THE ADHM CONSTRUCTION OF INSTANTONS ON NONCOMMUTATIVE SPACES

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Abstract. We present an account of the ADHM construction of instantons on Euclidean space-time $\mathbb{R}^4$ from the point of view of noncommutative geometry. We recall the main ingredients of the classical construction in a coordinate algebra format, which we then deform using a cocycle twisting procedure to obtain a method for constructing families of instantons on noncommutative space-time, parameterised by solutions to an appropriate set of ADHM equations. We illustrate the noncommutative construction in two special cases: the Moyal-Groenewold plane $\mathbb{R}^4_\hbar$ and the Connes-Landi plane $\mathbb{R}^4_\theta$.

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1. Introduction

There has been a great deal of interest in recent years in the construction of instanton gauge fields on space-times whose algebra of coordinate functions is noncommutative. In classical geometry, an instanton is a connection with anti-self-dual curvature on a smooth vector bundle over a four-dimensional manifold. The moduli space of instantons on a classical four-manifold is an important invariant of its differential structure [12] and it is only natural to try to generalise this idea to the study of the differential geometry of noncommutative four-manifolds.

In this article we study the construction of SU(2) instantons on the Euclidean four-plane $\mathbb{R}^4$ and its various noncommutative generalisations. The problem of constructing instantons on classical space-time was solved by the ADHM method of [2] and consequently it is known that the moduli space of (framed) SU(2) instantons on $\mathbb{R}^4$ with topological charge $k \in \mathbb{Z}$ is a manifold of dimension $8k-3$. In what follows we review the ADHM construction of instantons on classical $\mathbb{R}^4$ and its extension to noncommutative geometry. In particular, we study the construction of instantons on two explicit examples of noncommutative Euclidean space-time: the Moyal-Groenewold plane $\mathbb{R}^4_\hbar$, whose algebra of coordinate functions has commutation relations of the Heisenberg form, and the Connes-Landi plane $\mathbb{R}^4_\theta$ which arises as a localisation of the noncommutative four-sphere S^4_θ constructed in [9] (*cf.* also [10]).

In order to deform the ADHM construction we adopt the technique of 'functorial cocycle twisting', a very general method which can be used in particular to deform the coordinate algebra of any space carrying an action of a locally compact Abelian group. Both of the above examples of noncommutative space-times are obtained in this way as deformations of classical Euclidean space-time: the Moyal plane $\mathbb{R}^4_\hbar$ as a twist along a group of translational symmetries, the Connes-Landi plane $\mathbb{R}^4_\theta$ as a twist along a group of rotational symmetries.

Crucially, the twisting technique does not just deform space-time alone: its functorial nature means that it simultaneously deforms any and all constructions which are covariant under the chosen group of symmetries. In particular, the parameter spaces which occur in the ADHM construction also carry canonical actions of the relevant symmetry groups, whence their coordinate algebras are also deformed by the quantisation procedure. In this way, we obtain a method for constructing families of instantons which are parameterised by noncommutative spaces.

As natural as this may seem, it immediately leads to the conceptual problem of how to interpret these spaces of noncommutative parameters. Indeed, our quest in each case is for a moduli space of instantons, which is necessarily modeled on the space of *all* connections on a given vector bundle over space-time [3]. Even for noncommutative space-times, the set of such connections is an affine space and is therefore commutative [7]. To solve this problem, we adopt the strategy of [5] and incorporate into the ADHM construction the 'internal gauge symmetries' of noncommutative space-time [8], in order to 'gauge away' the noncommutativity and arrive at a classical space of parameters.

The paper is organised as follows. The remainder of §1 is dedicated to a brief review of the algebraic structures that we shall need, including in particular the notions of Hopf algebras and their comodules, together with the cocycle twisting construction itself.

Following this, in §2 and §3 we review the differential geometry of instantons from the point of view of coordinate algebras. We recall how to generalise these structures to incorporate the notions of noncommutative families of instantons and their gauge theory. In §4 we sketch the ADHM construction of instanton from the point of view of coordinate algebras, stressing how the method is covariant under the group of isometries of Euclidean space-time. It is precisely this covariance that we use in later sections to deform the ADHM construction by cocycle twisting.

This coordinate-algebraic version of the ADHM construction has in fact already been studied in some detail [4, 5]. In this sense, the first four sections of the present article consist mainly of review material. However, those earlier works studied the geometry of the ADHM construction from a somewhat abstract categorical point of view: in the present exposition, our goal is to give an explanation of the ADHM method in which we keep things as concrete as we are able. Moreover, the focus of these papers was on the construction of instantons on the Euclidean four-sphere S^4. In what follows we show how to adapt this technique to construct intantons on the local coordinate chart $\mathbb{R}^4$.

We illustrate the deformation procedure by giving the noncommutative ADHM construction in two special cases. In §5 we look at how the method behaves under quantisation to give an ADHM construction of instantons on the Moyal plane $\mathbb{R}^4_\hbar$. On the other hand, §6 addresses the issue of deforming the ADHM method to obtain a construction of instantons on the Connes-Landi plane $\mathbb{R}^4_\theta$. It is these latter two sections which contain the majority of our new results.

Indeed, an algorithm for the construction of instantons on the Moyal plane $\mathbb{R}^4_\hbar$ has been known for some time [21]. However, a good understanding of the noncommutative-geometric origins of this construction has so far been lacking and our goal is to shed some light upon this subject (although it is worth pointing out that a different approach to the twistor theory of $\mathbb{R}^4_\hbar$, from the point of view of noncommutative algebraic geometry, was carried out in [14]). Using the noncommutative twistor theory developed in [6], we give an explicit construction of families of instantons on Moyal space-time, from which we recover the well-known noncommutative ADHM equations of Nekrasov and Schwarz.

On the other hand, the noncommutative geometry of instantons and the ADHM construction on the Connes-Landi plane $\mathbb{R}^4_\theta$ was investigated in [15, 16, 4, 5]. As already mentioned, however, this abstract geometric characterisation of the instanton construction is in need of a more concrete description. In the present paper we derive an explicit set of ADHM equations whose solutions parameterise instantons on the Connes-Landi plane (although we do not claim at this stage that *all* such instantons arise in this way).

We stress that our intention throughout the following is to present the various geometrical aspects of the ADHM construction in concrete terms. This means that, throughout the paper, we work purely at the algebraic level, *i.e.* with algebras of coordinate functions on all relevant spaces. A more detailed approach, which in particular addresses all of the analytic aspects of the construction, will be presented elsewhere.

1.1. **Hopf algebraic structures.** Let H be a unital Hopf algebra. We denote its structure maps by

$$\Delta : H \to H \otimes H, \qquad \epsilon : H \to \mathbb{C}, \qquad S : H \to H$$

for the coproduct, counit and antipode, respectively. We use Sweedler notation for the coproduct, $\Delta h = h_{(1)} \otimes h_{(2)}$, as well as $(\Delta \otimes \mathrm{id}) \circ \Delta h = (\mathrm{id} \otimes \Delta) \circ \Delta h = h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$ and so on, with summation inferred.

We say that H is *coquasitriangular* if it is equipped with a convolution-invertible Hopf bicharacter $\mathcal{R} : H \otimes H \to \mathbb{C}$ obeying

$$g_{(1)}h_{(1)}\mathcal{R}(h_{(2)}, g_{(2)}) = \mathcal{R}(h_{(1)}, g_{(1)})h_{(2)}g_{(2)} \tag{1.1}$$

for all $h, g \in H$. Convolution-invertibility means that there exists a map $\mathcal{R}^{-1} : H \otimes H \to \mathbb{C}$ such that

$$\mathcal{R}(h_{(1)}, g_{(1)})\mathcal{R}^{-1}(h_{(2)}, g_{(2)}) = \mathcal{R}^{-1}(h_{(1)}, g_{(1)})\mathcal{R}(h_{(2)}, g_{(2)}) = \epsilon(g)\epsilon(h) \tag{1.2}$$

for all $g, h \in H$. Being a Hopf bicharacter means that

$$\mathcal{R}(fg, h) = \mathcal{R}(f, h_{(1)})\mathcal{R}(g, h_{(2)}), \qquad \mathcal{R}(f, gh) = \mathcal{R}(f_{(1)}, h)\mathcal{R}(f_{(2)}, g), \tag{1.3}$$

for all $f, g, h \in H$. If $\mathcal{R}$ also has the property that

$$\mathcal{R}(h_{(1)}, g_{(1)})\mathcal{R}(g_{(2)}, h_{(2)}) = \epsilon(g)\epsilon(h)$$

for all $g, h \in H$, then we say that H is *cotriangular*.

A vector space V is said to be a *left H-comodule* if it is equipped with a linear map $\Delta_V : V \to H \otimes V$ such that

$$(\Delta_V \otimes \mathrm{id}) \circ \Delta_V = (\Delta \otimes \mathrm{id}) \circ \Delta_V, \qquad (\epsilon \otimes \mathrm{id}) \circ \Delta_V = \mathrm{id}.$$

We shall often use the Sweedler notation $\Delta_V(v) = v^{(-1)} \otimes v^{(0)}$ for $v \in V$, again with summation inferred. If V, W are left H-comodules, a linear transformation $\sigma : V \to W$ is said to be a *left H-comodule map* if it satisfies

$$\Delta_W \circ \sigma = (\mathrm{id} \otimes \sigma) \circ \Delta_V.$$

Given a pair V, W of left H-comodules, the vector space $V \otimes W$ is also a left H-comodule when equipped with the tensor product H-coaction

$$\Delta_{V\otimes W}(v \otimes w) = v^{(-1)}w^{(-1)} \otimes (v^{(0)} \otimes w^{(0)}) \tag{1.4}$$

for each $v \in V$, $w \in W$. An algebra A is said to be a *left H-comodule algebra* if it is a left H-comodule equipped with a product $m : A \otimes A \to A$ which is an H-comodule map, meaning in this case that

$$(\mathrm{id} \otimes m) \circ \Delta_{A\otimes A} = \Delta_A \circ m.$$

Dually, a vector space V is said to be a *left H-module* if there is a linear map $\triangleright : H \otimes V \to V$, denoted $h \otimes v \mapsto h \triangleright v$, such that

$$h \triangleright (g \triangleright v) = (hg) \triangleright v, \qquad 1 \triangleright v = v$$

for all $v \in V$. An algebra A is said to be a *left H-module algebra* if it a left H-module equipped with a product $m : A \otimes A$ which obeys

$$h \triangleright (ab) = (h_{(1)} \triangleright a)(h_{(2)} \triangleright b) \qquad \text{for all} \quad a, b \in A.$$

In the special case where H is coquasitriangular, every left H-comodule V is also a left H-module when equipped with the canonical left H-action

$$\triangleright : H \otimes V \to V, \qquad h \triangleright v = \mathcal{R}(v^{(-1)}, h)v^{(0)} \tag{1.5}$$

for each $h \in H$, $v \in V$. The action (1.5) does not commute with the H-coaction on V; rather it obeys the 'crossed module' condition

$$h_{(1)}v^{(-1)} \otimes h_{(2)} \triangleright v^{(0)} = (h_{(1)} \triangleright v)^{(-1)}h_{(2)} \otimes (h_{(1)} \triangleright v)^{(0)} \tag{1.6}$$

for each $h \in H$ and $v \in V$. In particular, if A is a left H-comodule algebra then it is also a left H-module algebra when equipped with the canonical action (1.5).

To each left H-module algebra A there is an associated *smash product algebra* $A \rtimes H$, which is nothing other than the vector space $A \otimes H$ equipped with the product

$$(a \otimes h)(b \otimes g) = a(h_{(1)} \triangleright b) \otimes h_{(2)}g \tag{1.7}$$

for each $a, b \in A$, $h, g \in H$. The main example of this construction relevant to the present paper is the following.

Example 1.1. Let G be a locally compact Abelian Lie group with Pontryagin dual group $\widehat{G}$ and let $\gamma : \widehat{G} \to \mathrm{Aut}\, A$ be an action of $\widehat{G}$ on a unital $*$-algebra A by $*$-automorphisms. Then associated to this action there is the crossed product algebra $A \rtimes_\gamma \widehat{G}$. On the other hand, the $\widehat{G}$-action gives A the structure of a $C_0(G)$-module algebra and the Fourier transform on $\widehat{G}$ gives an identification of $A \rtimes_\gamma \widehat{G}$ with the smash product $A \rtimes C_0(G)$.

In this paper, our strategy is to keep things as explicit as possible, avoiding the technical details of analytic arguments and working purely at the algebraic level. For this reason, instead of using the function algebra $C_0(G)$, we prefer to work with the bialgebra $\mathcal{A}[G]$ of representative functions on G equipped with pointwise multiplication. In this setting, we can still make sense of Example 1.1: although the algebras $A \rtimes_\gamma \widehat{G}$ and $A \rtimes C_0(G)$ are defined analytically using completions of appropriate function algebras, we think of the smash product $A \rtimes \mathcal{A}[G]$ as an 'algebraic version' of the crossed product algebra $A \rtimes_\gamma \widehat{G}$.

This also explains our use throughout the paper of coactions of Hopf algebras in place of group actions: the construction of crossed products by group actions is not defined at the algebraic level, whence we need to replace it by the smash product algebra instead.

1.2. **Quantisation by cocycle twist.** Following [19], in this section we give a brief review of the deformation procedure that we shall use later in the paper to 'quantise' the ADHM construction of instantons.

Let H be a unital Hopf algebra whose antipode we assume to be invertible. A *two-cocycle* on H is a linear map $F : H \otimes H \to \mathbb{C}$ which is unital, convolution-invertible in the sense of Eq. (1.2) and obeys the condition $\partial F = 1$, *i.e.*

$$F(g_{(1)}, f_{(1)})F(h_{(1)}, g_{(1)}f_{(2)})F^{-1}(h_{(2)}g_{(3)}, f_{(3)})F^{-1}(h_{(3)}, g_{(4)}) = \epsilon(f)\epsilon(h)\epsilon(g) \tag{1.8}$$

for all $f, g, h \in H$. Given such an F, there is a twisted Hopf algebra H_F with the same coalgebra structure as H, but with modified product $\bullet_F$ and antipode S_F given respectively by

$$h \bullet_F g = F(h_{(1)}, g_{(1)})h_{(2)}g_{(2)}F^{-1}(h_{(3)}, g_{(3)}), \tag{1.9}$$

$$S_F(h) := U(h_{(1)})S(h_{(2)})U^{-1}(h_{(3)}), \tag{1.10}$$

for each $h, g \in H_F$, where on the right-hand sides we use the product and antipode of H and define $U(h) := F(h_{(1)}, Sh_{(2)})$. The cocycle condition (1.8) is sufficient to ensure that the product (1.9) is associative. In the case where H is a Hopf $*$-algebra we also need to impose upon the cocycle F the reality condition

$$\overline{F(h,g)} = F((S^2 g)^*, (S^2 h)^*). \tag{1.11}$$

In this situation the twisted Hopf algebra H_F acquires a deformed $*$-structure

$$h^{*_F} := \overline{V^{-1}(S^{-1}h_{(1)})}(h_{(2)})^* \overline{V(S^{-1}h_{(3)})} \tag{1.12}$$

for each $h \in H_F$, where $V(h) := U^{-1}(h_{(1)})U(S^{-1}h_{(2)})$.

