Scalars, Monads, and Categories

Dion Coumans
Inst. for Mathematics, Astrophysics and Particle Physics (IMAPP)
www.math.ru.nl/~coumans

Bart Jacobs
Inst. for Computing and Information Sciences (ICIS)
www.cs.ru.nl/~bart

Radboud University Nijmegen, The Netherlands

November 1, 2011

Abstract

The paper describes interrelations between: (1) algebraic structure on sets of scalars, (2) properties of monads associated with such sets of scalars, and (3) structure in categories (esp. Lawvere theories) associated with these monads. These interrelations will be expressed in terms of "triangles of adjunctions", involving for instance various kinds of monoids (non-commutative, commutative, involutive) and semirings as scalars. It will be shown to which kind of monads and categories these algebraic structures correspond via adjunctions.

1 Introduction

Scalars are the elements s used in scalar multiplication $s \cdot v$, yielding for instance a new vector for a given vector v. Scalars are elements in some algebraic structure, such as a field (for vector spaces), a ring (for modules), a group (for group actions), or a monoid (for monoid actions).

A categorical description of scalars can be given in a monoidal category $\mathbf{C}$, with tensor $\otimes$ and tensor unit I, as the homset $\mathbf{C}(I, I)$ of endomaps on I. In [15] it is shown that such homsets $\mathbf{C}(I, I)$ always form a commutative monoid; in [2, §3.2] this is called the 'miracle' of scalars. More recent work in the area of quantum computation has led to renewed interest in such scalars, see for instance [1, 2], where it is shown that the presence of biproducts makes this homset $\mathbf{C}(I, I)$ of scalars a semiring, and that daggers $\dagger$ make it involutive. These are first examples where categorical structure (a category which is monoidal or has biproducts or daggers) gives rise to algebraic structure (a set with a commutative monoid, semiring or involution structure). Such correspondences form the focus of this paper, not only those between categorical and algebraic structure, but also involving a third element, namely structure on endofunctors (especially monads). Such correspondences will be described in terms of triangles of adjunctions.

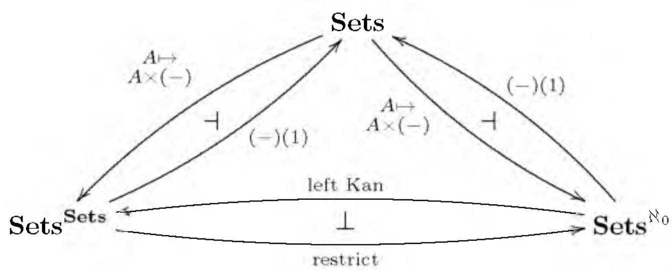

Figure 1: Basic triangle of adjunctions.

To start, we describe the basic triangle of adjunctions that we shall build on. At this stage it is meant as a sketch of the setting, and not as an exhaustive explanation. Let $\aleph_0$ be the category with natural numbers $n \in \mathbb{N}$ as objects. Such a number n is identified with the n-element set $\underline{n} = \{0, 1, \ldots, n-1\}$. Morphisms $n \to m$ in $\aleph_0$ are ordinary functions $\underline{n} \to \underline{m}$ between these finite sets. Hence there is a full and faithful functor $\aleph_0 \hookrightarrow \mathbf{Sets}$. The underline notation is useful to avoid ambiguity, but we often omit it when no confusion arises and write the number n for the set $\underline{n}$.

Now consider the triangle in Figure 1, with functor categories at the two bottom corners. We briefly explain the arrows (functors) in this diagram. The downward arrows $\mathbf{Sets} \to \mathbf{Sets}^{\mathbf{Sets}}$ and $\mathbf{Sets} \to \mathbf{Sets}^{\aleph_0}$ describe the functors that map a set $A \in \mathbf{Sets}$ to the functor $X \mapsto A \times X$. In the other, upward direction right adjoints are given by the functors $(-)(1)$ describing "evaluate at unit 1", that is $F \mapsto F(1)$. At the bottom the inclusion $\aleph_0 \hookrightarrow \mathbf{Sets}$ induces a functor $\mathbf{Sets}^{\mathbf{Sets}} \to \mathbf{Sets}^{\aleph_0}$ by restriction: F is mapped to the functor $n \mapsto F(n)$. In the reverse direction a left adjoint is obtained by left Kan extension [17, Ch. X]. Explicitly, this left adjoint maps a functor $F\colon \aleph_0 \to \mathbf{Sets}$ to the functor $\mathcal{L}(F)\colon \mathbf{Sets} \to \mathbf{Sets}$ given by:

$$\mathcal{L}(F)(X) \quad = \quad \Big(\coprod_{i \in \mathbb{N}} F(i) \times X^i\Big)/\sim,$$

where $\sim$ is the least equivalence relation such that, for each $f\colon n \to m$ in $\aleph_0$,

$$\kappa_m(F(f)(a), v) \quad \sim \quad \kappa_n(a, v \circ f), \qquad \text{where } a \in F(n) \text{ and } v \in X^m.$$

The adjunction on the left in Figure 1 is then in fact the composition of the other two. The adjunctions in Figure 1 are not new. For instance, the one at the bottom plays an important role in the description of analytic functors and species [14], see also [10, 3, 6]. The category of presheaves $\mathbf{Sets}^{\aleph_0}$ is used to provide a semantics for binding, see [7]. What is new in this paper is the systematic organisation of correspondences in triangles like the one in Figure 1 for various kinds of algebraic structures (instead of sets).

- There is a triangle of adjunctions for monoids, monads, and Lawvere theories, see Figure 2.

- This triangle restricts to commutative monoids, commutative monads, and symmetric monoidal Lawvere theories, see Figure 3.
- There is also a triangle of adjunctions for commutative semirings, commutative additive monads, and symmetric monoidal Lawvere theories with biproducts, see Figure 4.
- This last triangle restricts to involutive commutative semirings, involutive commutative additive monads, and dagger symmetric monoidal Lawvere theories with dagger biproducts, see Figure 5 below.

These four figures with triangles of adjunctions provide a quick way to get an overview of the paper (the rest is just hard work). The triangles capture fundamental correspondences between basic mathematical structures. As far as we know they have not been made explicit at this level of generality.

The paper is organised as follows. It starts with a section containing some background material on monads and Lawvere theories. The triangle of adjunctions for monoids, much of which is folklore, is developed in Section 3. Subsequently, Section 4 forms an intermezzo; it introduces the notion of additive monad, and proves that a monad T is additive if and only if in its Kleisli category $\mathcal{K}\ell(T)$ coproducts form biproducts, if and only if in its category $\mathit{Alg}(T)$ of algebras products form biproducts. These additive monads play a crucial role in Sections 5 and 6 which develop a triangle of adjunctions for commutative semirings. Finally, Section 7 introduces the refined triangle with involutions and daggers.

The triangles of adjunctions in this paper are based on many detailed verifications of basic facts. We have chosen to describe all constructions explicitly but to omit most of these verifications, certainly when these are just routine. Of course, one can continue and try to elaborate deeper (categorical) structure underlying the triangles. In this paper we have chosen not to follow that route, but rather to focus on the triangles themselves.

2 Preliminaries

We shall assume a basic level of familiarity with category theory, especially with adjunctions and monads. This section recalls some basic facts and fixes notation. For background information we refer to [4, 5, 17].

In an arbitrary category $\mathbf{C}$ we write finite products as $\times, 1$, where $1 \in \mathbf{C}$ is the final object. The projections are written as π_i and tupling as $\langle f_1, f_2 \rangle$. Finite coproducts are written as $+$ with initial object 0, and with coprojections κ_i and cotupling $[f_1, f_2]$. We write $!$, both for the unique map $X \to 1$ and the unique map $0 \to X$. A category is called distributive if it has both finite products and finite coproducts such that functors $X \times (-)$ preserve these coproducts: the canonical maps $0 \to X \times 0$, and $(X \times Y) + (X \times Z) \to X \times (Y + Z)$ are isomorphisms. Monoidal products are written as $\otimes, I$ where I is the tensor unit, with the familiar isomorphisms: $\alpha\colon X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$ for associativity, $\rho\colon X \otimes I \overset{\cong}{\Rightarrow} X$ and $\lambda\colon I \otimes X \overset{\cong}{\Rightarrow} X$ for unit, and in the symmetric case also $\gamma\colon X \otimes Y \overset{\cong}{\Rightarrow} Y \otimes X$ for swap.

We write $\mathbf{Mnd}(\mathbf{C})$ for the category of monads on a category $\mathbf{C}$. For convenience we write $\mathbf{Mnd}$ for $\mathbf{Mnd}(\mathbf{Sets})$. Although we shall use strength for monads mostly with respect to finite products $(\times, 1)$ we shall give the more general definition involving monoidal products $(\otimes, I)$. A monad T is called strong if it comes with a 'strength' natural transformation st with components $\mathsf{st}\colon T(X) \otimes Y \to T(X \otimes Y)$, commuting with unit η and multiplication μ, in the sense that $\mathsf{st} \circ \eta \otimes \mathrm{id} = \eta$ and $\mathsf{st} \circ \mu \otimes \mathrm{id} = \mu \circ T(\mathsf{st}) \circ \mathsf{st}$. Additionally, for the familiar monoidal isomorphisms ρ and α,

$$
\begin{tikzcd}
T(Y) \otimes I \arrow[r, "\mathsf{st}"] \arrow[dr, "\rho"'] & T(Y \otimes I) \arrow[d, "T(\rho)"] \\
& T(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X) \otimes (Y \otimes Z) \arrow[rr, "\mathsf{st}"] \arrow[d, "\alpha"'] & & T(X \otimes (Y \otimes Z)) \arrow[d, "T(\alpha)"] \\
(T(X) \otimes Y) \otimes Z \arrow[r, "\mathsf{st} \otimes \mathrm{id}"] & T(X \otimes Y) \otimes Z \arrow[r, "\mathsf{st}"] & T((X \otimes Y) \otimes Z)
\end{tikzcd}
$$

Also, when the tensor $\otimes$ is a cartesian product $\times$ we sometimes write these ρ and α for the obvious maps.

The category $\mathbf{StMnd}(\mathbf{C})$ has monads with strength (T, st) as objects. Morphisms are monad maps commuting with strength. The monoidal structure on $\mathbf{C}$ is usually clear from the context.

Lemma 1 *Monads on* $\mathbf{Sets}$ *are always strong w.r.t. finite products, in a canonical way, yielding a functor* $\mathbf{Mnd}(\mathbf{Sets}) = \mathbf{Mnd} \to \mathbf{StMnd} = \mathbf{StMnd}(\mathbf{Sets})$.

Proof For every functor $T\colon \mathbf{Sets} \to \mathbf{Sets}$, there exists a strength map $\mathsf{st}\colon T(X) \times Y \to T(X \times Y)$, namely $\mathsf{st}(u, y) = T(\lambda x.\, \langle x, y \rangle)(u)$. It makes the above diagrams commute, and also commutes with unit and multiplication in case T is a monad. Additionally, strengths commute with natural transformations $\sigma\colon T \to S$, in the sense that $\sigma \circ \mathsf{st} = \mathsf{st} \circ (\sigma \times \mathrm{id})$. $\square$

Given a general strength map $\mathsf{st}\colon T(X) \otimes Y \to T(X \otimes Y)$ in a *symmetric* monoidal category one can define a swapped $\mathsf{st}'\colon X \otimes T(Y) \to T(X \otimes Y)$ as $\mathsf{st}' = T(\gamma) \circ \mathsf{st} \circ \gamma$, where $\gamma\colon X \otimes Y \overset{\cong}{\Rightarrow} Y \otimes X$ is the swap map. There are now in principle two maps $T(X) \otimes T(Y) \rightrightarrows T(X \otimes Y)$, namely $\mu \circ T(\mathsf{st}') \circ \mathsf{st}$ and $\mu \circ T(\mathsf{st}) \circ \mathsf{st}'$. A strong monad T is called commutative if these two composites $T(X) \otimes T(Y) \rightrightarrows T(X \otimes Y)$ are the same. In that case we shall write dst for this (single) map, which is a monoidal transformation, see also [16]. The powerset monad $\mathcal{P}$ is an example of a commutative monad, with $\mathsf{dst}\colon \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$ given by $\mathsf{dst}(U, V) = U \times V$. Later we shall see other examples.

We write $\mathcal{K}\ell(T)$ for the Kleisli category of a monad T, with $X \in \mathbf{C}$ as objects, and maps $X \to T(Y)$ in $\mathbf{C}$ as arrows. For clarity we sometimes write a fat dot $\bullet$ for composition in Kleisli categories, so that $g \bullet f = \mu \circ T(g) \circ f$. The inclusion functor $\mathbf{C} \to \mathcal{K}\ell(T)$ is written as J, where $J(X) = X$ and $J(f) = \eta \circ f$. A map of monads $\sigma\colon T \to S$ yields a functor $\mathcal{K}\ell(\sigma)\colon \mathcal{K}\ell(T) \to \mathcal{K}\ell(S)$ which is the identity on objects, and maps an arrow f to $\sigma \circ f$. This functor $\mathcal{K}\ell(\sigma)$

commutes with the J's. One obtains a functor $\mathcal{K}\ell\colon \mathbf{Mnd}(\mathbf{C}) \to \mathbf{Cat}$, where $\mathbf{Cat}$ is the category of (small) categories.

We will use the following standard result.

