Poincaré inequality for Markov random fields via disagreement percolation

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December 14, 2010

Abstract

We consider Markov random fields of discrete spins on the lattice $\mathbb{Z}^d$. We use a technique of coupling of conditional distributions. If under the coupling the disagreement cluster is "sufficiently" subcritical, then we prove the Poincaré inequality. In the whole subcritical regime, we have a weak Poincaré inequality.

Keywords: Poincaré inequality, weak Poincaré inequality, Gibbs measures, Glauber dynamics, coupling.

1 Introduction

Concentration inequalities is an active field of research in probability, with applications in other areas of mathematics such as functional analysis, geometry of metric spaces, as well as in more applied areas such as combinatorics, optimization and computer science [10], [13], [5].

Gibbsian random fields on lattice spin systems provide examples of interacting random systems having at the same time non-trivial and natural (e.g. Markovian) dependence structure. They provide a good class of examples where the validity of concentration inequalities in the context of dependent random fields can be tested.

The relation between good mixing properties of Gibbs measures and exponential relaxation to equilibrium of the associated reversible Glauber dynamics is a thoroughly studied subject. Well-known results in this area were obtained by Zegarliński [17], Stroock and Zegarliński [15], Martinelli and Olivieri [12]. One of the main results in this area is the equivalence between the log-Sobolev inequality (implying exponential relaxation of the dynamics in L^∞) and the Dobrushin-Shlosman complete analyticity condition.

More recently, a direct relation between the Dobrushin uniqueness condition and Gaussian concentration estimates was proved in [9], and a more general relation between the existence of a coupling of a system of conditional distributions and Gaussian and moment inequalities in [3]. Besides the Dobrushin uniqueness condition, disagreement percolation technique appears here as a basic tool in constructing a good coupling of conditional distributions. The deviation of a function from its expectation is estimated in terms of the sum of the squares of the maximal variation, via martingale difference approach combined with coupling.

So far, no relation has been established between Gaussian concentration estimates or moment estimates (such as the variance inequality) of a Gibbs measure and relaxation properties of the associated reversible Glauber dynamics.

In this paper we show the correspondence between the existence of a good coupling of conditional distributions and the Poincaré inequality in the context of lattice Ising spin systems. In [4] this was proved in dimension one for a large class of Gibbs measures in the uniqueness regime. The extension to higher dimension which we deal with here (for finite-range potentials) presents new challenges. The Poincaré inequality estimates the variance of a function in terms of the sum of its expected quadratic variations (instead of maximal variation). Therefore, the Poincaré inequality gives much more information. In particular it is equivalent with relaxation of the corresponding reversible Glauber dynamics in L^2. The Poincaré inequality is strictly weaker than the log-Sobolev inequality. So in the complete analyticity regime, the Poincaré inequality is satisfied. A direct proof of the Poincaré inequality in the Dobrushin uniqueness regime can be found in [16].

Our result gives a direct road between "good" coupling of conditional distributions and the Poincaré inequality. By good coupling we mean that if in some region of the space we condition on two configurations that differ only in a single point, then we can couple the unconditioned spins such that the set of sites where we have a discrepancy in the coupling is small. Small here means: behaving as a subcritical percolation cluster, uniformly in the conditioning. The size of this region of discrepancies can be thought of as the analogue of the "coupling time" for processes. In order to derive the Poincaré inequality, we need the existence of an exponential moment of the disagreement cluster. This corresponds to a high-temperature condition which is slightly better than Dobrushin uniqueness, both for the ferromagnetic and antiferromagnetic case. We want to stress however that the main message of the paper is the direct link between coupling of conditional distributions and the Poincaré inequality, rather than finding an optimal region of β where the inequality holds.

In case the required exponential moment of the disagreement cluster does not exist, we still obtain the so-called weak Poincaré inequality which gives at least polynomial relaxation of the corresponding dynamics.

Our paper is organized as follows: in section 2 we introduce the basic ingredients and discuss coupling via disagreement percolation. In section 4 we prove the Poincaré inequality for small β and h close to zero, in section 5 we treat the case h large, in section 6 we prove the weak Poincaré inequality in the whole subcritical regime.

Acknowledgment. We thank Pierre Collet for fruitful discussions.

2 Setting

2.1 Configurations

We work in the context of Ising spin systems on a lattice, i.e., with state space $\Omega = \{-1,+1\}^{\mathbb{Z}^d}$ ($d \geq 2$). Elements of Ω are denoted σ, η, ξ, and are called spin configurations. We fix a "spiraling" enumeration of $\mathbb{Z}^d$

$$\mathbb{Z}^d = \{x_1, x_2, \ldots, x_n, \ldots\}.$$

such that x_{i+1} lies in the exterior boundary of $\{x_1, \ldots, x_i\}$. This enumeration induces an order and lattice intervals like

$$[1,i] = \{x_k, 1 \leq k \leq i\}.$$

We use the notation ξ_i^j, $1 \leq i \leq j \leq \infty$, for a configuration supported on the set $\{x_k, i \leq k \leq j\}$. We denote by $\xi_1^{i-1}+_i$ the concatenation of ξ_1^{i-1} with a 'plus' spin at site x_i. More generally, we write $\xi_V \xi_W$ for the concatenation of a configuration ξ_V supported on V with a configuration ξ_W supported on W.

2.2 Functions

For a function $f : \Omega \to \mathbb{R}$ we define the "discrete derivative" in the direction η_x at the configuration η to be

$$\nabla_x f(\eta) = f(\eta^x) - f(\eta),$$

where η^x denotes the configuration obtained from η by "flipping" the spin at site x, i.e., $\eta^x_y = \eta_y$ for all $y \neq x$ and $\eta^x_x = 1 - \eta_x$. For a finite subset $A \subset \mathbb{Z}^d$ we denote by σ^A the configuration obtained from σ by flipping all the spins in A, and

$$\nabla_A f(\sigma) = f(\sigma^A) - f(\sigma).$$

For an enumeration $A = \{y_1, \ldots, y_n\}$ of A, and $x \in A$, we denote by $A_{<x}$ the set of those elements in A preceding x (x not included). For the minimal element $x^* \in A$, in the chosen order of enumeration of A, $A_{<x^*} = \emptyset$ by definition.