If H is a coquasitriangular Hopf algebra, then so is H_F. In particular, if H is commutative then H_F is cotriangular with 'universal R-matrix' given by

$$\mathcal{R}(h,g) := F(g_{(1)}, h_{(1)})F^{-1}(h_{(2)}, g_{(2)}) \tag{1.13}$$

for all $h, g \in H$. Since as coalgebras H and H_F are the same, every left H-comodule is a left H_F-comodule and every H-comodule map is an H_F-comodule map. This means that there is an invertible functor which 'functorially quantises' any H-covariant construction to give an H_F-covariant one. As already mentioned, our strategy will be to apply this idea to the construction of instantons.

In passing from H to H_F, from each left H-comodule algebra A we automatically obtain a left H_F-comodule algebra A_F which as a vector space is the same as A but has the modified product

$$A_F \otimes A_F \to A_F, \qquad a \otimes b \mapsto a \cdot_F b := F(a^{(-1)}, b^{(-1)})a^{(0)}b^{(0)}. \tag{1.14}$$

If A is a left H-comodule algebra and a $*$-algebra such that the coaction is a $*$-algebra map, the twisted algebra A_F also has a new $*$-structure defined by

$$a^{*_F} := \overline{V^{-1}(S^{-1}a^{(-1)})}(a^{(0)})^* \tag{1.15}$$

for each $a \in A_F$.

2. The Twistor Fibration

The Penrose fibration $\mathbb{CP}^3 \to S^4$ is an essential component of the ADHM construction of instantons, since it encodes in its geometry the very nature of the anti-self-duality equations on S^4 [1]. Following [6, 4], in this section we sketch the details of the Penrose fibration from the point of view of coordinate algebras, then investigate what happens to the fibration upon passing to a local coordinate chart.

2.1. The Penrose fibration.

The $*$-algebra $\mathcal{A}[\mathbb{C}^4]$ of coordinate functions on the classical space $\mathbb{C}^4$ is the commutative unital $*$-algebra generated by the elements

$$\{z_j, z_l^* \mid j,l = 1, \ldots, 4\}.$$

The coordinate algebra $\mathcal{A}[S^7]$ of the seven-sphere S^7 is the quotient of $\mathcal{A}[\mathbb{C}^4]$ by the sphere relation

$$z_1^* z_1 + z_2^* z_2 + z_3^* z_3 + z_4^* z_4 = 1. \tag{2.1}$$

On the other hand, the coordinate algebra $\mathcal{A}[S^4]$ of the four-sphere S^4 is the commutative unital $*$-algebra generated by the elements x_1, x_1^*, x_2, x_2^* and $x_0 = x_0^*$ subject to the sphere relation

$$x_1^* x_1 + x_2^* x_2 + x_0^2 = 1.$$

There is a canonical inclusion of algebras $\mathcal{A}[S^4] \hookrightarrow \mathcal{A}[S^7]$ defined on generators by

$$x_1 = 2(z_1 z_3^* + z_2^* z_4), \quad x_2 = 2(z_2 z_3^* - z_1^* z_4), \quad x_0 = z_1 z_1^* + z_2 z_2^* - z_3 z_3^* - z_4 z_4^*, \tag{2.2}$$

and extended as a $*$-algebra map.Clearly one has

$$x_1^* x_1 + x_2^* x_2 + x_0^2 = (z_1^* z_1 + z_2^* z_2 + z_3^* z_3 + z_4^* z_4)^2 = 1, \tag{2.3}$$

so that the algebra inclusion is well-defined. This is just a coordinate algebra description of the principal bundle $S^7 \to S^4$ with structure group SU(2) (*cf.* [15] for further details of this construction).

The twistor space of the Euclidean four-sphere S^4 is nothing other than the complex projective space $\mathbb{CP}^3$. As a real six-dimensional manifold, twistor space $\mathbb{CP}^3$ may be identified with the set of all 4×4 Hermitian projector matrices of rank one, since each such matrix uniquely determines and is uniquely determined by a one-dimensional subspace of $\mathbb{C}^4$. Thus the coordinate $*$-algebra $\mathcal{A}[\mathbb{CP}^3]$ of $\mathbb{CP}^3$ has a defining matrix of (commuting) generators

$$\mathsf{q} := \begin{pmatrix} a_1 & u_1 & u_2 & u_3 \\ u_1^* & a_2 & v_3 & v_2 \\ u_2^* & v_3^* & a_3 & v_1 \\ u_3^* & v_2^* & v_1^* & a_4 \end{pmatrix}, \tag{2.4}$$

with $a_j^* = a_j$, $j = 1, \ldots, 4$ and $\operatorname{Tr} \mathsf{q} = \sum_j a_j = 1$, subject to the relations coming from the projection condition $\mathsf{q}^2 = \mathsf{q}$, that is to say $\sum_r \mathsf{q}_{jr} \mathsf{q}_{rl} = \mathsf{q}_{jl}$ for each $j, l = 1, \ldots, 4$.

There is a canonical inclusion of algebras $\mathcal{A}[\mathbb{CP}^3] \hookrightarrow \mathcal{A}[S^7]$ defined on generators by

$$\mathsf{q}_{jl} = z_j z_l^*, \qquad j, l = 1, \ldots, 4, \tag{2.5}$$

with the relation $\mathsf{q}_{11} + \mathsf{q}_{22} + \mathsf{q}_{33} + \mathsf{q}_{44} = 1$ coming from the sphere relation (2.1).

To determine the twistor fibration $\mathbb{CP}^3 \to S^4$ in our coordinate algebra framework, we need the map $\mathrm{J} : \mathcal{A}[\mathbb{C}^4] \to \mathcal{A}[\mathbb{C}^4]$ defined on generators by

$$\mathrm{J}(z_1, z_2, z_3, z_4) := (-z_2^*, z_1^*, -z_4^*, z_3^*) \tag{2.6}$$

and extended as a $*$-anti-algebra map. Equipping the algebra $\mathcal{A}[\mathbb{C}^4]$ with the map J thus identifies the space $\mathbb{C}^4$ with the quaternionic vector space $\mathbb{H}^2$ [18]. Accordingly, we define $\mathcal{A}[\mathbb{H}^2]$ to be the $*$-algebra $\mathcal{A}[\mathbb{C}^4]$ equipped with the quaternionic structure J.

Using the identification of generators (2.5), the map J extends to an automorphism of the algebra $\mathcal{A}[\mathbb{CP}^3]$ given by

$$\begin{array}{lllll} \mathrm{J}(a_1) = a_2, & \mathrm{J}(a_2) = a_1, & \mathrm{J}(a_3) = a_4, & \mathrm{J}(a_4) = a_3, & \mathrm{J}(u_1) = -u_1, \\ \mathrm{J}(v_1) = -v_1, & \mathrm{J}(u_2) = v_2^*, & \mathrm{J}(u_3) = -v_3^*, & \mathrm{J}(v_2) = u_2^*, & \mathrm{J}(v_3) = -u_3^* \end{array}$$

and extended as a $*$-anti-algebra map. It is straightforward to check that the subalgebra of $\mathcal{A}[\mathbb{CP}^3]$ fixed by this automorphism is precisely the four-sphere algebra $\mathcal{A}[S^4]$. Indeed, there is an algebra inclusion $\mathcal{A}[S^4] \hookrightarrow \mathcal{A}[\mathbb{CP}^3]$ defined on generators by

$$x_0 \mapsto 2(a_1 + a_2 - 1), \qquad x_1 \mapsto 2(u_2 + v_2^*), \qquad x_2 \mapsto 2(v_3 - u_3^*), \tag{2.7}$$

which is just a coordinate algebra description of the Penrose fibration $\mathbb{CP}^3 \to S^4$.

2.2. **Localisation of the twistor bundle.** Next we look at what happens to the fibration $\mathcal{A}[S^4] \hookrightarrow \mathcal{A}[\mathbb{CP}^3]$ when we pass to a local chart of S^4 by removing a point. Indeed, it is well-known that in making such a localisation the twistor bundle $\mathbb{CP}^3 \to S^4$ becomes isomorphic to the trivial fibration $\mathbb{R}^4 \times \mathbb{CP}^1$ over $\mathbb{R}^4$. In this section we illustrate this fact using the langauge of coordinate algebras.

By definition, the localisation $\mathcal{A}_0[S^4]$ of $\mathcal{A}[S^4]$ is the commutative unital $*$-algebra

$$\mathcal{A}_0[S^4] := \mathcal{A}[\tilde{x}_1, \tilde{x}_1^*, \tilde{x}_2, \tilde{x}_2^*, \tilde{x}_0, (1+\tilde{x}_0)^{-1} \mid \tilde{x}_1^*\tilde{x}_1 + \tilde{x}_2^*\tilde{x}_2 + \tilde{x}_0^2 = 1,\ (1+\tilde{x}_0)(1+\tilde{x}_0)^{-1} = 1].$$

It is the algebra obtained from $\mathcal{A}[S^4]$ by adjoining an inverse $(1+x_0)^{-1}$ to the function $1+x_0$; geometrically this corresponds to 'deleting' the point $(x_1, x_2, x_0) = (0, 0, -1)$ from the spectrum of the (smooth completion of the) algebra $\mathcal{A}[S^4]$, with $\mathcal{A}_0[S^4]$ being the algebra of coordinate functions on the resulting space. On the other hand, the coordinate algebra $\mathcal{A}[\mathbb{R}^4]$ of the Euclidean four-plane $\mathbb{R}^4$ is the commutative unital $*$-algebra generated by the elements

$$\{\zeta_j, \zeta_l^* \mid j, l = 1, 2\}. \tag{2.8}$$

Defining $|\zeta|^2 := \zeta_1^*\zeta_1 + \zeta_2^*\zeta_2$, the element $(1+|\zeta|^2)^{-1}$ clearly belongs to the (smooth completion of the) algebra $\mathcal{A}[\mathbb{R}^4]$ and so we have the following result.

Lemma 2.1. *The map $\mathcal{A}_0[S^4] \to \mathcal{A}[\mathbb{R}^4]$ defined on generators by*

$$\tilde{x}_1 \mapsto 2\zeta_1(1+|\zeta|^2)^{-1}, \qquad \tilde{x}_2 \mapsto 2\zeta_2(1+|\zeta|^2)^{-1}, \qquad \tilde{x}_0 \mapsto (1-|\zeta|^2)(1+|\zeta|^2)^{-1} \tag{2.9}$$

is a $*$-algebra isomorphism.

Proof. The inverse of (2.9) is the map $\mathcal{A}[\mathbb{R}^4] \to \mathcal{A}_0[S^4]$ given on generators by

$$\zeta_1 \mapsto \tilde{x}_1(1+\tilde{x}_0)^{-1}, \qquad \zeta_2 \mapsto \tilde{x}_2(1+\tilde{x}_0)^{-1} \tag{2.10}$$

and extended as a $*$-algebra map. Thus we have an isomorphism of vector spaces. One checks that the elements $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_0$ satisfy the same relation as the generators x_1, x_2, x_0 of the algebra $\mathcal{A}[S^4]$. The difference is that the point determined by the coordinate values $(x_1, x_2, x_0) = (0, 0, -1)$ is not in the spectrum of the (smooth) algebra generated by the $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_0$. In this way, we obtain $\mathbb{R}^4$ as a local chart of the four-sphere S^4, with the identification (2.9) defining the 'inverse stereographic projection'. The point $(0, 0, -1)$ will henceforth be called the *point at infinity.* □

At the level of twistor space $\mathbb{CP}^3$, passing to the local chart $\mathbb{R}^4$ by removing the point at infinity corresponds to removing the fibre $\mathbb{CP}^1$ over that point: we refer to this copy of $\mathbb{CP}^1$ as the *line at infinity* and denote it by ℓ_∞. Under the algebra inclusion (2.7), inverting the element $1+x_0$ in $\mathcal{A}[S^4]$ is equivalent to inverting the element $a_1 + a_2$ in $\mathcal{A}[\mathbb{CP}^3]$. We denote by $\mathcal{A}_0[\mathbb{CP}^3]$ the resulting localised algebra, *i.e.*

$$\mathcal{A}_0[\mathbb{CP}^3] := \mathcal{A}[\mathsf{q}_{jl}, (a_1+a_2)^{-1} \mid \textstyle\sum_r \mathsf{q}_{jr}\mathsf{q}_{rl} = \mathsf{q}_{jl},\ \operatorname{Tr}\mathsf{q} = 1,\ (a_1+a_2)(a_1+a_2)^{-1} = 1].$$

We now show that this algebra is isomorphic to the algebra of coordinate functions on the Cartesian product $\mathbb{R}^4 \times \mathbb{CP}^1$.

The coordinate algebra $\mathcal{A}[\mathbb{CP}^1]$ is the commutative unital $*$-algebra generated by the entries of the matrix

$$\tilde{\mathsf{q}} := \begin{pmatrix} \tilde{a}_1 & \tilde{u}_1 \\ \tilde{u}_1^* & \tilde{a}_2 \end{pmatrix} \tag{2.11}$$

subject to the relations $\tilde{\mathsf{q}}^2 = \tilde{\mathsf{q}}^* = \tilde{\mathsf{q}}$ and $\mathrm{Tr}\,\tilde{\mathsf{q}} = 1$, that is to say $\tilde{a}_1\tilde{a}_2 = \tilde{u}_1^*\tilde{u}_1$, $\tilde{a}_1^* = \tilde{a}_1$, $\tilde{a}_2^* = \tilde{a}_2$ and $\tilde{a}_1 + \tilde{a}_2 = 1$. With the coordinate algebra $\mathcal{A}[\mathbb{R}^4]$ of (2.8), we have the following result.

Lemma 2.2. *There is a $*$-algebra isomorphism $\mathcal{A}[\mathbb{R}^4] \otimes \mathcal{A}[\mathbb{CP}^1] \cong \mathcal{A}_0[\mathbb{CP}^3]$ defined on generators by*

$$\zeta_1 \otimes 1 \mapsto (a_1 + a_2)^{-1}(u_2 + v_2^*), \qquad \zeta_2 \otimes 1 \mapsto (a_1 + a_2)^{-1}(v_3 - u_3^*),$$

$$1 \otimes \tilde{a}_1 \mapsto (a_1 + a_2)^{-1}a_1, \qquad 1 \otimes \tilde{u}_1 \mapsto (a_1 + a_2)^{-1}u_1, \qquad 1 \otimes \tilde{a}_2 \mapsto (a_1 + a_2)^{-1}a_2$$

and extended as a $*$-algebra map.