Lemma 2 *For $T \in \mathbf{Mnd}(\mathbf{C})$, consider the generic statement "if $\mathbf{C}$ has $\diamondsuit$ then so does $\mathcal{K}\ell(T)$ and $J\colon \mathbf{C} \to \mathcal{K}\ell(T)$ preserves $\diamondsuit$'s", where $\diamondsuit$ is some property. This holds for:*

(i). $\diamondsuit$ = (finite coproducts $+, 0$), or in fact any colimits;

(ii). $\diamondsuit$ = (monoidal products $\otimes, I$), in case the monad T is commutative;

Proof Point *(i)* is obvious; for *(ii)* one defines the tensor on morphisms in $\mathcal{K}\ell(T)$ as:

$$\big(X \xrightarrow{f} T(U)\big) \otimes \big(Y \xrightarrow{g} T(V)\big) \quad = \quad \big(X \otimes Y \xrightarrow{f\otimes g} T(U) \otimes T(V) \xrightarrow{\mathsf{dst}} T(U \otimes V)\big).$$

Then: $J(f) \otimes J(g) = \mathsf{dst} \circ ((\eta \circ f) \otimes (\eta \circ g)) = \eta \circ (f \otimes g) = J(f \otimes g)$. $\square$

As in this lemma we sometimes formulate results on monads in full generality, *i.e.* for arbitrary categories, even though our main results—see Figures 2, 3, 4 and 5—only deal with monads on **Sets**. These results involve algebraic structures like monoids and semirings, which we interpret in the standard set-theoretic universe, and not in arbitrary categories. Such greater generality is possible, in principle, but it does not seem to add enough to justify the additional complexity.

Often we shall be interested in a "finitary" version of the Kleisli construction, corresponding to the Lawvere theory [18, 12] associated with a monad. For a monad $T \in \mathbf{Mnd}$ on $\mathbf{Sets}$ we shall write $\mathcal{K}\ell_{\mathbb{N}}(T)$ for the category with natural numbers $n \in \mathbb{N}$ as objects, regarded as finite sets $\underline{n} = \{0, 1, \ldots, n-1\}$. A map $f\colon n \to m$ in $\mathcal{K}\ell_{\mathbb{N}}(T)$ is then a function $\underline{n} \to T(\underline{m})$. This yields a full inclusion $\mathcal{K}\ell_{\mathbb{N}}(T) \hookrightarrow \mathcal{K}\ell(T)$. It is easy to see that a map $f\colon n \to m$ in $\mathcal{K}\ell_{\mathbb{N}}(T)$ can be identified with an n-cotuple of elements $f_i \in T(m)$, which may be seen as m-ary terms/operations.

By the previous lemma the category $\mathcal{K}\ell_{\mathbb{N}}(T)$ has coproducts given on objects simply by the additive monoid structure $(+, 0)$ on natural numbers. There are obvious coprojections $n \to n + m$, using $\underline{n+m} \cong \underline{n} + \underline{m}$. The identities $n+0 = n = 0+n$ and $(n+m)+k = n+(m+k)$ are in fact the familiar monoidal isomorphisms. The swap map is an isomorphism $n + m \cong m + n$ rather than an identity $n + m = m + n$.

In general, a Lawvere theory is a small category $\mathbf{L}$ with natural numbers $n \in \mathbb{N}$ as objects, and $(+, 0)$ on $\mathbb{N}$ forming finite coproducts in $\mathbf{L}$. It forms a categorical version of a term algebra, in which maps $n \to m$ are understood as n-tuples of terms t_i each with m free variables. Formally a Lawvere theory involves a functor $\aleph_0 \to \mathbf{L}$ that is the identity on objects and preserves finite coproducts "on the nose" (up-to-identity) as opposed to up-to-isomorphism. A morphism of Lawvere theories $F\colon \mathbf{L} \to \mathbf{L}'$ is a functor that is the identity on objects and strictly preserves finite coproducts. This yields a category **Law**.

$$
\begin{tikzcd}
& \mathbf{Mon} \arrow[dl, bend right, "\mathcal{A}"'] \arrow[dr, "\mathcal{K}\ell_{\mathbb{N}}\mathcal{A}"'] \arrow[dl, phantom, shift right=2, "\dashv"] \arrow[dr, phantom, shift left=2, "\dashv"] & \\
\mathbf{Mnd} \arrow[ur, "\mathcal{E}\cong\mathcal{H}\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[rr, bend right=10, "\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Law} \arrow[ul, bend right, "\mathcal{H}"'] \arrow[ll, bend right=10, "\mathcal{T}"']
\end{tikzcd}
$$

$$
\text{where} \quad \begin{cases}
\mathcal{A}(M) = M \times (-) & \text{action monad} \\
\mathcal{E}(T) = T(1) & \text{evaluation at singleton set } 1 \\
\mathcal{H}(\mathbf{L}) = \mathbf{L}(1,1) & \text{endo-homset of } 1 \in \mathbf{L} \\
\mathcal{K}\ell_{\mathbb{N}}(T) & \text{Kleisli category restricted to objects } n \in \mathbb{N} \\
\mathcal{T}(\mathbf{L}) = T_{\mathbf{L}} & \text{monad associated with Lawvere theory } \mathbf{L}.
\end{cases}
$$

Figure 2: Basic relations between monoids, monads and Lawvere theories.

Corollary 3 *The finitary Kleisli construction* $\mathcal{K}\ell_{\mathbb{N}}$ *for monads on* **Sets***, yields a functor* $\mathcal{K}\ell_{\mathbb{N}} : \mathbf{Mnd} \to \mathbf{Law}$. □

3 Monoids

The aim of this section is to replace the category **Sets** of sets at the top of the triangle in Figure 1 by the category **Mon** of monoids $(M, \cdot, 1)$, and to see how the corners at the bottom change in order to keep a triangle of adjunctions. Formally, this can be done by considering monoid objects in the three categories at the corners of the triangle in Figure 1 (see also [7, 6]) but we prefer a more concrete description. The results in this section, which are summarised in Figure 2, are not claimed to be new, but are presented in preparation of further steps later on in this paper.

We start by studying the interrelations between monoids and monads. In principle this part can be skipped, because the adjunction on the left in Figure 2 between monoids and monads follows from the other two by composition. But we do make this adjunction explicit in order to completely describe the situation.

The following result is standard. We only sketch the proof.

Lemma 4 *Each monoid* M *gives rise to a monad* $\mathcal{A}(M) = M \times (-)$: **Sets** $\to$ **Sets**. *The mapping* $M \mapsto \mathcal{A}(M)$ *yields a functor* **Mon** $\to$ **Mnd**.

Proof For a monoid $(M, \cdot, 1)$ the unit map $\eta \colon X \to M \times X = \mathcal{A}(M)$ is $x \mapsto (1, x)$. The multiplication $\mu \colon M \times (M \times X) \to M \times X$ is $(s, (t, x)) \mapsto (s \cdot t, x)$. The standard strength map $\mathsf{st} \colon (M \times X) \times Y \to M \times (X \times Y)$ is given by $\mathsf{st}((s, x), y) = (s, (x, y))$. Each monoid map $f \colon M \to N$ gives rise to a map of monads with components $f \times \mathrm{id}\colon M \times X \to N \times X$. These components commute with strength. $\square$

The monad $\mathcal{A}(M) = M \times (-)$ is called the 'action monad', as its category of Eilenberg-Moore algebras consists of M-actions $M \times X \to X$ and their morphisms. The monoid elements act as scalars in such actions.

Conversely, each monad (on **Sets**) gives rise to a monoid. In the following lemma we prove this in more generality. For a category $\mathbf{C}$ with finite products, we denote by $\mathbf{Mon}(\mathbf{C})$ the category of monoids in $\mathbf{C}$, *i.e.* the category of objects M in $\mathbf{C}$ carrying a monoid structure $1 \to M \leftarrow M \times M$ with structure preserving maps between them.

Lemma 5 *Each strong monad T on a category $\mathbf{C}$ with finite products, gives rise to a monoid $\mathcal{E}(T) = T(1)$ in $\mathbf{C}$. The mapping $T \mapsto T(1)$ yields a functor $\mathbf{StMnd}(\mathbf{C}) \to \mathbf{Mon}(\mathbf{C})$*

Proof For a strong monad $(T, \eta, \mu, \mathsf{st})$, we define a multiplication on $T(1)$ by $\mu \circ T(\pi_2) \circ \mathsf{st}\colon T(1) \times T(1) \to T(1)$, with unit $\eta_1\colon 1 \to T(1)$. Each monad map $\sigma\colon T \to S$ gives rise to a monoid map $T(1) \to S(1)$ by taking the component of σ at 1. $\square$

The swapped strength map st' gives rise to a swapped multiplication on $T(1)$, namely $\mu \circ T(\pi_1) \circ \mathsf{st}'\colon T(1) \times T(1) \to T(1)$, again with unit η_1. It corresponds to $(a, b) \mapsto b \cdot a$ instead of $(a, b) \mapsto a \cdot b$ like in the lemma. In case T is a commutative monad, the two multiplications coincide as we prove in Lemma 10.

The functors defined in the previous two Lemmas 4 and 5 form an adjunction. This result goes back to [19].

Lemma 6 *The pair of functors $\mathcal{A}\colon \mathbf{Mon} \rightleftarrows \mathbf{Mnd}\colon \mathcal{E}$ forms an adjunction $\mathcal{A} \dashv \mathcal{E}$, as on the left in Figure 2.*

Proof For a monoid M and a (strong) monad T on **Sets** there are (natural) bijective correspondences:

$$\frac{\mathcal{A}(M) \xrightarrow{\ \sigma\ } T \quad \text{in } \mathbf{Mnd}}{\overline{} \; M \xrightarrow[f]{} T(1) \quad \text{in } \mathbf{Mon}}$$

Given σ one defines a monoid map $\overline{\sigma}\colon M \to T(1)$ as:

$$\overline{\sigma} = \Big(M \xrightarrow[\cong]{\rho^{-1}} M \times 1 = \mathcal{A}(M)(1) \xrightarrow{\sigma_1} T(1)\Big),$$

where $\rho^{-1} = \langle \mathrm{id}, ! \rangle$ in this cartesian case. Conversely, given f one gets a monad map $\overline{f}\colon \mathcal{A}(M) \to T$ with components:

$$\overline{f}_X = \Big(M \times X \xrightarrow{f \times \mathrm{id}} T(1) \times X \xrightarrow{\mathsf{st}} T(1 \times X) \xrightarrow[\cong]{T(\lambda)} T(X)\Big),$$

where $\lambda = \pi_2 \colon 1 \times X \overset{\cong}{\Rightarrow} X$. Straightforward computations show that these assignments indeed give a natural bijective correspondence. $\square$

Notice that, for a monoid M, the counit of the above adjunction is the projection $(\mathcal{E} \circ \mathcal{A})(M) = \mathcal{A}(M)(1) = M \times 1 \overset{\cong}{\Rightarrow} M$. Hence the adjunction is a reflection.

We now move to the bottom of Figure 2. The finitary Kleisli construction yields a functor from the category of monads to the category of Lawvere theories (Corollary 3). This functor has a left adjoint, as is proven in the following two standard lemmas.

Lemma 7 *Each Lawvere theory $\mathbf{L}$, gives rise to a monad $T_{\mathbf{L}}$ on $\mathbf{Sets}$, which is defined by*

$$T_{\mathbf{L}}(X) \quad = \quad \Big(\textstyle\coprod_{i \in \mathbb{N}} \mathbf{L}(1,i) \times X^i\Big)/\!\sim, \tag{1}$$

where $\sim$ is the least equivalence relation such that, for each $f \colon i \to m$ in $\aleph_0 \hookrightarrow \mathbf{L}$,

$$\kappa_m(f \circ g, v) \quad \sim \quad \kappa_i(g, v \circ f), \qquad \textit{where } g \in \mathbf{L}(1,i) \textit{ and } v \in X^m.$$

Finally, the mapping $\mathbf{L} \to T_{\mathbf{L}}$ yields a functor $\mathcal{T} \colon \mathbf{Law} \to \mathbf{Mnd}$.

Proof For a Lawvere theory $\mathbf{L}$, the unit map $\eta \colon X \to T_{\mathbf{L}}(X) = \Big(\coprod_{i \in \mathbb{N}} \mathbf{L}(1,i) \times X^i\Big)/\!\sim$ is given by

$$x \quad \mapsto \quad [\kappa_1(id_1, x)].$$

The multiplication $\mu \colon T^2_{\mathbf{L}}(X) \to T_{\mathbf{L}}(X)$ is given by:

$$\begin{aligned}\mu([\kappa_i(g,v)]) \quad &= \quad [\kappa_j((g_0 + \cdots + g_{i-1}) \circ g, [v_0, \ldots, v_{i-1}])] \\ & \qquad \text{where } g \colon 1 \to i, \text{ and } v \colon i \to T_{\mathbf{L}}(X) \text{ is written as} \\ & \qquad\qquad v(a) = \kappa_{j_a}(g_a, v_a), \text{ for } a < i, \\ & \qquad \text{and } j = j_0 + \cdots + j_{i-1}.\end{aligned}$$

It is straightforward to show that this map μ is well-defined and that η and μ indeed define a monad structure on $T_{\mathbf{L}}$.