Elementary telescoping yields the estimate

$$|\nabla_A f(\sigma)| \leq \sum_{x \in A} \left|\nabla_x f(\sigma^{A_{<x}})\right|.$$

Notice that if $A \subset B$ then we have the inequality

$$\sum_{x \in A} |\nabla_x f(\sigma^{A_{<x}})| \leq \sum_{x \in B} |\nabla_x f(\sigma^{B_{<x}})|$$

in an order where we enumerate B by first enumerating A and then the elements of $B \backslash A$.

The variation in direction σ_x is defined as

$$\delta_x f = \sup_{\eta \in \Omega} (f(\eta^x) - f(\eta)).$$

The collection $\{\delta_x f : x \in \mathbb{Z}^d\}$ is denoted by δf, and

$$\|\delta f\|_2^2 = \sum_{x \in \mathbb{Z}^d} (\delta_x f)^2 .$$

2.3 Markov random fields

Let $\mathbb{X} = \{X_x, x \in \mathbb{Z}^d\}$ be a Markov random field of "Ising spins", i.e., X_x takes values in $\{-1, +1\}$. In accordance with the previous section, we use the notations X_1^i, X_V, $X_V \xi_W$, etc.

The conditional probabilities of $\mathbb{X}$ are thus given by

$$\mathbb{P}\big(X_x = +1 | X_{\mathbb{Z}^d \setminus x} = \sigma_{\mathbb{Z}^d \setminus x}\big) = \frac{e^{\beta h} e^{\beta J \sum_{y \sim x} \sigma_y}}{2 \cosh \left(\beta h + \beta J \sum_{y \sim x} \sigma_y\right)}. \tag{1}$$

In this formula $x \sim y$ means that x and y are nearest neighbors, $J \in \mathbb{R}$ is the coupling strength and $h \geq 0$ is interpreted as a uniform magnetic field. Without loss of generality we can assume that $|J| = 1$. The case $J = 1$ is the Ising ferromagnet whereas the case $J = -1$ is the Ising anti-ferromagnet.

An easy consequence of (1) is the following uniform bound on the Radon-Nikodym derivative w.r.t. spin-flip:

$$\left\| \frac{\mathrm{d}\mathbb{P}^x}{\mathrm{d}\mathbb{P}} \right\|_\infty \leq e^{2\beta h + 4\beta d} =: e^c \tag{2}$$

where $\mathbb{P}^x$ denotes the image measure of $\mathbb{P}$ under spin-flip at lattice site x. From the previous estimate we deduce that, for a finite subset $A \subset \mathbb{Z}^d$,

$$\left\| \frac{\mathrm{d}\mathbb{P}^A}{\mathrm{d}\mathbb{P}} \right\|_\infty \leq e^{|A|(2\beta h + 4\beta d)}, \tag{3}$$

where $\mathbb{P}^A$ is the image measure of $\mathbb{P}$ under simultaneous flips of all the spins in A.

2.4 Glauber dynamics

In this section we review some well-known facts about Glauber dynamics. Much more information can be found in [11], chapter 3.

Given a random field $\mathbb{X}$ with distribution $\mathbb{P}$, the natural Glauber dynamics associated to it is a Markovian spin-flip dynamics that flips the spin configuration σ with rate $c(x, \sigma)$ at lattice site x. This is the Markov process $\{\sigma_t : t \geq 0\}$ with generator acting on the core of local functions given by

$$Lf(\sigma) = \sum_{x \in \mathbb{Z}^d} c(x, \sigma) \nabla_x f(\sigma). \tag{4}$$

We denote by S_t the associated semigroup generated by L, i.e.,

$$S_t f(\sigma) = \mathbb{E}_\sigma(f(\sigma_t)).$$

The rates $c(x, \sigma)$ are assumed to be local, uniformly bounded away from zero and uniformly bounded from above, i.e., there exist $0 < \delta < M < \infty$ such that for all $x \in \mathbb{Z}^d$, and $\sigma \in \Omega$,

$$\delta < c(x, \sigma) < M. \tag{5}$$

Moreover, we assume the so-called detailed balance relation between $c(x,\sigma)$ and $\mathbb{P}$ which reads, informally,

$$c(x,\sigma)\mathbb{P}(\sigma) = c(x,\sigma^x)\mathbb{P}(\sigma^x).$$

This is formally rewritten as

$$\frac{c(x,\sigma)}{c(x,\sigma^x)} = \frac{\mathrm{d}\mathbb{P}^x}{\mathrm{d}\mathbb{P}}(\sigma) \tag{6}$$

i.e., the lhs of (6) is a (and hence the unique) continuous (as a function of σ) version of the Radon-Nikodym derivative of $\mathbb{P}$ w.r.t. spin-flip at site x (i.e., the rhs).

Several choices for the rates are possible, one common choice is the heat-bath dynamics where

$$c(x,\sigma) = \mathbb{P}\big(X_x = -\sigma_x | X_{\mathbb{Z}^d\setminus x} = \sigma_{\mathbb{Z}^d\setminus x}\big).$$

The condition (6) ensures that $\mathbb{P}$ is a reversible measure for the Markov process with generator (4), i.e., the closure of L is a self-adjoint operator on $L^2(\mathbb{P})$.

The Dirichlet form associated to the rates $c(x,\sigma)$ is given by

$$\mathscr{E}_c(f,f) = 2\langle f(-L)f\rangle = \sum_{x\in\mathbb{Z}^d}\int c(x,\sigma)(\nabla_x f)^2\ \mathbb{P}(\mathrm{d}\sigma). \tag{7}$$

where $\langle\cdot\rangle$ denotes inner product in $L^2(\mathbb{P})$. We say that the Glauber dynamics has a spectral gap if for all f local functions with $\int f\mathrm{d}\mathbb{P} = 0$,

$$\mathscr{E}_c(f,f) \geq \kappa\|f\|_2^2.$$

This implies that the $(-L)$ has simple eigenvalue zero and that the $L^2(\mathbb{P})$ spectrum has κ as a lower bound. This in turn implies the estimate

$$\mathrm{Var}(S_t f) \leq e^{-\kappa t}\|f\|_2^2$$

i.e., exponential relaxation to equilibrium in $L^2(\mathbb{P})$-sense.

