for each stump \(\sigma\), the set \(\mathcal{P}(\sigma, \mathcal{E}_n)\) is a subset of the set Almost\(^*(\mathcal{E}_n)\) and that Brouwer’s Thesis implies that the set Almost\(^*(\mathcal{E}_n)\) id \(s\) subset of the set \(\bigcup_{\sigma \in \mathcal{Stp}} \mathcal{P}(\sigma, \mathcal{E}_n)\). This result will be proven again in Theorem 9.2.

We let CS be the set of all \(\beta\) coding an countable spread, that is the set of all \(\beta\) such that

(i) for all \(s, \beta(s) = 0\) if and only if, for some \(n\), \(\beta(s \cdot \langle n \rangle) = 0\), and

(ii) for some \(\delta, F_\delta\) is a subset of \(\mathcal{E}_n\).

In the above definition, one might think of strengthening (ii) as follows: “for some \(\delta\), for each \(n\), \(\delta^n\) belongs to \(F_\delta\), and \(F_\delta\) coincides with \(\mathcal{E}_n\).” This would not be a real strengthening. For suppose, for some \(\delta\), \(F_\delta\) is a subset of \(\mathcal{E}_n\). As in the proof of Theorem 2.3(i), let \(\gamma\) be a function from \(\mathcal{N}\) to \(\mathcal{N}\) retracting \(\mathcal{N}\) onto \(F_\delta\). Let \(\delta'\) satisfy: for each \(n\), \((\delta')^n = \gamma(\delta^n)\). Note that for each \(n\), \((\delta')^n\) belongs to \(F_\gamma\), and \(F_\gamma\) coincides with \(\mathcal{E}_{n'}\).

For each stump \(\sigma\) we let PCS\(_\sigma\) be the set of all \(\beta\) coding a \(\sigma\)-perhaps-countable spread, that is the set of all \(\beta\) such that

(i) for all \(s, \beta(s) = 0\) if and only if, for some \(n\), \(\beta(s \cdot \langle n \rangle) = 0\), and

(ii) for some \(\delta, F_\delta\) is a subset of \(\mathcal{P}(\sigma, \mathcal{E}_n)\).

We let ACS be the set of all \(\beta\) coding an almost-countable spread, that is the set of all \(\beta\) such that

(i) for all \(s, \beta(s) = 0\) if and only if, for some \(n\), \(\beta(s \cdot \langle n \rangle) = 0\), and

(ii) for some \(\delta, F_\delta\) is a subset of Almost\(^*(\mathcal{E}_n)\).

It is easy to see that the sets PCS\(_\sigma\) and the set ACS belong to the class \(\Sigma^1_2\).

Note that a classical mathematician, having seen Section 2, and in particular its conclusion that the set UNC is analytic, see Theorem 2.3 would expect the sets PCS\(_\sigma\) and the set ACS to be in the class \(\Pi^1_1\).

Recall that Fin is the set of all \(\alpha\) in Cantor space \(\mathcal{C}\) such that there exists \(n\) such that, for all \(m\), if \(m > n\), then \(\alpha(m) = 0\).

Note that the set Fin and also all its perhaps extensions are shift-invariant subsets of \(\mathcal{C}\), that is: for every stump \(\sigma\), for every \(\alpha\) in \(\mathcal{C}\), \(\alpha\) belongs to the set \(\mathcal{P}(\sigma, \text{Fin})\) if and only if \((0) \ast \alpha\) belongs to \(\mathcal{P}(\sigma, \text{Fin})\) if and only if \((1) \ast \alpha\) belongs to \(\mathcal{P}(\sigma, \text{Fin})\).

Note that there exists \(\delta\) such that Fin = \(\mathcal{E}_n\). For instance, let \(\delta\) satisfy: for each \(s\), if \(s\) belongs to the set Bin of all natural numbers coding a finite binary sequence, then \(\delta^s = s \ast \underline{0}\) and, if not, then \(\delta^s = \underline{1}\).

We define a function \(\sigma \mapsto C B^1_\sigma\) associating to every stump \(\sigma\) a subset of \(\mathcal{N}\), as follows, by induction on the set of stumps:
(i) For each stump $\sigma$, if $\sigma(0) \neq 0$, then $CB^*_{\sigma} = \{0\} \cup \{\overline{m} \ast (1) \ast 0 | n \in \mathbb{N}\}$.

(ii) For each stump $\sigma$, if $\sigma(0) = 0$, then

$$CB^*_{\sigma} = \{0\} \cup \{\overline{m} \ast (1) \ast 0 | n \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} \{\overline{p} \ast (1) \ast 0 | CB^*_{\sigma \ast p} \}.$$

The sets $CB^*_{\sigma}$ are called very special Cantor-Bendixson-sets. We introduced the special Cantor-Bendixson-sets $CB^*_{\sigma}$ in Subsection 3.5. It is not difficult to prove that, for each stump $\sigma$, there exists a stump $\tau$ such that $CB^*_{\tau} = CB^*_{\sigma}$, that is, every very special Cantor-Bendixson-set is a special Cantor-Bendixson-set.

Let $X$ be a located and closed subset of $\mathcal{N}$ and let $Y$ be a subset of $\mathcal{N}$. $X$ embeds into $Y$ if and only if there exists $\gamma$ coding a function from $X$ into $Y$ that is strongly injective, that is, for all $\alpha$, $\beta$ in $X$, if $\alpha \neq \beta$, then $\gamma(\alpha) \neq \gamma(\beta)$.

**Theorem 9.1.**

(i) For every $\beta$ in $CS$, for every stump $\sigma$, if $\sigma(0) \neq 0$, then $F_\beta$ embeds into $CB^*_{\sigma} \cap Fin$.

(ii) For every stump $\sigma$, for every $\beta$ in $PCS_\sigma$, $F_\beta$ embeds into $CB^*_{\sigma} \cap \mathcal{P}(\sigma, Fin)$.

(iii) For every $\beta$ in $ACS$, $F_\beta$ embeds into $Almost^* Fin$.

**Proof.** (i) Let $\sigma$ be a stump satisfying $\sigma(0) \neq 0$, that is, $\sigma$ is or codes the empty stump. Assume that $\beta$ belongs to $CS$. Find $\delta$ such that $F_\beta$ is a subset of $En_\delta$. Using the Second Axiom of Continuous Choice, Axiom [3] find $\gamma$ coding a continuous function from $F_\beta$ to $\mathbb{N}$ such that, for every $\alpha$ in $F_\beta$, $\alpha$ coincides with $\delta(\alpha)$. Note that, for each $\alpha$ in $F_\beta$, $\delta(\alpha)$ is the only element of $F_\beta$ passing through $\overline{m}n$ where $m$ is the least $j$ such that $\gamma(\alpha_j) \neq 0$. We let $\varepsilon$ be the code of a continuous function from $F_\beta$ to $\mathcal{N}$ such that, for each $\alpha$, $\varepsilon(\alpha) = \overline{\mathcal{P}}(\gamma(\alpha)) \ast (1) \ast 0$. It is clear that $\gamma$ defines a strongly injective function from $F_\beta$ to $CB^*_{\sigma} \cap Fin$.