For each morphism of Lawvere theories $F \colon \mathbf{L} \to \mathbf{K}$, one may define a monad morphism $\mathcal{T}(F) \colon T_{\mathbf{L}} \to T_{\mathbf{K}}$ with components $\mathcal{T}(F)_X \colon [\kappa_i(g,v)] \mapsto [\kappa_i(F(g),v)]$. This yields a functor $\mathcal{T} \colon \mathbf{Law} \to \mathbf{Mnd}$. Checking the details is left to the reader. $\square$

Lemma 8 *The pair of functors $\mathcal{T} \colon \mathbf{Law} \rightleftarrows \mathbf{Mnd} \colon \mathcal{K}\ell_{\mathbb{N}}$ forms an adjunction $\mathcal{T} \dashv \mathcal{K}\ell_{\mathbb{N}}$, as at the bottom in Figure 2.*

Proof For a Lawvere theory $\mathbf{L}$ and a monad T there are (natural) bijective correspondences:

$$\frac{\mathcal{T}(\mathbf{L}) \xrightarrow{\ \sigma\ } T \qquad \text{in } \mathbf{Mnd}}{\overline{\mathbf{L} \xrightarrow[F]{} \mathcal{K}\ell_{\mathbb{N}}(T) \qquad \text{in } \mathbf{Law}}}$$

Given σ, one defines a **Law**-map $\overline{\sigma}\colon \mathbf{L} \to \mathcal{K}\ell_{\mathbb{N}}(T)$ which is the identity on objects and sends a morphism $f\colon n \to m$ in $\mathbf{L}$ to the morphism

$$n \xrightarrow{\lambda i<n.\,[\kappa_m(f\circ\kappa_i,\mathrm{id}_m)]} \mathcal{T}(\mathbf{L})(m) \xrightarrow{\sigma_m} T(m)$$

in $\mathcal{K}\ell_{\mathbb{N}}(T)$.

Conversely, given F, one defines a monad morphism $\overline{F}$ with components $\overline{F}_X\colon \mathcal{T}(\mathbf{L})(X) \to T(X)$ given, for $i \in \mathbb{N}$, $g\colon 1 \to i \in \mathbf{L}$ and $v \in X^i$, by:

$$[\kappa_i(g, v)] \quad \mapsto \quad (T(v) \circ F(g))(*),$$

where $*$ is the unique element of 1. □

Finally, we consider the right-hand side of Figure 2. For each category $\mathbf{C}$ and object X in $\mathbf{C}$, the homset $\mathbf{C}(X, X)$ is a monoid, where multiplication is given by composition with the identity as unit. The mapping $\mathbf{L} \mapsto \mathcal{H}(\mathbf{L}) = \mathbf{L}(1, 1)$, defines a functor $\mathbf{Law} \to \mathbf{Mon}$. This functor is right adjoint to the composite functor $\mathcal{K}\ell_{\mathbb{N}} \circ \mathcal{A}$.

Lemma 9 *The pair of functors $\mathcal{K}\ell_{\mathbb{N}} \circ \mathcal{A}\colon \mathbf{Mon} \rightleftarrows \mathbf{Law}\colon \mathcal{H}$ forms an adjunction $\mathcal{K}\ell_{\mathbb{N}} \circ \mathcal{A} \dashv \mathcal{H}$, as on the right in Figure 2.*

Proof For a monoid M and a Lawvere theory $\mathbf{L}$ there are (natural) bijective correspondences:

$$\begin{array}{cl} \mathcal{K}\ell_{\mathbb{N}}\mathcal{A}(M) \xrightarrow{F} \mathbf{L} & \text{in } \mathbf{Law} \\ \hline\hline M \xrightarrow[f]{} \mathcal{H}(\mathbf{L}) & \text{in } \mathbf{Mon} \end{array}$$

Given F one defines a monoid map $\overline{F}\colon M \to \mathcal{H}(\mathbf{L}) = \mathbf{L}(1, 1)$ by

$$s \mapsto F(1 \xrightarrow{\langle \lambda x.\, s, ! \rangle} M \times 1).$$

Note that $1 \xrightarrow{\langle \lambda x.\, s, ! \rangle} M \times 1 = \mathcal{A}(M)(1)$ is an endomap on 1 in $\mathcal{K}\ell_{\mathbb{N}}\mathcal{A}(M)$. Since F is the identity on objects it sends this endomap to an element of $\mathbf{L}(1, 1)$.

Conversely, given a monoid map $f\colon M \to \mathbf{L}(1, 1)$ one defines a **Law**-map $\overline{f}\colon \mathcal{K}\ell_{\mathbb{N}}\mathcal{A}(M) \to \mathbf{L}$. It is the identity on objects and sends a morphism $h\colon n \to m$ in $\mathcal{K}\ell_{\mathbb{N}}\mathcal{A}(M)$, *i.e.* $h\colon n \to M \times m$ in **Sets**, to the morphism

$$\overline{f}(h) = \left(n \xrightarrow{\left[\kappa_{h_2(i)} \circ f(h_1(i))\right]_{i<n}} m\right).$$

Here we write $h(i) \in M \times m$ as pair $(h_1(i), h_2(i))$. We leave further details to the reader. □

Given a monad T on **Sets**, $\mathcal{HK}\ell_{\mathbb{N}}(T) = \mathcal{K}\ell_{\mathbb{N}}(T)(1,1) = \mathbf{Sets}(1, T(1))$ is a monoid, where the multiplication is given by

$$(1 \xrightarrow{a} T(1)) \cdot (1 \xrightarrow{b} T(1)) = \big(1 \xrightarrow{a} T(1) \xrightarrow{T(b)} T^2(1) \xrightarrow{\mu} T(1)\big).$$

The functor $\mathcal{E} : \mathbf{Mnd}(\mathbf{C}) \to \mathbf{Mon}(\mathbf{C})$, defined in Lemma 5 also gives a multiplication on $\mathbf{Sets}(1, T(1)) \cong T(1)$, namely $\mu \circ T(\pi_2) \circ \mathsf{st} \colon T(1) \times T(1) \to T(1)$. These two multiplications coincide as is demonstrated in the following diagram,

$$\begin{tikzcd}
1 \arrow[r, "a"'] \arrow[rrr, bend left, "\langle a,b\rangle"] & T(1) \arrow[r, "\rho^{-1}"] \arrow[dr, "T(\rho^{-1})"] \arrow[ddrr, bend right, "T(b)"'] & T(1) \times 1 \arrow[r, "\mathrm{id}\times b"'] \arrow[d, "\mathsf{st}"] & T(1) \times T(1) \arrow[d, "\mathsf{st}"] \arrow[ddr, bend left] & \\
 & & T(1 \times 1) \arrow[r, "T(\mathrm{id}\times b)"'] & T(1 \times T(1)) \arrow[d, "T(\lambda)"] & \\
 & & & T^2(1) \arrow[r, "\mu"] & T(1)
\end{tikzcd}$$

In fact, $\mathcal{E} \cong \mathcal{HK}\ell_{\mathbb{N}}$, which completes the picture from Figure 2.

3.1 Commutative monoids

In this subsection we briefly summarize what will change in the triangle in Figure 2 when we restrict ourselves to commutative monoids (at the top). This will lead to commutative monads, and to tensor products. The latter are induced by Lemma 2. The new situation is described in Figure 3. For the adjunction between commutative monoids and commutative monads we start with the following basic result.

Lemma 10 *Let T be a commutative monad on a category $\mathbf{C}$ with finite products. The monoid $\mathcal{E}(T) = T(1)$ in $\mathbf{C}$ from Lemma 5 is then commutative.*

Proof Recall that the multiplication on $T(1)$ is given by $\mu \circ T(\lambda) \circ \mathsf{st} \colon T(1) \times T(1) \to T(1)$ and commutativity of the monad T means $\mu \circ T(\mathsf{st}') \circ \mathsf{st} = \mu \circ T(\mathsf{st}) \circ \mathsf{st}'$ where $\mathsf{st}' = T(\gamma) \circ \mathsf{st} \circ \gamma$, for the swap map γ, see Section 2. Then:

$$\begin{array}{rcl}
\mu \circ T(\lambda) \circ \mathsf{st} \circ \gamma & = & \mu \circ T(T(\lambda) \circ \mathsf{st}') \circ \mathsf{st} \circ \gamma \\
 & = & T(\lambda) \circ \mu \circ T(\mathsf{st}') \circ \mathsf{st} \circ \gamma \\
 & = & T(\rho) \circ \mu \circ T(\mathsf{st}) \circ \mathsf{st}' \circ \gamma \quad \text{by commutativity of } T, \\
 & & \qquad \text{and because } \rho = \lambda \colon 1 \times 1 \to 1 \\
 & = & \mu \circ T(T(\rho) \circ \mathsf{st} \circ \gamma) \circ \mathsf{st} \\
 & = & \mu \circ T(\rho \circ \gamma) \circ \mathsf{st} \\
 & = & \mu \circ T(\lambda) \circ \mathsf{st}.
\end{array}$$

$\square$

The proof of the next result is easy and left to the reader.

Lemma 11 *A monoid M is commutative (Abelian) if and only if the associated monad $\mathcal{A}(M) = M \times (-)\colon$ **Sets** $\rightarrow$ **Sets** is commutative (as described in Section 2).* $\square$

Next, we wish to define an appropriate category **SMLaw** of Lawvere theories with symmetric monoidal structure $(\otimes, I)$. In order to do so we need to take a closer look at the category $\aleph_0$ described in the introduction. Recall that $\aleph_0$ has $n \in \mathbb{N}$ as objects whilst morphisms $n \to m$ are functions $\underline{n} \to \underline{m}$ in **Sets**, where, as described earlier $\underline{n} = \{0, 1, \ldots, n-1\}$. This category $\aleph_0$ has a monoidal structure, given on objects by multiplication $n \times m$ of natural numbers, with $1 \in \mathbb{N}$ as tensor unit. Functoriality involves a (chosen) coordinatisation, in the following way. For $f\colon n \to p$ and $g\colon m \to q$ in $\aleph_0$ one obtains $f \otimes g\colon n \times m \to p \times q$ as a function:

$$f \otimes g \quad = \quad \mathsf{co}_{p,q}^{-1} \circ (f \times g) \circ \mathsf{co}_{n,m} \;:\; \underline{n \times m} \longrightarrow \underline{p \times q},$$

where co is a coordinatisation function

$$\underline{n \times m} = \{0, \ldots, (n \times m) - 1\} \xrightarrow[\cong]{\mathsf{co}_{n,m}} \{0, \ldots, n-1\} \times \{0, \ldots, m-1\} = \underline{n} \times \underline{m},$$

given by

$$\mathsf{co}(c) = (a, b) \quad \Leftrightarrow \quad c = a \cdot m + b. \tag{2}$$

We may write the inverse $\mathsf{co}^{-1}\colon \overline{n} \times \overline{m} \to \overline{n \times m}$ as a small tensor, as in $a \otimes b = \mathsf{co}^{-1}(a, b)$. Then: $(f \otimes b)(a \otimes b) = f(a) \otimes g(b)$. The monoidal isomorphisms in $\aleph_0$ are then obtained from **Sets**, as in

$$\gamma^{\aleph_0} = \Big(\underline{n \times m} \xrightarrow{\mathsf{co}} \underline{n} \times \underline{m} \xrightarrow{\gamma^{\mathbf{Sets}}} \underline{m} \times \underline{n} \xrightarrow{\mathsf{co}^{-1}} \underline{m \times n}\Big).$$

Thus $\gamma^{\aleph_0}(a \otimes b) = b \otimes a$. Similarly, the associativity map $\alpha^{\aleph_0}\colon n \otimes (m \otimes k) \to (n \otimes m) \otimes k$ is determined as $\alpha^{\aleph_0}(a \otimes (b \otimes c)) = (a \otimes b) \otimes c$. The maps $\rho\colon n \times 1 \to n$ in $\aleph_0$ are identities.

This tensor $\otimes$ on $\aleph_0$ distributes over sum: the canonical distributivity map $(n \otimes m) + (n \otimes k) \to n \otimes (m + k)$ is an isomorphism. Its inverse maps $a \otimes b \in n \otimes (m + k)$ to $a \otimes b \in \underline{n \times m}$ if $b < m$, and to $a \otimes (b - m) \in \underline{n \times k}$ otherwise.

We thus define the objects of the category **SMLaw** to be symmetric monoidal Lawvere theories $\mathbf{L} \in$ **Law** for which the map $\aleph_0 \to \mathbf{L}$ strictly preserves the monoidal structure that has just been described via multiplication $(\times, 1)$ of natural numbers; additionally the coproduct structure must be preserved, as in **Law**. Morphisms in **SMLaw** are morphisms in **Law** that strictly preserve this tensor structure. We note that for $\mathbf{L} \in$ **SMLaw** we have a distributivity $n \otimes m + n \otimes k \xrightarrow{\cong} n \otimes (m + k)$, since this isomorphism lies in the range of the functor $\aleph_0 \to \mathbf{L}$.

$$\begin{tikzcd}
& \mathbf{CMon} \arrow[dl, bend right, "\mathcal{A}"'] \arrow[dr, "\mathcal{K}\ell_{\mathbb{N}}\mathcal{A}"] & \\
\mathbf{CMnd} \arrow[ur, "\mathcal{E}\cong\mathcal{H}\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[rr, bend right=10, "\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{SMLaw} \arrow[ul, bend right, "\mathcal{H}"'] \arrow[ll, bend right=10, "\mathcal{T}"']
\end{tikzcd}$$

Figure 3: Commutative version of Figure 2, with commutative monoids, commutative monads and symmetric monoidal Lawvere theories.