Defining the quadratic form

$$\mathscr{E}(f,f) = \sum_{x\in\mathbb{Z}^d}\int(\nabla_x f)^2\mathrm{d}\mathbb{P}.$$

we have by (5) the estimate

$$\delta\mathscr{E}(f,f) \leq \mathscr{E}_c(f,f) \leq M\mathscr{E}(f,f).$$

Hence, estimating the variance of a function in terms of the quadratic form $\mathscr{E}(f,f)$ is equivalent with estimating the variance in terms of the Dirichlet form (7) and therefore gives relevant information about the presence of a spectral gap and hence L^2-relaxation properties of the associated Glauber dynamics.

2.5 Coupling of conditional probabilities

We write $\mathbb{P}_{\xi_1^i}$ for the conditional distribution of $X_{[i+1,\infty)}$ given $X_1^i = \xi_1^i$.

Remark 2.1. *Notice that we have the same bound* (3) *for the measure* $\mathbb{P}_{\xi_1^i}$*, when* $A \subset [1,i]^c$*, uniformly in* ξ*.*

We denote by $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}$ a coupling of the distributions $\mathbb{P}_{\xi_1^{i-1}+_i}$ and $\mathbb{P}_{\xi_1^{i-1}-_i}$. This coupling is a distribution of a random field

$$\{(Y_x, Z_x), x \in [i+1,\infty)\} \quad \text{on} \quad \big(\{-1,+1\} \times \{-1,+1\}\big)^{[i+1,\infty)}.$$

Similarly we write $\widehat{\mathbb{P}}_{X_1^{i-1}+_i,X_1^{i-1}-_i}$. We define the random set of discrepancies

$$\mathscr{C}_i = \{x_k : k \geq i, Y_{x_k} \neq Z_{x_k}\}.$$

The distribution of this set depends of course on the choice of the coupling.

The coupling $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}$ which we will use throughout this paper is the one used in [1]. It is constructed as follows: the couple $(Y_{x_{i+1}}, Z_{x_{i+1}})$ is generated according to the optimal coupling of $\mathbb{P}_{\xi_1^{i-1}+_i}(X_{x_{i+1}} = \cdot)$ and $\mathbb{P}_{\xi_1^{i-1}-_i}(X_{x_{i+1}} = \cdot)$. Having generated (Y_{x_k}, Z_{x_k}) for $i+1 \leq k \leq j$, we generate the couple $(Y_{x_{j+1}}, Z_{x_{j+1}})$ according to the optimal coupling of $\mathbb{P}_{\xi_1^{i-1}Y_{i+1}^j}(X_{x_{j+1}} = \cdot)$ and $\mathbb{P}_{\xi_1^{i-1}Z_{i+1}^j}(X_{x_{j+1}} = \cdot)$.

By the Markov character of the random field $\mathbb{X}$, the sets of discrepancies $\mathscr{C}_i$ are almost-surely (nearest-neighbor) connected. So we can think of the $\mathscr{C}_i$'s as "percolation clusters" containing x_i; see [1, 6]. If these clusters behave as sub-critical percolation clusters, then we say that we are in the "good coupling regime". We then expect to obtain corresponding good relaxation properties of the natural Glauber dynamics associated to $\mathbb{P}$.

2.6 Subcritical disagreement percolation

We suppose that, under the coupling $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}$, the disagreement clusters $\mathscr{C}_i$ are dominated by independent subcritical site-percolation clusters, uniformly in the conditioning ξ. In fact, we shall need more than subcriticality. We believe that it is an artefact of our method and that the Poincaré inequality holds in the entire subcritical regime.

We denote by $\mathbb{P}_p$ the distribution of independent site-percolation with parameter $0 \leq p < 1$ and by p_c the corresponding critical value. Let $\mathfrak{C}_i$ be the open cluster containing x_i. In our model (1), by the construction of the coupling, we have domination by independent clusters, i.e., for any finite subset $A \subset \mathbb{Z}^d$

$$\sup_i \sup_\xi \widehat{\mathbb{P}}_{\xi_1^{i-1}+_i, \xi_1^{i-1}-_i} (\mathscr{C}_i \supset A) \leq \mathbb{P}_p(\mathfrak{C}_i \supset A), \tag{8}$$

with

$$p = p(\beta, h) = e^{-2\beta h} \ \left(e^{4\beta d} - e^{-4\beta d}\right). \tag{9}$$

In particular,

$$\sup_i \sup_\xi \widehat{\mathbb{P}}_{\xi_1^{i-1}+_i, \xi_1^{i-1}-_i} (|\mathscr{C}_i| \geq n) \leq \mathbb{P}_p(|\mathfrak{C}| \geq n),$$

where $\mathfrak{C} = \mathfrak{C}_0$. Our subcriticality assumption reads as follows:

$$\mathbb{E}_p\left(|\mathfrak{C}| e^{c|\mathfrak{C}|}\right) < \infty, \tag{10}$$

where c is defined in (2). This condition is satisfied for β sufficiently small or h sufficiently large; see below for the precise region of (β, h).
By the uniform bound (8), the coupling $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i, \xi_1^{i-1}-_i}$ can be realized in two stages. Having generated Y_{x_k}, Z_{x_k} for $k = i+1, \ldots, i+n$, we first generate $Y_{x_{i+n+1}}$. Then we flip an independent coin with success probability $1-p$ (corresponding to certain agreement) given by (9). Given that we have success, we put $Z_{x_{i+n+1}} = Y_{x_{i+n+1}}$. If we do not have success, then we possibly choose $Z_{x_{i+n+1}} = Y_{x_{i+n+1}}$ or $Z_{x_{i+n+1}} \neq Y_{x_{i+n+1}}$ in order to obtain the correct distribution of the coupling. The crucial point here is that the cluster of failures (=no success), which we denote $\tilde{\mathscr{C}}_i$, is a cluster that, *given the realization of* Y, is *independent of* Y and contains the cluster of disagreement $\mathscr{C}_i$. Therefore, in events that depend in a monotone way on the cluster of disagreements $\mathscr{C}_i$, we can replace it by $\mathfrak{C}_i$, the cluster of failures.