(ii) We use induction on the set of stumps. Note that, if $\sigma(0) \neq 0$, we may refer to (i). Let $\sigma$ be a non-empty stump and assume the statement has been proven for every immediate substump of $\sigma$. Assume that $\beta$ belongs to $PCS_\sigma$. Find $\delta$ such that $F_\beta$ is a subset of $\mathcal{P}(\sigma, En_\delta)$. Note that, for every $\alpha$ in $F_\beta$, there exists $n$ such that, if $\delta(\alpha) \neq \alpha$, then there exists $m$ such that $\alpha$ belongs to $\mathcal{P}(\sigma^n, En_\delta)$. Using the (generalized) First Axiom of Continuous Choice, Axiom [3] find $\gamma$ coding a continuous function from $F_\beta$ to $\mathbb{N}$ such that, for each $\alpha$ in $F_\beta$, if $\delta(\alpha) \neq \alpha$, then there exists $m$ such that $\alpha$ belongs to $\mathcal{P}(\sigma^n, En_\delta)$. We define $s$ in $\mathbb{N}$ to be wrong if $\beta(s) = 0$ and there exists $m \leq length(s)$ such that $m$ is the least $i$ such that $\gamma(\overline{m}n) \neq 0$ and $\delta(\overline{m}n) \neq 1$ does not pass through $s$.

Now assume $s$ is a wrong element of $\mathbb{N}$ such that, for every $n$, if $n < length(s)$, then $\overline{m}n$ is not wrong. So $s$ is a wrong element of $\mathbb{N}$ of minimal length. Let us denote the set of all $\alpha$ in $F_\beta$ that pass through $s$ by $F_\beta \cap s$. Note that, for all $\alpha$ in $F_\beta \cap s$, $\alpha \neq \delta(\alpha)$ and there exists $m$ such that $\alpha$ belongs to $\mathcal{P}(\sigma^n, En_\delta)$. Also note that $F_\beta \cap s$ is a spread. Applying the First Axiom of Continuous Choice, Axiom [3] find $\zeta$ coding a function from $F_\beta \cap s$ to $\mathbb{N}$ such that for all $n$, if $n \leq length(s)$, then $\zeta(\overline{m}n) = 0$, and, for all $\alpha$ in $F_\beta \cap s$, $\alpha$ belongs to $\mathcal{P}(\sigma^{\zeta(\overline{m}n)}, En_\delta)$.

Applying the Second Axiom of Countable Choice, Axiom [3] we find $\rho$ such that, for each $s$, is $s$ a wrong element of $\mathbb{N}$ of minimal length, then $\rho^s$ codes a function from $F_\beta \cap s$ to $\mathbb{N}$ such that for all $n$, if $n \leq length(s)$, then $\rho^s(\overline{m}n) = 0$, and, for all $\alpha$ in $F_\beta \cap s$, $\alpha$ belongs to $\mathcal{P}(\sigma^{\rho^s(\overline{m}n)}, En_\delta)$. We define $t$ in $\mathbb{N}$ to be right if there exists an initial part $s$ of $t$ such that $s$ is a wrong element of $\mathbb{N}$ of minimal length and $\rho^s(t) \neq 0$ and for all $n$, if $n < length(t)$, then $\rho^s(\overline{m}n) = 0$.

Using the induction hypothesis and the Second Axiom of Countable Choice, Axiom [3] we find $\eta$ in $\mathcal{N}$ such that, for every $t$, for each $m$, if $t$ is right, and $m$ is the least
Finally, we let \( \tau \) be the code of a continuous function from \( F_\beta \) to \( \mathcal{N} \) such that, for each \( \alpha \) in \( F_\beta \),

1. for each \( \alpha \) in \( F_\beta \), for all \( n \), if \( n \) is the least \( i \) such that \( \gamma(i) \neq 0 \), then \( \tau|_\alpha \) passes through \( \overline{\mathcal{P}(\mathcal{P}(\alpha|_n)1)} \).
2. for each \( \alpha \) in \( F_\beta \), for each \( m \), if \( n \) is the least \( i \) such that \( \gamma(i) \neq 0 \) and \( m > n \) and there is no \( i \leq m \) such that \( \gamma(i) \) is wrong, then \( \tau|_\alpha \) passes through \( \overline{\mathcal{P}(\mathcal{P}(\alpha|_n)1)} \).
3. for each \( \alpha \) in \( F_\beta \), for each \( m \), for each \( k \), if \( n \) is the least \( i \) such that \( \gamma(i) \neq 0 \) and \( m > n \) and \( m > k \) and \( k \) is the least \( i \) such that \( \rho(i) \neq 0 \), then \( \tau|_\alpha \) passes through \( \overline{\mathcal{P}(\mathcal{P}(\alpha|_n)1)} \).

We now define \( \tau \) such that \( \tau(i) = \overline{\mathcal{P}(\mathcal{P}(\alpha|_n)1)} \). For, given \( \alpha \) such that \( \beta(s) = 0 \), let \( \alpha \) be an element of \( F_\beta \) that passes through \( s \). Define \( \gamma = \text{length}(s) \), that is, for each \( i \), \( \gamma(i) = \text{length}(s) \). Find \( i \) such that \( \gamma(i) \neq 0 \) and conclude: \( \delta^i \) passes through \( s \).

We now define \( \varepsilon \) in \( \mathcal{N} \) such that, for each \( s \), if \( \beta(s) = 0 \), then \( \varepsilon(s) \) equals the least \( i \) such that \( \delta^i \) passes through \( s \). Note that, for all \( s \), for all \( n \), if \( \beta(s \ast n) = 0 \), then \( \varepsilon(s) \leq \varepsilon(s \ast n) \). We then define \( \eta \) in \( \mathcal{N} \) such that \( \eta(\langle i \rangle) = \langle i \rangle \) and, for all \( s \), for all \( n \), if \( \beta(s \ast n) = 0 \), then: \( \varepsilon(s \ast n) = \varepsilon(s) \) and \( \eta(s \ast n) = \eta(s) \ast 0 \), or \( \varepsilon(s \ast n) \neq \varepsilon(s) \) and \( \eta(s \ast n) = \eta(s) \ast 0 \). Finally, we let \( \gamma \) be an element of \( \mathcal{N} \) coding a function from \( F_\beta \) to \( \mathcal{N} \) such that, for every \( \alpha \) in \( F_\beta \), for every \( n \), \( \gamma|_\alpha \) passes through \( \eta(\gamma(i)) \).

It is easily seen that \( \gamma \) is a strongly injective function from \( F_\beta \) to Cantor space \( C \).

We now prove that, for all \( \alpha \) in \( F_\beta \), \( \gamma|_\alpha \) belongs to \( \text{Almost}^* \text{Fin} \).

Let \( \alpha \) belong to \( F_\beta \). Suppose that \( \zeta \) is strictly increasing, that is, for all \( n \), \( \zeta(n) < \zeta(n + 1) \). Note that, for all \( n \), there exists \( m > n \) such that \( \zeta|_\alpha(m) = 0 \). We define \( \zeta' \) as follows. \( \zeta'(0) = \zeta(0) \) and, for each \( n \), \( \zeta'(n + 1) = \zeta(p) \) where \( p \) is the least \( j \) such that there exists \( m \) satisfying: \( \zeta'(n) < m < \zeta'(j) \) and \( \zeta'\alpha(m) = 0 \). We define \( \eta \) in \( \mathcal{N} \) as follows. For each \( p \), \( \eta(p) \) is the number of elements of the set \( \{ m \in C \mid \gamma|_\alpha(m) = 0 \} \). Note that \( \eta \) is strictly increasing and that, for each \( p \), for each \( \alpha' \) in \( F_\beta \), if \( \alpha' \) passes through \( \gamma|_\alpha(p) \), then \( \gamma|_\alpha \) passes through \( \gamma|_\alpha(p) \). Note that, for each \( p \), if \( \gamma|_\alpha(\zeta'(p)) = 1 \), then \( \varepsilon(\gamma(\eta(p))) \neq \varepsilon(\gamma(\eta(p - 1))) \).