Proof. We need to show that this map is an isomorphism of vector spaces which respects the algebra relations in $\mathcal{A}[\mathbb{R}^4] \otimes \mathcal{A}[\mathbb{CP}^1]$. Using the expressions (2.2) and (2.5), we find in $\mathcal{A}[S^7]$ the identities

$$2(a_1 + a_2)z_3 = x_1^*z_1 + x_2^*z_2, \qquad 2(a_1 + a_2)z_3^* = x_1z_1^* + x_2z_2^*,$$

$$2(a_1 + a_2)z_4 = x_1z_2 - x_2z_1, \qquad 2(a_1 + a_2)z_4^* = x_1^*z_2^* - x_2^*z_1^*.$$

In the localisation where $2(a_1 + a_2) = 1 + x_0$ is invertible, these expressions combined with the identifications $\mathsf{q}_{ij} = z_iz_j^*$ define the inverse of the map stated in the lemma, so that we have a vector space isomorphism. The algebra $\mathcal{A}[\mathbb{CP}^1]$ generated by $\tilde{a}_1$, $\tilde{a}_2$, $\tilde{u}_1$ and $\tilde{u}_1^*$ is identified with the subalgebra of $\mathcal{A}[\mathbb{CP}^3]$ generated by the localised upper left 2×2 block of the matrix (2.4), *i.e.* the subalgebra generated by the elements $(a_1 + a_2)^{-1}\mathsf{q}_{ij}$ for $i, j = 1, 2$. It is easy to check that the relations in $\mathcal{A}[\mathbb{CP}^1]$ are automatically preserved by this identification. To check that the trace relation $\mathrm{Tr}\,\mathsf{q} = \sum_j \mathsf{q}_{jj} = 1$ in $\mathcal{A}_0[\mathbb{CP}^3]$ also holds in $\mathcal{A}[\mathbb{R}^4] \otimes \mathcal{A}[\mathbb{CP}^1]$, one first computes that

$$(a_1 + a_2)^{-1}(z_1^*z_1 + z_2^*z_2) \mapsto 1 \otimes 1, \qquad (a_1 + a_2)^{-1}(z_3^*z_3 + z_4^*z_4) \mapsto (\zeta_1^*\zeta_1 + \zeta_2^*\zeta_2) \otimes 1,$$

so that the trace relation holds if and only if $(a_1 + a_2)^{-1} \mapsto (1 + |\zeta|^2)$, which is certainly true. Moreover, in $\mathcal{A}[\mathbb{CP}^3]$ there are relations of the form

$$\mathsf{q}_{ij}\mathsf{q}_{kl} = z_iz_j^*z_kz_l^* = z_iz_l^*z_kz_j^* = \mathsf{q}_{il}\mathsf{q}_{kj} \tag{2.12}$$

for $i, j, k, l = 1, \ldots, 4$. By adding together various linear combinations and using the trace relation $\mathrm{Tr}\,\mathsf{q} = 1$, one finds that the relations (2.12) are equivalent to the projector relations $\mathsf{q}^2 = \mathsf{q}$. Hence it follows that the projector relations in $\mathcal{A}_0[\mathbb{CP}^3]$ are equivalent to the remaining relations in $\mathcal{A}[\mathbb{R}^4] \otimes \mathcal{A}[\mathbb{CP}^1]$. □

In this way, we see that there is a canonical inclusion of algebras $\mathcal{A}[\mathbb{R}^4] \hookrightarrow \mathcal{A}_0[\mathbb{CP}^3]$ in the obvious way; this is a coordinate algebra description of the localised twistor fibration $\mathbb{R}^4 \times \mathbb{CP}^1 \to \mathbb{R}^4$. Moreover, using the isomorphism in Lemma 2.2, the quaternionic structure J of Eq. (2.6) is well-defined on the algebra $\mathcal{A}_0[\mathbb{CP}^3]$, with $\mathcal{A}[\mathbb{R}^4]$ being the J-invariant subalgebra.

2.3. **Symmetries of the twistor fibration.** In later sections we shall obtain deformations of the twistor fibration by the cocycle twisting of §1.2; for this we need a group of symmetries acting upon the twistor bundle. Here we describe the general strategy that we shall adopt.

We write $\mathrm{M}(2,\mathbb{H})$ for the algebra of 2×2 matrices with quaternion entries. The algebra $\mathcal{A}[\mathrm{M}(2,\mathbb{H})]$ of coordinate functions on $\mathrm{M}(2,\mathbb{H})$ is the commutative unital $*$-algebra generated by the entries of the 4×4 matrix

$$A=\begin{pmatrix}\mathrm{a}_{ij} & \mathrm{b}_{ij}\\ \mathrm{c}_{ij} & \mathrm{d}_{ij}\end{pmatrix}=\begin{pmatrix}\alpha_1 & -\alpha_2^* & \beta_1 & -\beta_2^*\\ \alpha_2 & \alpha_1^* & \beta_2 & \beta_1^*\\ \gamma_1 & -\gamma_2^* & \delta_1 & -\delta_2^*\\ \gamma_2 & \gamma_1^* & \delta_2 & \delta_1^*\end{pmatrix}. \tag{2.13}$$

We think of this matrix as being generated by a set of quaternion-valued functions, writing

$$\mathbf{a}=(\mathrm{a}_{ij})=\begin{pmatrix}\alpha_1 & -\alpha_2^*\\ \alpha_2 & \alpha_1^*\end{pmatrix}$$

and similarly for the other entries $\mathbf{b},\mathbf{c},\mathbf{d}$. The $*$-structure on this algebra is evident from the matrix (2.13). We equip $\mathcal{A}[\mathrm{M}(2,\mathbb{H})]$ with the matrix coalgebra structure

$$\Delta(A_{ij})=\sum\nolimits_r A_{ir}\otimes A_{rj},\qquad \epsilon(A_{ij})=\delta_{ij}\qquad \text{for}\quad i,j=1,\ldots,4.$$

Dual to the canonical action of $\mathrm{M}(2,\mathbb{H})$ on $\mathbb{C}^4\simeq\mathbb{H}^2$ there is a left coaction defined by

$$\Delta_L:\mathcal{A}[\mathbb{C}^4]\to\mathcal{A}[\mathrm{M}(2,\mathbb{H})]\otimes\mathcal{A}[\mathbb{C}^4],\qquad z_j\mapsto\sum\nolimits_r A_{jr}\otimes z_r, \tag{2.14}$$

extended as a $*$-algebra map. This coaction commutes with the quaternionic structure (2.6), in the sense that

$$(\mathrm{id}\otimes \mathrm{J})\circ\Delta_L=\Delta_L\circ \mathrm{J},$$

so that we have a coaction $\Delta_L:\mathcal{A}[\mathbb{H}^2]\to\mathcal{A}[\mathrm{M}(2,\mathbb{H})]\otimes\mathcal{A}[\mathbb{H}^2]$ (*cf.* [18]).

The Hopf algebra $\mathcal{A}[\mathrm{GL}(2,\mathbb{H})]$ of coordinate functions on the group $\mathrm{GL}(2,\mathbb{H})$ is obtained by adjoining to $\mathcal{A}[\mathrm{M}(2,\mathbb{H})]$ an invertible group-like element D obeying the relation $D^{-1}=\det A$, where $\det A$ is the determinant of the matrix (2.13). This yields a left coaction

$$\Delta_L:\mathcal{A}[\mathbb{H}^2]\to\mathcal{A}[\mathrm{GL}(2,\mathbb{H})]\otimes\mathcal{A}[\mathbb{H}^2],$$

also defined by the formula (2.14) and extended as a $*$-algebra map.

The group $\mathrm{GL}(2,\mathbb{H})$ is the group of conformal symmetries of the twistor fibration $\mathbb{CP}^3\to S^4$ [23, 20]. However, since we are interested in the *localised* twistor bundle described in Lemma 2.2, we work instead with the localised group of symmetries $\mathrm{GL}^+(2,\mathbb{H})$, which is just the 'coordinate patch' of $\mathrm{GL}(2,\mathbb{H})$ in which the 2×2 block $\mathbf{a}$ is assumed to be invertible. The coordinate algebra $\mathcal{A}[\mathrm{GL}^+(2,\mathbb{H})]$ of this localisation is obtained by adjoining to $\mathcal{A}[\mathrm{GL}(2,\mathbb{H})]$ an invertible element $\widetilde{D}$ which obeys the relation $\widetilde{D}^{-1}=\det\mathbf{a}$. The coaction of $\mathcal{A}[\mathrm{GL}^+(2,\mathbb{H})]$ on $\mathcal{A}[\mathbb{H}^2]$ is once again defined by the formula (2.14). We refer to [6] for further details of this construction.

Throughout the paper, our strategy will be to deform the localised twistor fibration and its associated geometry using the action of certain subgroups of $\mathrm{GL}^+(2,\mathbb{H})$. In dual terms, we suppose H to be a commutative Hopf $*$-algebra obtained *via* a Hopf algebra projection

$$\pi : \mathcal{A}[\mathrm{GL}^+(2,\mathbb{H})] \to H. \tag{2.15}$$

This determines a left coaction of H on $\mathcal{A}[\mathbb{H}^2]$ by projection of the coaction (2.14), namely

$$\Delta_\pi : \mathcal{A}[\mathbb{H}^2] \to H \otimes \mathcal{A}[\mathbb{H}^2], \qquad \Delta_\pi := (\pi \otimes \mathrm{id}) \circ \Delta_L, \tag{2.16}$$

which makes $\mathcal{A}[\mathbb{H}^2]$ into a left H-comodule $*$-algebra. Moreover, we assume that this coaction respects the defining relations of the localised twistor algebra $\mathcal{A}_0[\mathbb{CP}^3]$ given in Lemma 2.2, whence it makes $\mathcal{A}_0[\mathbb{CP}^3]$ and $\mathcal{A}[\mathbb{R}^4]$ into left H-comodule $*$-algebras in such a way that the algebra inclusion $\mathcal{A}[\mathbb{R}^4] \hookrightarrow \mathcal{A}_0[\mathbb{CP}^3]$ is a left H-comodule map.

3. Families of Instantons and Gauge Theory

We are now ready to study differential structures on the twistor fibration. In this section we recall the basic theory of anti-self-dual connections on Euclidean space $\mathbb{R}^4$ from the point of view of noncommutative geometry. Following [18, 5], we then generalise this by recalling what it means to have a *family* of anti-self-dual connections on $\mathbb{R}^4$ and when such families are gauge equivalent. These notions will pave the way for the algebraic formulation of the ADHM construction to follow.

3.1. Differential structures and instantons.

As discussed, our intention is to present the construction of connections and gauge fields in an entirely H-covariant framework, from which all of our deformed versions will immediately follow by functorial cocycle twisting. First of all we discuss the various differential structures that we shall need.

We write $\Omega(\mathbb{C}^4)$ for the canonical differential calculus on $\mathcal{A}[\mathbb{C}^4]$. It is the graded differential algebra generated by the degree zero elements z_j, z_l^*, $j,l = 1,\ldots,4$, and the degree one elements $\mathrm{d}z_j, \mathrm{d}z_l^*$, $j,l = 1,\ldots,4$, subject to the relations

$$\mathrm{d}z_j \wedge \mathrm{d}z_l + \mathrm{d}z_l \wedge \mathrm{d}z_j = 0, \qquad \mathrm{d}z_j \wedge \mathrm{d}z_l^* + \mathrm{d}z_l^* \wedge \mathrm{d}z_j = 0$$

for $j,l = 1,\ldots,4$. The exterior derivative d on $\Omega(\mathbb{C}^4)$ is defined by $\mathrm{d} : z_j \to \mathrm{d}z_j$ and extended uniquely using a graded Leibniz rule. There is also an involution on $\Omega(\mathbb{C}^4)$ given by graded extension of the map $z_j \mapsto z_j^*$.

The story is similar for the canonical differential calculus $\Omega(\mathbb{R}^4)$. It is generated by the degree zero elements ζ_j, ζ_l^* and the degree one elements $\mathrm{d}\zeta_j$, $\mathrm{d}\zeta_l^*$, $j,l = 1,2$, subject to the relations

$$\mathrm{d}\zeta_j \wedge \mathrm{d}\zeta_l + \mathrm{d}\zeta_l \wedge \mathrm{d}\zeta_j = 0, \qquad \mathrm{d}\zeta_j \wedge \mathrm{d}\zeta_l^* + \mathrm{d}\zeta_l^* \wedge \mathrm{d}\zeta_j = 0.$$

With $\pi : \mathcal{A}[\mathrm{GL}^+(2,\mathbb{H})] \to H$ a choice of Hopf algebra projection as in Eq. (2.15), we assume throughout that the differential calculi $\Omega(\mathbb{C}^4)$ and $\Omega(S^4)$ are (graded) left H-comodule algebras such that the exterior derivative d is a left H-comodule map, *i.e.* the coaction (2.16) obeys

$$\Delta_\pi(\mathrm{d}z_j) = (\mathrm{id} \otimes \mathrm{d})\Delta_\pi(z_j), \qquad j = 1,\ldots,4.$$

In this way, the H-coactions on $\Omega(\mathbb{C}^4)$ and on $\Omega(\mathbb{R}^4)$ are given by extending the coaction on $\mathcal{A}[\mathbb{C}^4]$.

Next we come to discuss vector bundles over $\mathbb{R}^4$. Of course, the fact that $\mathbb{R}^4$ is contractible means that the K-theory of the algebra $\mathcal{A}[\mathbb{R}^4]$ is trivial, *i.e.* all finitely generated projective modules $\mathcal{E}$ over $\mathcal{A}[\mathbb{R}^4]$ have the form $\mathcal{E} = \mathcal{A}[\mathbb{R}^4]^N$ for N a positive integer and are equipped with a canonical $\mathcal{A}[\mathbb{R}^4]$-valued Hermitian structure $\langle \cdot | \cdot \rangle$. A connection on $\mathcal{E}$ is a linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}[\mathbb{R}^4]} \Omega^1(\mathbb{R}^4)$ satisfying the Leibniz rule

$$\nabla(\xi x) = (\nabla \xi)x + \xi \otimes \mathrm{d}x \qquad \text{for all } \ \xi \in \mathcal{E}, \ x \in \mathcal{A}[\mathbb{R}^4].$$

The connection ∇ is said to be compatible with the Hermitian structure on $\mathcal{E}$ if it obeys

$$\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = \mathrm{d}\langle \xi | \eta \rangle \qquad \text{for all } \ \xi, \eta \in \mathcal{E}, \ x \in \mathcal{A}[\mathbb{R}^4].$$

Since $\mathcal{E}$ is necessarily free as an $\mathcal{A}[\mathbb{R}^4]$-module, any compatible connection ∇ can be written $\nabla = \mathrm{d} + \alpha$, where α is a skew-adjoint element of $\mathrm{Hom}_{\mathcal{A}[\mathbb{R}^4]}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}[\mathbb{R}^4]} \Omega^1(\mathbb{R}^4))$.

The curvature of ∇ is the $\mathrm{End}_{\mathcal{A}[\mathbb{R}^4]}(\mathcal{E})$-valued two-form

$$F := \nabla^2 = \mathrm{d}\alpha + \alpha^2.$$

The Euclidean metric on $\mathbb{R}^4$ determines the Hodge $*$-operator on $\Omega(\mathbb{R}^4)$, which on two-forms is a linear map $* : \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ such that $*^2 = \mathrm{id}$. Since H coacts by conformal transformations on $\mathcal{A}[\mathbb{R}^4]$, there is an H-covariant splitting of two-forms

$$\Omega^2(\mathbb{R}^4) = \Omega^2_+(\mathbb{R}^4) \oplus \Omega^2_-(\mathbb{R}^4)$$

into self-dual and anti-self-dual components, *i.e.* the ± 1 eigenspaces of the Hodge operator. The curvature ∇^2 of a connection ∇ is said to be *anti-self-dual* if it satisfies the equation $*F = -F$.

Definition 3.1. A compatible connection ∇ on $\mathcal{E}$ is said to be an *instanton* if its curvature $F = \nabla^2$ is an anti-self-dual two-form.

The *gauge group* of $\mathcal{E}$ is defined to be

$$\mathcal{U}(\mathcal{E}) := \left\{ U \in \mathrm{End}_{\mathcal{A}[\mathbb{R}^4]}(\mathcal{E}) \mid \langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle \text{ for all } \xi, \eta \in \mathcal{E} \right\}.$$

It acts upon the space of compatible connections by

$$\nabla \mapsto \nabla^U := U \nabla U^*$$

for each compatible connection ∇ and each element U of $\mathcal{U}(\mathcal{E})$. We say that a pair of connections ∇_1, ∇_2 on $\mathcal{E}$ are *gauge equivalent* if they are related by such a gauge transformation U. The curvatures of gauge equivalent connections are related by $F^U = (\nabla^U)^2 = UFU^*$. Note in particular that if ∇ has anti-self-dual curvature then so does the gauge-transformed connection ∇^U.