2.7 Sufficient conditions on β

A sufficient condition for (10) to hold is that

$$\sum_{n=0}^{\infty} n \ p^n \left(2d-1\right)^n e^{cn} < \infty,$$

where c is the constant appearing in (2) and p is defined in (9). In turn the above series is finite if

$$e^{4\beta d} - e^{-4\beta d} < \frac{e^{-4\beta d}}{2d-1},$$

which gives

$$\beta < \frac{1}{8d}\log\Big(\frac{2d}{2d-1}\Big). \tag{11}$$

Notice that this condition is independent of h.
The Dobrushin uniqueness condition in this context reads

$$e^{4\beta d} - e^{-4\beta d} < \frac{1}{2d}.$$

One can easily verify that (11) is slightly better for β large and low d.

3 The Poincaré inequality and related variance inequalities

The general idea of concentration inequalities is to give an estimate of the probability of a deviation event $\{|f-\mathbb{E}(f)| > a\}$, in terms of a quantity that measures the influence on f of variations of the spin configuration at different sites. Usually, such estimates are obtained via Chebychev's inequality, by estimating moments of $|f-\mathbb{E}(f)|$, such as the variance of f, or higher order moments, exponential moments etc., in terms of a norm measuring the variability of f. In this paper we concentrate on estimates of the variance.

3.1 Uniform variance estimate

The norm

$$\|\delta f\|_2 = \sum_{x\in\mathbb{Z}^d} (\delta_x f)^2$$

measures the influence of spin-flips on f in a uniform way, i.e., for each x the worst influence is computed.

The first inequality measures the variance in terms of $\|\delta f\|_2^2$.

Definition 3.1. *We say that a random field $\mathbb{X}$ satisfies the uniform variance inequality if there exists $C>0$, such for all $f:\Omega\to\mathbb{R}$, $f\in L^2(\mathbb{P})$, we have*

$$\mathbb{E}((f-\mathbb{E}(f))^2) \le C\|\delta f\|_2^2 \tag{12}$$

The uniform variance inequality estimates the variance in terms of the rather "rough" norm $\|\delta f\|_2^2$. Surprisingly, it is still a powerful inequality with many useful applications, such as almost-sure central limit theorems, convergence of the empirical distribution in a strong (Kantorovich) distance, etc. See [2] for a list of applications.

Examples where the uniform variance inequality is satisfied include high-temperature Gibbsian random fields (where it follows from the much stronger log-Sobolev inequality) and plus phase of the Ising model at low enough temperatures, see [3].

3.2 Poincaré inequality

The quadratic form

$$\mathscr{E}(f,f) = \sum_{x\in\mathbb{Z}^d} \int (\nabla_x f)^2 d\mathbb{P}$$

measures the influence of spin-flips on f, taking into account the distribution of the spin-configuration, i.e., large differences between $f(\sigma^x)$ and $f(\sigma)$ are weighted less if they correspond to exceptional configurations (in the sense of the measure $\mathbb{P}$). We have the obvious inequality $\mathscr{E}(f,f) \le \|\delta f\|_2^2$, therefore, estimating the variance in terms of $\mathscr{E}(f,f)$ is clearly better, and, as we will see in examples below, this difference can be substantial.

Definition 3.2. *We say that the random field $\mathbb{X}$ satisfies the Poincaré inequality if there exists a constant $C_P > 0$ such that for all $f \in L^2(\mathbb{P})$*

$$\int \big(f - \mathbb{E}(f)\big)^2 d\mathbb{P} \le C_P\ \mathscr{E}(f,f)\ . \tag{13}$$

The Poincaré inequality is strictly stronger than the uniform variance inequality. Moreover, contrary to the uniform variance estimate, the Poincaré inequality gives exponentially fast decay to equilibrium for the associated Glauber dynamics in $L^2(\mathbb{P})$. Indeed, (13) implies

$$\mathrm{Var}(f) \le \frac{1}{\delta} C_P \mathscr{E}_c(f,f) = 2\langle f, (-L) f\rangle$$

from which one easily sees that $(-L)$ has a spectral gap in $L^2(\mathbb{P})$ of at least $\kappa = 2\delta/C_P$, which implies the relaxation estimate

$$\mathrm{Var}(S_t f) \le e^{-\kappa t}\|f\|_2^2$$

3.3 Weak Poincaré inequality

Finally, the variance can be estimated in terms of a combination of $\|\delta f\|_2$ and $\mathscr{E}(f,f)$. The idea here is that if the Poincaré inequality does not hold, it can be due to "bad events" which have relatively small probability (e.g. large disagreement clusters). The idea is then to estimate the variance by $\mathscr{E}(f,f)$

on the good configurations and by $\|\delta f\|_2^2$ on the bad configurations. This leads to the weak Poincaré inequality, initially introduced by Röckner and Wang [14]. This inequality contains enough information to conclude relaxation properties of the associated Glauber dynamics, but now with $\mathrm{Var}(S_t f)$ estimated with a stronger norm than the $L^2(\mathbb{P})$-norm.

Definition 3.3. *The measure $\mathbb{P}$ satisfies the weak Poincaré inequality if there exists a decreasing function $\alpha : (0,\infty) \to (0,\infty)$ such that for all bounded measurable functions $f : \Omega \to \mathbb{R}$ we have, for all $r > 0$*

$$\int \big(f - \mathbb{E}(f)\big)^2 \mathrm{d}\mathbb{P} \le \alpha(r)\ \mathscr{E}(f,f) + r\|\delta f\|_2^2 \ .$$

For functions $f \in L^2(\mathbb{P})$ with $\int f \mathrm{d}\mathbb{P} = 0$, the weak Poincaré inequality implies the relaxation estimate

$$\mathrm{Var}(S_t f) \le \xi(t)\left(\|f\|_2^2 + \|\delta f\|_2^2\right)$$

where $\xi(t) \to 0$ as $t \to \infty$ is determined by α:

$$\xi(t) = \inf\left\{r > 0 : -\frac{1}{\delta}\alpha(r)\log r \le 2t\right\},\ t > 0.$$

where $\delta > 0$ is the lower-bound on the spin-flip rates. In the case when $\alpha(r) \le Cr^{-\kappa}$ for $C, \kappa > 0$, we get $\xi(t) \le \left(1 + \frac{1}{\kappa}\right)^{1+\frac{1}{\kappa}} \left(\frac{2t\delta}{C}\right)^{-\frac{1}{\kappa}}$. We refer the reader to [14] for more background and details.