By Lemma 2 we know that the Kleisli category $\mathcal{K}\ell(T)$ is symmetric monoidal if T is commutative. In order to see that also the finitary Kleisli category $\mathcal{K}\ell_{\mathbb{N}}(T) \in \mathbf{Law}$ is symmetric monoidal, we have to use the coordinatisation map described in (2). For $f\colon n \to p$ and $g\colon m \to q$ in $\mathcal{K}\ell_{\mathbb{N}}(T)$ we then obtain $f \otimes g\colon n \times m \to p \times q$ as

$$f \otimes g = \Big(\underline{n \times m} \xrightarrow{\mathsf{co}} \underline{n} \times \underline{m} \xrightarrow{f \times g} T(\underline{p}) \times T(\underline{q}) \xrightarrow{\mathsf{dst}} T(\underline{p} \times \underline{q}) \xrightarrow{T(\mathsf{co}^{-1})} T(\underline{p \times q})\Big).$$

We recall from [15] (see also [1, 2]) that for a monoidal category $\mathbf{C}$ the homset $\mathbf{C}(I, I)$ of endomaps on the tensor unit forms a commutative monoid. This applies in particular to Lawvere theories $\mathbf{L} \in \mathbf{SMLaw}$, and yields a functor $\mathcal{H}\colon \mathbf{SMLaw} \to \mathbf{Mon}$ given by $\mathcal{H}(\mathbf{L}) = \mathbf{L}(1, 1)$, where $1 \in \mathbf{L}$ is the tensor unit. Thus we almost have a triangle of adjunctions as in Figure 3. We only need to check the following result.

Lemma 12 *The functor $\mathcal{T}\colon \mathbf{Law} \to \mathbf{Mnd}$ defined in* (1) *restricts to $\mathbf{SMLaw} \to \mathbf{CMnd}$. Further, this restriction is left adjoint to $\mathcal{K}\ell_{\mathbb{N}}\colon \mathbf{CMnd} \to \mathbf{SMLaw}$.*

Proof For $\mathbf{L} \in \mathbf{SMLaw}$ we define a map

$$\begin{array}{ccc}
\mathcal{T}(\mathbf{L})(X) \times \mathcal{T}(\mathbf{L})(Y) & \xrightarrow{\qquad\mathsf{dst}\qquad} & \mathcal{T}(\mathbf{L})(X \times Y) \\
\big([\kappa_i(g, v)], [\kappa_j(h, w)]\big) & \longmapsto & [\kappa_{i \times j}(g \otimes h, (v \times w) \circ \mathsf{co}_{i,j})],
\end{array}$$

where $g\colon 1 \to i$ and $h\colon 1 \to j$ in $\mathbf{L}$ yield $g \otimes h\colon 1 = 1 \otimes 1 \to i \otimes j = i \times j$, and co is the coordinatisation function (2). Then one can show that both $\mu \circ \mathcal{T}(\mathbf{L})(\mathsf{st}') \circ \mathsf{st}$ and $\mu \circ \mathcal{T}(\mathbf{L})(\mathsf{st}) \circ \mathsf{st}'$ are equal to dst. This makes $\mathcal{T}(\mathbf{L})$ a commutative monad.

In order to check that the adjunction $\mathcal{T} \dashv \mathcal{K}\ell_{\mathbb{N}}$ restricts, we only need to verify that the unit $\mathbf{L} \to \mathcal{K}\ell_{\mathbb{N}}(\mathcal{T}(\mathbf{L}))$ strictly preserves tensors. This is easy. $\square$

4 Additive monads

Having an adjunction between commutative monoids and commutative monads (Figure 3) raises the question whether we may also define an adjunction between commutative semirings and some specific class of monads. It will appear that so-called additive commutative monads are needed here. In this section we will define and study such additive (commutative) monads and see how they relate to biproducts in their Kleisli categories and categories of algebras.

We consider monads on a category $\mathbf{C}$ with both finite products and coproducts. If, for a monad T on $\mathbf{C}$, the object $T(0)$ is final—*i.e.* satisfies $T(0) \cong 1$—then 0 is both initial and final in the Kleisli category $\mathcal{K}\ell(T)$. Such an object that is both initial and final is called a *zero object*.

Also the converse is true, if $0 \in \mathcal{K}\ell(T)$ is a zero object, then $T(0)$ is final in $\mathbf{C}$. Although we don't use this in the remainder of this paper, we also mention a related result on the category of Eilenberg-Moore algebras. The proofs are simple and are left to the reader.

Lemma 13 *For a monad T on a category $\mathbf{C}$ with finite products $(\times, 1)$ and coproducts $(+, 0)$, the following statements are equivalent.*

(i). $T(0)$ *is final in* $\mathbf{C}$*;*

(ii). $0 \in \mathcal{K}\ell(T)$ *is a zero object;*

(iii). $1 \in \mathcal{A}lg(T)$ *is a zero object.* □

A zero object yields, for any pair of objects X, Y, a unique "zero map" $0_{X,Y} \colon X \to 0 \to Y$ between them. In a Kleisli category $\mathcal{K}\ell(T)$ for a monad T on $\mathbf{C}$, this zero map $0_{X,Y} \colon X \to Y$ is the following map in $\mathbf{C}$

$$0_{X,Y} = \Big(X \xrightarrow{!_X} 1 \cong T(0) \xrightarrow{T(!_Y)} T(Y)\Big). \tag{3}$$

For convenience, we make some basic properties of this zero map explicit.

Lemma 14 *Assume $T(0)$ is final, for a monad T on $\mathbf{C}$. The resulting zero maps $0_{X,Y} \colon X \to T(Y)$ from (3) make the following diagrams in $\mathbf{C}$ commute*

$$\begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "0"'] & T^2(Y) \arrow[d, "\mu"] & T(X) \arrow[l, "T(0)"'] \arrow[dl, "0"] \\ & T(Y) & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "0"] \arrow[d, "f"'] \arrow[dr, "0"] & T(Y) \arrow[d, "T(f)"] \\ Y \arrow[r, "0"'] & T(Z) \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "0"] \arrow[dr, "0"'] & T(Y) \arrow[d, "\sigma_Y"] \\ & S(Y) \end{tikzcd}$$

where $f \colon Y \to Z$ is a map in $\mathbf{C}$ and $\sigma \colon T \to S$ is a map of monads. □

Still assuming that $T(0)$ is final, the zero map (3) enables us to define a canonical map

$$\mathsf{bc} \stackrel{\text{def}}{=} \Big(T(X+Y) \xrightarrow{\langle \mu \circ T(p_1), \mu \circ T(p_2)\rangle} T(X) \times T(Y)\Big), \tag{4}$$

where

$$p_1 \stackrel{\text{def}}{=} \Big(X + Y \xrightarrow{[\eta, 0_{Y,X}]} T(X) \Big), \qquad p_2 \stackrel{\text{def}}{=} \Big(X + Y \xrightarrow{[0_{X,Y}, \eta]} T(Y) \Big). \qquad (5)$$

Here we assume that the underlying category $\mathbf{C}$ has both finite products and finite coproducts. The abbreviation "bc" stands for "bicartesian", since this maps connects the coproducts and products. The auxiliary maps p_1, p_2 are sometimes called projections, but should not be confused with the (proper) projections π_1, π_2 associated with the product $\times$ in $\mathbf{C}$.

We continue by listing a series of properties of this map bc that will be useful in what follows.

Lemma 15 *In the context just described, the map* $\mathsf{bc}\colon T(X+Y) \to T(X) \times T(Y)$ *in* (4) *has the following properties.*

(i). This bc *is a natural transformation, and it commutes with any monad map* $\sigma\colon T \to S$*, as in:*

$$\begin{tikzcd}
T(X+Y) \arrow[r, "\mathsf{bc}"] \arrow[d, "T(f+g)"'] & T(X) \times T(Y) \arrow[d, "T(f)\times T(g)"] \\
T(U+V) \arrow[r, "\mathsf{bc}"] & T(U) \times T(V)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X+Y) \arrow[r, "\mathsf{bc}"] \arrow[d, "\sigma_{X+Y}"'] & T(X) \times T(Y) \arrow[d, "\sigma_X \times \sigma_Y"] \\
S(X+Y) \arrow[r, "\mathsf{bc}"] & S(X) \times S(Y)
\end{tikzcd}$$

(ii). It also commutes with the monoidal isomorphisms (for products and coproducts in $\mathbf{C}$*):*

$$\begin{tikzcd}
T(X+0) \arrow[r, "\mathsf{bc}"] \arrow[dr, "\cong", "T(\rho)"'] & T(X) \times T(0) \arrow[d, "\cong"', "\rho"] \\
& T(X)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(X+Y) \arrow[r, "\mathsf{bc}"] \arrow[d, "T([\kappa_2,\kappa_1])"', "\cong"] & T(X) \times T(Y) \arrow[d, "\cong"', "\langle \pi_2, \pi_1 \rangle"] \\
T(Y+X) \arrow[r, "\mathsf{bc}"] & T(Y) \times T(X)
\end{tikzcd}$$

$$\begin{tikzcd}
T((X+Y)+Z) \arrow[r, "\mathsf{bc}"] \arrow[d, "T(\alpha)"', "\cong"] & T(X+Y) \times T(Z) \arrow[r, "\mathsf{bc} \times id"] & (T(X) \times T(Y)) \times T(Z) \arrow[d, "\cong"', "\alpha"] \\
T(X+(Y+Z)) \arrow[r, "\mathsf{bc}"] & T(X) \times T(Y+Z) \arrow[r, "id \times \mathsf{bc}"] & T(X) \times (T(Y) \times T(Z))
\end{tikzcd}$$

(iii). The map bc *interacts with* η *and* μ *in the following manner:*

$$\begin{tikzcd}
X+Y \arrow[d, "\eta"'] \arrow[dr, "\langle p_1, p_2 \rangle"] & \\
T(X+Y) \arrow[r, "\mathsf{bc}"'] & T(X) \times T(Y)
\end{tikzcd}$$

$$
\begin{tikzcd}
T^2(X+Y) \arrow[r, "\mu"] \arrow[d, "T(\mathsf{bc})"'] & T(X+Y) \arrow[dd, "\mathsf{bc}"] \\
T(T(X)\times T(Y)) \arrow[d, "\langle T(\pi_1), T(\pi_2)\rangle"'] & \\
T^2(X)\times T^2(Y) \arrow[r, "\mu\times\mu"] & T(X)\times T(Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
T(T(X)+T(Y)) \arrow[r, "\mathsf{bc}"] \arrow[d, "T({[T(\kappa_1), T(\kappa_2)]})"'] & T^2(X)\times T^2(Y) \arrow[dd, "\mu\times\mu"] \\
T^2(X+Y) \arrow[d, "\mu"'] & \\
T(X+Y) \arrow[r, "\mathsf{bc}"] & T(X)\times T(Y)
\end{tikzcd}
$$

(iv). If **C** *is a distributive category,* bc *commutes with strength* st *as follows:*

$$
\begin{tikzcd}
T(X+Y)\times Z \arrow[r, "\mathsf{bc}\times\mathrm{id}"] \arrow[d, "\mathsf{st}"'] & (T(X)\times T(Y))\times Z \arrow[r, "dbl"] & (T(X)\times Z)\times (T(Y)\times Z) \arrow[d, "\mathsf{st}\times\mathsf{st}"] \\
T((X+Y)\times Z) \arrow[r, "\cong"] & T((X\times Z)+(Y\times Z)) \arrow[r, "\mathsf{bc}"] & T(X\times Z)\times T(Y\times Z)
\end{tikzcd}
$$

where dbl is the "double" map $\langle \pi_1\times\mathrm{id}, \pi_2\times\mathrm{id}\rangle\colon (A\times B)\times C \to (A\times C)\times (B\times C)$.

Proof These properties are easily verified, using Lemma 14 and the fact that the projections p_i are natural, both in **C** and in $\mathcal{K}\ell(T)$. $\square$

The definition of the map bc also makes sense for arbitrary set-indexed (co)products (see [13]), but here we only consider finite ones. Such generalised bc-maps also satisfy (suitable generalisations of) the properties in Lemma 15 above.

We will study monads for which the canonical map bc is an isomorphism. Such monads will be called 'additive monads'.

Definition 16 *A monad T on a category* **C** *with finite products $(\times, 1)$ and finite coproducts $(+, 0)$ will be called* additive *if $T(0)\cong 1$ and if the canonical map $\mathsf{bc}\colon T(X+Y)\to T(X)\times T(Y)$ from (4) is an isomorphism.*

We write **AMnd**(**C**) for the category of additive monads on **C** with monad morphism between them, and similarly **ACMnd**(**C**) for the category of additive and commutative monads on **C**.

A basic result is that additive monads T induce a commutative monoid structure on objects $T(X)$. This result is sometimes taken as definition of additivity of monads (*cf.* [9]).

Lemma 17 *Let T be an additive monad on a category* **C** *and X an object of* **C**. *There is an addition $+$ on $T(X)$ given by*

$$
+ \stackrel{def}{=} \Big(T(X)\times T(X) \xrightarrow{\mathsf{bc}^{-1}} T(X+X) \xrightarrow{T(\nabla)} T(X) \Big),
$$

where $\nabla = [\mathrm{id}, \mathrm{id}]$. Then:

(i). this $+$ is commutative and associative,

(ii). and has unit $0_{1,X} : 1 \to T(X)$;

(iii). this monoid structure is preserved by maps $T(f)$ as well as by multiplication μ;

(iv). the mapping $(T, X) \mapsto (T(X), +, 0_{1,X})$ yields a functor $\mathcal{A}d\colon \mathbf{AMnd}(\mathbf{C}) \times \mathbf{C} \to \mathbf{CMon}(\mathbf{C})$.