We claim that there exists \( i \) such that \( \varepsilon(\gamma(\eta(p))) = \varepsilon(\gamma(\eta(p - 1))) \). In order to prove this claim we define \( \rho \) in \( \mathcal{N} \) as follows. Note that, for each \( i \), if \( j < \varepsilon(\gamma(\eta(i + 1))) \), then \( \delta^i \neq \zeta \). Therefore, for each \( i \), if \( j < \varepsilon(\gamma(\eta(i + 1))) \), we let \( \rho(i) \) be the least \( j \) such that \( \delta^i j \neq \gamma(i) \), and, if not, then \( \rho(i) = 0 \). Using the fact that \( \alpha \) belongs to \( \text{Almost}^* \text{Fin} \), we find \( i \) such that \( \delta^i(\rho(i)) = \gamma(i) \). Note that \( \rho(i) = 0 \) and \( i \geq \varepsilon(\gamma(\eta(i + 1))) \) and there exists \( p \leq i \) such that \( \varepsilon(\gamma(\eta(p))) = \varepsilon(\gamma(\eta(p - 1))) \).

We may conclude that there exists \( p \) such that \( \gamma|_\alpha(\zeta'(p)) = 0 \) and that there exists \( p \) such that \( \gamma|_\alpha(\zeta'(p)) = 0 \).

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We have shown that $\gamma|\alpha$ belongs to Almost$^+$ Fin.

The set of the hereditarily repetitive stumps has been introduced in Section 1, just before Theorem 1.3. One may verify that, for every hereditarily repetitive stump $\sigma$, for each $n$, the set $\overline{CB}_n^\sigma$ embeds into the set $\overline{CB}_n^\sigma \cap \overline{\mathbb{N}}$, that is, the set of all elements of $\overline{CB}_n^\sigma$ passing through $\overline{\mathbb{N}}$, see [46], Theorem 3.9(i).

In the proof of the next theorem, we shall use the fact that, for all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $\overline{CB}_n^\sigma$ embeds into $\overline{CB}_n^\tau$. This fact is proven without much difficulty, see [46], Theorem 3.9(ii).

Theorem 9.2.

(i) For each stump $\sigma$, the set $\overline{CB}_n^\sigma$ is a $\sigma$-perihaps-countable spread.

(ii) For each hereditarily repetitive stump $\sigma$, the set $\overline{CB}_n^\sigma|\Fin$ does not embed into $\mathbb{P}(\sigma, \Fin)$.

(iii) For each hereditarily repetitive stump $\sigma$, the set $\overline{CB}_n^\sigma$ is not a $\sigma$-perihaps-countable spread.

Proof. (i) For each stump $\sigma$, the set $\overline{CB}_n^\sigma$ is easily seen to be a countable set and its closure $\overline{CB}_n^\sigma$ coincides with the set $\Fin(\sigma, \Fin)$, see Theorem 1.3(ii). The latter fact has been established in [46], Theorem 3.22. The proof is given there for the Cantor-Bendixson-sets $CB_n$ but works as well for the special Cantor-Bendixson-sets $\overline{CB}_n^\sigma$, as the proof is by induction on the set of stumps and only the starting clause has to be adapted for the case of the special Cantor-Bendixson-sets introduced in Subsection 3.5 of this paper. The starting statement is trivial, however, as in the case of the Cantor-Bendixson-sets themselves.

(ii) We again use induction on the set of stumps.

First assume that $\sigma$ is a stump satisfying $\sigma(0) \neq 0$. Note that $CB_1^\sigma$ is the set $\{0\} \cup \{\overline{m} \cdot (1) \ast \overline{n} \mid n \in \mathbb{N}\}$. Suppose that $\gamma$ codes a continuous function from $\overline{CB}_1^\sigma$ to $\mathbb{N}$ embedding $\overline{CB}_1^\sigma$ into $\mathbb{P}(\sigma, \Fin) = \Fin$. Note that, for all $\alpha$ in $\overline{CB}_1^\sigma$, there exists $n$ such that, for all $m > n$, $(\gamma|\alpha)(n) = 0$. Using Brouwer’s Continuity Principle, Axiom [4] we find $p, m$, such that, for all $\alpha$ in $\overline{CB}_1^\sigma$, if $\overline{\alpha}p = \overline{\alpha}m$, then, for all $n$, if $n > m$, then $((\gamma|\alpha))(n) = 0$. Find $q \geq p$ such that $\overline{(\gamma|\alpha)(m + 1) \subseteq \gamma|\alpha}q$. It follows that, for every $\alpha$, if $\overline{\alpha}q = \overline{\gamma|\alpha}q$, then $\gamma|\alpha = \overline{\gamma|\alpha}$, so $\gamma$ is not strongly injective.

Now assume that $\sigma$ is a hereditarily repetitive stump such that $\sigma(0) = 0$ and, for each $n$, $\overline{CB}_n^\sigma$ does not embed into $\mathbb{P}(\sigma^n, \Fin)$. Suppose that $\gamma$ codes a continuous function from $\overline{CB}_n^\sigma$ to Cantor space $\mathcal{C}$ embedding $\overline{CB}_n^\sigma$ into $\mathbb{P}(\sigma, \Fin)$. Note that, for all $\alpha$ in $\overline{CB}_n^\sigma$, there exists $m$, such that, for all $n$, if $n > m$ and $(\gamma|\alpha)(n) \neq 0$, then there exists $k$ such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \Fin)$. Using Brouwer’s Continuity Principle, Axiom [4] we find $p, m$, such that, for all $\alpha$ in $\overline{CB}_n^\sigma$, if $\overline{\alpha}p = \overline{\alpha}m$, then, for all $n$, if $n > m$, then there exists $k$ such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \Fin)$. Find $q \geq p$ such that $\overline{(\gamma|\alpha)(m + 1) \subseteq \gamma|\alpha}q$. Now consider $\alpha_0 := \overline{m} \ast (1) \ast \overline{0}$ and $\alpha_1 := \overline{m} \ast (n + 1) \ast (1) \ast \overline{0}$. Note that $\alpha_0 \neq \alpha_1$, and for both $i < 2$, $\overline{\alpha}_i = \overline{\alpha}_q$. As $\gamma$ is strongly injective, $\gamma|\alpha_0 \neq \gamma|\alpha_1$, and there exist $i < 2$ and $n > m$ such that $(\gamma|\alpha_i)(n) \neq 0$. Find $i < 2$ and $n > m$ such that $(\gamma|\alpha_i)(n) \neq 0$. Find $r > q$ such that $\overline{(\gamma|\alpha_i)(n + 1) \subseteq \gamma|\alpha_i}r$. Note that, for every $\alpha$ in $\overline{CB}_n^\sigma$, if $\alpha$ passes through $\overline{\alpha_i}r$, then there exists $k$ such that $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \Fin)$. Using Brouwer’s Continuity Principle, Axiom [4] again, find $t \geq r$ and $k$ such that, for every $\alpha$ in $\overline{CB}_n^\sigma$, if $\alpha$ passes through $\overline{\alpha_i}t$, then $\gamma|\alpha$ belongs to $\mathbb{P}(\sigma^k, \Fin)$. Note that $\gamma$ embeds $\overline{CB}_n^\sigma$ into $\overline{\Fin(\sigma^k, \Fin)}$. On the other hand, $\Fin(\sigma^k, \Fin) \leq \sigma$ and, therefore, $\overline{CB}_n^\sigma$ embeds into $\overline{CB}_n^\sigma$. Also, as $\sigma$ is hereditarily repetitive, $\overline{CB}_n^\sigma$ embeds into $\overline{CB}_n^\sigma|\Fin$. Combining these facts, we find that $\overline{CB}_n^\sigma$ embeds into $\overline{\Fin(\sigma^k, \Fin)}$, and this contradicts the induction hypothesis.
(iii) This is an immediate consequence of (ii) and Theorem 9.1(ii).