3.4 Examples

Here we illustrate with some simple examples that the Poincaré inequality is much stronger than the uniform variance inequality. The example is a representant of a whole class of functions for which the effect of spin-flip is only "typically small", which gives a good estimate of $\mathscr{E}(f,f)$, but where the uniform variation $\delta_i f$ is always of order one.

Let $d = 1$ and $\mathbb{P}$ be a translation invariant probability measure on configurations $\sigma \in \Omega = \{-1,+1\}^{\mathbb{Z}}$ such that there exists $0 < \theta < 1$ with

$$\mathbb{P}(\sigma_1 = \alpha_1, \ldots, \sigma_n = \alpha_n) \le \theta^n \tag{14}$$

for all $n \in \mathbb{N}$, $\alpha_1, \ldots, \alpha \in \{-1, 1\}$ Examples of such $\mathbb{P}$ are translation-invariant Gibbs measures.

Consider for $n \in \mathbb{N}, k < n$

$$f_k(\sigma_1, \ldots, \sigma_n) = |\{i \in \{1, \ldots, n-k\} : \sigma_i = \sigma_{i+1} = \cdots = \sigma_{i+k} = +1\}|$$

i.e., the number of lattice intervals of size k, contained in $[1,n]$ and filled with plus spins.

We have

$$\nabla_r f_k(\sigma) = \sum_{j\in[1,n-k]:r\in[j,j+k]} (\mathbb{1}\{\sigma_r=-1\} - \mathbb{1}\{\sigma_r=+1\}) \prod_{i\in[j,j+k],i\neq r} \mathbb{1}\{\sigma_i=+1\}$$

which gives

$$\sum_r \int (\nabla_r f_k)^2 d\mathbb{P} \le 2k\theta^k$$

and hence

$$\mathscr{E}(f_k, f_k) \le 2k(n-k)\theta^k.$$

Therefore, if $\mathbb{P}$ satisfies the Poincaré inequality (which is the case e.g. for Gibbs measures in one dimension in the uniqueness regime, [4]) then

$$\mathrm{Var}(f_k) \le C_P 2k(n-k)\theta^k$$

Choosing now $k = c\log(n)$, and putting $\theta = e^{-\alpha}$ we find that

$$\mathrm{Var}(f_{c\log n}) \le 2c\log(n)(n - c\log(n))n^{-\alpha c}.$$

Hence if $\alpha c > 1$, $\mathrm{Var}(f_{c\log n})$ goes to zero as $n \to \infty$. It is immediate from (14) that $\alpha > c$ the first moment $\mathbb{E}(f_{c\log(n)})$ converges to zero as $n \to \infty$. Therefore, $\alpha c > 1$ implies that $f_{c\log n}$ converges to zero in $L^2(\mathbb{P})$ (and hence in probability) as $n \to \infty$.

On the other hand, it is clear that $\delta_i(f) = 1$ for all $i = 1, \ldots, n$, therefore the uniform variance estimate gives $\mathrm{Var}(f_k) \le Cn$, which is not useful here.

One can consider similar quantities like the number of clusters of size k of plus-spins, the number of self-overlaps of size k, etc. Such quantities will have small $\mathscr{E}(f,f)$ (for measures satisfying (14)) and large $\|\delta f\|_2^2$.

4 Poincaré inequality for the case $h = 0$

We start with the following result.

Theorem 4.1. *Consider the Markov random defined in* (1) *with $h = 0$. For β chosen such that*

$$\mathbb{E}_p\left(|\mathfrak{C}|e^{c|\mathfrak{C}|}\right) < \infty,$$

the Poincaré inequality (13) *holds.*

In section 5 below (Theorem 5.1), we will give a complementary result which covers the case of large β and (correspondingly) large h.

Proof. The proof is divided in four steps.
Step 1 (Martingale decomposition).
Let $f : \Omega \to \mathbb{R}$ be a bounded measurable function. Define

$$\Delta_i = \Delta_i(X_1^i) = \mathbb{E}(f|\mathscr{F}_i) - \mathbb{E}(f|\mathscr{F}_{i-1})$$

where $\mathscr{F}_i$ is the sigma-field generated by $\{X_{x_k} : 1 \leq k \leq i\}$ for $i \geq 1$ and where $\mathscr{F}_0$ is the trivial sigma-field $\{\emptyset, \Omega\}$. Then we have

$$\mathrm{Var}(f) = \sum_{i\in\mathbb{N}} \mathbb{E}(\Delta_i^2).$$

Step 2 (Coupling representation of Δ_i)
We have (using that spins can take only two values)

$$\begin{aligned}
|\Delta_i| &= \left| \int \mathrm{d}\mathbb{P}_{X_1^{i-1}}(\xi_i) \int \mathrm{d}\widehat{\mathbb{P}}_{X_1^i, X_1^{i-1}\xi_i}(\sigma_{i+1}^\infty, \eta_{i+1}^\infty) \left(f(X_1^{i-1} X_i \sigma_{i+1}^\infty) - f(X_1^{i-1}\xi_i \eta_{i+1}^\infty) \right) \right| \\
&\leq \int \left| f(X_1^{i-1} +_i \sigma_{i+1}^\infty) - f(X_1^{i-1} -_i \eta_{i+1}^\infty) \right| \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma_{i+1}^\infty, \eta_{i+1}^\infty) \\
&= \int \left| f(X_1^{i-1} +_i \sigma_{i+1}^\infty) - f(X_1^{i-1} -_i \eta_{i+1}^\infty) \right| \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma_i^\infty, \eta_i^\infty) \\
&= \sum_{A \ni x_i} \int \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma_i^\infty, \eta_i^\infty) \times \\
&\qquad \mathbb{1}\{\mathscr{C}_i = A\} \left| f(X_1^{i-1}\eta_A \sigma_{(A\cup[1,i-1])^c}) - f(X_1^{i-1}\sigma_A \sigma_{(A\cup[1,i-1])^c}) \right|, \qquad (15)
\end{aligned}$$

where $\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}$ is the coupling of conditional probabilities defined in subsection 2.5. Notice that the sum over A runs over *finite* connected subsets of $\mathbb{Z}^d$ containing x_i since $\mathscr{C}_i$ is dominated by a subcritical percolation cluster.