Proof The first three statements follow by the properties of bc from Lemma 15. For instance, 0 is a (right) unit for $+$ as demonstrated in the following diagram.

$$\begin{tikzcd}
T(X) \arrow[r, "\rho^{-1}", "\cong"'] \arrow[dr, "\cong", "T(\rho^{-1})"'] & T(X) \times T(0) \arrow[r, "\mathrm{id}\times T(!)"] \arrow[d, "\mathsf{bc}^{-1}"] & T(X) \times T(X) \arrow[d, "\mathsf{bc}^{-1}"] \arrow[dd, bend left=60, "+"] \\
& T(X+0) \arrow[r, "T(\mathrm{id}+!)"] \arrow[dr, "\cong", "T(\rho)"'] & T(X+X) \arrow[d, "T(\nabla)"] \\
& & T(X)
\end{tikzcd}$$

Regarding (iv) we define, for a pair of morphisms $\sigma : T \to S$ in $\mathbf{AMnd}(\mathbf{C})$ and $f\colon X \to Y$ in $\mathbf{C}$,

$$\mathcal{A}d((\sigma, f)) = \sigma \circ T(f)\colon T(X) \to S(Y),$$

which is equal to $S(f) \circ \sigma$ by naturality of σ. Preservation of the unit by $\mathcal{A}d((\sigma, f))$ follows from Lemma 14. The following diagram demonstrates that addition is preserved.

$$\begin{tikzcd}
T(X) \times T(X) \arrow[r, "T(f)\times T(f)"] \arrow[d, "\mathsf{bc}^{-1}"'] & T(Y) \times T(Y) \arrow[r, "\sigma\times\sigma"] \arrow[d, "\mathsf{bc}^{-1}"] & S(Y) \times S(Y) \arrow[dd, "\mathsf{bc}^{-1}"] \\
T(X+X) \arrow[r, "T(f+f)"] \arrow[dr, "\sigma"'] \arrow[dd, "T(\nabla)"'] & T(Y+Y) \arrow[dr, "\sigma"] & \\
& S(X+X) \arrow[r, "S(f+f)"] \arrow[d, "S(\nabla)"'] & S(Y+Y) \arrow[d, "S(\nabla)"] \\
T(X) \arrow[r, "\sigma"] & S(X) \arrow[r, "S(f)"] & S(Y)
\end{tikzcd}$$

where we use point (i) of Lemma 15 and the naturality of σ. It is easily checked that this mapping defines a functor. $\square$

By Lemma 2, for a monad T on a category $\mathbf{C}$ with finite coproducts, the Kleisli construction yields a category $\mathcal{K}\ell(T)$ with finite coproducts. Below we will prove that, under the assumption that $\mathbf{C}$ also has products, these coproducts form biproducts in $\mathcal{K}\ell(T)$ if and only if T is additive. Again, as in Lemma 13, a related result holds for the category $Alg(T)$.

Definition 18 *A* category with biproducts *is a category* $\mathbf{C}$ *with a zero object* $0 \in \mathbf{C}$*, such that, for any pair of objects* $A_1, A_2 \in \mathbf{C}$*, there is an object* $A_1 \oplus A_2 \in \mathbf{C}$ *that is both a product with projections* $\pi_i : A_1 \oplus A_2 \to A_i$ *and a coproduct with coprojections* $\kappa_i : A_i \to A_1 \oplus A_2$*, such that*

$$\pi_j \circ \kappa_i \quad = \quad \begin{cases} \mathrm{id}_{A_i} & \text{if } i = j \\ 0_{A_i, A_j} & \text{if } i \neq j. \end{cases}$$

Theorem 19 *For a monad* T *on a category* $\mathbf{C}$ *with finite products* $(\times, 1)$ *and coproducts* $(+, 0)$*, the following are equivalent.*

(i). T is additive;

(ii). the coproducts in $\mathbf{C}$ form biproducts in the Kleisli category $\mathcal{K}\ell(T)$;

(iii). the products in $\mathbf{C}$ yield biproducts in the category of Eilenberg-Moore algebras $Alg(T)$.

Here we shall only use this result for Kleisli categories, but we include the result for algebras for completeness.

Proof First we assume that T is additive and show that $(+, 0)$ is a product in $\mathcal{K}\ell(T)$. As projections we take the maps p_i from (5). For Kleisli maps $f\colon Z \to T(X)$ and $g\colon Z \to T(Y)$ there is a tuple via the map bc, as in

$$\langle f, g\rangle_{\mathcal{K}\ell} \stackrel{\text{def}}{=} \Big(Z \xrightarrow{\langle f,g\rangle} T(X) \times T(Y) \xrightarrow{\mathsf{bc}^{-1}} T(X+Y)\Big).$$

One obtaines $p_1 \bullet \langle f, g\rangle_{\mathcal{K}\ell} = \mu \circ T(p_1) \circ \mathsf{bc}^{-1} \circ \langle f, g\rangle = \pi_1 \circ \mathsf{bc} \circ \mathsf{bc}^{-1} \circ \langle f, g\rangle = \pi_1 \circ \langle f, g\rangle = f$. Remaining details are left to the reader.

Conversely, assuming that the coproduct $(+, 0)$ in $\mathbf{C}$ forms a biproduct in $\mathcal{K}\ell(T)$, we have to show that the bicartesian map $\mathsf{bc}\colon T(X+Y) \to T(X) \times T(Y)$ is an isomorphism. As $+$ is a biproduct, there exist projection maps $q_i\colon X_1 + X_2 \to X_i$ in $\mathcal{K}\ell(T)$ satisfying

$$q_j \bullet \kappa_i \quad = \quad \begin{cases} \mathrm{id}_{X_i} & \text{if } i = j \\ 0_{X_i, X_j} & \text{if } i \neq j. \end{cases}$$

From these conditions it follows that $q_i = p_i$, where p_i is the map defined in (5). The ordinary projection maps $\pi_i\colon T(X_1) \times T(X_2) \to T(X_i)$ are maps $T(X_1) \times T(X_2) \to X_i$ in $\mathcal{K}\ell(T)$. Hence, as $+$ is a product, there exists a unique map $h\colon T(X_1) \times T(X_2) \to X_1 + X_2$ in $\mathcal{K}\ell(T)$, *i.e.* $h\colon T(X_1) \times T(X_2) \to T(X_1 + X_2)$ in $\mathbf{C}$, such that $p_1 \bullet h = \pi_1$ and $p_2 \bullet h = \pi_2$. It is readily checked that this map h is the inverse of bc.

To prove the equivalence of *(i)* and *(iii)*, first assume that the monad T is additive. In the category $Alg(T)$ of algebras there is the standard product

$$\Big(T(X) \xrightarrow{\alpha} X\Big) \times \Big(T(Y) \xrightarrow{\beta} Y\Big) \stackrel{\text{def}}{=} \Big(T(X \times Y) \xrightarrow{\langle \alpha \circ T(\pi_1), \beta \circ T(\pi_2)\rangle} X \times Y\Big).$$

In order to show that $\times$ also forms a coproduct in $Alg(T)$, we first show that for an arbitrary algebra $\gamma \colon T(Z) \to Z$ the object Z carries a commutative monoid structure. We do so by adapting the structure $(+, 0)$ on $T(Z)$ from Lemma 17 to $(+_Z, 0_Z)$ on Z via

$$\begin{array}{rcl} +_Z & \stackrel{\text{def}}{=} & \Big(Z \times Z \xrightarrow{\eta \times \eta} T(Z) \times T(Z) \xrightarrow{+} T(Z) \xrightarrow{\gamma} Z\Big) \\ 0_Z & \stackrel{\text{def}}{=} & \Big(1 \xrightarrow{0} T(Z) \xrightarrow{\gamma} Z\Big) \end{array}$$

This monoid structure is preserved by homomorphisms of algebras. Now, we can form coprojections $k_1 = \langle \mathrm{id}, 0_Y \circ\, !\rangle \colon X \to X \times Y$, and a cotuple of algebra homomorphisms $(TX \xrightarrow{\alpha} X) \xrightarrow{f} (TZ \xrightarrow{\gamma} Z)$ and $(TY \xrightarrow{\beta} X) \xrightarrow{g} (TZ \xrightarrow{\gamma} Z)$ given by

$$[f, g]_{Alg} \stackrel{\text{def}}{=} \Big(X \times Y \xrightarrow{f \times g} Z \times Z \xrightarrow{+_Z} Z\Big).$$

Again, remaining details are left to the reader.

Finally, to show that *(iii)* implies *(i)*, consider the algebra morphisms:

$$\Big(T^2(X_i) \xrightarrow{\mu} T(X_i)\Big) \xrightarrow{T(\kappa_i)} \Big(T^2(X_1 + X_2) \xrightarrow{\mu} T(X_1 + X_2)\Big).$$

The free functor $\mathbf{C} \to Alg(T)$ preserves coproducts, so these $T(\kappa_i)$ form a coproduct diagram in $Alg(T)$. As $\times$ is a coproduct in $Alg(T)$, by assumption, the cotuple $[T(\kappa_1), T(\kappa_2)] \colon T(X_1) \times T(X_2) \to T(X_1 + X_2)$ in $Alg(T)$ is an isomorphism. The coprojections $\ell_i \colon T(X_i) \to T(X_1) \times T(X_2)$ satisfy $\ell_1 = \langle \pi_1 \circ \ell_1, \pi_2 \circ \ell_2\rangle = \langle \mathrm{id}, 0\rangle$, and similarly, $\ell_2 = \langle 0, \mathrm{id}\rangle$. Now we compute:

$$\begin{array}{rcl} \mathsf{bc} \circ [T(\kappa_1), T(\kappa_2)] \circ \ell_1 & = & \langle \mu \circ T(p_1), \mu \circ T(p_2)\rangle \circ T(\kappa_1) \\ & = & \langle \mu \circ T(p_1 \circ \kappa_1), \mu \circ T(p_2 \circ \kappa_1)\rangle \\ & = & \langle \mu \circ T(\eta), \mu \circ T(0)\rangle \\ & = & \langle \mathrm{id}, 0\rangle \\ & = & \ell_1. \end{array}$$

Similarly, $\mathsf{bc} \circ [T(\kappa_1), T(\kappa_2)] \circ \ell_2 = \ell_2$, so that $\mathsf{bc} \circ [T(\kappa_1), T(\kappa_2)] = \mathrm{id}$, making bc an isomorphism. □

It is well-known (see for instance [15, 1]) that a category with finite biproducts $(\oplus, 0)$ is enriched over commutative monoids: each homset carries a commutative monoid structure $(+, 0)$, and this structure is preserved by pre- and post-composition. The addition operation $+$ on homsets is obtained as

$$f + g \stackrel{\text{def}}{=} \Big(X \xrightarrow{\langle \mathrm{id}, \mathrm{id} \rangle} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{[\mathrm{id}, \mathrm{id}]} Y \Big). \tag{6}$$

The zero map is neutral element for this addition. One can also describe a monoid structure on each object X as

$$X \oplus X \xrightarrow{[\mathrm{id}, \mathrm{id}]} X \xleftarrow{\;0\;} 0. \tag{7}$$

We have just seen that the Kleisli category of an additive monad has biproducts, using the addition operation from Lemma 17. When we apply the sum description (7) to such a Kleisli category its biproducts, we obtain precisely the original addition from Lemma 17, since the codiagonal $\nabla = [\mathrm{id}, \mathrm{id}]$ in the Kleisli category is given $T(\nabla) \circ \mathsf{bc}^{-1}$.

4.1 Additive commutative monads

In the remainder of this section we focus on the category $\mathbf{ACMnd}(\mathbf{C})$ of monads that are both additive and commutative on a distributive category $\mathbf{C}$. As usual, we simply write $\mathbf{ACMnd}$ for $\mathbf{ACMnd}(\mathbf{Sets})$. For $T \in \mathbf{ACMnd}(\mathbf{C})$, the Kleisli category $\mathcal{K}\ell(T)$ is both symmetric monoidal—with $(\times, 1)$ as monoidal structure, see Lemma 2—and has biproducts $(+, 0)$. Moreover, it is not hard to see that this monoidal structure distributes over the biproducts via the canonical map $(Z \times X) + (Z \times Y) \to Z \times (X + Y)$ that can be lifted from $\mathbf{C}$ to $\mathcal{K}\ell(T)$.

We shall write $\mathbf{SMBLaw} \hookrightarrow \mathbf{SMLaw}$ for the category of symmetric monoidal Lawvere theories in which $(+, 0)$ form not only coproducts but biproducts. Notice that a projection $\pi_1 \colon n + m \to n$ is necessarily of the form $\pi_1 = [\mathrm{id}, 0]$, where $0 \colon m \to n$ is the zero map $m \to 0 \to n$. The tensor $\otimes$ distributes over $(+, 0)$ in $\mathbf{SMBLaw}$, as it already does so in $\mathbf{SMLaw}$. Morphisms in $\mathbf{SMBLaw}$ are functors that strictly preserve all the structure.

The following result extends Corollary 3.

Lemma 20 *The (finitary) Kleisli construction on a monad yields a functor* $\mathcal{K}\ell_{\mathbb{N}} \colon \mathbf{ACMnd} \to \mathbf{SMBLaw}$.

Proof It follows from Theorem 19 that $(+, 0)$ form biproducts in $\mathcal{K}\ell_{\mathbb{N}}(T)$, for T an additive commutative monad (on $\mathbf{Sets}$). This structure is preserved by functors $\mathcal{K}\ell_{\mathbb{N}}(\sigma)$, for $\sigma \colon T \to S$ in $\mathbf{ACMnd}$. □

We have already seen in Lemma 12 that the functor $\mathcal{T} \colon \mathbf{Law} \to \mathbf{Mnd}$ defined in Lemma 7 restricts to a functor between symmetric monoidal Lawvere theories and commutative monads. We now show that it also restricts to a functor between symmetric monoidal Lawvere theories with biproducts and commutative additive monads. Again, this restriction is left adjoint to the finitary Kleisli construction.