We observe *a posteriori* that the above definitions do not depend on the commutativity of the algebras $\mathcal{A}[\mathbb{R}^4]$ and $\Omega(\mathbb{R}^4)$, so that they continue to make sense even if we allow for deformations of the algebras $\mathcal{A}[\mathbb{R}^4]$ and $\Omega(\mathbb{R}^4)$.

3.2. **Noncommutative families of instantons.** Having given the definition of an instanton on $\mathbb{R}^4$, we now come to discuss what it means to have a *family* of instantons over $\mathbb{R}^4$. In the following, we let A be an arbitrary (possibly noncommutative) unital $*$-algebra.

Definition 3.2. A *family of Hermitian vector bundles* over $\mathbb{R}^4$ parameterised by the algebra A is a finitely generated projective right module $\mathcal{E}$ over the algebra $A \otimes \mathcal{A}[\mathbb{R}^4]$ equipped with an $A \otimes \mathcal{A}[\mathbb{R}^4]$-valued Hermitian structure $\langle \cdot | \cdot \rangle$.

By definition, any such module $\mathcal{E}$ is given by a self-adjoint idempotent $\mathsf{P} \in \mathrm{M}_N(A \otimes \mathcal{A}[\mathbb{R}^4])$, *i.e.* an $N \times N$ matrix with entries in $A \otimes \mathcal{A}[\mathbb{R}^4]$ satisfying $\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^*$; the corresponding module is $\mathcal{E} := \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N$. Although Definition 3.2 is given in terms of an arbitrary algebra A, it is motivated by the case where A is the (commutative) coordinate algebra of some underlying classical space X. In this situation, for each point $x \in X$ there is an evaluation map $\mathrm{ev}_x : A \to \mathbb{C}$ and the object

$$\mathcal{E}_x := (\mathrm{ev}_x \otimes \mathrm{id})\mathsf{P}((A \otimes \mathcal{A}[\mathbb{R}^4])^N)$$

is a finitely generated projective right $\mathcal{A}[\mathbb{R}^4]$-module corresponding to a vector bundle over $\mathbb{R}^4$. In this way, the projection P defines a family of Hermitian vector bundles parameterised by the space X. When the algebra A is noncommutative, there need not be enough evaluation maps available, but we may nevertheless work with the whole family at once.

Next we come to say what it means to have a family of connections over $\mathbb{R}^4$. We write $A \otimes \Omega^1(\mathbb{R}^4)$ for the tensor product bimodule over the algebra $A \otimes \mathcal{A}[\mathbb{R}^4]$ and extend the exterior derivative d on $\mathcal{A}[\mathbb{R}^4]$ to $A \otimes \mathcal{A}[\mathbb{R}^4]$ as $\mathrm{id} \otimes \mathrm{d}$.

Definition 3.3. A *family of connections* parameterised by the algebra A consists of a family of Hermitian vector bundles $\mathcal{E} := \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N$ over $\mathcal{A}[\mathbb{R}^4]$, together with a linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{A \otimes \mathcal{A}[\mathbb{R}^4]} (A \otimes \Omega^1(\mathbb{R}^4)) \simeq \mathcal{E} \otimes_{\mathcal{A}[\mathbb{R}^4]} \Omega^1(\mathbb{R}^4)$$

obeying the Leibniz rule

$$\nabla(\xi x) = (\nabla \xi) x + \xi \otimes (\mathrm{id} \otimes \mathrm{d}) x$$

for all $\xi \in \mathcal{E}$ and $x \in A \otimes \mathcal{A}[\mathbb{R}^4]$. The family is said to be *compatible* with the Hermitian structure if it obeys $\langle \nabla \xi | \eta \rangle + \langle \xi | \nabla \eta \rangle = (\mathrm{id} \otimes \mathrm{d}) \langle \xi | \eta \rangle$ for all $\xi \in \mathcal{E}$ and $x \in A \otimes \mathcal{A}[\mathbb{R}^4]$.

It is clear that a family of connections parameterised by $A = \mathbb{C}$ (*i.e.* by a one-point space) is just a connection in the usual sense. In general, a given family of Hermitian vector bundles $\mathcal{E} = \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N$ always carries the family of Grassmann connections defined by

$$\nabla_0 = \mathsf{P} \circ (\mathrm{id} \otimes \mathrm{d}).$$

It follows that any family of connections can be written in the form $\nabla = \nabla_0 + \alpha$, where α is a skew-adjoint element of

$$\mathrm{End}_{A \otimes \mathcal{A}[\mathbb{R}^4]}(\mathcal{E}, \mathcal{E} \otimes_{A \otimes \mathcal{A}[\mathbb{R}^4]} (A \otimes \Omega^1(\mathbb{R}^4))) \simeq \mathrm{End}_{A \otimes \mathcal{A}[\mathbb{R}^4]}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}[\mathbb{R}^4]} \Omega^1(\mathbb{R}^4)).$$

Definition 3.4. Let $\mathcal{E} := \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N$ be a family of Hermitian vector bundles parameterised by the algebra A. The *gauge group* of $\mathcal{E}$ is

$$\mathcal{U}(\mathcal{E}) := \{ U \in \mathrm{End}_{A \otimes \mathcal{A}[\mathbb{R}^4]}(\mathcal{E}) \mid \langle U\xi | U\eta \rangle = \langle \xi | \eta \rangle \text{ for all } \xi, \eta \in \mathcal{E} \}.$$

We say that two families of compatible connections ∇_1, ∇_2 on $\mathcal{E}$ are *equivalent families* and write $\nabla_1 \sim \nabla_2$ if they are related by the action of the unitary group, *i.e.* there exists $U \in \mathcal{U}(\mathcal{E})$ such that $\nabla_2 = U \nabla_1 U^*$.

More generally, if ∇_1 and ∇_2 are two families of connections on $\mathcal{E}$ parameterised by algebras A_1 and A_2 respectively, we say that $\nabla_1 \sim \nabla_2$ if there exists an algebra B and algebra maps $\phi_1 : A_1 \to B$ and $\phi_2 : A_2 \to B$ such that $\phi_1^* \nabla_1 \sim \phi_2^* \nabla_2$ in the above sense. Here $\phi_i^* \nabla_i$ is the connection on $\mathcal{E} \otimes_{A_i} B$ naturally induced by ∇_i (*cf.* [5] for more details).

In the case where $A = \mathbb{C}$, *i.e.* for a family parameterised by a one-point space, the above relation reduces to the usual definition of gauge equivalence of connections. In the case where the families ∇_1, ∇_2 are Grassmann families associated to projections $\mathsf{P}_1, \mathsf{P}_2 \in \mathrm{M}_N(A \otimes \mathcal{A}[\mathbb{R}^4])$, equivalence means that $\mathsf{P}_2 = U\mathsf{P}_1U^*$ for some unitary U.

Lemma 3.5. *With* $\mathcal{E} = \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N$, *there exists* $\mathsf{P}_A \in \mathrm{M}_N(A)$ *such that there is an algebra isomorphism*

$$\mathrm{End}_{A\otimes\mathcal{A}[\mathbb{R}^4]}(\mathcal{E}) \simeq \mathrm{End}_A(\mathsf{P}_A(A^N)) \otimes \mathcal{A}[\mathbb{R}^4]$$

and hence an isomorphism $\mathcal{U}(\mathcal{E}) \simeq \mathcal{U}(\mathrm{End}_A(\mathsf{P}_A(A^N)) \otimes \mathcal{A}[\mathbb{R}^4])$ *of gauge groups.*

Proof. Since $\mathbb{R}^4$ is topologically trivial, there is an isomorphism of K-groups $\mathrm{K}_0(A \otimes \mathcal{A}[\mathbb{R}^4]) \cong \mathrm{K}_0(A)$. It follows that for each projection $\mathsf{P} \in \mathrm{M}_N(A \otimes \mathcal{A}[\mathbb{R}^4])$ there exists a projection $\mathsf{P}_A \in \mathrm{M}_N(A)$ such that P and $\mathsf{P}_A \otimes 1$ are equivalent projections in $\mathrm{M}_N(A \otimes \mathcal{A}[\mathbb{R}^4])$. This implies that there is an isomorphism

$$\mathcal{E} = \mathsf{P}(A \otimes \mathcal{A}[\mathbb{R}^4])^N \simeq \mathsf{P}_A(A^N) \otimes \mathcal{A}[\mathbb{R}^4] \tag{3.1}$$

of right $A \otimes \mathcal{A}[\mathbb{R}^4]$-modules, from which the result follows immediately. □

With these ideas in mind, finally we arrive at the following definition of a family of instantons over $\mathbb{R}^4$ parameterised by a (possibly noncommutative) $*$-algebra A.

Definition 3.6. A *family of instantons* over $\mathbb{R}^4$ is a family of compatible connections ∇ over $\mathbb{R}^4$ whose curvature $F := \nabla^2$ obeys the anti-self-duality equation

$$(\mathrm{id} \otimes *)F = -F,$$

where $*$ is the Hodge operator on $\Omega^2(\mathbb{R}^4)$.

4. The ADHM Construction

Next we review the ADHM construction of instantons on the classical Euclidean four-plane $\mathbb{R}^4$. We present the construction in a coordinate algebra format which is covariant under the coaction of a given Hopf algebra of symmetries, paving the way for a deformation by the cocycle twisting of §1.2.

4.1. **The space of classical monads.**

We begin by describing the input data for the ADHM construction of instantons. Although the ADHM construction is capable of constructing instanton bundles of arbitrary rank, in this paper we restrict our attention to the construction of vector bundles with rank two.

Definition 4.1. Let $k \in \mathbb{Z}$ be a fixed positive integer. A *monad* over $\mathcal{A}[\mathbb{C}^4]$ is a sequence of free right $\mathcal{A}[\mathbb{C}^4]$-modules,

$$0 \to \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4] \xrightarrow{\sigma_z} \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4] \xrightarrow{\tau_z} \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4] \to 0, \tag{4.1}$$

where $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ are complex vector spaces of dimensions k, $2k+2$ and k respectively, such that the maps σ_z, τ_z are linear in the generators $z_1, \ldots, z_4$ of $\mathcal{A}[\mathbb{C}^4]$. The first and last terms of the sequence are required to be exact, so that the only non-trivial cohomology is in the middle term.

Given a monad (4.1), its cohomology $\mathcal{E} := \operatorname{Ker}\tau_z/\operatorname{Im}\sigma_z$ is a finitely-generated projective right $\mathcal{A}[\mathbb{C}^4]$-module and hence defines a vector bundle over $\mathbb{C}^4$. In fact, since the maps σ_z, τ_z are assumed linear in the coordinate functions $z_1, \ldots, z_4$, this vector bundle is well-defined over the projective space $\mathbb{CP}^3$ [22].

With respect to ordered bases $(u_1, \ldots, u_k)$, $(v_1, \ldots, v_{2k+2})$ and $(w_1, \ldots, w_k)$ for the vector spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$ respectively, the maps σ_z and τ_z have the form

$$(4.2)\qquad \sigma_z : u_b \otimes Z \mapsto \sum\nolimits_{a,j} M^j_{ab} \otimes v_a \otimes z_j Z, \qquad \tau_z : v_c \otimes Z \mapsto \sum\nolimits_{d,j} N^j_{dc} \otimes w_d \otimes z_j,$$

where $Z \in \mathcal{A}[\mathbb{C}^4]$ and the quantities $M^j := (M^j_{ab})$ and $N^j := (N^j_{dc})$, $j = 1, \ldots, 4$, are complex matrices with $a, c = 1, \ldots, 2k+2$ and $b, d = 1, \ldots, k$. In more compact notation, σ_z and τ_z may be written

$$(4.3)\qquad \sigma_z = \sum\nolimits_j M^j \otimes z_j, \qquad \tau_z = \sum\nolimits_j N^j \otimes z_j.$$

It is immediate from the formulæ (4.2) that the composition $\tau_z \circ \sigma_z$ is given by

$$\tau_z \circ \sigma_z : \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4] \to \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4], \qquad u_b \otimes Z \mapsto \sum\nolimits_{j,l,c,d} N^j_{dc} M^l_{cb} \otimes w_d \otimes z_j z_l Z,$$

with respect to the bases $(u_1, \ldots, u_k)$ and $(w_1, \ldots, w_k)$ of $\mathcal{H}$ and $\mathcal{L}$. It follows that the condition $\tau_z \circ \sigma_z = 0$ is equivalent to requiring that

$$(4.4)\qquad \sum\nolimits_r \left(N^j_{dr} M^l_{rb} + N^l_{dr} M^j_{rb}\right) = 0$$

for all $j, l = 1, \ldots, 4$ and $b, d = 1, \ldots, k$.

Introducing the conjugate matrix elements ${M^j_{ab}}^*$ and ${N^l_{dc}}^*$, we use the compact notation $(M^{j\dagger})_{ab} = {M^j_{ba}}^*$ and $(N^{l\dagger})_{cd} = {N^l_{dc}}^*$. Then given a monad (4.1), the corresponding *conjugate monad* is defined to be

$$(4.5)\qquad 0 \to \mathcal{L}^* \otimes \mathrm{J}(\mathcal{A}(\mathbb{C}^4_\theta))^* \xrightarrow{\tau^*_{\mathrm{J}(z)}} \mathcal{K}^* \otimes \mathrm{J}(\mathcal{A}(\mathbb{C}^4_\theta))^* \xrightarrow{\sigma^*_{\mathrm{J}(z)}} \mathcal{H}^* \otimes \mathrm{J}(\mathcal{A}(\mathbb{C}^4_\theta))^*,$$

where $\tau^*_{\mathrm{J}(z)}$ and $\sigma^*_{\mathrm{J}(z)}$ are the 'adjoint' maps defined by

$$\sigma^*_{\mathrm{J}(z)} = \sum\nolimits_j M^{j\dagger} \otimes \mathrm{J}(z_j)^*, \qquad \tau^*_{\mathrm{J}(z)} = \sum\nolimits_j N^{j\dagger} \otimes \mathrm{J}(z_j)^*$$

and J is the quaternionic involution defined in Eq. (2.6) (*cf.* [4] for further explanation). If a given monad (4.1) is isomorphic to its conjugate (4.5) then we say that it is *self-conjugate*. A necessary and sufficient condition for a monad to be self-conjugate is that the maps σ_z and τ_z should obey $\tilde{\tau}^*_{J(z)} = -\tilde{\sigma}_z$ and $\tilde{\sigma}^*_{J(z)} = \tilde{\tau}_z$, equivalently that the matrices M^j, N^l should satisfy the reality conditions

$$(4.6)\qquad N^1 = M^{2\dagger}, \quad N^2 = -M^{1\dagger}, \quad N^3 = M^{4\dagger}, \quad N^4 = -M^{3\dagger}.$$

This is for fixed maps σ_z, τ_z. In dual terms, by allowing σ_z, τ_z to vary, we think of the elements M^j_{ab}, N^j_{dc} as coordinate functions on the space of all possible pairs of $\mathcal{A}[\mathbb{C}^4]$-module maps

$$(4.7)\qquad \sigma_z : \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4] \to \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4], \qquad \tau_z : \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4] \to \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4].$$

Imposing the conditions (4.4) and (4.6), we obtain coordinate functions on the space M_k of all self-conjugate monads with index $k \in \mathbb{Z}$.

Definition 4.2. We write $\mathcal{A}[\mathsf{M}_k]$ for the commutative $*$-algebra generated by the coordinate functions M^j_{ab}, N^j_{dc} subject to the relations (4.4) and the $*$-structure (4.6).