Let $\delta, \alpha$ belong to $N$ and let $\sigma$ be a stump. We define:

$\sigma$ secures that $\alpha$ belongs to $\text{Almost}^*(En_\delta)$ if and only if

for each $\gamma$, there exists $n$ such that $\overline{\gamma}(n+1)$ is admitted by $\sigma$ and $\overline{\gamma}(n) = \overline{\delta(\gamma(n))}$. 

Note that item (iv) of the next theorem is a boundedness result, to be compared with Theorems 8.2(ii) and 8.9.

Theorem 9.3.

(i) For every stump $\sigma$, for all $\delta$, for all $\alpha$, if $\sigma$ secures that $\alpha$ belongs to $\text{Almost}^*(En_\delta)$, then $\alpha$ belongs to $\mathbb{P}(\sigma, En_\delta)$.

(ii) For all $\delta$, for all $\alpha$, if $\alpha$ belongs to $\text{Almost}^*(En_\delta)$, then there exists a stump $\sigma$ such that $\alpha$ belongs to $\mathbb{P}(\sigma, En_\delta)$.

(iii) For all $\delta$, the set $\text{Almost}^*(En_\delta)$ coincides with the set $\bigcup_{\sigma \in \text{Stp}} \mathbb{P}(\sigma, En_\delta)$.

(iv) For every $\beta$, for every $\delta$, if for every $s$, $\beta(s) = 0$ if and only if, for some $n$, $\beta(s s(n)) = 0$, and $F_\beta$ is a subset of $\text{Almost}^*(En_\delta)$, then there exists a stump $\tau$ such that $F_\beta$ is a subset of $\mathbb{P}(\tau, En_\delta)$.

(v) The set $\text{ACS}$ coincides with the set $\bigcup_{\sigma \in \text{Stp}} \mathbb{P}(\sigma, En_\delta)$.

Proof. (i) We use the principle of induction on the set of stumps, Axiom 8(iii). Note that, if $\sigma(0) \neq 0$, that is, $\sigma$ codes the empty stump, then the statement to be proven is true for trivial reasons. Now assume that $\sigma$ is a stump satisfying $\sigma(0) = 0$ and the statement has been verified for every immediate substump $\sigma^n$ of $\sigma$. Let $\delta, \alpha$ be elements of $N$ such that $\sigma$ secures that $\alpha$ belongs to $\text{Almost}(En_\delta)$. Suppose that $\alpha \neq \delta^0$. Find $m$ such that $\overline{\sigma}m \neq \overline{\delta^m}$. Note that, for each $\gamma$, if $\gamma(0) = m$, then there exists $n > 0$ such that $\overline{\gamma}(n+1)$ is admitted by $\sigma$ and $\overline{\gamma}(n) = \overline{\delta^n(\gamma(n))}$. It follows that, for each $\gamma$, there exists $n$ such that $\overline{\gamma}(n+1)$ is admitted by $\sigma^n$ and $\overline{\gamma}(n) = \overline{\delta^n(\gamma(n))}$, that is, $\sigma^n$ secures that $\alpha$ belongs to $\text{Almost}^*(En_\delta)$, where $\delta^n$ satisfies: for each $n$, $(\delta^n)^n = \delta^{n+1}$. Using the induction hypothesis, we conclude that, if $\alpha \neq \delta^0$, then, for some $m$, $\alpha$ belongs to $\mathbb{P}(\sigma^m, En_\delta)$. As $En_\delta$ is a subset of $En_\delta$, for each $m$, $\mathbb{P}(\sigma^m, En_\delta)$ is a subset of $\mathbb{P}(\sigma^m, En_\delta)$. It follows that $\alpha$ belongs to $\mathbb{P}(\sigma, En_\delta)$.

(ii) Note that, if $\alpha$ belongs to $\text{Almost}(En_\delta)$, then, according to Brouwer's Thesis, Axiom 9 there exists a stump $\sigma$ that secures that $\alpha$ belongs to $\text{Almost}(En_\delta)$, and apply (i).

(iii) is an easy consequence of (ii).

(iv) Let $\beta, \delta$ satisfy the assumptions. Let $\varepsilon$ be the code of a continuous function from $N$ to $N$ retracting $N$ onto the set $F_\delta$ that is, for all $\alpha$, $\varepsilon|\alpha$ belongs to $F_\delta$, and, if $\alpha$ belongs to $F_\delta$, then $\varepsilon|\alpha = \alpha$. Note that, for each $\alpha$, there exists $n$, such that $\varepsilon|\alpha|\alpha_2(n) = \overline{\alpha_2(\alpha_2(n))}$. Let $B$ be the set of all $s$ such that $\text{length}(s)$ is even and, for some $n < \text{length}(s)$, $\text{length}(\varepsilon|s_1) \geq s_1(n)$ and $\varepsilon|s_1(s_1(n)) = \overline{\varepsilon|s_1(s_1(n))}$. Note that $B$ is a bar in $N$, and, using Brouwer's Thesis, Axiom 9 find a stump $\tau$ such that the set of all elements in $B$ that are admitted by $\tau$ is a bar in $N$. We now prove that, for every $\alpha$ in $F_\delta$, there exists a stump $\sigma$ securing that $\alpha$ belongs to $\text{Almost}(En_\delta)$ such that $\sigma \leq^* \tau$, that is: $\sigma$ embeds into $\tau$. (The notion $\leq^*$ has been defined in Subsection 8.1.)

Let $\alpha$ be an element of $F_\delta$. We let $\varepsilon$ be the element of $N$ satisfying: for each $s$, $\text{length}(\varepsilon(s)) = 2\text{length}(s)$ and $\alpha$ passes through $(\varepsilon(s))_1$ and $(\varepsilon(s))_2 = s$. Now let $\sigma$ in $N$ satisfy: for each $s$, $\sigma(s) = \tau(\varepsilon(s))$. One easily verifies that $\sigma$ is a stump securing that $\alpha$ belongs to $\text{Almost}(En_\delta)$ and that $\varepsilon$ embeds $\sigma$ into $\tau$. It follows from (i) that $\alpha$
belongs to $P(\sigma, En_\delta)$, and, as $\sigma \leq \tau$, also to $P(\tau, En_\delta)$. We thus see that $F_\beta$ is a subset of $P(\tau, En_\delta)$.

(v) is an easy consequence of (iv). □

We now may draw a conclusion similar to a conclusion Cantor found by an indirect argument, see [8], Theorem C, in [9], page 220.

**Corollary 9.4.** Let $X$ be a located and closed subset of $\mathcal{N}$. If $X$ is almost-countable, then $X$ is reducible in Cantor’s sense, that is: there exists a stump $\sigma$ such that $\text{Der}(\sigma, X) = \emptyset$.

**Proof.** Let $X$ be an almost-countable located and closed subset of $\mathcal{N}$. Using Theorem [9.3 (v)], we find a stump $\sigma$ such that $X$ is $\sigma$-perhaps-countable. Using Theorem [9.1 (ii)], we find that $X$ embeds into $\overline{CB_\sigma}^*$. We determine a stump $\tau$ such that $\overline{CB_\tau}^*$ coincides with $\overline{CB_\sigma}^*$. We now use Theorem [1.8 (ii)] and the easy observation that, for each subset $Y$ of $\mathcal{N}$, for each non-empty stump $\sigma$, the set $\text{Der}(\sigma, Y)$ coincides with the set $\text{Der}(\sigma, Y)$. We conclude $\text{Der}(\tau, \overline{CB_\tau}^*) = \emptyset$. Now determine a strongly injective function $\gamma$ from $X$ into $\overline{CB_\tau}^*$. One may prove, by induction on the set of stumps, that, for each stump $\sigma$, $\gamma$ embeds the set $\text{Der}(\sigma, X)$ into the set $\text{Der}(\sigma, \overline{CB_\tau}^*)$. It follows that $\gamma$ embeds the set $\text{Der}(\tau, X)$ into the empty set, and that $\text{Der}(\tau, X) = \emptyset$. □

The converse of Corollary 9.4 that is, the statement that every reducible located and closed subset of $X$ is almost-countable, does not seem to be true.