In the sequel, we simply write $\sigma_V \xi_W \eta$ for $\sigma_V \xi \eta_{(V\cup W)^c}$ to alleviate notations.

Step 3 (Telescoping and domination by independent clusters).

Start again from (15) and telescope the disagreement cluster:

$$\begin{aligned}
|\Delta_i| &\leq \int |\nabla_{\mathscr{C}_i} f(X_1^{i-1}\sigma^{\mathscr{C}_i})\ \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma,\eta) \\
&\leq \int \sum_{x\in\mathscr{C}_i} \left|\nabla_x f(X_1^{i-1}\sigma^{(\mathscr{C}_i)_{<x}})\right|\ \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma,\eta) \\
&\leq \int \sum_{x\in\tilde{\mathscr{C}}_i} \left|\nabla_x f(X_1^{i-1}\sigma^{(\tilde{\mathscr{C}}_i)_{<x}})\right|\ \mathrm{d}\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma,\eta) \\
&= \widetilde{\mathbb{E}} \int \sum_{x\in\tilde{\mathscr{C}}_i} \left|\nabla_x f(X_1^{i-1}\sigma^{(\tilde{\mathscr{C}}_i)_{<x}})\right|\ \mathrm{d}\mathbb{P}_{X_1^{i-1}+_i}(\sigma) \\
&= \sum_{A\ni x_i}\sum_{x\in A} \mathbb{P}_p(\mathfrak{C}_i = A)\left|\nabla_x f(X_1^{i-1}\sigma^{A_{<x}})\right|\ \mathrm{d}\mathbb{P}_{X_1^{i-1}+_i}(\sigma).
\end{aligned}$$

In the third inequality the expectation is over the "failure cluster" $\tilde{\mathscr{C}}_i$ only, which is independent of σ. This independence gives the factorization in the last equality, by decomposing over the realization of this cluster (which is finite with $\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}$ probability one under the subcriticality assumption).

Step 4 (Change of measure).
Using now the bound (3) and the remark in the beginning of subsection 2.5, we further estimate, using

$$|\Delta_i| \leq \sum_{A\ni x_i}\sum_{x\in A}\mathbb{P}_p(\mathfrak{C}_i = A)\ e^{c|A|}\int |\nabla_x f(X_1^{i-1} +_i \sigma)|\ \mathrm{d}\mathbb{P}_{X_1^{i-1}+_i}(\sigma)$$

where c is defined in (2).
Define the finite number (by the subcriticality assumption (10))

$$K := \sum_{A\ni 0} |A|\ \mathbb{P}_p(\mathfrak{C} = A)\ e^{c|A|} = \mathbb{E}_p\big(|\mathfrak{C}|e^{c|\mathfrak{C}|}\big).$$

Then, using the elementary inequality

$$\left(\sum_k a_k b_k\right)^2 \leq \sum_k a_k \sum_k a_k b_k^2$$

for $a_k, b_k \geq 0$, we obtain

$$\begin{aligned}
\sum_{i\in\mathbb{N}} \mathbb{E}(\Delta_i^2) &\leq Ke^{2c}\sum_{i\in\mathbb{N}}\sum_{A\ni x_i}\sum_{x\in A} e^{c|A|}\mathbb{P}_p(\mathfrak{C}_i = A)\int (\nabla_x f)^2 \mathrm{d}\mathbb{P} \\
&= K^2 e^{2c}\mathscr{E}(f,f),
\end{aligned}$$

where the extra factor e^c arises from removing the plus in the conditioning in $\mathbb{P}_{X_1^{i-1}+_i}$. This finishes the proof of Theorem 4.1 $\square$

5 Non-zero magnetic field

In this section we show how to prove the Poincaré inequality under a subcricality condition different from Theorem 4.1. It is strictly worse in the case $h = 0$ (since it uses Cauchy-Schwarz to seperate the realization of the disagreement cluster from the gradient of f) but can be used for β large and h large, where the condition (10) fails.

Theorem 5.1. *Suppose that* p *given in* (9) *is such that*

$$\sum_n n(2d-1)^n e^{c'n} \mathbb{P}_p(|\mathfrak{C}| \geq n)^{1/2} < \infty, \tag{16}$$

where

$$c' = 4\beta d. \tag{17}$$

Then the Poincaré inequality holds.

For (16) to hold, it is sufficient that

$$(2d-1)p^{\frac{1}{2}}e^{c'} < 1$$

which gives

$$(2d-1)^2 e^{-2\beta h}(e^{12\beta d} - e^{4\beta d}) < 1.$$

This is satisfied for β small enough or h large enough.

Proof. The telescoping and coupling steps are the same as in the proof of Theorem 1. So we arrive at

$$|\Delta_i| \leq \sum_{A \ni x_i} \sum_{x \in A} \int d\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma_i^\infty, \eta_i^\infty) \mathbb{1}\{\mathscr{C}_i = A\} \big|\nabla_x f(X_1^{i-1}\sigma_{A_{<x}}\eta)\big|.$$

Now we use Cauchy-Schwarz inequality to obtain

$$|\Delta_i| \leq \sum_{A \ni x_i} \sum_{x \in A} \left(\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}\big(\mathscr{C}_i = A\big)\right)^{1/2} \times \\ \left(\int d\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}(\sigma_i^\infty, \eta_i^\infty)\Big(\nabla_x f(X_1^{i-1}\sigma_{A_{<x}}\eta)\Big)^2\right)^{1/2}. \tag{18}$$

Step 4 (Change of measure). In the r.h.s. of (18) we integrate over the "composite" configuration $\sigma_{A_{<x}}\eta$ under the coupling $\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}$. To recover the measure $\mathbb{P}$ (see later) we need to replace $\sigma_{A_{<x}}$ by $\eta_{A_{<x}}$. The cost of this replacement is independent of h and is estimated in the following lemma where $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}$ is the coupling introduced above.