Lemma 21 *The functor* $\mathcal{T}\colon \mathbf{SMLaw} \to \mathbf{CMnd}$ *from Lemma 12 restricts to* $\mathbf{SMBLaw} \to \mathbf{ACMnd}$. *Further, this restriction is left adjoint to the finitary Kleisli construction* $\mathcal{K}\ell_{\mathbb{N}}\colon \mathbf{ACMnd} \to \mathbf{SMBLaw}$.

Proof First note that $T_{\mathbf{L}}(0)$ is final:

$$T_{\mathbf{L}}(0) \quad = \quad \coprod_i \mathbf{L}(1, i) \times 0^i \quad \cong \quad \mathbf{L}(1, 0) \times 0^0 \quad \cong \quad 1,$$

where the last isomorphism follows from the fact that $(+, 0)$ is a biproduct in $\mathbf{L}$ and hence 0 is terminal. The resulting zero map $0_{X,Y}\colon X \to T(Y)$ is given by

$$x \quad \mapsto \quad [\kappa_0(!\colon 1 \to 0, !\colon 0 \to Y)].$$

To prove that the bicartesian map $\mathsf{bc}\colon T_{\mathbf{L}}(X+Y) \to T_{\mathbf{L}}(X) \times T_{\mathbf{L}}(Y)$ is an isomorphism, we introduce some notation. For $[\kappa_i(g, v)] \in T_{\mathbf{L}}(X+Y)$, where $g\colon 1 \to i$ and $v\colon i \to X+Y$, we form the pullbacks (in $\mathbf{Sets}$)

$$\begin{tikzcd}
i_X \arrow[r] \arrow[d, "v_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & i \arrow[d, "v"] & i_Y \arrow[l] \arrow[d, "v_Y"] \arrow[dl, phantom, "\llcorner", very near start] \\
X \arrow[r, "\kappa_1"] & X+Y & Y \arrow[l, "\kappa_2"']
\end{tikzcd}$$

By universality of coproducts we can write $i = i_X + i_Y$ and $v = v_X + v_Y\colon i_X + i_Y \to X + Y$. Then we can also write $g = \langle g_X, g_Y\rangle\colon 1 \to i_X + i_Y$. Hence, for $[k_i(g, v)] \in T_{\mathbf{L}}(X+Y)$,

$$\mathsf{bc}([\kappa_i(g, v)]) \quad = \quad \big([\kappa_{i_X}(g_X, v_X)],\, [\kappa_{i_Y}(g_Y, v_Y)]\big). \tag{8}$$

It then easily follows that the map $T_{\mathbf{L}}(X) \times T_{\mathbf{L}}(Y) \to T_{\mathbf{L}}(X+Y)$ defined by

$$([\kappa_i(g, v)], [\kappa_j(h, w)]) \quad \mapsto \quad [\kappa_{i+j}(\langle g, h\rangle, v+w)]$$

is the inverse of bc.

Checking that the unit of the adjunction $\mathcal{T}\colon \mathbf{SMLaw} \leftrightarrows \mathbf{CMnd}\colon \mathcal{K}\ell_{\mathbb{N}}$ preserves the product structure is left to the reader. This proves that also the restricted functors form an adjunction. □

In the next two sections we will see how additive commutative monads and symmetric monoidal Lawvere theories with biproducts relate to commutative semirings.

5 Semirings and monads

This section starts with some clarification about semirings and modules. Then it shows how semirings give rise to certain "multiset" monads, which are both commutative and additive. It is shown that the "evaluate at unit 1"-functor yields a map in the reverse direction, giving rise to an adjunction, as before.

A commutative semiring in **Sets** consists of a set S together with two commutative monoid structures, one additive $(+, 0)$ and one multiplicative $(\cdot, 1)$, where the latter distributes over the former: $s \cdot 0 = 0$ and $s \cdot (t+r) = s \cdot t + s \cdot r$. For more information on semirings, see [8]. Here we only consider commutative ones. Typical examples are the natural numbers $\mathbb{N}$, or the non-negative rationals $\mathbb{Q}_{\geq 0}$, or the reals $\mathbb{R}_{\geq 0}$.

One way to describe semirings categorically is by considering the additive monoid $(S, +, 0)$ as an object of the category **CMon** of commutative monoids, carrying a multiplicative monoid structure $I \xrightarrow{1} S \xleftarrow{\cdot} S \otimes S$ in this category **CMon**. The tensor guarantees that multiplication is a bihomomorphism, and thus distributes over additions.

In the present context of categories with finite products we do not need to use these tensors and can give a direct categorical formulation of such semirings, as a pair of monoids $1 \xrightarrow{0} S \xleftarrow{+} S \times S$ and $1 \xrightarrow{1} S \xleftarrow{\cdot} S \times S$ making the following distributivity diagrams commute.

$$
\begin{tikzcd}
S \times 1 \arrow[r, "\mathrm{id} \times 0"] \arrow[d, "!"'] & S \times S \arrow[d, "\cdot"] \\
1 \arrow[r, "0"] & S
\end{tikzcd}
\qquad
\begin{tikzcd}
(S \times S) \times S \arrow[r, "dbl"] \arrow[d, "+ \times \mathrm{id}"'] & (S \times S) \times (S \times S) \arrow[r, "\cdot \times \cdot"] & S \times S \arrow[d, "+"] \\
S \times S \arrow[rr, "\cdot"] & & S
\end{tikzcd}
$$

where $dbl = \langle \pi_1 \times \mathrm{id}, \pi_2 \times \mathrm{id} \rangle$ is the doubling map that was also used in Lemma 15. With the obvious notion of homomorphism between semirings this yields a category $\mathbf{CSRng}(\mathbf{C})$ of (commutative) semirings in a category $\mathbf{C}$ with finite products.

Associated with a semiring S there is a notion of module over S. It consists of a commutative monoid $(M, 0, +)$ together with a (multiplicative) action $\star \colon S \times M \to M$ that is an additive bihomomorphism, that is, the action preserves the additive structure in each argument separately. We recall that the properties of an action are given categorically by $\star \circ (\cdot \times \mathrm{id}) = \star \circ (\mathrm{id} \times \star) \circ \alpha^{-1} \colon (S \times S) \times M \to M$ and $\star \circ (1 \times \mathrm{id}) = \pi_2 \colon 1 \times M \to M$. The fact that $\star$ is an additive bihomomorphism is expressed by

$$
\begin{tikzcd}
S \times (M \times M) \arrow[r, "dbl'"] \arrow[dd, "\mathrm{id} \times +"'] & (S \times M) \times (S \times M) \arrow[d, "\star \times \star"] & (S \times S) \times M \arrow[l, "dbl"'] \arrow[dd, "+ \times \mathrm{id}"] \\
& M \times M \arrow[d, "+"] & \\
S \times M \arrow[r, "\star"] & M & S \times M \arrow[l, "\star"']
\end{tikzcd}
$$

where dbl' is the obvious duplicator of S. Preservation of zeros is simply $\star \circ (0 \times \mathrm{id}) = 0 \circ \pi_1 \colon 1 \times M \to M$ and $\star \circ (\mathrm{id} \times 0) = 0 \circ \pi_2 \colon S \times 1 \to M$.

We shall assemble such semirings and modules in one category $\mathcal{M}od(\mathbf{C})$ with triples $(S, M, \star)$ as objects, where $\star \colon S \times M \to M$ is an action as above. A morphism $(S_1, M_1, \star_1) \to (S_2, M_2, \star_2)$ consists of a pair of morphisms $f \colon S_1 \to S_2$ and $g \colon M_1 \to M_2$ in $\mathbf{C}$ such that f is a map of semirings, f is a map of monoids, and the actions interact appropriately: $\star_2 \circ (f \times g) = g \circ \star_1$.

5.1 From semirings to monads

To construct an adjunction between semirings and additive commutative monads we start by defining, for each commutative semiring S, the so-called multiset monad on S and show that this monad is both commutative and additive.

Definition 22 *For a semiring S, define a "multiset" functor $M_S\colon \mathbf{Sets} \to \mathbf{Sets}$ on a set X by*

$$M_S(X) \quad = \quad \{\varphi\colon X \to S \mid \mathrm{supp}(\varphi) \text{ is finite}\},$$

where $\mathrm{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}$ is called the support of φ. For a function $f\colon X \to Y$ one defines $M_S(f)\colon M_S(X) \to M_S(Y)$ by:

$$M_S(f)(\varphi)(y) \quad = \quad \textstyle\sum_{x \in f^{-1}(y)} \varphi(x).$$

Such a multiset $\varphi \in M_S(X)$ may be written as formal sum $s_1x_1 + \cdots + s_kx_k$, where $\mathrm{supp}(\varphi) = \{x_1, \ldots, x_k\}$ and $s_i = \varphi(x_i) \in S$ describes the "multiplicity" of the element x_i. In this notation one can write the application of M_S on a map f as $M_S(f)(\sum_i s_ix_i) = \sum_i s_if(x_i)$. Functoriality is then obvious.

Lemma 23 *For each semiring S, the multiset functor M_S forms a commutative and additive monad, with unit and multiplication:*

$$\begin{array}{ccc} X & \xrightarrow{\eta} & M_S(X) \\ x & \longmapsto & 1x \end{array} \qquad\qquad \begin{array}{ccc} M_S(M_S(X)) & \xrightarrow{\mu} & M_S(X) \\ \sum_i s_i\varphi_i & \longmapsto & \lambda x \in X.\, \sum_i s_i\varphi_i(x). \end{array}$$

Proof The verification that M_S with these η and μ indeed forms a monad is left to the reader. We mention that for commutativity and additivity the relevant maps are given by:

$$\begin{array}{ccc} M_S(X) \times M_S(Y) & \xrightarrow{\mathsf{dst}} & M_S(X \times Y) \\ (\varphi, \psi) & \longmapsto & \lambda(x, y).\, \varphi(x) \cdot \psi(y) \end{array} \qquad \begin{array}{ccc} M_S(X + Y) & \xrightarrow{\mathsf{bc}} & M_S(X) \times M_S(Y) \\ \chi & \longmapsto & (\chi \circ \kappa_1, \chi \circ \kappa_2). \end{array}$$

Clearly, bc is an isomorphism, making M_S additive. $\square$

Lemma 24 *The assignment $S \mapsto M_S$ yields a functor $\mathcal{M}\colon$ **CSRng** $\to$ **ACMnd**.*

Proof Every semiring homomorphism $f\colon S \to R$, gives rise to a monad morphism $\mathcal{M}(f)\colon M_S \to M_R$ with components defined by $\mathcal{M}(f)_X(\sum_i s_ix_i) = \sum_i f(s_i)x_i$. It is left to the reader to check that $\mathcal{M}(f)$ is indeed a monad morphism. $\square$

For a semiring S, the category $\mathit{Alg}(M_S)$ of algebras of the multiset monad M_S is (equivalent to) the category $\mathcal{M}od_S(\mathbf{C}) \hookrightarrow \mathcal{M}od(\mathbf{C})$ of modules over S. This is not used here, but just mentioned as background information.

5.2 From monads to semirings

A commutative additive monad T on a category $\mathbf{C}$ gives rise to two commutative monoid structures on $T(1)$, namely the multiplication defined in Lemma 10 and the addition defined in Lemma 17 (considered for $X = 1$). In case the category $\mathbf{C}$ is distributive these two operations turn $T(1)$ into a semiring.

Lemma 25 *Each commutative additive monad T on a distributive category $\mathbf{C}$ with terminal object 1 gives rise to a semiring $\mathcal{E}(T) = T(1)$ in $\mathbf{C}$. The mapping $T \mapsto \mathcal{E}(T)$ yields a functor $\mathbf{ACMnd}(\mathbf{C}) \to \mathbf{CSRng}(\mathbf{C})$.*

Proof For a commutative additive monad T on $\mathbf{C}$, addition on $T(1)$ is given by $T(\nabla) \circ \mathsf{bc}^{-1} \colon T(1) \times T(1) \to T(1)$ with unit $0_{1,1} \colon 1 \to T(1)$ as in Lemma 17, the multiplication is given by $\mu \circ T(\lambda) \circ \mathsf{st} \colon T(1) \times T(1) \to T(1)$ with unit $\eta_1 \colon 1 \to T(1)$ as in Lemma 10.

It was shown in the lemmas just mentioned that both addition and multiplication define a commutative monoid structure on $T(1)$. The following diagram proves distributivity of multiplication over addition.

$$
\begin{tikzcd}
(T(1) \times T(1)) \times T(1) \arrow[r, "\mathsf{bc}^{-1} \times id"] \arrow[d, "dbl"'] & T(1+1) \times T(1) \arrow[r, "T(\nabla) \times id"] \arrow[d, "\mathsf{st}"'] & T(1) \times T(1) \arrow[d, "\mathsf{st}"] \\
(T(1) \times T(1)) \times (T(1) \times T(1)) \arrow[d, "\mathsf{st} \times \mathsf{st}"'] & T((1+1) \times T(1)) \arrow[r, "T(\nabla \times id)"] \arrow[d, "\cong"'] & T(1 \times T(1)) \arrow[ddd, "T(\lambda)"] \\
T(1 \times T(1)) \times T(1 \times T(X)) \arrow[r, "\mathsf{bc}^{-1}"] \arrow[d, "T(\lambda) \times T(\lambda)"'] & T(1 \times T(1) + 1 \times T(1)) \arrow[d, "T(\lambda+\lambda"'] & \\
T^2(1) \times T^2(1) \arrow[r, "\mathsf{bc}^{-1}"] \arrow[dd, "\mu \times \mu"'] & T(T(1) + T(1)) \arrow[d, "T([T(\kappa_1), T(\kappa_2)])"'] \arrow[dr, "T(\nabla)"] & \\
 & T^2(1+1) \arrow[r, "T^2(\nabla)"] \arrow[d, "\mu"'] & T^2(1) \arrow[d, "\mu"] \\
T(1) \times T(1) \arrow[r, "\mathsf{bc}^{-1}"] & T(1+1) \arrow[r, "T(\nabla)"] & T(1)
\end{tikzcd}
$$

Here we rely on Lemma 15 for the commutativity of the upper and lower square on the left.