Remark 4.3. For each point $x \in \mathsf{M}_k$ there is an evaluation map

$$\epsilon_x : \mathcal{A}[\mathsf{M}_k] \to \mathbb{C}$$

and the complex matrices $(\epsilon_x \otimes \mathrm{id})\sigma_z$ and $(\epsilon_x \otimes \mathrm{id})\tau_z$ define a self-conjugate monad over $\mathbb{C}^4$,

$$0 \to \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4] \xrightarrow{(\mathrm{ev}_x \otimes \mathrm{id})\sigma_z} \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4] \xrightarrow{(\mathrm{ev}_x \otimes \mathrm{id})\tau_z} \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4] \to 0. \tag{4.8}$$

As already remarked, the cohomology $\mathcal{E} = \mathrm{Ker}, \tau_z / \mathrm{Im}\, \sigma_z$ of a monad (4.1) defines a vector bundle E over $\mathbb{CP}^3$; the self-conjugacy condition (4.6) ensures that E arises *via* pull-back along the Penrose fibration $\mathbb{CP}^3 \to S^4$. This means that the bundle E is trivial upon restriction to each of the fibres of the Penrose fibration and, in particular, to the fibre ℓ_∞ over the 'point at infinity'.

As already mentioned, we wish to view our monads as being covariant under a certain coaction of a Hopf algebra H. Recall that $\mathcal{A}[\mathbb{C}^4]$ is already a left H-comodule algebra, with H-coaction defined using the projection $\pi : \mathcal{A}[\mathrm{GL}^+(2, \mathbb{H})] \to H$ and the formula (2.16). It automatically follows that the free modules $\mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4]$, $\mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4]$ and $\mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4]$ are also left H-comodules whose $\mathcal{A}[\mathbb{C}^4]$-module structures are H-equivariant. It remains to address the requirement that the module maps σ_z and τ_z should be H-equivariant as well.

Lemma 4.4. *The maps $\sigma_z := \sum_j M^j \otimes z_j$ and $\tau_z := \sum_j N^j \otimes z_j$ are H-comodule maps if and only if the coordinate functions M^j_{ab}, N^l_{dc} carry the left H-coaction*

$$M^j_{ab} \mapsto \sum\nolimits_r \pi(S(A_{rj})) \otimes M^r_{ab}, \qquad N^j_{dc} \mapsto \sum\nolimits_r \pi(S(A_{rj})) \otimes N^r_{dc} \tag{4.9}$$

for each $j = 1, \ldots, 4$ and $a, c = 1, \ldots, 2k+2$, $b, d = 1, \ldots, k$.

Proof. Upon inspection of Eq. (4.2) we see that σ_z cannot possibly be an intertwiner for the H-coactions on $\mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4]$ and $\mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4]$ unless we also allow for a coaction of H on the algebra $\mathcal{A}[\mathsf{M}_k]$ as well. Since the definition of σ_z depends only upon the generators M^j_{ab}, it is enough to check equivariance only on these generators. It is immediate that, for fixed a, b, the four-dimensional H-comodule spanned by the generators M^j_{ab}, $j = 1, \ldots, 4$, must be conjugate to the four-dimensional comodule spanned by the generators $z_1, \ldots, z_4$, giving the coaction as stated. Indeed, we verify that

$$\begin{aligned} \sum\nolimits_r M^r \otimes z_r \mapsto & \sum\nolimits_{r,s} \pi(S(A_{sr})A_{rs}) \otimes M^s \otimes z_s \\ &= \sum\nolimits_s \pi(\epsilon(A_{ss})) \otimes M^s \otimes z_s \\ &= \sum\nolimits_s 1 \otimes M^s \otimes z_s, \end{aligned}$$

as required. The same analysis applies to the map τ_z. □

By extending it as a $*$-algebra map, the formula (4.9) equips $\mathcal{A}[\mathsf{M}_k]$ with the structure of a left H-comodule $*$-algebra. This will be of paramount importance in later sections when we come to deform the ADHM construction.

4.2. **The construction of instantons on $\mathbb{R}^4$.** For self-conjugate monads, the maps of interest are the $(2k+2)\times k$ algebra-valued matrices

$$\sigma_z = M^1 \otimes z_1 + M^2 \otimes z_2 + M^3 \otimes z_3 + M^4 \otimes z_4,$$
$$\sigma_{\mathrm{J}(z)} = -M^1 \otimes z_2^* + M^2 \otimes z_1^* - M^3 \otimes z_4^* + M^4 \otimes z_3^*.$$

In terms of these generators, the monad condition $\tau_z\sigma_z = 0$ becomes $\sigma^*_{\mathrm{J}(z)}\sigma_z = 0$. By polarisation of this identity, one also finds that $\sigma^*_{\mathrm{J}(z)}\sigma_{\mathrm{J}(z)} = \sigma^*_z\sigma_z$. The identification of the vector space $\mathcal{K}$ with its dual $\mathcal{K}^*$ means that the module $\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4]$ acquires a bilinear form given by

$$(\xi,\eta) := \langle \mathrm{J}\xi|\eta\rangle = \sum\nolimits_a (\mathrm{J}\xi)^*_a\eta_a \tag{4.10}$$

for $\xi = (\xi_a)$ and $\eta = (\eta_a) \in \mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4]$, with $\langle\,\cdot\,|\,\cdot\,\rangle$ the canonical Hermitian structure on $\mathcal{K}\otimes\mathcal{A}(\mathbb{C}^4_\theta)$. The monad condition $\sigma^*_{\mathrm{J}(z)}\sigma_z = 0$ implies that the columns of the matrix σ_z are orthogonal with respect to the form $(\,\cdot\,,\,\cdot\,)$.

Let us introduce the notation

$$\rho^2 := \sigma^*_z\sigma_z = \sigma^*_{\mathrm{J}(z)}\sigma_{\mathrm{J}(z)}, \tag{4.11}$$

a $k\times k$ matrix with entries in the algebra $\mathcal{A}[\mathsf{M}_k]\otimes\mathcal{A}[\mathbb{C}^4]$. In order to proceed, we need this matrix ρ^2 to be invertible, although of course this is not the case in general. Thus we need to slightly enlarge the matrix algebra $\mathrm{M}_k(\mathbb{C})\otimes\mathcal{A}[\mathsf{M}_k]\otimes\mathcal{A}[\mathbb{C}^4]$ by adjoining an inverse element ρ^{-2} for ρ^2. Doing so is equivalent to deleting a collection of points from the parameter space M_k, corresponding to the so-called 'instantons of zero-size' [12]. We henceforth assume that this has been done, although we do not change our notation.

We collect together the matrices σ_z, $\sigma_{\mathrm{J}(z)}$ into the $(2k+2)\times 2k$ matrix

$$\mathsf{V} := \begin{pmatrix}\sigma_z & \sigma_{\mathrm{J}(z)}\end{pmatrix}, \tag{4.12}$$

which by the definition of ρ^2 obeys

$$\mathsf{V}^*\mathsf{V} = \rho^2\begin{pmatrix}\mathbb{1}_k & 0\\ 0 & \mathbb{1}_k\end{pmatrix},$$

where $\mathbb{1}_k$ denotes the $k\times k$ identity matrix. We form the matrix

$$\mathsf{Q} := \mathsf{V}\rho^{-2}\mathsf{V}^* = \sigma_z\rho^{-2}\sigma^*_z + \sigma_{\mathrm{J}(z)}\rho^{-2}\sigma^*_{\mathrm{J}(z)} \tag{4.13}$$

and for convenience we denote

$$Q_z := \sigma_z\rho^{-2}\sigma^*_z, \qquad Q_{\mathrm{J}(z)} := \sigma_{\mathrm{J}(z)}\rho^{-2}\sigma^*_{\mathrm{J}(z)}. \tag{4.14}$$

Immediately we have the following result.

Proposition 4.5. *The quantity $\mathsf{Q} := \mathsf{V}\rho^{-2}\mathsf{V}^*$ is a $(2k+2)\times(2k+2)$ projection, $\mathsf{Q}^2 = \mathsf{Q} = \mathsf{Q}^*$, with entries in the algebra $\mathcal{A}[\mathsf{M}_k]\otimes\mathcal{A}[\mathbb{R}^4]$ and trace equal to $2k$.*

Proof. That Q is a projection is a direct consequence of the fact that $\mathsf{V}^*\mathsf{V} = \rho^2$. The matrices Q_z and $Q_{\mathrm{J}(z)}$ are also projections: in fact they are orthogonal projections, since $Q_zQ_{\mathrm{J}(z)} = 0$. Moreover, both matrices Q_z and $Q_{\mathrm{J}(z)}$ have entries whose $\mathcal{A}[\mathbb{C}^4]$-components have the form $z_j^*z_l$ for $j,l = 1,\ldots,4$. From the proof of Lemma 2.2, we know that we can rewrite each of these expressions in terms of generators of the algebra $\mathcal{A}[\mathbb{R}^4]\otimes\mathcal{A}[\mathbb{CP}^1]$.

Since the matrix sum Q has entries which are J-invariant, it follows that the $\mathcal{A}[\mathbb{C}^4]$-components of these entries must lie in the J-invariant subalgebra of $\mathcal{A}[\mathbb{R}^4] \otimes \mathcal{A}[\mathbb{CP}^1]$, which is just $\mathcal{A}[\mathbb{R}^4]$. For the trace, we compute that

$$\begin{aligned}\operatorname{Tr} Q_z &= \sum\nolimits_{\mu} (\sigma_z \rho^{-2} \sigma_z^*)_{\mu\mu} = \sum\nolimits_{\mu,r,s} (\sigma_z)_{\mu r} (\rho^{-2})_{rs} (\sigma_z^*)_{s\mu} = \sum\nolimits_{\mu,r,s} (\rho^{-2})_{rs} (\sigma_z)_{\mu r} (\sigma_z)^*_{s\mu} \\ &= \sum\nolimits_{\mu,r,s} (\rho^{-2})_{rs} (\sigma_z)^*_{s\mu} (\sigma_z)_{\mu r} = \sum\nolimits_{r,s} (\rho^{-2})_{rs} (\sigma_z^* \sigma_z)_{sr} = \operatorname{Tr} \mathbb{1}_k = k.\end{aligned}$$

A similar computation establishes that $Q_{\mathrm{J}(z)}$ also has trace equal to k, whence the trace of Q is $2k$ by linearity. □

From the projection Q we construct the complementary projection $\mathsf{P} := \mathbb{1}_{2k+2} - \mathsf{Q}$, also having entries in the algebra $\mathcal{A}[\mathsf{M}_k] \otimes \mathcal{A}[\mathbb{R}^4]$. It is immediate that the trace of P is equal to two, so it follows that the finitely generated projective right $\mathcal{A}[\mathsf{M}_k] \otimes \mathcal{A}[\mathbb{R}^4]$-module

$$\mathcal{E} := \mathsf{P}(\mathcal{A}[\mathsf{M}_k] \otimes \mathcal{A}[\mathbb{R}^4])^{2k+2}$$

defines a family of rank two vector bundles over $\mathbb{R}^4$ parameterised by the space M_k of self-conjugate monads.

We equip this family of vector bundles with the family of Grassmann connections $\nabla := \mathsf{P} \circ (\mathrm{id} \otimes \mathrm{d})$. Immediately we obtain the following result.

Proposition 4.6. *The curvature $F = \mathsf{P}\,((\mathrm{id} \otimes \mathrm{d})\mathsf{P})^2$ of the family of Grassmann connections ∇ is anti-self-dual, that is to say $(\mathrm{id} \otimes *)F = -F$.*

Proof. By applying $\mathrm{id} \otimes \mathrm{d}$ to the relation $\rho^{-2}\rho^2 = \mathbb{1}_k$ and using the Leibniz rule, one finds that $(\mathrm{id} \otimes \mathrm{d})\rho^{-2} = -\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\rho^2)\rho^{-2}$ (this is a standard formula for calculating the derivative of a matrix-valued function). Using this, one finds that

$$(\mathrm{id} \otimes \mathrm{d})(\mathsf{V}\rho^{-2}\mathsf{V}^*) = \mathsf{P}((\mathrm{id} \otimes \mathrm{d})\mathsf{V})\rho^{-2}\mathsf{V}^* + \mathsf{V}\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\mathsf{V}^*)\mathsf{P},$$

and hence in turn that

$$\begin{aligned}((\mathrm{id} \otimes \mathrm{d})\mathsf{P}) \wedge ((\mathrm{id} \otimes \mathrm{d})\mathsf{P}) = \mathsf{P}((\mathrm{id} \otimes \mathrm{d})\mathsf{V})\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\mathsf{V}^*)\mathsf{P} \\ + \mathsf{V}\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\mathsf{V}^*)\mathsf{P}((\mathrm{id} \otimes \mathrm{d})\mathsf{V})\rho^{-2}\mathsf{V}^*,\end{aligned}$$

where we have used the facts that $\rho^{-2}\mathsf{V}^*\mathsf{P} = 0 = \mathsf{P}\mathsf{V}\rho^{-2}$. The second term in the above expression is identically zero when acting on any element in the image $\mathcal{E}$ of P, whence the curvature F of the family ∇ works out to be

$$\begin{aligned}F &= \mathsf{P}\,((\mathrm{id} \otimes \mathrm{d})\mathsf{P})^2 \\ &= \mathsf{P}((\mathrm{id} \otimes \mathrm{d})\mathsf{V})\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\mathsf{V}^*)\mathsf{P} \\ &= \mathsf{P}\left(((\mathrm{id} \otimes \mathrm{d})\sigma_z)\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\sigma_z^*) + ((\mathrm{id} \otimes \mathrm{d})\sigma_{\mathrm{J}(z)})\rho^{-2}((\mathrm{id} \otimes \mathrm{d})\sigma^*_{\mathrm{J}(z)})\right)\mathsf{P}.\end{aligned}$$

It is clear by inspection that on twistor space $\mathcal{A}_0[\mathbb{CP}^3]$ this F is a horizontal two-form of type $(1,1)$ and it is known [1] that such a two-form is necessarily the pull-back of an anti-self-dual two-form on $\mathbb{R}^4$. □

Thus we have reproduced the ADHM construction of instantons on $\mathbb{R}^4$ in our coordinate algebra framework: as usual, we must now address the question of the extent to which the construction depends on the choice of bases for the vector spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ that we made in §4.1.

It is clear that we are free to act on the $\mathcal{A}[\mathbb{C}^4]$-module $\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4]$ by a unitary element of the matrix algebra $\mathrm{M}_{2k+2}(\mathbb{C})\otimes\mathcal{A}[\mathbb{C}^4]$. In order to preserve the instanton construction, we must do so in a way which preserves the bilinear form $(\cdot\,,\cdot)$ of Eq. (4.10) determined by the identification of $\mathcal{K}$ with its dual $\mathcal{K}^*$. It follows that the map σ_z in Eq. (4.2) is defined up to a unitary transformation $U\in\mathrm{End}_{\mathcal{A}[\mathbb{C}^4]}(\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4])$ which commutes with the quaternion structure J, namely the elements of the group

$$\mathrm{Sp}(\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4]) := \Big\{U\in\mathrm{End}_{\mathcal{A}(\mathbb{C}^4_\theta)}(\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4]) \mid \langle U\xi|U\xi\rangle = \langle\xi|\xi\rangle,\ \mathrm{J}(U\xi)=U\mathrm{J}(\xi)\Big\}.$$

Similarly, we are free to change basis in the module $\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4]$, whence the map τ_z in Eq. (4.2) is defined up to an invertible transformation $W\in\mathrm{GL}(\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4])$. Given $U\in\mathrm{Sp}(\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4])$ and $W\in\mathrm{GL}(\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4])$, the available freedom in the ADHM construction is to map $\sigma_z\mapsto U\sigma_z W$.