Lemma 5.1. *Let A be a finite subset of $\mathbb{Z}^d$ containing x_i and let $x \in A$. Let $\mathbb{P}_1$ be the distribution of $Z_{A_{<x}} Y_{(A_{<x})^c}$ and $\mathbb{P}_2$ be the distribution of $\{Y_x, x \in \mathbb{Z}^d\}$. Then $\mathbb{P}_1$ is absolutely continuous with respect to $\mathbb{P}_2$ and*

$$\left\| \frac{\mathrm{d}\mathbb{P}_1}{\mathrm{d}\mathbb{P}_2} \right\|_\infty \le e^{c'|A|}$$

where c' is defined in (17).

Proof. Let $\Lambda \subset \mathbb{Z}^d$ finite, large enough to contain A. We have by construction of the coupling $\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}$ (see subsection 2.5):

$$\begin{aligned}
&\frac{\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}\big(Z_{A_{<x}} = \sigma_{A_{<x}}, Y_{\Lambda\setminus A_{<x}} = \eta_{\Lambda\setminus A_{<x}}\big)}{\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}\big(Y_{A_{<x}} = \sigma_{A_{<x}}, Y_{\Lambda\setminus A_{<x}} = \eta_{\Lambda\setminus A_{<x}}\big)} \\
&= \sum_{\zeta_{A_{<x}}} \mathbb{P}_{\xi_1^{i-1}-_i}(\sigma_{A_{<x}})\ \mathbb{P}_{\xi_1^{i-1}+_i\zeta_{A_{<x}}}(\eta_{\Lambda\setminus A_{<x}}) \times \\
&\qquad\qquad \frac{\widehat{\mathbb{P}}_{\xi_1^{i-1}+_i,\xi_1^{i-1}-_i}\big(Z_{A_{<x}} = \zeta_{A_{<x}} \big| Y_{A_{<x}} = \sigma_{A_{<x}}\big)}{\mathbb{P}_{\xi_1^{i-1}+_i\sigma_{A_{<x}}}\big(\eta_{\Lambda\setminus A_{<x}}\big)} \\
&\le \sup_\zeta \frac{\mathbb{P}_{\xi_1^{i-1}+_i\zeta_{A_{<x}}}\big(\eta_{\Lambda\setminus A_{<x}}\big)}{\mathbb{P}_{\xi_1^{i-1}+_i\sigma_{A_{<x}}}\big(\eta_{\Lambda\setminus A_{<x}}\big)} \\
&\le e^{c'|\partial A_{<x}|} \le e^{c'|A|}.
\end{aligned}$$

We conclude by letting $\Lambda \uparrow \mathbb{Z}^d$. $\square$

Returning to (18) and using the preceding lemma we get

$$\begin{aligned}
|\Delta_i| &\le \sum_{A \ni x_i} \sum_{x \in A} \Big(\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}\big(\mathscr{C}_i = A\big)\Big)^{1/2}\ e^{c'|A|} \times \\
&\qquad\qquad \left(\int \mathrm{d}\mathbb{P}_{X_1^{i-1}-_i}(\eta) \big(\nabla_x f(X_1^{i-1}\eta)\big)^2 \right)^{1/2} \\
&\le e^c \sum_{A \ni x_i} \sum_{x \in A} \Big(\widehat{\mathbb{P}}_{X_1^{i-1}+_i, X_1^{i-1}-_i}\big(\mathscr{C}_i = A\big)\Big)^{1/2}\ e^{c'|A|} \times \\
&\qquad\qquad \left(\int \mathrm{d}\mathbb{P}_{X_1^i}(\eta) \big(\nabla_x f(X_1^i\eta)\big)^2 \right)^{1/2}, \qquad (19)
\end{aligned}$$

where for the second inequality we used that, under the measure $\mathbb{P}$, the cost of flip at a single site is bounded by e^c (see (2)).

Step 5 (Domination by independent clusters). Using (8) we get from (19)

$$|\Delta_i| \leq e^c \sum_{A \ni x_i} \sum_{x \in A} \big(\mathbb{P}_p(|\mathfrak{C}| \geq |A|)\big)^{1/2} e^{c'|A|} \times \\ \left(\int d\mathbb{P}_{X_1^i}(\eta) \big(\nabla_x f(X_1^i \eta)\big)^2 \right)^{1/2}. \qquad (20)$$

Now let

$$K' = \sum_{A \ni x_i} \sum_{x \in A} \mathbb{P}_p(|\mathfrak{C}| \geq |A|)^{1/2} e^{c'|A|} = \sum_{A \ni 0} |A| \, \mathbb{P}_p(|\mathfrak{C}| \geq |A|)^{1/2} e^{c'|A|}.$$

By assumption (16) K' is finite. Using once more the elementary inequality for $a_k, b_k \geq 0$

$$\left(\sum_k a_k b_k^{1/2} \right)^2 \leq \sum_k a_k \times \sum_k a_k b_k$$

we deduce from (20) that

$$\begin{aligned} \sum_i \mathbb{E}(\Delta_i^2) &\leq e^{2c} K' \sum_i \sum_{A \ni x_i} \sum_{x \in A} \mathbb{P}_p\big(|\mathfrak{C}| \geq |A|\big)^{1/2} e^{c'|A|} \int \big(\nabla_x f\big)^2 d\mathbb{P} \\ &= e^{2c} K' \sum_x \left(\int \big(\nabla_x f\big)^2 d\mathbb{P} \right) \sum_{A \ni x} |A| \, \mathbb{P}_p\big(|\mathfrak{C}| \geq |A|\big)^{1/2} e^{c'|A|} \\ &= C_P \sum_x \int \big(\nabla_x f\big)^2 \, d\mathbb{P} \end{aligned}$$

where

$$C_P := e^{2c} K'^2.$$

This finishes the proof of Theorem 16. $\square$

6 Weak Poincaré inequality

If the assumption (10) fails, but $p < p_c$ (where p_c denotes the critical value for independent site percolation) then we are still in the uniqueness regime (i.e., the conditional probabilities (1) admit a unique Gibbs measure) and expect suitable decay properties of the Glauber dynamics.