In a distributive category $0 \cong 0 \times X$, for every object X. In particular $T(0 \times T(1)) \cong T(0) \cong 1$ is final. This is used to obtain commutativity of the upper-left square of the following diagram proving $0 \cdot s = 0$:

$$\begin{tikzcd}
T(1) \arrow[r, "\cong"] \arrow[d, "!"] & T(0) \times T(1) \arrow[r, "T(!) \times id"] \arrow[d, "\mathsf{st}"] & T(1) \times T(1) \arrow[d, "\mathsf{st}"] \\
T(0) \arrow[r, "\cong"] \arrow[dr, "T(!)"] \arrow[ddrr, bend right, "T(!)"'] & T(0 \times T(1)) \arrow[r, "T(! \times id)"] \arrow[d, "T(\lambda)"] & T(1 \times T(1)) \arrow[d, "T(\lambda)"] \\
& T^2(1) \arrow[r, "id"] & T^2(1) \arrow[d, "\mu"] \\
& & T(1)
\end{tikzcd}$$

For a monad morphism $\sigma\colon T \to S$, we define $\mathcal{E}(\sigma) = \sigma_1 \colon T(1) \to S(1)$. By Lemma 5, σ_1 commutes with the multiplicative structure. As $\sigma_1 = T(id) \circ \sigma_1 = \mathcal{A}d((\sigma, id))$, it follows from Lemma 17 that σ_1 also commutes with the additive structure and is therefore a **CSRng**-homomorphism. $\square$

5.3 Adjunction between monads and semirings

The functors defined in the Lemmas 24 and 25, considered on $\mathbf{C} = \mathbf{Sets}$, form an adjunction $\mathcal{M}\colon \mathbf{CSRng} \leftrightarrows \mathbf{ACMnd} \colon \mathcal{E}$. To prove this adjunction we first show that each pair (T, X), where T is a commutative additive monad on a category $\mathbf{C}$ and X an object of $\mathbf{C}$, gives rise to a module on $\mathbf{C}$ as defined at the beginning of this section.

Lemma 26 *Each pair (T, X), where T is a commutative additive monad on a category $\mathbf{C}$ and X is an object of $\mathbf{C}$, gives rise to a module $\mathcal{M}od(T, X) = (T(1), T(X), \star)$. Here $T(1)$ is the commutative semiring defined in Lemma 25 and $T(X)$ is the commutative monoid defined in Lemma 17. The action map is given by $\star = T(\lambda) \circ dst \colon T(1) \times T(X) \to T(X)$. The mapping $(T, X) \mapsto \mathcal{M}od(T, X)$ yields a functor $\mathbf{ACMnd}(\mathbf{C}) \times \mathbf{C} \to \mathcal{M}od(\mathbf{C})$.*

Proof Checking that $\star$ defines an appropriate action requires some work but is essentially straightforward, using the properties from Lemma 15. For a pair of maps $\sigma\colon T \to S$ in $\mathbf{ACMnd}(\mathbf{C})$ and $g\colon X \to Y$ in $\mathbf{C}$, we define a map $\mathcal{M}od(\sigma, g)$ by

$$(T(1), T(X), \star^T) \xrightarrow{(\sigma_1, \sigma_Y \circ T(g))} (S(1), S(Y), \star^S).$$

Note that, by naturality of σ, one has $\sigma_Y \circ T(g) = S(g) \circ \sigma_X$. It easily follows that this defines a $\mathcal{M}od(\mathbf{C})$-map and that the assignment is functorial. $\square$

Lemma 27 *The pair of functors $\mathcal{M}\colon \mathbf{CSRng} \leftrightarrows \mathbf{ACMnd} \colon \mathcal{E}$ forms an adjunction, $\mathcal{M} \dashv \mathcal{E}$.*

Proof For a semiring S and a commutative additive monad T on **Sets** there are (natural) bijective correspondences:

$$\frac{M_S = \mathcal{M}(S) \xrightarrow{\ \sigma\ } T \qquad \text{in } \mathbf{CAMnd}}{S \xrightarrow[f]{} \mathcal{E}(T) = T(1) \ \text{ in } \mathbf{CSRng}}$$

Given $\sigma\colon M_S \to T$, one defines a semiring map $\overline{\sigma}\colon S \to T(1)$ by

$$\overline{\sigma} = \Big(S \xrightarrow{\lambda s.(\lambda x.s)} M_S(1) \xrightarrow{\sigma_1} T(1)\Big).$$

Conversely, given a semiring map $f\colon S \to T(1)$, one gets a monad map $\overline{f}\colon M_S \to T$ with components:

$$M_S(X) \xrightarrow{\overline{f}_X} T(X) \quad \text{given by} \quad \textstyle\sum_i s_i x_i \longmapsto \sum_i f(s_i) \star \eta(x_i),$$

where the sum on the right hand side is the addition in $T(X)$ defined in Lemma 17 and $\star$ is the action of $T(1)$ on $T(X)$ defined in Lemma 26.

Showing that $\overline{f}$ is indeed a monad morphism requires some work. In doing so one may rely on the properties of the action and on Lemma 17. The details are left to the reader. □

Notice that the counit of the above adjunction $\mathcal{E}\mathcal{M}(S) = M_S(1) \to S$ is an isomorphism. Hence this adjunction is in fact a reflection.

6 Semirings and Lawvere theories

In this section we will extend the adjunction between commutative monoids and symmetric monoidal Lawvere theories depicted in Figure 3 to an adjunction between commutative semirings and symmetrical monoidal Lawvere theories with biproducts, *i.e.* between the categories **CSRng** and **SMBLaw**.

6.1 From semirings to Lawvere theories

Composing the multiset functor $\mathcal{M}\colon \mathbf{CSRng} \to \mathbf{ACMnd}$ from the previous section with the finitary Kleisli construction $\mathcal{K}\ell_{\mathbb{N}}$ yields a functor from **CSRng** to **SMBLaw**. This functor may be described in an alternative (isomorphic) way by assigning to every semiring S the Lawvere theory of matrices over S, which is defined as follows.

Definition 28 *For a semiring S, the Lawvere theory $\mathcal{M}at(S)$ of* matrices over S *has, for $n, m \in \mathbb{N}$ morphisms (in* **Sets***) $n \times m \to S$, i.e. $n \times m$ matrices over S, as morphisms $n \to m$. The identity $\mathrm{id}_n\colon n \to n$ is given by the identity matrix:*

$$\mathrm{id}_n(i,j) \quad = \quad \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

The composition of $g\colon n \to m$ and $h\colon m \to p$ is given by matrix multiplication:

$$(h \circ g)(i,k) = \textstyle\sum_j g(i,j) \cdot h(j,k).$$

The coprojections $\kappa_1\colon n \to n+m$ and $\kappa_2\colon m \to n+m$ are given by

$$\kappa_1(i,j) = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{otherwise.} \end{cases} \qquad \kappa_2(i,j) = \begin{cases} 1 & \textit{if } j \geq n \textit{ and } j-n=i \\ 0 & \textit{otherwise.} \end{cases}$$

Lemma 29 *The assignment $S \mapsto \mathcal{M}at(S)$ yields a functor* $\mathbf{CSRng} \to \mathbf{Law}$. *The two functors $\mathcal{M}at\mathcal{E}$ and $\mathcal{K}\ell_{\mathbb{N}}\colon$ $\mathbf{ACMnd} \to \mathbf{Law}$ are naturally isomorphic.*

Proof A map of semirings $f\colon S \to R$ gives rise to a functor $\mathcal{M}at(f)\colon \mathcal{M}at(S) \to \mathcal{M}at(R)$ which is the identity on objects and which acts on morphisms by post-composition: $h\colon n \times m \to S$ in $\mathcal{M}at(S)$ is mapped to $f \circ h\colon n \times m \to T$ in $\mathcal{M}at(T)$. It is easily checked that $\mathcal{M}at(f)$ is a morphism of Lawvere theories and that the assigment is functorial.

To prove the second claim we define two natural transformations. First we define $\xi\colon \mathcal{M}at\mathcal{E} \to \mathcal{K}\ell_{\mathbb{N}}$ with components $\xi_T\colon \mathcal{M}at(T(1)) \to \mathcal{K}\ell_{\mathbb{N}}(T)$ that are the identity on objects and send a morphism $h\colon n \times m \to T(1)$ in $\mathcal{M}at(T(1))$ to the morphism $\xi_T(h)$ in $\mathcal{K}\ell_{\mathbb{N}}(T)$ given by

$$\xi_T(h) = \Big(n \xrightarrow{\langle h(-,j)\rangle_{j \in m}} T(1)^m \xrightarrow{\mathsf{bc}_m^{-1}} T(m)\Big),$$

where bc_m^{-1} is the inverse of the generalised bicartesian map

$$\mathsf{bc}_m = \Big(T(m) \;=\; T(\textstyle\coprod_m 1) \longrightarrow T(1)^m\Big).$$

And secondly, in the reverse direction, we define $\theta\colon \mathcal{K}\ell_{\mathbb{N}} \to \mathcal{M}at\mathcal{E}$ with components $\theta_T\colon \mathcal{K}\ell_{\mathbb{N}}(T) \to \mathcal{M}at(T(1))$ that are the identity on objects and send a morphism $g\colon n \to T(m)$ in $\mathcal{K}\ell_{\mathbb{N}}(T)$ to the morphism $\theta_T(g)\colon n \times m \to T(1)$ in $\mathcal{M}at(T(1))$ given by

$$\theta_T(g)(i,j) \quad = \quad (\pi_j \circ \mathsf{bc}_m \circ g)(i). \tag{9}$$

It requires some work, but is relatively straightforward to check that the components ξ_T and θ_T are $\mathbf{Law}$-maps. To prove preservation of the composition by ξ_T and θ_T one uses the definition of addition and multiplication in $T(1)$ and (generalisations of) the properties of the map bc listed in Lemma 15. A short computation shows that the functors are each other's inverses. The naturality of both ξ and θ follows from (a generalisation of) point (i) of Lemma 15. $\square$

The pair of functors $\mathcal{M}\colon$ $\mathbf{CSRng} \leftrightarrows \mathbf{ACMnd}$ $\colon \mathcal{E}$ forms a reflection, $\mathcal{E}\mathcal{M} \cong \mathrm{id}$ (Lemma 27). Combining this with the previous proposition, it follows that also the functors $\mathcal{M}at, \mathcal{K}\ell_{\mathbb{N}}\mathcal{M}\colon$ $\mathbf{CSRng} \to \mathbf{Law}$ are naturally isomorphic. Hence, the functor $\mathcal{M}at\colon \mathbf{CSRng} \to \mathbf{Law}$ may be viewed as a functor from commutative semirings to symmetric monoidal Lawvere theories with biproducts. For a commutative semiring S the projection maps $\pi_1\colon n{+}m \to n$ and $\pi_2\colon n{+}m \to m$ in $\mathcal{M}at(S)$ are defined in a similar way as the coprojection maps from Definition 28. For a pair of maps $g\colon m \to p$, $h\colon n \to q$, the tensor product $g \otimes h\colon (m \times n) \to (p \times q)$ is the map $g \otimes h\colon (m \times n) \times (p \times q) \to S$ defined as

$$(g \otimes h)((i_0, i_1), (j_0, j_1)) \quad = \quad g(i_0, j_0) \cdot h(i_1, j_1),$$

where $\cdot$ is the multiplication from S.

6.2 From Lawvere theories to semirings

In Section 3.1, just after Lemma 11, we have already seen that the homset $\mathbf{L}(1,1)$ of a Lawvere theory $\mathbf{L} \in \mathbf{SMLaw}$ is a commutative monoid, with multiplication given by composition of endomaps on 1. In case $\mathbf{L}$ also has biproducts we have, by (6), an addition on this homset, which is preserved by composition. Combining those two monoid structures yields a semiring structure on $\mathbf{L}(1,1)$. This is standard, see *e.g.* [1, 15, 11]. The assignment of the semiring $\mathbf{L}(1,1)$ to a Lawvere theory $\mathbf{L} \in \mathbf{SMBLaw}$ is functorial and we denote this functor, as in Section 3.1, by $\mathcal{H}\colon \mathbf{SMBLaw} \to \mathbf{CSRng}$.

6.3 Adjunction between semirings and Lawvere theories

Our main result is the adjunction on the right in the triangle of adjunctions for semirings, see Figure 4.

Lemma 30 *The pair of functors $\mathcal{M}at\colon \mathbf{CSRng} \rightleftarrows \mathbf{SMBLaw}\colon \mathcal{H}$, forms an adjunction $\mathcal{M}at \dashv \mathcal{H}$.*

Proof For $S \in \mathbf{CSRng}$ and $\mathbf{L} \in \mathbf{SMBLaw}$ there are (natural) bijective correspondences:

$$\begin{array}{cl} \mathcal{M}at(S) \xrightarrow{F} \mathbf{L} & \text{in } \mathbf{SMBLaw} \\ \hline\hline S \xrightarrow[f]{} \mathcal{H}(\mathbf{L}) & \text{in } \mathbf{CSRng} \end{array}$$

Given F one defines a semiring map $\overline{F}\colon S \to \mathcal{H}(\mathbf{L}) = \mathbf{L}(1,1)$ by

$$s \mapsto F(1 \times 1 \xrightarrow{\lambda x.\, s} S).$$

Note that $1 \times 1 \xrightarrow{\lambda x.\, s} S$ is an endomap on 1 in $\mathcal{M}at(S)$ which is mapped by F to an element of $\mathbf{L}(1,1)$.