Proposition 4.7. *For all $W\in\mathrm{GL}(\mathcal{H})$ the projection $\mathsf{P}=\mathbb{I}_{2k+2}-\mathsf{Q}$ is invariant under the transformation $\sigma_z\mapsto\sigma_z W$. For all $U\in\mathrm{Sp}(\mathcal{K})$, under the transformation $\sigma_z\mapsto U\sigma_z$ the projection P of transforms as $\mathsf{P}\mapsto U\mathsf{P}U^*$.*

Proof. One first checks that $\rho^2\mapsto(\sigma_z W)^*(\sigma_z W)=W^*\rho^2 W$, so that

$$\mathsf{Q}_z\mapsto\sigma_z W(W^*\rho^2W)^{-1}W^*\sigma_z^* = \sigma_z W(W^{-1}\rho^{-2}(W^*)^{-1})W^*\sigma_z^*=\mathsf{Q}_z,$$

whence the projection P is unchanged. Replacing σ_z by $U\sigma_z$ leaves ρ^2 invariant (since U is unitary) and so has the effect that

$$\mathsf{Q}_z\mapsto U\sigma_z\rho^{-2}\sigma_z^*U^* = U\mathsf{Q}_zU^*,$$

whence it follows that P is mapped to $U\mathsf{P}U^*$. □

In this way, such changes of module bases result in gauge equivalent families of instantons. However, from the point of view of constructing equivalence classes of connections it is in fact sufficient to consider the effect of the subgroups of 'constant' module automorphisms, *i.e.* those generated changes of basis in the vector spaces $\mathcal{H}$, $\mathcal{K}$ and $\mathcal{L}$, described by the group $\mathrm{Sp}(\mathcal{K})=\mathrm{Sp}(k+1)\subset\mathrm{Sp}(\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4])$ and the group $\mathrm{GL}(k,\mathbb{R})\subset\mathrm{GL}(\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4])$ [2].

Although it is beyond our scope to prove this here, we note that the algebra $\mathcal{A}[\mathsf{M}_k]$ has a total of $4k(2k+2)$ generators and $5k(k-1)$ constraints (determined by the orthogonality relations $\sigma^*_{J(z)}\sigma_z=0$); the $\mathrm{Sp}(k+1)$ symmetries impose a further $(k+1)(2(k+1)+1)$ constraints and the $\mathrm{GL}(k,\mathbb{R})$ a further k^2 constraints. This elementary argument yields that the construction has

$$(8k^2+8k)-5k(k-1)-(3k^2+5k+3)=8k-3$$

degrees of freedom, in precise agreement with the dimension of the moduli space computed in [3].

Definition 4.8. We say that a pair of self-conjugate monads are *equivalent* if they are related by a change of bases of the vector spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ of the above form, *i.e.* by a pair of linear transformations $U\in\mathrm{Sp}(k+1)$ and $W\in\mathrm{GL}(k,\mathbb{R})$. We denote by $\sim$ the resulting equivalence relation on the space M_k of self-conjugate monads.

5. The Moyal-Groenewold Noncommutative Plane $\mathbb{R}^4_\hbar$

The Moyal noncommutative space-time $\mathbb{R}^4_\hbar$ is arguably one of the best-known and most widely-studied examples of a noncommutative space. In this section we analyse the construction of instantons on this space from the point of view of cocycle twisting. In this section we show how to deform Euclidean space-time $\mathbb{R}^4$ and its associated geometric structure into that of the Moyal-Groenewold space-time; then we look at what happens to the ADHM construction of instantons under the deformation procedure.

5.1. **A Moyal-deformed family of monads.** In order to deform the ADHM construction of instantons, we need to choose a Hopf algebra H of symmetries together with a two-cocycle F by which to perform the twisting. For our twisting Hopf algebra we take $H = \mathcal{A}[\mathbb{R}^4]$, the algebra of coordinate functions on the additive group $\mathbb{R}^4$. It is the commutative unital $*$-algebra

$$\mathcal{A}[\mathbb{R}^4] = \mathcal{A}[t_j, t_j^* \mid j = 1, 2] \tag{5.1}$$

equipped with the Hopf algebra structure

$$\Delta(t_j) = 1 \otimes t_j + t_j \otimes 1, \qquad \epsilon(t_j) = 0, \qquad S(t_j) = -t_j, \tag{5.2}$$

with Δ, ϵ extended as $*$-algebra maps and S extended as a $*$-anti-algebra map. In order to deform the twistor fibration, we have to equip our various algebras with left H-comodule algebra structures, which we achieve using the discussion of §2.3. There is a Hopf algebra projection from $\mathcal{A}[\mathrm{GL}^+(2, \mathbb{H})]$ onto H, defined on generators by

$$\pi : \mathcal{A}[\mathrm{GL}^+(2, \mathbb{H})] \to H, \qquad \begin{pmatrix} \alpha_1 & -\alpha_2^* & \beta_1 & -\beta_2^* \\ \alpha_2 & \alpha_1^* & \beta_2 & \beta_1^* \\ \gamma_1 & -\gamma_2^* & \delta_1 & -\delta_2^* \\ \gamma_2 & \gamma_1^* & \delta_2 & \delta_1^* \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ t_1^* & t_2^* & 1 & 0 \\ -t_2 & t_1 & 0 & 1 \end{pmatrix} \tag{5.3}$$

and extended as a $*$-algebra map. Using Eq. (2.16), this projection determines a left H-coaction $\Delta_\pi : \mathcal{A}[\mathbb{C}^4] \to H \otimes \mathcal{A}[\mathbb{C}^4]$ by

$$z_1 \mapsto 1 \otimes z_1, \qquad z_3 \mapsto t_1^* \otimes z_1 + t_2^* \otimes z_2 + 1 \otimes z_3, \tag{5.4}$$

$$z_2 \mapsto 1 \otimes z_2, \qquad z_4 \mapsto -t_2 \otimes z_1 + t_1 \otimes z_2 + 1 \otimes z_4, \tag{5.5}$$

extended as a $*$-algebra map. Using the identification of generators (2.10), the coordinate algebra $\mathcal{A}[\mathbb{R}^4]$ of Euclidean space therefore carries the coaction

$$\mathcal{A}[\mathbb{R}^4] \to H \otimes \mathcal{A}[\mathbb{R}^4], \qquad \zeta_1 \mapsto 1 \otimes \zeta_1 + t_1 \otimes 1, \quad \zeta_2 \mapsto 1 \otimes \zeta_2 + t_2 \otimes 1, \tag{5.6}$$

making $\mathcal{A}[\mathbb{R}^4]$ into a left H-comodule $*$-algebra.

Let $(\partial_j{}^l)$, $j, l = 1, 2$, be the Lie algebra of translation generators dual to H. Writing

$$\tau := (\tau_r{}^s) = \begin{pmatrix} t_1^* & t_2^* \\ -t_2 & t_1 \end{pmatrix}, \qquad r, s = 1, 2,$$

this means that there is a non-degenerate pairing

$$\langle \partial_j{}^l, \tau_r{}^s \rangle = \delta_j^s \, \delta_r^l, \qquad j, l, r, s = 1, 2,$$

which extends to an action on products of the generators t_j, t_l^* by differentiation and evaluation at zero. Using this pairing, we define a twisting two-cocycle by

$$F : H \otimes H \to \mathbb{C}, \qquad F(h, g) = \Big\langle \exp\Big(\tfrac{1}{2}\mathrm{i}\Theta_r{}^{r'}{}_s{}^{s'} \partial_{r'}{}^r \otimes \partial_{s'}{}^s\Big), h \otimes g \Big\rangle$$

for $h, g \in H$, where $\Theta = (\Theta_r{}^{r'}{}_s{}^{s'})$, $r, r', s, s' = 1, 2$, is a real 4×4 anti-symmetric matrix with rows rr' and columns ss', which we may choose to have the canonical form

$$\Theta = \hbar \begin{pmatrix} 0 & 0 & 0 & \alpha \\ 0 & 0 & \beta & 0 \\ 0 & -\beta & 0 & 0 \\ -\alpha & 0 & 0 & 0 \end{pmatrix}$$

for non-zero real constants α, β and $\hbar > 0$ a deformation parameter. We assume for simplicity that $\alpha + \beta \neq 0$. This F is multiplicative (*i.e.* it is a Hopf bicharacter in the sense of Eq. (1.3)) and so it is determined by its values on the generators $(\tau_r{}^s)$, $r, s = 1, 2$. One computes in particular that

$$F(t_1^*, t_1) = \tfrac{1}{2}\alpha \mathrm{i}\hbar, \qquad F(t_2^*, t_2) = -\tfrac{1}{2}\beta \mathrm{i}\hbar,$$

with F evaluating as zero on all other pairs of generators. From the formulæ (1.9)–(1.10) and (1.12), one immediately finds that $H = H_F$ as a Hopf $*$-algebra. However, the effect of the twisting on the H-comodule algebras $\mathcal{A}[\mathbb{C}^4]$ and $\mathcal{A}[\mathbb{R}^4]$ is not trivial, as shown by the following lemmata.

Lemma 5.1. *The algebra relations in the H-comodule algebra $\mathcal{A}[\mathbb{C}^4]$ are twisted into*

$$(5.7) \qquad [z_3, z_4] = \mathrm{i}\hbar(\alpha + \beta) z_1 z_2, \qquad [z_3^*, z_4^*] = \mathrm{i}\hbar(\alpha + \beta) z_1^* z_2^*,$$

$$(5.8) \qquad [z_3, z_3^*] = \mathrm{i}\hbar\alpha z_1 z_1^* - \mathrm{i}\hbar\beta z_2 z_2^*, \qquad [z_4, z_4^*] = \mathrm{i}\hbar\beta z_1 z_1^* - \mathrm{i}\hbar\alpha z_2 z_2^*,$$

with all other relations left unchanged. In particular, the generators z_1, z_2 and their conjugates remain central in the deformed algebra.

Proof. The cocycle-twisted product on the H-comodule algebra $\mathcal{A}[\mathbb{C}^4]$ is defined by the formula (1.14); the corresponding algebra relations can be expressed using the 'universal R-matrix' of Eq. (1.13), namely

$$(5.9) \qquad z_j \cdot_F z_l = \mathcal{R}(z_l{}^{(-1)}, z_j{}^{(-1)}) z_l{}^{(0)} \cdot_F z_j{}^{(0)}, \qquad z_j \cdot_F z_l^* = \mathcal{R}(z_l^{*(-1)}, z_j{}^{(-1)}) z_l^{*(0)} \cdot_F z_j{}^{(0)};$$

One finds in particular that the R-matrix has the values

$$(5.10) \qquad \mathcal{R}(t_1^*, t_1) = 2F^{-1}(t_1^*, t_1) = -\mathrm{i}\hbar\alpha, \qquad \mathcal{R}(t_2^*, t_2) = 2F^{-1}(t_2^*, t_2) = \mathrm{i}\hbar\beta,$$

and gives zero when evaluated on all other pairs of generators. By explicitly computing Eqs. (5.9) (and omitting the product symbol $\cdot_F$), one finds the relations as stated in the lemma. We denote by $\mathcal{A}[\mathbb{C}^4_\hbar]$ the $*$-algebra generated by $\{z_j, z_j^* \mid j = 1, \dots, 4\}$ modulo the relations (5.7)–(5.8). This makes $\mathcal{A}[\mathbb{C}^4_\hbar]$ into a left H_F-comodule $*$-algebra. □

Lemma 5.2. *The algebra relations in the H-comodule algebra $\mathcal{A}[\mathbb{R}^4]$ are twisted into*

$$(5.11) \qquad [\zeta_1^*, \zeta_1] = \mathrm{i}\hbar\alpha, \qquad [\zeta_2^*, \zeta_2] = -\mathrm{i}\hbar\beta, \qquad j, l = 1, 2,$$

with vanishing commutators between all other pairs of generators.

Proof. The product in $\mathcal{A}[\mathbb{R}^4]$ is twisted using the formula (1.14). Once again omitting the product symbol $\cdot_F$, the corresponding algebra relations are computed to be those as stated. We denote by $\mathcal{A}[\mathbb{R}^4_\hbar]$ the algebra generated by ζ_1, ζ_2, ζ_1^*, ζ_2^*, modulo the relations (5.11). This makes $\mathcal{A}[\mathbb{R}^4_\hbar]$ into a left H_F-comodule $*$-algebra. □

Remark 5.3. Since the generators z_1, z_2 and their conjugates z_1^*, z_2^* remain central in the algebra $\mathcal{A}[\mathbb{C}^4_\hbar]$, we see immediately from Lemma 2.2 that the localised twistor algebra has the form $\mathcal{A}[\mathbb{R}^4_\hbar] \otimes \mathcal{A}[\mathbb{CP}^1]$. Only the base $\mathbb{R}^4_\hbar$ of the twistor fibration is deformed; the typical fibre $\mathbb{CP}^1$ remains classical.

The canonical differential calculi described in §3.1 are also deformed using this cocycle twisting procedure. The relations in the deformed calculi are given in the following lemmata.

Lemma 5.4. *The twisted differential calculus $\Omega(\mathbb{C}^4_\hbar)$ is generated by the degree zero elements z_j, z_l^* and the degree one elements $\mathrm{d}z_j, \mathrm{d}z_l^*$ for $j,l = 1,\ldots,4$, subject to the bimodule relations between functions and one-forms*

$$[z_3, \mathrm{d}z_4] = \mathrm{i}\hbar(\alpha+\beta)z_1\mathrm{d}z_2, \qquad [z_3^*, \mathrm{d}z_4^*] = \mathrm{i}\hbar(\alpha+\beta)z_1^*\mathrm{d}z_2^*,$$
$$[z_4, \mathrm{d}z_3] = -\mathrm{i}\hbar(\alpha+\beta)z_2\mathrm{d}z_1, \qquad [z_4^*, \mathrm{d}z_3^*] = -\mathrm{i}\hbar(\alpha+\beta)z_2^*\mathrm{d}z_1^*,$$
$$[z_3, \mathrm{d}z_3^*] = \mathrm{i}\hbar\alpha z_1\mathrm{d}z_1^* - \mathrm{i}\hbar\beta z_2\mathrm{d}z_2^*, \qquad [z_4, \mathrm{d}z_4^*] = \mathrm{i}\hbar\beta z_1\mathrm{d}z_1^* - \mathrm{i}\hbar\alpha z_2\mathrm{d}z_2^*,$$

and the anti-commutation relations between one-forms

$$\{\mathrm{d}z_3, \mathrm{d}z_4\} = \mathrm{i}\hbar(\alpha+\beta)\mathrm{d}z_1\mathrm{d}z_2, \qquad \{\mathrm{d}z_3^*, \mathrm{d}z_4^*\} = \mathrm{i}\hbar(\alpha+\beta)\mathrm{d}z_1^*\mathrm{d}z_2^*,$$
$$\{\mathrm{d}z_3, \mathrm{d}z_3^*\} = \mathrm{i}\hbar\alpha\mathrm{d}z_1\mathrm{d}z_1^* - \mathrm{i}\hbar\beta\mathrm{d}z_2\mathrm{d}z_2^*, \qquad \{\mathrm{d}z_4, \mathrm{d}z_4^*\} = \mathrm{i}\hbar\beta\mathrm{d}z_1\mathrm{d}z_1^* - \mathrm{i}\hbar\alpha\mathrm{d}z_2\mathrm{d}z_2^*,$$

with all other relations undeformed.