We show that in this regime the so-called weak Poincaré inequality holds, which gives polynomial relaxation to equilibrium.

Theorem 6.1. *Suppose that* p *(defined in (9)) satisfies* $p < p_c$. *Then the weak Poincaré inequality is satisfied. Moreover, there exists* $C, \kappa > 0$ *such that*

$$\alpha(r) \le C r^{-\kappa}.$$

Proof. The proof follows the lines of the proof of Theorem 1, so we sketch where we start to deviate from it: In the estimation of the variance, the contribution involving $\|\delta f\|_2^2$ will arise by cutting the cluster of disagreement at some order of magnitude N.

The sum in (10) is now possibly infinite, so we define

$$K_N = \sum_{n=0}^{N} n\, e^{cn}\, \mathbb{P}_p\left(|\mathfrak{C}| \ge n\right).$$

Following the line of proof of Theorem 4.1, we follow the change of measure road for realizations of the cluster $\mathfrak{C}_i = A$ of cardinality less than or equal to N, and for A with $|A| > N$ we use the uniform estimate

$$\sup_{\eta} |f(\eta^A) - f(\eta)| \le \sum_{x \in A} \delta_x f.$$

Next estimate

$$\begin{aligned}
&\sum_{i \in \mathbb{N}} \Big(\sum_{A \ni x_i, |A| > N} \sum_{x \in A} \mathbb{P}_p(\mathfrak{C}_i = A) \sum_{x \in A} (\delta_x f) \Big)^2 \\
\le\ & \left(\mathbb{E}_p(|\mathfrak{C}| \mathbb{1}\{|\mathfrak{C}| > N\})\right)^2 \|\delta f\|_2^2.
\end{aligned}$$

This gives the inequality

$$\mathrm{Var}(f) \le 2e^c K_N^2 \mathscr{E}(f,f) + 2\left(\mathbb{E}_p(|\mathfrak{C}| \mathbb{1}\{|\mathfrak{C}| > N\})\right)^2 \|\delta f\|_2^2.$$

The constant in front of $\mathscr{E}(f,f)$ blows up at most exponentially in N, i.e., we have the estimate

$$2e^c K_N^2 \le C_1 e^{aN}$$

where C_1, a are strictly positive and (β, h)-dependent. The constant in front of $\|\delta f\|_2^2$ is exponentially small in the whole subcritical regime, i.e.,

$$2\left(\mathbb{E}_p(|\mathfrak{C}| \mathbb{1}\{|\mathfrak{C}| > N\})\right)^2 \le C_2 e^{-bN}$$

where C_2, b are strictly positive and (β, h)-dependent. Therefore we can take

$$\alpha(r) \le C_1 \left(\frac{r}{C_2}\right)^{-\frac{a}{b}}.$$

□

References

[1] J. van den Berg, C. Maes. Disagreement percolation in the study of Markov fields. Ann. Probab. **22**, 749–763 (1994).

[2] J.R. Chazottes, P. Collet, B. Schmitt. Statistical consequences of the Devroye inequality for processes. Applications to a class of non-uniformly hyperbolic dynamical systems. Nonlinearity **18**, 23412364 (2005).

[3] J.R. Chazottes, P. Collet, C. Külske, and F. Redig. Concentration inequalities for random fields via coupling. Probab. Theory & Related Fields **137**, 201–225 (2007).

[4] J.-R. Chazottes, P. Collet, F. Redig. Coupling, concentration inequalities, and stochastic dynamics. J. Math. Phys. **49** (2008), no. 12, 125214, 22 pp.

[5] D.P. Dubhashi, A. Panconesi. *Concentration of measure for the analysis of randomized algorithms.* Cambridge University Press, Cambridge, (2009).

[6] H.-O. Georgii, O. Häggström, C. Maes. The random geometry of equilibrium phases. In "Phase transitions and critical phenomena", Vol. 18, 1–142, Phase Transit. Crit. Phenom., 18, Academic Press, San Diego, CA, 2001.

[7] G. Grimmett. *Percolation.* Second edition. Fundamental Principles of Mathematical Sciences vol. 321. Springer-Verlag, Berlin, 1999.

[8] A. Guionnet, B. Zegarliński. Lectures on logarithmic Sobolev inequalities. Séminaire de Probabilits, XXXVI, 1–134, Lecture Notes in Math., 1801, Springer, Berlin, 2003.

[9] C. Külske. Concentration inequalities for functions of Gibbs fields with application to diffraction and random Gibbs measures. Comm. Math. Phys. **239** (2003), no. 1-2, 29–51.

[10] M. Ledoux. *The concentration of measure phenomenon.* Mathematical Surveys and Monographs, **89**. American Mathematical Society, Providence, RI, (2001).

[11] T.M. Liggett. *Interacting particle systems.* Reprint of the 1985 original. Classics in Mathematics. Springer-Verlag, Berlin, (2005).

[12] F. Martinelli, E. Olivieri. Approach to equilibrium of Glauber dynamics in the one phase region. II. The general case. Comm. Math. Phys. **161** (1994), no. 3, 487–514.

[13] Y. Ollivier. Ricci curvature of Markov chains on metric spaces. J. Funct. Anal. **256**, 810864, (2009).

[14] M. Röckner, F.-Y. Wang. Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. J. Funct. Anal. **185** (2001), no. 2, 564–603.

[15] D. Stroock, B. Zegarliński. The logarithmic Sobolev inequality for discrete spin systems on a lattice. Comm. Math. Phys. **149** (1992), no. 1, 175–193.

[16] L. Wu. Poincaré and transportation inequalities for Gibbs measures under the Dobrushin uniqueness condition. Ann. Probab. **34**, 1960–1989 (2006).

[17] B. Zegarliński. Dobrushin uniqueness theorem and logarithmic Sobolev inequalities. J. Funct. Anal. **105** (1992), no. 1, 77–111.