**Theorem 9.5.**

(i) For all located and closed subsets $X, Y$ of $\mathcal{N}$, if $X$ is compact and $X$ embeds into $Y$, then there exists a (continuous) function from $Y$ to $X$ mapping $Y$ onto $X$.

(ii) For all located and closed subsets $X, Y$ of $\mathcal{N}$, if there exists a surjective (continuous) function from $Y$ onto $X$, then $X$ embeds into $Y$.

(iii) For every non-empty compact, located and closed subset $X$ of $\mathcal{N}$, the following conditions are equivalent:

(a) $X$ is almost-countable.

(b) There exists a stump $\sigma$ such that $X$ embeds into $\overline{CB_\sigma}^*$.

(c) There exists a stump $\sigma$ such that $\overline{CB_\sigma}^*$ maps onto $X$.

**Proof.** (i) Let $X, Y$ be located and closed subsets of $\mathcal{N}$ and let $X$ be compact. Suppose $\gamma$ codes a strongly injective (continuous) function from $X$ to $Y$. Note that, for each $n$, for each $\alpha$ in $X$, there exists $m$ such that $\text{length}(\gamma(\langle m \rangle)) \geq n$. Using the Fan Theorem, Theorem [1.4] we conclude that there exists $m$ such that, for each $\alpha$ in $X$, $\text{length}(\gamma(\langle m \rangle)) \geq n$. Let $\beta$ be an element of $\mathcal{N}$ such that, for each $n$, $\beta(n)$ equals the least $m$ such that for each $\alpha$ in $X$, $\text{length}(\gamma(\langle m \rangle)) \geq n$. Note that, for each $s$, there exists $\alpha$ in $X$ such that $\gamma|\alpha$ passes through $s$ if and only if there exists $t$ such that $t$ contains an element of $X$ and $s \subseteq \gamma|t$ and $\text{length}(t) = \text{length}(\beta(\text{length}(s)))$. Define $\delta$ in $\mathcal{C}$ such that, for each $s$, $\delta(s) = 0$ if and only if there exists $\alpha$ in $X$ such that $\gamma|\alpha$ passes through $s$. Note that $F_\delta$ is a compact located and closed subset of $\mathcal{N}$ and that, for each $\alpha$ in $X$, $\gamma|\alpha$ belongs to $F_\delta$.

Now assume $\varepsilon$ belongs to $F_\delta$ We intend to show there exists $\alpha$ in $X$ such that $\gamma|\alpha = \varepsilon$. First, find $\zeta$ in $\mathcal{C}$ such that, for each $s$, $\zeta(s) = 0$ if and only if there exists $\alpha$ in $X$ passing through $s$. We claim:

For all $s$, $t$, if $\zeta(s) = \zeta(t) = 0$ and $s \bot t$, then either: for every $\alpha$ in $X \cap s$, $\gamma|\alpha \neq \varepsilon$, or: for every $\alpha$ in $X \cap t$, $\gamma|\alpha \neq \varepsilon$.

We prove this claim as follows. Note that, for every $\alpha$ in $X \cap s$, for every $\eta$ in $X \cap t$, $\gamma|\alpha \neq \gamma|\eta$, and, therefore: either $\gamma|\alpha \neq \varepsilon$ or $\gamma|\eta \neq \varepsilon$, and: there exists $n$ such that
\(\gamma | (\exists n) \perp \exists n\) or there exists \(n\) such that \(\gamma | (\exists u) \perp \exists u\). Using the Fan Theorem, \([14]\) we find finite subsets \(B, C\) of \(\mathbb{N}\) such that \(B\) is a bar in \(X \cap s\) and \(C\) is a bar in \(X \cap t\), and, for all \(u\) in \(B\), for all \(v\) in \(C\), \(\varepsilon\) does not pass through \(\gamma | u\) or \(\varepsilon\) does not pass through \(\gamma | v\). As \(B\) and \(C\) are finite, we may infer: either: for all \(u\) in \(B\), \(\varepsilon\) does not pass through \(\gamma | u\) or: for all \(v\) in \(C\), \(\varepsilon\) does not pass through \(\gamma | v\), and therefore: either: for every \(\alpha\) in \(X \cap s\), \(\gamma | \alpha \neq \varepsilon\), or: for every \(\alpha\) in \(X \cap t\), \(\gamma | \alpha \neq \varepsilon\). This ends the proof of our claim.

Using the just proven result and the fact that \(X\) is compact, we conclude: for each \(n\), there exists \(s\) such that \(\text{length}(s) = n\) and \(\zeta(s) = 0\) and, for all \(t\), if \(\text{length}(t) = n\) and \(\zeta(t) = 0\) and \(t \neq s\), then for every \(\alpha\) in \(X \cap t\), \(\gamma | \alpha \neq \varepsilon\). We define \(\lambda\) in \(\mathcal{N}\) such that, for each \(n\), \(\text{length}(\lambda(n)) = n\) and \(\zeta(\lambda(n)) = 0\) and, for all \(t\), if \(\zeta(t) = 0\) and \(\text{length}(t) = n\) and \(t \neq \lambda(n)\), then for every \(\alpha\) in \(X \cap t\), \(\gamma | \alpha \neq \varepsilon\).

We claim: for each \(n\), not: for every \(\alpha\) in \(X \cap \lambda(n)\), \(\gamma | \alpha \neq \varepsilon\). For suppose we find \(n\) such that for every \(\alpha\) in \(X \cap \lambda(n)\), \(\gamma | \alpha \neq \varepsilon\). It follows that, for every \(\alpha\) in \(X\), there exists \(n\) such that \(\gamma(\alpha)n \neq \exists n\) and then, again by the Fan Theorem, Theorem \([14]\) there exists \(n\) such that, for all \(\alpha\) in \(X\), \(\gamma | \alpha n \neq \exists n\), contradicting the assumption: \(\delta(\exists n) = 0\).

It now will be clear that, for each \(n\), \(\lambda(n) \subseteq \lambda(n + 1)\). Let \(\alpha\) be the element of \(X\) passing through every \(\lambda(n)\). Assume \(\gamma | \alpha \neq \varepsilon\). Then there exists \(n\) such that \(\exists n\) does not pass through \(\gamma | \exists n\), and, therefore, there exists \(n\) such that \(\exists n\) does not pass through \(\gamma | \exists n\). Contradiction. Therefore, \(\gamma | \alpha\) coincides with \(\varepsilon\).

We now define a function \(\mu\) from \(\mathbb{N}\) to \(\mathbb{N}\), by induction on the length of the argument, as follows. \(\mu(0) = 0\) and, for every \(e\), \(\mu(e) = p + 1\) if and only if \(\text{length}(\mu(e)) = n\) and \(\delta(e * (q)) = 0\) and \(\zeta((\mu(e) * (p)) = 0\) and, for every \(t\), if \(\text{length}(t) = \text{length}(e)\) and \(\zeta(t) = 0\), and \(t \perp (\mu(e) * (p))\), then \(\gamma | t \perp e * (q)\). Using the Fan Theorem, one may prove that \(\mu\) is the code of a continuous function from \(F_3\) to \(F_\varepsilon\) satisfying: for all \(\alpha\) in \(F_\varepsilon\), \(\mu(\gamma | \alpha) = \alpha\).