Proof. One views the classical calculus $\Omega(\mathbb{C}^4)$ as a left H-comodule algebra and accordingly computes the deformed product using the twisting cocycle F. Since the exterior derivative d commutes with the H-coaction (5.4), it is straightforward to observe that the (anti-)commutation relations in the deformed calculus $\Omega(\mathbb{C}^4_\hbar)$ are just the same as the algebra relations in $\mathcal{A}[\mathbb{C}^4_\hbar]$ but with d inserted appropriately. □

Lemma 5.5. *The twisted differential calculus $\Omega(\mathbb{R}^4_\hbar)$ is generated by the degree zero elements $\zeta_1, \zeta_1^*, \zeta_2, \zeta_2^*$ and the degree one elements $\mathrm{d}\zeta_1, \mathrm{d}\zeta_1^*, \mathrm{d}\zeta_2, \mathrm{d}\zeta_2^*$. The relations in the calculus are not deformed.*

Proof. Once again, the classical calculus $\Omega(\mathbb{R}^4)$ is deformed as a twisted left H-comodule algebra. Although the products of functions and differential forms in the calculus are indeed twisted, one finds that the extra terms which appear in the twisted product all vanish in the expressions for the (anti-)commutators (*cf.* [6] for full details). □

In particular, we see that the vector space $\Omega^2(\mathbb{R}^4_\hbar)$ is the same as it is classically. Since the coaction of H on $\mathcal{A}[\mathbb{R}^4]$ is by isometries, the Hodge $*$-operator $* : \Omega^2(\mathbb{R}^4) \to \Omega^2(\mathbb{R}^4)$ commutes with the H-coaction in the sense that

$$\Delta_\pi(*\omega) = (\mathrm{id} \otimes *)\Delta_\pi(\omega), \qquad \omega \in \Omega^2(\mathbb{R}^4),$$

so there is also a Hodge operator $*_\hbar : \Omega^2(\mathbb{R}^4_\hbar) \to \Omega^2(\mathbb{R}^4_\hbar)$ defined by the same formula as in the classical case. In particular, this means that the decomposition of $\Omega^2(\mathbb{R}^4_\hbar)$ into self-dual and anti-self-dual two-forms,

$$\Omega^2(\mathbb{R}^4_\hbar) = \Omega^2_+(\mathbb{R}^4_\hbar) \oplus \Omega^2_-(\mathbb{R}^4_\hbar),$$

is identical at the level of vector spaces to the corresponding decomposition in the classical case.

The above lemmata are really just special cases of the cocycle twisting procedure; recall that in fact our 'quantisation map' applies to every suitable H-covariant construction, in particular to the coordinate algebra $\mathcal{A}[\mathsf{M}_k]$ of the space of self-conjugate monads. We view $\mathcal{A}[\mathsf{M}_k]$ as a left H-comodule algebra according to Lemma 4.4 and write $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ for the corresponding cocycle-twisted H_F-comodule algebra.

Proposition 5.6. *The coordinate $*$-algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ is generated by the matrix elements M^j_{ab}, N^l_{dc} for $a, c = 1, \ldots, k$ and $b, d = 1, \ldots, 2k+2$, modulo the relations*

$$[M^1_{ab}, M^2_{rs}] = \mathrm{i}\hbar(\alpha-\beta) M^3_{ab} M^4_{rs}, \qquad [M^1_{ab}, M^1_{rs}{}^*] = \mathrm{i}\hbar\alpha M^3_{ab} M^3_{rs}{}^* + \mathrm{i}\hbar\beta M^4_{ab} M^4_{rs}{}^*,$$
$$[M^1_{ab}{}^*, M^2_{rs}{}^*] = \mathrm{i}\hbar(\alpha-\beta) M^3_{ab}{}^* M^4_{rs}{}^*, \qquad [M^2_{ab}, M^2_{rs}{}^*] = -\mathrm{i}\hbar\beta M^3_{ab} M^3_{rs}{}^* - \mathrm{i}\hbar\alpha M^4_{ab} M^4_{rs}{}^*$$

and the $*$-structure (4.6). The generators M^3, M^4, M^{3*}, M^{4*} are central in the algebra.

Proof. From Lemma 4.4 we read off the H-coaction on generators M^j, $j = 1, \ldots, 4$, obtaining

$$M^1 \mapsto 1 \otimes M^1 - t_1^* \otimes M^3 + t_2 \otimes M^4, \qquad M^3 \mapsto 1 \otimes M^3,$$
$$M^2 \mapsto 1 \otimes M^2 - t_2^* \otimes M^3 - t_1 \otimes M^4, \qquad M^4 \mapsto 1 \otimes M^4,$$

which we extend as a $*$-algebra map. The deformed relations follow immediately from an application of the twisting formula (1.14). The coaction of H on $\mathcal{A}[\mathsf{M}_k]$ does not depend on the matrix indices of the generators M^j, N^l, $j, l = 1, \ldots, 4$, hence neither do the twisted commutation relations. Similar computations yield the other commutation relations as stated. In terms of the deformed product, the relations (4.4) are twisted into the relations

$$\sum\nolimits_r N^j_{dr} M^l_{rb} + N^l_{dr} M^j_{rb} + \mathrm{i}\hbar(\alpha+\beta)(\delta^{j1}\delta^{l2} - \delta^{j2}\delta^{l1}) = 0$$

for each $b, d = 1, \ldots, k$, where δ^{rs} is the Kronecker delta symbol. □

We think of $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ as the coordinate algebra of a noncommutative space $\mathsf{M}_{k;\hbar}$ of monads on $\mathbb{C}^4_\hbar$. Although we do not have as many evaluation maps on $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ as we did in the classical case, we can nevertheless work with the whole family $\mathsf{M}_{k;\hbar}$ at once.

5.2. The construction of instantons on $\mathbb{R}^4_\hbar$.

From the noncommutative space of monads $\mathrm{M}_{k;\hbar}$ we may proceed as in §4.2 to construct families of instantons, now on the Moyal space $\mathbb{R}^4_\hbar$.

Let $\widehat{\mathbb{R}}^4$ denote the Pontryagin dual to the additive group $\mathbb{R}^4$ used in Eq. (5.1). Given a pair of complex numbers $\vec{c} := (c_1, c_2) \in \mathbb{C}^2 \simeq \widehat{\mathbb{R}}^4$ we define unitary elements $\vec{u} = (u_1, u_2)$ of the algebra H_F by

$$u_1 = \exp(\mathrm{i}(c_1 t_1 + c_1^* t_1^*)), \quad u_2 = \exp(\mathrm{i}(c_2 t_2 + c_2^* t_2^*)). \tag{5.12}$$

It is straightforward to check that u_1, u_2 are group-like elements of (the smooth completion of) the Hopf algebra H_F, *i.e.* they transform as $\Delta(u_j) = u_j \otimes u_j$ under the coproduct $\Delta : H_F \to H_F \otimes H_F$.

Lemma 5.7. *There is a canonical left action of H_F on the algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ given by*

$$\begin{aligned} u_1 \triangleright M^1 &= M^1 - \hbar\alpha c_1 M^3, & u_1 \triangleright M^2 &= M^2 + \hbar\alpha c_1^* M^4, \\ u_1 \triangleright M^{1*} &= M^{1*} - \hbar\alpha c_1^* M^{3*}, & u_1 \triangleright M^{2*} &= M^{2*} + \hbar\alpha c_1 M^{4*}, \\ u_2 \triangleright M^1 &= M^1 + \hbar\beta c_2^* M^4, & u_2 \triangleright M^2 &= M^2 + \hbar\beta c_2 M^3, \\ u_2 \triangleright M^{1*} &= M^{1*} + \hbar\beta c_2 M^{4*}, & u_2 \triangleright M^{2*} &= M^2 + \hbar\beta c_2^* M^{3*}, \end{aligned}$$

with $u_j \triangleright M^l = M^l$ and $u_j \triangleright M^{l*} = M^{l*}$ for $l = 3, 4$.

Proof. Recall from the proof of Proposition 5.6 that $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ is a left H_F-comodule algebra; it is therefore also a left H_F-module algebra according to the formula (1.5). Evaluating the R-matrix by expanding the exponentials as power series, one finds that

$$\begin{aligned} \mathcal{R}(t_1, u_1) &= \mathcal{R}(t_1, \mathrm{i}c_1^* t_1^*) = -\hbar\alpha c_1^*, & \mathcal{R}(t_1^*, u_1) &= \mathcal{R}(t_1^*, \mathrm{i}c_1 t_1) = \hbar\alpha c_1, \\ \mathcal{R}(t_2, u_2) &= \mathcal{R}(t_1, \mathrm{i}c_2^* t_2^*) = -\hbar\beta c_2^*, & \mathcal{R}(t_2^*, u_2) &= \mathcal{R}(t_2^*, \mathrm{i}c_2 t_2) = \hbar\beta c_2, \end{aligned}$$

with all other combinations evaluating as zero. Using the fact that the unitaries u_j are group-like elements of the Hopf algebra H_F, one finds the actions to be as stated. □

In turn, there is an infinitesimal version of the H_F-action on $\mathcal{A}[\mathsf{M}_{k;\hbar}]$, given by

$$\begin{aligned} &t_1 \triangleright M^1 = \mathrm{i}\hbar\alpha M^3, \quad t_1^* \triangleright M^{1*} = -\mathrm{i}\hbar\alpha M^{3*}, \quad t_1^* \triangleright M^2 = -\mathrm{i}\hbar\alpha M^4, \quad t_1 \triangleright M^{2*} = \mathrm{i}\hbar\alpha M^{4*}, \\ &t_2^* \triangleright M^1 = -\mathrm{i}\hbar\beta M^4, \quad t_2 \triangleright M^{1*} = \mathrm{i}\hbar\beta M^{4*}, \quad t_2 \triangleright M^2 = -\mathrm{i}\hbar\beta M^3, \quad t_2^* \triangleright M^{2*} = \mathrm{i}\hbar\beta M^{3*}, \end{aligned}$$

with $t_j \triangleright M^l = 0$ and $t_j \triangleright M^{l*} = 0$ for all other possible combinations of generators. Either way, we obtain a group action

$$\gamma : \widehat{\mathbb{R}}^4 \to \operatorname{Aut} \mathcal{A}[\mathsf{M}_{k;\hbar}]$$

of the Pontryagin dual $\widehat{\mathbb{R}}^4$ on the coordinate algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ by $*$-automorphisms. We also form the smash product algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$ associated to the above H_F-action, whose multiplication is defined by the formula (1.7). With the coproduct $\Delta(t_j) = 1 \otimes t_j + t_j \otimes 1$ on H_F, we find in particular the formula

$$(M^j_{ab} \otimes t_r)(M^l_{cd} \otimes t_s) = M^j_{ab} M^l_{cd} \otimes t_r t_s + M^j_{ab}(t_r \triangleright M^l_{cd}) \otimes t_s,$$

with similar expressions for products involving the conjugate generators M^{j*}. The corresponding algebra relations between such elements are given by

$$[M^j_{ab} \otimes t_r, M^l_{cd} \otimes t_s] = [M^j_{ab}, M^l_{cd}] \otimes t_r t_s + M^j_{ab}(t_r \triangleright M^l_{cd}) \otimes t_s - M^l_{cd}(t_s \triangleright M^j_{ab}) \otimes t_r \tag{5.13}$$

for $j, l = 1, \dots, 4$ and $r, s = 1, 2$, with similar formulæ occurring when the generators M^j and t_r are replaced by their conjugates. Of course, these relations are just a small part of the full algebra structure in the smash product $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$, but those in Eq. (5.13) are the ones that we will need later on in the paper.

Remark 5.8. This situation is a special case of Example 1.1. Recall that, upon making suitable completions of our algebras, we can think of the smash product algebra

$$\mathcal{A}[\mathsf{M}_{k;\hbar}]\rtimes H_F = \mathcal{A}[\mathsf{M}_{k;\hbar}]\rtimes\mathcal{A}[\mathbb{R}^4]$$

as being equivalent to the crossed product algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]\rtimes_\gamma \widehat{\mathbb{R}}^4$.

Thanks to the functorial nature of the cocycle twisting, *mutatis mutandis* the ADHM construction goes through as described in §4.2. In the following, we highlight the main differences which arise as a consequence of the quantisation procedure. The next lemma takes care of an important technical point: as well as twisting the relations in the algebras $\mathcal{A}[\mathsf{M}_k]$ and $\mathcal{A}[\mathbb{C}^4]$, we also have to deform the cross-relations in the tensor product algebra $\mathcal{A}[\mathsf{M}_k]\otimes\mathcal{A}[\mathbb{C}^4]$.

Lemma 5.9. *The algebra structure of the twisted tensor product algebra* $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{C}^4_\hbar]$ *is given by the relations in the respective subalgebras* $\mathcal{A}[\mathsf{M}_{k;\hbar}]$ *and* $\mathcal{A}[\mathbb{C}^4_\hbar]$ *determined above, together with the cross-relations*

$$\begin{aligned}
z_3M^1 &= M^1z_3 - \mathrm{i}\hbar\beta M^4z_2, & z_3M^2 &= M^2z_3 - \mathrm{i}\hbar\alpha M^4z_1,\\
z_4M^1 &= M^1z_4 + \mathrm{i}\hbar\alpha M^3z_2, & z_4M^2 &= M^2z_4 + \mathrm{i}\hbar\beta M^3z_1,\\
z_3^*M^1 &= M^1z_3^* + \mathrm{i}\hbar\alpha M^3z_1^*, & z_3^*M^2 &= M^2z_3^* - \mathrm{i}\hbar\beta M^3z_2^*,\\
z_4^*M^1 &= M^1z_4^* + \mathrm{i}\hbar\beta M^4z_1^*, & z_4^*M^2 &= M^2z_4^* - \mathrm{i}\hbar\alpha M^4z_2^*
\end{aligned}$$

and their conjugates. The generators z_1, z_2, M^3, M^4 *are central.*

Proof. The classical algebra $\mathcal{A}[\mathsf{M}_k]\otimes\mathcal{A}[\mathbb{C}^4]$ is a left comodule $*$-algebra under the tensor product H_F-coaction defined by Eq. (1.4). The twisted product is determined by the formula (1.14), with the non-trivial cross-terms in the deformed algebra being the ones stated in the lemma. We denote the deformed algebra by $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{C}^4_\hbar]$, with the symbol $\underline{\otimes}$ to remind us that the tensor product algebra structure is not the usual one, but has been twisted as well. □

Just as in the classical situation, we have a pair of matrices σ_z and τ_z,

$$\sigma_z = \sum\nolimits_j M^j\otimes z_j, \qquad \tau_z = \sum\nolimits_j N^j\otimes z_j,$$

whose entries this time live in the twisted algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{C}^4_\hbar]$. The resulting matrix $\mathsf{V} := \begin{pmatrix}\sigma_z & \sigma_{J(z)}\end{pmatrix}$ is a $2k\times(2k+2)$ matrix with entries in $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{C}^4_\hbar]$. We set $\rho^2 := \mathsf{V}^*\mathsf{V}$. From the projection $\mathsf{Q} := \mathsf{V}\rho^{-2}\mathsf{V}^*$ we construct the complementary matrix $\mathsf{P} := \mathbb{1}_{2k+2} - \mathsf{Q}$, which has entries in the algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{R}^4_\hbar]$.