Conversely, given f one defines a $\mathbf{SMBLaw}$-map $\overline{f}\colon \mathcal{M}at(S) \to \mathbf{L}$ which sends a morphism $h\colon n \to m$ in $\mathcal{M}at(S)$, *i.e.* $h\colon n \times m \to S$ in $\mathbf{Sets}$, to the following morphism $n \to m$ in $\mathbf{L}$, forming an n-cotuple of m-tuples

$$\overline{f}(h) = \left(n \xrightarrow{\left[\langle f(h(i,j))\rangle_{j<m}\right]_{i<n}} m\right).$$

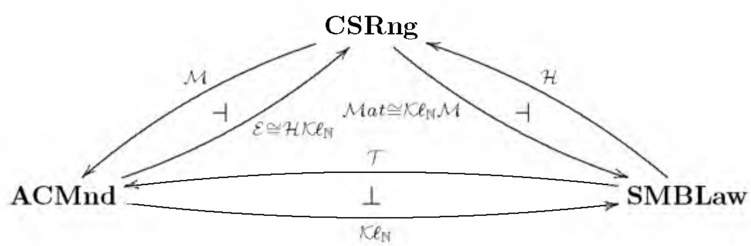

Figure 4: Triangle of adjunctions starting from commutative semirings, with commutative additive monads, and symmetric monoidal Lawvere theories with biproducts.

It is readily checked that $\overline{F}\colon S \to \mathbf{L}(1,1)$ is a map of semirings. To show that $\overline{f}\colon \mathcal{M}at(S) \to \mathbf{L}$ is a functor one has to use the definition of the semiring structure on $\mathbf{L}(1,1)$ and the properties of the biproduct on $\mathbf{L}$. One easily verifies that $\overline{f}$ preserves the biproduct. To show that it also preserves the monoidal structure one has to use that, for $s,t \in \mathbf{L}(1,1)$, $s \otimes t = t \circ s\,(= s \circ t)$. $\square$

The results of Section 5 and 6 are summarized in Figure 4.

7 Semirings with involutions

In this final section we enrich our approach with involutions. Actually, such involutions could have been introduced for monoids already. We have not done so for practical reasons: involutions on semirings give the most powerful results, combining daggers on categories with both symmetric monoidal and biproduct structure.

An involutive semiring (in **Sets**) is a semiring $(S,+,0,\cdot,1)$ together with a unary operation $*$ that preserves the addition and multiplication, *i.e.* $(s+t)^* = s^* + t^*$ and $0^* = 0$, and $(s \cdot t)^* = s^* \cdot t^*$ and $1^* = 1$, and is involutive, *i.e.* $(s^*)^* = s$. The complex numbers with conjugation form an example. We denote the category of involutive semirings, with homomorphisms that preserve all structure, by **ICSRng**.

The adjunction $\mathcal{M}\colon \mathbf{CSRng} \leftrightarrows \mathbf{ACMnd}\colon \mathcal{E}$ considered in Lemma 27 may be restricted to an adjunction between involutive semirings and so-called involutive commutative additive monads (on **Sets**), which are commutative additive monads T together with a monad morphism $\zeta\colon T \to T$ satisfying $\zeta \circ \zeta = id$. We call ζ an involution on T, just as in the semiring setting. A morphism between such monads (T,ζ) and (T',ζ'), is a monad morphism $\sigma\colon T \to T'$ preserving the involution, *i.e.* satisfying $\sigma \circ \zeta = \zeta' \circ \sigma$. We denote the category of involutive commutative additive monads by **IACMnd**.

Lemma 31 *The functors $\mathcal{M}\colon \mathbf{CSRng} \leftrightarrows \mathbf{ACMnd}\colon \mathcal{E}$ from Lemma 24 and Lemma 25 restrict to a pair of functors $\mathcal{M}\colon \mathbf{ICSRng} \leftrightarrows \mathbf{IACMnd}\colon \mathcal{E}$. The restricted functors form an adjunction $\mathcal{M} \vdash \mathcal{E}$.*

Proof Given a semiring S with involution $*$, we may define an involution ζ on the multiset monad $\mathcal{M}(S) = M_S$ with components

$$\zeta_X : M_S(X) \to M_S(X), \qquad \sum_i s_i x_i \mapsto \sum_i s_i^* x_i.$$

Conversely, for an involutive monad (T, ζ), the map ζ_1 gives an involution on the semiring $\mathcal{E}(T) = T(1)$.

A simple computation shows that the unit and the counit of the adjunction $\mathcal{M}\colon \mathbf{CSRng} \leftrightarrows \mathbf{ACMnd}\colon \mathcal{E}$ from Lemma 27 preserve the involution (on semirings and on monads respectively). Hence the restricted functors again form an adjunction. $\square$

The adjunction $\mathcal{M}at\colon \mathbf{CSRng} \rightleftarrows \mathbf{SMBLaw}\colon \mathcal{H}$ from Lemma 30 may also be restricted to involutive semirings. To do so, we have to consider dagger categories. A dagger category is a category $\mathbf{C}$ with a functor $\dagger\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{C}$ that is the identity on objects and satisfies, for all morphisms $f\colon X \to Y$, $(f^\dagger)^\dagger = f$. The functor $\dagger$ is called a dagger on $\mathbf{C}$. Combining this dagger with the categorical structure we studied in Section 6 yields a so-called dagger symmetric monoidal category with dagger biproducts, that is, a category $\mathbf{C}$ with a symmetric monoidal structure $(\otimes, I)$, a biproduct structure $(\oplus, 0)$ and a dagger $\dagger$, such that, for all morphisms f and g, $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$, all the coherence isomorphisms α, ρ and γ are dagger isomorphisms and, with respect to the biproduct structure, $\kappa_i = \pi_i^\dagger$, where a dagger isomorphism is an isomorphism f satisfying $f^{-1} = f^\dagger$. Further details may be found in [1, 2, 11].

We will denote the category of dagger symmetric monoidal Lawvere theories with dagger biproducts such that the monoidal structure distributes over the biproduct structure by **DSMBLaw**. Morphisms in **DSMBLaw** are maps in **SMBLaw** that (strictly) commute with the daggers.

Lemma 32 *The functors* $\mathcal{M}at\colon \mathbf{CSRng} \rightleftarrows \mathbf{SMBLaw}\colon \mathcal{H}$ *defined in Section 6 restrict to a pair of functors* $\mathcal{M}at\colon \mathbf{ICSRng} \rightleftarrows \mathbf{DSMBLaw}\colon \mathcal{H}$*. The restricted functors form an adjunction,* $\mathcal{M}at \dashv \mathcal{H}$*.*

Proof For an involutive semiring S, we may define a dagger on the Lawvere theory $\mathcal{M}at(S)$ by assigning to a morphism $f\colon n \to m$ in $\mathcal{M}at(S)$ the morphism $f^\dagger\colon m \to n$ given by

$$f^\dagger(i, j) \quad = \quad f(j, i)^*. \tag{10}$$

Some short and straightforward computations show that the functor $\dagger$ is indeed a dagger on $\mathcal{M}at(S)$, which interacts appropriately with the monoidal and biproduct structure.

For a dagger symmetric monoidal Lawvere theory $\mathbf{L}$ with dagger biproduct, it easily follows from the properties of the dagger that this functor induces an involution on the semiring $\mathcal{H}(\mathbf{L}) = \mathbf{L}(1, 1)$, namely via $s \mapsto s^\dagger$.

The unit and the counit of the adjunction $\mathcal{M}at\colon \mathbf{CSRng} \leftrightarrows \mathbf{SMBLaw}$ from Lemma 30 preserve the involution and the dagger respectively. Hence, also the restricted functors form an adjunction. $\square$

To complete our last triangle of adjunctions, recall that, for the Lawvere theory associated with a (involutive commutative additive) monad T, $\mathcal{K}\ell_{\mathbb{N}}(T) \cong \mathcal{M}at(\mathcal{E}(T))$, see Proposition 29. Hence, using the previous two lemmas, the finitary Kleisli construction restricts to a functor $\mathcal{K}\ell_{\mathbb{N}} \colon \mathbf{IACMnd} \to \mathbf{DSMBLaw}$. For the other direction we use the following result.

Lemma 33 *The functor $\mathcal{T} \colon \mathbf{SMBLaw} \to \mathbf{ACMnd}$ from Lemma 12 restricts to $\mathbf{DSMBLaw} \to \mathbf{IACMnd}$, and yields a left adjoint to $\mathcal{K}\ell_{\mathbb{N}} \colon \mathbf{IACMnd} \to \mathbf{DSMBLaw}$.*

Proof To start, for a Lawvere theory $\mathbf{L} \in \mathbf{DSMBLaw}$ with dagger $\dagger$ we have to define an involution $\zeta \colon T_{\mathbf{L}} \to T_{\mathbf{L}}$. For a set X this involves a map

$$\begin{array}{rcl} T_{\mathbf{L}}(X) = \big(\coprod_{i\in\mathbb{N}} \mathbf{L}(1,i) \times X^i\big)/\sim & \xrightarrow{\zeta_X} & \big(\coprod_{i\in\mathbb{N}} \mathbf{L}(1,i) \times X^i\big)/\sim \;= T_{\mathbf{L}}(X) \\ {[\kappa_i(g,v)]} & \longmapsto & [\kappa_i(\langle g_0^\dagger, \ldots, g_{i-1}^\dagger\rangle, v)], \end{array}$$

where $g \colon 1 \to i$ is written as $g = \langle g_0, \ldots, g_{i-1}\rangle$ using that $i = 1 + \cdots + 1$ is not only a sum, but also a product. Clearly, ζ is natural, and satisfies $\zeta \circ \zeta = \mathrm{id}$. This ζ is also a map of monads; commutatution with multiplication μ requires commutativity of composition in the homset $\mathbf{L}(1,1)$.

The unit of the adjunction $\eta \colon \mathbf{L} \to \mathcal{K}\ell_{\mathbb{N}}(T_{\mathbf{L}}) \cong \mathcal{M}at(T_{\mathbf{L}}(1))$ commutes with daggers, since for $f \colon n \to m$ in $\mathbf{L}$ we get $\eta(f)^\dagger = \eta(f^\dagger)$ via the following argument in $\mathcal{M}at(T_{\mathbf{L}}(1))$. For $i < n$ and $j < m$,

$$\begin{array}{rcll} \eta(f)^\dagger(i,j) & = & \eta(f)(i,j)^* & \text{by (10)} \\ & = & \pi_j \mathsf{bc}_m(\kappa_m(f \circ \kappa_i, \mathrm{id}_m))^* & \text{by (9)} \\ & = & \kappa_1(\pi_j \circ f \circ \kappa_i, \mathrm{id}_1)^* & \text{by definition of } \mathsf{bc}\text{, see (8)} \\ & = & \kappa_1(\pi_j \circ f \circ \kappa_i, \mathrm{id})^\dagger & \text{since } (-)^* = (-)^\dagger \text{ on } T_{\mathbf{L}}(1) \\ & = & \kappa_1((\pi_j \circ f \circ \kappa_i)^\dagger, \mathrm{id}) & \\ & = & \kappa_1(\kappa_i^\dagger \circ f^\dagger \circ \pi_j^\dagger, \mathrm{id}) & \\ & = & \kappa_1(\pi_i \circ f^\dagger \circ \kappa_j, \mathrm{id}) & \\ & = & \eta(f^\dagger)(i,j). & \end{array}$$

□

In the definition of the involution ζ on the monad $T_{\mathbf{L}}$ in this proof we have used that $+$ is a (bi)product in the Lawvere theory $\mathbf{L}$, namely when we decompose the map $g \colon 1 \to i$ into its components $\pi_a \circ g \colon 1 \to 1$ for $a < i$. We could have avoided this biproduct structure by first taking the dagger $g^\dagger \colon i \to 1$, and then precomposing with coprojections $g^\dagger \circ \kappa_a \colon 1 \to 1$. Again applying daggers, cotupling, and taking the dagger one gets the same result. This is relevant if one wishes to consider involutions/daggers in the context of monoids, where products in the corresponding Lawvere theories are lacking.

By combining the previous three lemmas we obtain another triangle of adjunctions in Figure 5. This concludes our survey of the interrelatedness of scalars, monads and categories.

$$
\begin{tikzcd}
 & \mathbf{ICSRng} \arrow[dl, bend right, "\mathcal{M}"'] \arrow[dr, bend left=15, "\mathcal{M}at\cong\mathcal{K}\ell_{\mathbb{N}}\mathcal{M}"'] & \\
\mathbf{IACMnd} \arrow[ur, bend right=15, "\mathcal{E}\cong\mathcal{H}\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[ur, phantom, "\dashv"] \arrow[rr, bend right, "\mathcal{K}\ell_{\mathbb{N}}"'] \arrow[rr, phantom, "\perp"] & & \mathbf{DSMBLaw} \arrow[ul, bend right, "\mathcal{H}"'] \arrow[ul, phantom, "\dashv"] \arrow[ll, bend right, "\mathcal{T}"']
\end{tikzcd}
$$

Figure 5: Triangle of adjunctions starting from involutive commutative semirings, with involutive commutative additive monads, and dagger symmetric monoidal Lawvere theories with dagger biproducts.

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