Assume that \(Y\) is a located and closed subset of \(\mathcal{N}\) containing \(F_\varepsilon\). Let \(\rho\) be the code of a continuous function from \(Y\) to \(F_3\) retracting \(Y\) onto \(F_3\). Let \(\nu\) be the code of a continuous function from \(Y\) to \(F_\varepsilon = X\) such that, for all \(\alpha\) in \(Y\), \(\nu | \alpha = \mu(\rho | \alpha)\). Note that \(\nu\) codes a surjective function from \(Y\) to \(X\).

(ii) Let \(X, Y\) be located and closed subsets of \(\mathcal{N}\) and let \(\gamma\) be the code of a continuous function from \(X\) to \(Y\) that maps \(X\) onto \(Y\). Let \(\rho\) be a function from \(\mathcal{N}\) to \(Y\) that retracts \(\mathcal{N}\) onto \(Y\). Note that, for every \(\alpha\), there exists \(\beta\) in \(X\) such that \(\rho | \alpha = \gamma | \beta\). Applying the Second Axiom of Continuous Choice, Axiom \([7]\) we find \(\delta\) coding a continuous function from \(\mathcal{N}\) to \(X\) such that, for every \(\alpha\), \(\rho | \alpha = \gamma | (\delta | \alpha)\), and, in particular, for every \(\alpha\) in \(Y\), \(\alpha = \rho | \alpha = \gamma | (\delta | \alpha)\).

We only have to verify that \(\delta\) is strongly injective. Suppose \(\alpha, \beta\) belong to \(Y\) and \(\alpha \neq \beta\). Find \(n\) such that \(\exists m \neq \exists n\). Find \(m\) such that \(\exists m \subseteq (\delta | \alpha)\). Find \(p\) such that \(\exists m \subseteq (\delta | \beta)\). Note that \((\delta | \alpha)\) must be different from \((\delta | \beta)\).

(iii) The equivalence of (b) and (c) follows from (i) and (ii).

The equivalence of (a) and (b) follows from Theorem \([9, 11, 12]\) and Theorem \([9, 24]\).

G. Ronzitti, on page 63 of her Ph.D. dissertation \([26]\) and in the last definition of her paper \([27]\), suggested to call a located and closed subset \(X\) of \(\mathcal{N}\) countable if \(X\) satisfies condition (c) of Theorem \([9, 13, 12]\): there exists a stamp \(\sigma\) such that \(CB^*_\sigma\) maps onto \(X\). Theorem \([9, 13]\) shows this suggestion makes sense if we restrict ourselves to compact located and closed subsets of \(\mathcal{N}\). It seems that Theorem \([9, 13]\) does not extend to located and closed subsets \(X\) of \(\mathcal{N}\) that we do not know to be compact.

9.2. The collapse of the projective hierarchy. One may parametrize the class \(\Sigma^1_2\), of the (existential) projections of the co-analytic subsets of \(\mathcal{N}\) as follows. For each \(\beta\), let \(PCA_{\beta}\) be the set \(\exists x(CA_{\beta})\), that is the set of all \(\alpha\) in \(\mathcal{N}\) such that, for some \(\gamma\), the
pair \((\alpha, \gamma)\) belongs to \(CA_\beta\). A subset \(X\) of \(\mathcal{N}\) belongs to the class \(\Sigma^1_2\) if and only there exists \(\beta\) such that \(X\) coincides with \(PCA_\beta\).

One may parametrize the class \(\Pi^1_1\) of the co-projections of the analytic subsets of \(\mathcal{N}\) as follows. For each \(\beta\), let \(CPA_\beta\) be the set \(Un(A_\beta)\), that is the set of all \(\alpha\) in \(\mathcal{N}\) such that, for all \(\gamma\), the pair \((\alpha, \gamma)\) belongs to \(A_\beta\). A subset \(X\) of \(\mathcal{N}\) belongs to the class \(\Pi^1_1\) if and only there exists \(\beta\) such that \(X\) coincides with \(CPA_\beta\).

Note that, for each \(\beta\), every element of \(PCA_\beta\) is apart from every element of \(CPA_\beta\).

We define subsets \(US^1_2\) and \(UP^1_2\) of \(\mathcal{N}\), as follows. \(US^1_2\) is the set of all \(\alpha\) such that \(\alpha_{11}\) belongs to \(PCA_\alpha\), and \(UP^1_2\) is the set of all \(\alpha\) such that \(\alpha_{11}\) belongs to \(CPA_\alpha\). \(US^1_2\), \(UP^1_2\) are called the cataloguing sets of \(\Sigma^1_2\), \(\Pi^1_2\), respectively.

Note that every element of \(US^1_2\) is apart from every element of \(UP^1_2\).

A subset \(X\) of \(\mathcal{N}\) is called positively projective if and only if it is obtained from a closed subset of \(\mathcal{N}\) by finitely many applications of the operations of existential and universal projection.

In Section 6, we defined, for each \(s\) in \(\mathbb{N}\), elements \(s_I\) and \(s_{II}\) of \(\mathbb{N}\) by:

\[
\text{length}(s_I) = \text{the least } k \text{ such that } \text{length}(s) \geq 2k \text{ and, for each } n, \text{ if } n < \text{length}(s_I), \text{ then } s_I(n) = s(2n), \text{ and} \\
\text{length}(s_{II}) = \text{the least } k \text{ such that } \text{length}(s) \geq 2k + 1 \text{ and, for each } n, \text{ if } n < \text{length}(s_{II}), \text{ then } s_{II}(n) = s(2n + 1).
\]

For each \(s\) in \(\mathbb{N}\), we now define: \(s_{I,I} := (s_I)_I\) and \(s_{I,II} := (s_I)_{II}\).

Recall that \(S^*\) denotes the element of \(\mathcal{N}\) satisfying: for all \(n\), \(S^*(n) = n + 1\).

**Theorem 9.6 (Properties of the classes \(\Sigma^1_2\) and \(\Pi^1_1\)).** (i) For every subset \(X\) of \(\mathcal{N}\), \(X\) belongs to \(\Sigma^1_2\) if and only if, for some \(\alpha\), \(X\) coincides with \(US^1_2\alpha\) if and only if \(X\) reduces to \(E^1_2\).

(ii) For every subset \(X\) of \(\mathcal{N}\), \(X\) belongs to \(\Pi^1_1\) if and only if, for some \(\alpha\), \(X\) coincides with \(UP^1_2\alpha\) if and only if \(X\) reduces to \(A^1_2\).

(iii) Both \(\Sigma^1_2\) and \(\Pi^1_1\) are subclasses of \(\Sigma^2_2\) as well as of \(\Pi^1_2\).

(iv) For every sequence \(X_0, X_1, \ldots\) of subsets of \(\mathcal{N}\), if, for each \(n\), \(X_n\) belongs to \(\Sigma^1_2\), then both \(\bigcup_{n \in \mathbb{N}} X_n\) and \(\bigcap_{n \in \mathbb{N}} X_n\) belong to \(\Sigma^2_2\).

(v) For every sequence \(X_0, X_1, \ldots\) of subsets of \(\mathcal{N}\), if, for each \(n\), \(X_n\) belongs to \(\Pi^1_1\), then \(\bigcap_{n \in \mathbb{N}} X_n\) belongs to \(\Pi^1_2\).

(vi) For every subset \(X\) of \(\mathcal{N}\), if \(X\) belongs to \(\Sigma^1_2\), then \(Ex(X)\) belongs to \(\Sigma^1_2\).

(vii) For every subset \(X\) of \(\mathcal{N}\), if \(X\) belongs to \(\Pi^1_2\), then \(Un(X)\) belongs to \(\Pi^1_2\).

**Proof.** (i) The proof of the first statement is left to the reader. Let \(X\) be a subset of \(\mathcal{N}\) that belongs to the class \(\Sigma^1_2\). We prove that \(X\) reduces to \(E^1_2\). Find \(\beta\) in \(\mathcal{N}\) such that \(X\) is the set \(PCA_\beta = Ex(CA_\beta)\). Note that, for every \(\alpha\) in \(\mathcal{N}\), \(\alpha\) belongs to \(X\) if and only if \(\exists \gamma \exists n [\beta(\langle (\alpha, \gamma), \delta n \rangle) \neq 0]\). Note that, for every \(\alpha\), \(\alpha\) belongs to \(E^1_2\) if and only if \(\exists \gamma \exists n [\alpha(\langle (\gamma, \delta n \rangle) \neq 0].\) Let \(\eta\) be a function from \(\mathcal{N}\) to \(\mathcal{N}\) such that, for each \(\alpha\), for each \(s\), \((\eta \alpha)(s) \neq 0\) if and only if there exists \(u \leq s\) such that \(\beta(u) \neq 0\) and \(\alpha\) passes through \(u_{I,I}\) and \(u_{I,II} \subseteq s(1)\) and \(u_{I,II} \subseteq s(0)\). One verifies easily that \(\eta\) reduces \(X\) to \(E^1_2\).

(ii) The statement is obviously true.
(iii) The proof of the first statement is left to the reader. Let $X$ be a subset of $\mathcal{N}$ that belongs to the class $\Pi^1_2$. We prove that $X$ reduces to $A^1_2$. Find $\beta$ in $\mathcal{N}$ such that $X$ is the set $CPA_\beta = \bigcap_n (A^1_\beta)$. Note that, for every $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $X$ if and only if $\forall \gamma \forall n [\beta(\langle \alpha, \gamma \rangle, \sigma^\gamma n) = 0]$. Note that, for every $\alpha$, $\alpha$ belongs to $A^1_\beta$ if and only if $\forall \delta \exists \gamma \forall n [\alpha(\langle \delta^\gamma n, \sigma^\gamma n \rangle) = 0]$. Let $\eta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $s$, $(\eta(\alpha)(s) \neq 0$ if and only if there exists $u \leq s$ such that $\beta(u) \neq 0$ and $\alpha$ passes through $u_{1,1}$ and $u_{1,11} \subseteq s(1)$ and $u_{11} \subseteq s(0)$. One verifies easily that $\eta$ reduces $X$ to $A^1_\alpha$.

(iv) Let $X_0, X_1, \ldots$ be an infinite sequence of elements of $\Sigma^1_2$.

Using the Second Axiom of Countable Choice, Axiom 4 we find $\beta$ such that, for each $n$, $X_n$ coincides with $PCA_{\beta^n} = Ex(CA_{\beta^n})$. Note that, for all $\alpha$, $\alpha$ belongs to $\bigcap_n X_n$ if and only if, for each $n$, there exists $\gamma$ such that $\langle \alpha, \gamma \rangle$ belongs to $CA_{\beta^n}$. Using the Second Axiom of Countable Choice again, we conclude that, for all $\alpha$, $\alpha$ belongs to $\bigcap_n X_n$ if and only there exists $\gamma$ such that, for all $n$, $\langle \alpha, \gamma \rangle$ belongs to $CA_{\beta^n}$. As the class $\Pi^1_2$ is closed under the operation of countable intersection, see Theorem 7.1 the set of all $\delta$ such that, for all $n$, $\langle \delta_1, (\delta_{11})^n \rangle$ belongs to $CA_{\beta^n}$, is a class of the member $\Pi^1_2$ and the set $\bigcap_n X_n$ is a member of the class $\Sigma^1_2$.

We also observe that, for all $\alpha$, $\alpha$ belongs to $\bigcup_n X_n$ if and only if there exist $n, \gamma$ such that $\langle \alpha, \gamma \rangle$ belongs to $CA_{\beta^n}$, if and only if there exists $\gamma$ such that $\langle \alpha, \gamma \rangle$ belongs to $CA_{\beta^n}$ if and only if there exists $\gamma$ such that $\langle \beta^n \rangle, \langle \alpha, \gamma \rangle \in S^*$ belongs to $U\Pi^1_1$. It follows that the set $\bigcup_n X_n$ is a member of the class $\Sigma^1_2$.

(v) Let $X_0, X_1, \ldots$ be an infinite sequence of elements of $\Pi^1_2$. Using the Second Axiom of Countable Choice, Axiom 4 we find $\beta$ such that, for each $n$, $X_n$ coincides with $PCA_{\beta^n} = \bigcap_n (A^1_{\beta^n})$. We observe that, for all $\alpha$, $\alpha$ belongs to $\bigcap_n X_n$ if and only if, for all $n$, there exists $\gamma$ such that $\langle \alpha, \gamma \rangle$ belongs to $A_{\beta^n}$ if and only if, for all $\gamma$, $\langle \alpha, \gamma \rangle \in S^*$ belongs to $A_{\beta^n}$ if and only if, for all $\gamma$, $\langle \beta^n, \gamma \rangle \in S^*$ belongs to $U\Pi^1_1$. It follows that the set $\bigcap_n X_n$ is a member of the class $\Sigma^1_2$.

(vi) Let $X$ be an element of the class $\Sigma^1_2$. Find $Y$ in $\Pi^1_1$ such that $X = Ex(Y)$. Note that, for all $\alpha$, $\alpha$ belongs to $Y$ if and only if there exists $\beta$ such that $\langle \alpha, \beta \rangle$ belongs to $Y$ if and only if there exist $\beta$ such that $\langle \alpha, \beta \rangle$ belongs to $Y$ if and only if there exist $\beta$ such that $\langle \alpha, \beta_1, \beta_{11} \rangle$ belongs to $Y$. It follows that the set $Ex(X)$ is a member of the class $\Sigma^1_2$.

(vii) Let $X$ be an element of the class $\Pi^1_1$. Find $Y$ in $\Pi^1_1$ such that $X = Un(Y)$. Note that, for all $\alpha$, $\alpha$ belongs to $Un(X)$ if and only if, for all $\beta$, $\langle \alpha, \beta \rangle$ belongs to $Y$ if and only if, for all $\beta$, $\langle \alpha, \beta_1, \beta_{11} \rangle$ belongs to $Y$. It follows that the set $Un(X)$ is a member of the class $\Pi^1_2$.

\begin{theorem} \textup{(Breakdown of the Projective Hierarchy).}

(i) The set $A^1_2$ reduces to the set $E^1_2$ and, therefore: $\Pi^1_2$ is a subclass of $\Sigma^1_2$.
(ii) For every subset $X$ of $\mathcal{N}$, if $X$ belongs to $\Sigma^1_2$, then $Un(X)$ belongs to $\Sigma^1_2$.
(iii) Every (positively) projective set belongs to the class $\Sigma^1_2$.

\end{theorem}

\begin{proof}
(i) Note that, for each $\alpha$, $\alpha$ belongs to $A^1_2$ if and only if, for every $\beta$, there exists $\gamma$ such that, for each $n$, $\alpha(\langle \gamma, (\gamma \beta)^n \rangle) = 0$. Using the Second Axiom of Countable Choice, we conclude that $\alpha$ belongs to $A^1_2$ if and only if there exists $\varepsilon$ in $A^1_2$ such that $\varepsilon(0) = 0$ and for each $\beta$, $\alpha(\langle \varepsilon, (\beta \beta)^n \rangle) = 0$ if and only if there exists $\varepsilon$ such that

\[\exists n \in \mathbb{N} \forall \beta \in A^1_2 \langle \varepsilon, (\beta \beta)^n \rangle = 0 .\]

(ii) For every subset $X$ of $\mathcal{N}$, if $X$ belongs to $\Sigma^1_2$, then $Un(X)$ belongs to $\Sigma^1_2$.

(iii) Every (positively) projective set belongs to the class $\Sigma^1_2$.

\end{proof}
$\varepsilon(0) \neq 0$ and for all $\beta$ there exists $n$ such that $\varepsilon(\beta n) \neq 0$ and for all $s$, $\alpha(s) \neq 0$ if and only if $s(1)$ is not an initial part of $\varepsilon(s(0))$.

This shows that the set $A^1_2$ is a member of the class $\Sigma^1_2$.

(ii) Let $X$ be a subset of $\mathcal{N}$ that is the co-projection of a set in the class $\Sigma^1_2$. Find $\beta$ in $\mathcal{N}$ such that $X$ is the set $\mathcal{C}PA\beta = U(\mathcal{C}A\beta)$. Note that, for every $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $X$ if and only if $\forall \varepsilon \exists \delta \exists \gamma \exists n[\beta(\langle \langle \alpha, \gamma \rangle, \delta \rangle, \varepsilon n) \neq 0]$ and only if $\exists \zeta[\zeta(\langle \langle \rangle \rangle) = 0 \land \forall \varepsilon \exists n[\zeta(\beta n) \neq 0] \land \forall \varepsilon \gamma \exists n[\beta(\langle \langle \alpha, \gamma \rangle, \delta \rangle, \varepsilon n) \neq 0] \land \forall \varepsilon \gamma \forall \delta \exists n[\beta(\langle \langle \alpha, \gamma \rangle, \delta \rangle, \varepsilon n) \neq 0 \lor \zeta(\langle \langle \rangle \rangle(\eta n)) \neq \beta n]]$.

It will be clear that $X$ belongs to the class $\Sigma^1_2$.

(iii) Use (ii) and Theorem 9.6.

9.3. The previous Theorem shows that, in intuitionistic mathematics, $\Sigma^1_2$ is the class of all positively projective sets. Many difficult questions remain, for instance, if $\Pi^1_2$ is a proper subclass of $\Sigma^1_2$ and if the class $\Pi^1_2$ is closed under the operation of disjunction.

We were unable to answer these questions.

Note that the existential projection of a positively Borel set is analytic. It is not true however, that the universal projection of a positively Borel set is always co-analytic, for the simple reason that some positively Borel sets, like $D^2(A_1)$, see Theorem 7.14 are not co-analytic.

From a classical point of view, Theorem 9.7 is a great surprise. The next theorem shows which conclusion the intuitionistic mathematician draws from the classical diagonal-oral arguments establishing the projective hierarchy. Note that these conclusions are, if one reads them classically, contradictions, and, for the classical mathematician, would refute Theorem 9.7 and the Second Axiom of Continuous Choice, Axiom 4 on which it is based.

**Lemma 9.8.** For every function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ there exists $\alpha$ such that $\alpha$ belongs to $E^1_2$ if and only if $\gamma|\alpha$ belongs to $E^1_2$ and also: $\alpha$ belongs to $A^1_2$ if and only if $\gamma|\alpha$ belongs to $A^1_2$.

**Proof.** Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$. We construct $\alpha$ in $C$ in such a way that for all $\beta, \gamma$, there exists $n$ such that $\alpha(\langle \gamma n, \beta n \rangle) = 1$ if and only if there exists $n$ such that $(\gamma|\alpha)(\langle \gamma n, \beta n \rangle) \neq 0$.

We define $\alpha$ by induction, as follows. Let $n$ be a natural number and suppose we already decided about $\alpha(0), \ldots, \alpha(n - 1)$. We now say: $\alpha(n) = 1$ if and only if there exists $b, c, p, q, j$ such that length($b$) = length($c$) and $n = \langle b, c \rangle$ and length($p$) = length($q$) and $p, q$ are initial parts of $b, c$ respectively and $j < n$ and $\gamma|\langle p q, \langle \beta j \rangle \rangle > 1$ and, for all $k < j$, $\gamma|\langle p q, \langle \beta k \rangle \rangle = 0$.

It is easy to see that $\alpha$ satisfies the requirements, and that $\alpha$ belongs to $E^1_2$ if and only if $f|\alpha$ belongs to $E^1_2$ and: $\alpha$ belongs to $A^1_2$ if and only if $f|\alpha$ belongs to $A^1_2$. \hfill $\Box$

Note that the classical mathematician would conclude, from Lemma 9.8, that the sets $A^1_2$ and $E^1_2$ do not reduce to each other.

**Theorem 9.9 (The Ruins of the Classical Projective Hierarchy: de Morgan’s nightmares).**

(i) There exists $\gamma$ belonging to neither one of $US^1_2$, $UP^1_2$.

(ii) There exists $\alpha$ belonging to neither one of $E^1_2$, $A^1_2$.

**Proof.** (i) We let $DP^1_2$ be the set of all $\alpha$ such that $\langle \alpha, \alpha \rangle$ belongs to $UP^1_2$. According to Theorem 9.7 $DP^1_2$ is a member of $\Sigma^1_2$ and we may find $\beta$ such that $DP^1_2$ coincides with $US^1_2[\beta]$, therefore, for every $\alpha$, $\langle \alpha, \alpha \rangle$ belongs to $UP^1_2$ if and only if $\langle \beta, \beta \rangle$ belongs to $US^1_2$. Define $\gamma := \langle \beta, \beta \rangle$ and observe that $\gamma$ does not belong to either $US^1_2$ or $UP^1_2$, as every element of $US^1_2$ is apart from every element of $UP^1_2$. \hfill $\Box$
(ii) Let \( \gamma \) be a function from \( N \) to \( N \) reducing \( A^1_2 \) to \( E^1_2 \) and apply Lemma 9.8 keeping in mind that every element of \( E^1_2 \) is apart from every element of \( A^1_2 \).

Let \( \alpha \) be an element of \( N \) not belonging to either \( E^1_2 \) or \( A^1_2 \). Note that

(i) \( \neg \exists \beta \forall \gamma \exists m [\alpha((\beta m, \gamma n)) \neq 0] \) and
(ii) \( \neg \forall \beta \exists \gamma \forall n [\alpha(\beta m, \gamma n)] = 0 \) and
(iii) \( \forall \beta \forall \gamma \forall n [\alpha(\beta m, \gamma n)) = 0 \lor \alpha(\beta m, \gamma n)) \neq 0] \).

This example shows that, in intuitionistic mathematics it is possible that statements

(i) \( \neg \exists x \forall y \exists z [P(x, y, z)] \) and
(ii) \( \neg \forall x \exists y \forall z [\neg P(x, y, z)] \) and
(iii) \( \forall x \forall y \forall z [P(x, y, z) \lor \neg P(x, y, z)] \)

are simultaneously true. The example depends on the Second Axiom of Continuous Choice, Axiom 7. Another example, depending only on Brouwer’s Continuity Principle, Axiom 4, has been given in [45], Section 5.5:

(i) \( \neg \exists \alpha \forall n \exists m [\alpha(n) = 0 \land \alpha(m) \neq 0] \) and
(ii) \( \neg \forall \alpha \exists n \forall m [\alpha(n) = 0 \land \alpha(m) = 0] \) and
(iii) \( \forall \alpha \forall n \forall m [(\alpha(n) = 0 \land \alpha(m) \neq 0) \lor (\alpha(n) \neq 0 \lor \alpha(m) = 0)] \).

It has been observed by J.R. Moschovakis that, in the context of intuitionistic arithmetic, Church’s Thesis causes the collapse of the (positively) arithmetical hierarchy, just as the Second Axiom of Continuous Choice, Axiom 7, causes the collapse of the (positively) projective hierarchy, see [23].
References


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