It is clear that this matrix P is a self-adjoint idempotent, $\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^*$, but it does not define an honest family of projections in the sense of Definition 3.2. Recall that, to define such a family, we need a matrix with entries in an algebra of the form $A\otimes\mathcal{A}[\mathbb{R}^4_\hbar]$ for some 'parameter algebra' A, whereas the quantisation procedure has produced a projection Q with entries in a *twisted* tensor product $\mathcal{A}[\mathsf{M}_{k;\hbar}]\underline{\otimes}\mathcal{A}[\mathbb{R}^4_\hbar]$. We may nevertheless recover a genuine family of projections using the following lemma, in which we use the Sweedler notation $Z\mapsto Z^{(-1)}\otimes Z^{(0)}$ for the left coaction $\mathcal{A}[\mathbb{C}^4_\hbar]\to H_F\otimes\mathcal{A}[\mathbb{C}^4_\hbar]$ defined in Eqs. (5.4)–(5.5).

Lemma 5.10. *There is a canonical $*$-algebra map*

$$\mu : \mathcal{A}[\mathsf{M}_{k;\hbar}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\hbar] \to (\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\hbar]$$

defined by $\mu(M \otimes Z) = M \otimes Z^{(-1)} \otimes Z^{(0)}$ *for each* $M \in \mathcal{A}[\mathsf{M}_{k;\hbar}]$ *and* $Z \in \mathcal{A}[\mathbb{C}^4_\hbar]$.

Proof. This follows from a straightforward verification. One checks that

$$\begin{aligned}
\mu(M \otimes Z)\mu(M' \otimes Z') &= (M \otimes Z^{(-1)} \otimes Z^{(0)})(M' \otimes Z'^{(-1)} \otimes Z'^{(0)}) \\
&= M({Z^{(-1)}}_{(1)} \triangleright M') \otimes {Z^{(-1)}}_{(2)} Z'^{(-1)} \otimes Z^{(0)} Z'^{(0)} \\
&= \mathcal{R}(M'^{(-1)}, {Z^{(-1)}}_{(1)}) M M'^{(0)} \otimes {Z^{(-1)}}_{(2)} Z'^{(-1)} \otimes Z^{(0)} Z'^{(0)} \\
&= \mathcal{R}(M'^{(-1)}, Z^{(-1)}) M M'^{(0)} \otimes Z(Z^{(0)(-1)}) Z'^{(-1)} \otimes (Z^{(0)(0)}) Z'^{(0)} \\
&= \mu\left(\mathcal{R}(M'^{(-1)}, Z^{(-1)}) M M'^{(0)} \otimes Z^{(0)} Z'\right) \\
&= \mu\left((M \otimes Z)(M' \otimes Z')\right)
\end{aligned}$$

so that μ is an algebra map, as well as

$$\begin{aligned}
(\mu(M \otimes Z))^* &= (M \otimes Z^{(-1)} \otimes Z^{(0)})^* = (M \otimes Z^{(-1)})^* \otimes Z^{(0)*} \\
&= \mathcal{R}(M^{(-1)*}, ({Z^{(-1)}}_{(1)})^*)(M^{(0)*} \otimes ({Z^{(-1)}}_{(2)})^*) \otimes Z^{(0)*} \\
&= \mu\left(\mathcal{R}(M^{(-1)*}, Z^{(-1)*})(M^{(0)*} \otimes Z^{(0)*})\right) \\
&= \mu\left((M \otimes Z)^*\right)
\end{aligned}$$

so that μ respects the $*$-structures as well. □

Remark 5.11. Lemma 5.10 is an example of Majid's 'bosonisation' construction [19], which converts noncommutative 'braid statistics' (in our case described by the twisted tensor product $\underline{\otimes}$) into commutative 'ordinary statistics' (described by the usual tensor product $\otimes$).

As a consequence of Lemma 5.10, we find that there are maps

$$\tilde{\sigma}_z : \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4_\hbar] \to (\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4_\hbar], \tag{5.14}$$

$$\tilde{\tau}_z : \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4_\hbar] \to (\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4_\hbar], \tag{5.15}$$

defined by composing σ_z and τ_z with the map μ. Explicitly, these maps are given by

$$\tilde{\sigma}_z := \sum\nolimits_r M^r \otimes {z_r}^{(-1)} \otimes {z_r}^{(0)}, \qquad \tilde{\tau}_z := \sum\nolimits_r N^r \otimes {z_r}^{(-1)} \otimes {z_r}^{(0)},$$

which are respectively $k \times (2k+2)$ and $(2k+2) \times k$ matrices with entries in the noncommutative algebra $(\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\hbar]$. With this in mind, we form the $(2k+2) \times 2k$ matrix $\widetilde{\mathsf{V}} := \begin{pmatrix} \tilde{\sigma}_z & \tilde{\sigma}_{J(z)} \end{pmatrix}$.

Proposition 5.12. *The* $(2k+2) \times (2k+2)$ *matrix* $\widetilde{\mathsf{Q}} = \widetilde{\mathsf{V}}\tilde{\rho}^{-2}\widetilde{\mathsf{V}}^*$ *is a projection,* $\widetilde{\mathsf{Q}}^2 = \widetilde{\mathsf{Q}} = \widetilde{\mathsf{Q}}^*$, *with entries in the algebra* $(\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\hbar]$ *and trace equal to* $2k$.

Proof. The fact that $\widetilde{\mathsf{Q}}$ is a projection follows from the fact that Q is a projection and that $\mu : \mathcal{A}[\mathsf{M}_{k;\hbar}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\hbar] \to (\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\hbar]$ is a $*$-algebra map. By construction, the entries of the matrix ρ^2 are central in the algebra $(\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\hbar]$ (this follows from the fact that its matrix entries are coinvariant under the left H_F-coaction), from which it follows that the trace computation in Proposition 4.5 is valid in the noncommutative case as well [5]. □

From the projection $\widetilde{\mathsf{Q}}$ we construct as before the complementary projection $\widetilde{\mathsf{P}} := \mathbb{1}_{2k+2} - \widetilde{\mathsf{Q}}$; it has entries in the algebra $(\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{R}^4_\hbar]$ and has trace equal to two. In analogy with Definition 3.2, the finitely-generated projective module

$$\mathcal{E} := \mathsf{P}\left((\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{R}^4_\hbar]\right)^{2k+2}$$

defines a family of rank two vector bundles over $\mathbb{R}^4_\hbar$ parameterised by the noncommutative algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$. We equip this family of vector bundles with the family of Grassmann connections associated to the projection P.

Proposition 5.13. *The curvature $F = \mathsf{P}((\mathrm{id} \otimes \mathrm{d})\mathsf{P})^2$ of the Grassmann family of connections $\nabla := (\mathrm{id} \otimes \mathrm{d}) \circ \mathsf{P}$ is anti-self-dual.*

Proof. From Lemma 5.5 we know that the space of two-forms $\Omega^2(\mathbb{R}^4_\hbar)$ and the Hodge $*$-operator $*_\hbar : \Omega^2(\mathbb{R}^4_\hbar) \to \Omega^2(\mathbb{R}^4_\hbar)$ are undeformed and equal to their classical counterparts; similarly for the decomposition $\Omega^2(\mathbb{R}^4_\hbar) = \Omega^2_+(\mathbb{R}^4_\hbar) \oplus \Omega^2_-(\mathbb{R}^4_\hbar)$ into self-dual and anti-self-dual two-forms. This identification of the 'quantum' with the 'classical' spaces of two-forms survives the tensoring with the parameter space $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$, which yields that $(\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F) \otimes \Omega^2_\pm(\mathbb{R}^4_\hbar)$ and $(\mathcal{A}[\mathsf{M}_k] \otimes H) \otimes \Omega^2_\pm(\mathbb{R}^4)$ are isomorphic as vector spaces. Computing the curvature F in exactly the same way as in Proposition 4.6, we see that it must be anti-self-dual, since the same is true in the classical case. □

5.3. **The Moyal-deformed ADHM equations.** The noncommutative ADHM construction of the previous section produced families of instantons on $\mathbb{R}^4$ parameterised by the noncommutative algebra $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$. We interpret the latter as an algebra of coordinate functions on some underlying 'quantum' parameter space, within which we shall seek a subspace of classical parameters. To this end, we introduce elements of $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$ defined by

$$\widetilde{M}^1_{ab} := M^1_{ab} \otimes 1 + M^3_{ab} \otimes \tfrac{1}{2}t_1^* - M^4_{ab} \otimes \tfrac{1}{2}t_2, \qquad \widetilde{M}^3_{ab} := M^3_{ab} \otimes 1,$$
$$\widetilde{M}^2_{ab} := M^2_{ab} \otimes 1 + M^3_{ab} \otimes \tfrac{1}{2}t_2^* + M^4_{ab} \otimes \tfrac{1}{2}t_1, \qquad \widetilde{M}^4_{ab} := M^4_{ab} \otimes 1$$

for each $a = 1, \ldots, k$ and $b = 1, \ldots, 2k+2$, together with their conjugates $\widetilde{M}^j_{ab}{}^*$, $j = 1, \ldots, 4$.

Definition 5.14. We write $\mathcal{A}[\mathfrak{M}(k;\hbar)]$ for the subalgebra of $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$ generated by the elements $\widetilde{M}^j_{ab}$, $\widetilde{M}^l_{dc}{}^*$, $j, l = 1, \ldots, 4$.

Proposition 5.15. *The algebra $\mathcal{A}[\mathfrak{M}(k;\hbar)]$ is a commutative $*$-subalgebra of the smash product $\mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$.*

Proof. This follows from direct computation. The generators $\widetilde{M}^3$ and $\widetilde{M}^4$ are clearly central. On the other hand, we also have

$$\widetilde{M}^1\widetilde{M}^2 = M^1M^2 \otimes 1 + M^1M^3 \otimes \tfrac{1}{2}t_2^* + M^1M^4 \otimes \tfrac{1}{2}t_1 + M^3M^2 \otimes \tfrac{1}{2}t_1^*$$
$$- M^4M^2 \otimes \tfrac{1}{2}t_2 - \tfrac{1}{2}\mathrm{i}\hbar\alpha M^3M^4 \otimes 1 + \tfrac{1}{2}\mathrm{i}\hbar\beta M^4M^3 \otimes 1,$$
$$\widetilde{M}^2\widetilde{M}^1 = M^2M^1 \otimes 1 + M^3M^1 \otimes \tfrac{1}{2}t_2^* + M^4M^1 \otimes \tfrac{1}{2}t_1 + M^2M^3 \otimes \tfrac{1}{2}t_1^*$$
$$- M^2M^4 \otimes \tfrac{1}{2}t_2 + \tfrac{1}{2}\mathrm{i}\hbar\alpha M^4M^3 \otimes 1 - \tfrac{1}{2}\mathrm{i}\hbar\beta M^3M^4 \otimes 1,$$

from which it follows that the commutator is given by

$$[\widetilde{M}^1, \widetilde{M}^2] = [M^1, M^2] \otimes 1 - \mathrm{i}\hbar(\alpha - \beta) M^3 M^4 \otimes 1 = 0.$$

All other commutators are shown to vanish in the same way. □

Although we have made a change of generators, this does not affect the family of instantons constructed in the previous section. In order to show this, let

$$\triangleright' : H_F \otimes \mathcal{A}[\mathfrak{M}(k;\hbar)] \to \mathcal{A}[\mathfrak{M}(k;\hbar)]$$

be the left action of H_F on $\mathcal{A}[\mathfrak{M}(k;\hbar)]$ defined on generators by

$$\begin{aligned} t_1 \triangleright' \widetilde{M}^1 &= \mathrm{i}\hbar\alpha \widetilde{M}^3, & t_1^* \triangleright' \widetilde{M}^{1*} &= -\mathrm{i}\hbar\alpha \widetilde{M}^{3*}, \\ t_1^* \triangleright' \widetilde{M}^2 &= -\mathrm{i}\hbar\alpha \widetilde{M}^4, & t_1 \triangleright' \widetilde{M}^{2*} &= \mathrm{i}\hbar\alpha \widetilde{M}^{4*}, \\ t_2^* \triangleright' \widetilde{M}^1 &= -\mathrm{i}\hbar\beta \widetilde{M}^4, & t_2 \triangleright' \widetilde{M}^{1*} &= \mathrm{i}\beta \widetilde{M}^{4*}, \\ t_2 \triangleright' \widetilde{M}^2 &= -\mathrm{i}\hbar\beta \widetilde{M}^3, & t_2^* \triangleright' \widetilde{M}^{2*} &= \mathrm{i}\hbar\beta \widetilde{M}^{3*}, \end{aligned}$$

together with $t_j \triangleright' \widetilde{M}^l = 0$ and $t_j \triangleright' \widetilde{M}^{l*} = 0$ for $l = 3, 4$. Let us write $\mathcal{A}[\mathfrak{M}(k;\hbar)] \rtimes H_F$ for the smash product algebra associated to the action $\triangleright'$.

Theorem 5.16. *There is a $*$-algebra isomorphism $\phi : \mathcal{A}[\mathfrak{M}(k;\hbar)] \rtimes H_F \to \mathcal{A}[\mathsf{M}_{k;\hbar}] \rtimes H_F$ defined for each $h \in H_F$ by*

$$\begin{aligned} \widetilde{M}^1_{ab} \otimes h &\mapsto M^1_{ab} \otimes h + M^3_{ab} \otimes \tfrac{1}{2} t_1^* h - M^4_{ab} \otimes \tfrac{1}{2} t_2 h, & \widetilde{M}^3_{ab} \otimes h &\mapsto M^3_{ab} \otimes h, \\ \widetilde{M}^2_{ab} \otimes h &\mapsto M^2_{ab} \otimes h + M^3_{ab} \otimes \tfrac{1}{2} t_2^* h - M^4_{ab} \otimes \tfrac{1}{2} t_1 h, & \widetilde{M}^4_{ab} \otimes h &\mapsto M^4_{ab} \otimes h \end{aligned}$$

and extended as a $*$-algebra map.

Proof. It is clear that this map is an isomorphism of vector spaces with inverse

$$\begin{aligned} M^1_{ab} \otimes h &\mapsto \widetilde{M}^1_{ab} \otimes h - \widetilde{M}^3_{ab} \otimes \tfrac{1}{2} t_1^* h + \widetilde{M}^4_{ab} \otimes \tfrac{1}{2} t_2 h, & M^3_{ab} \otimes h &\mapsto \widetilde{M}^3_{ab} \otimes h, \\ M^2_{ab} \otimes h &\mapsto \widetilde{M}^2_{ab} \otimes h - \widetilde{M}^3_{ab} \otimes \tfrac{1}{2} t_2^* h - \widetilde{M}^4_{ab} \otimes \tfrac{1}{2} t_1 h, & M^4_{ab} \otimes h &\mapsto \widetilde{M}^4_{ab} \otimes h. \end{aligned}$$

By definition, the map ϕ is a $*$-algebra homomorphism on the subalgebra $\mathcal{A}[\mathfrak{M}(k;\hbar)]$, so we just have to check that it preserves the cross-relations between $\mathcal{A}[\mathfrak{M}(k;\hbar)]$ and the subalgebra H_F. This is straightforward to verify: one has for example that

$$\begin{aligned} \phi(1 \otimes t_1)\phi(\widetilde{M}^1 \otimes 1) &= (1 \otimes t_1)(M^1_{ab} \otimes 1 + M^3_{ab} \otimes \tfrac{1}{2} t_1^* - M^4_{ab} \otimes \tfrac{1}{2} t_2) \\ &= (M^1_{ab} \otimes t_1 + M^3_{ab} \otimes \tfrac{1}{2} t_1^* t_1 - M^4_{ab} \otimes \tfrac{1}{2} t_2 t_1) + (t_1 \triangleright M^1) \otimes 1 \\ &= \phi(\widetilde{M}^1 \otimes t_1 + (t_1 \triangleright' \widetilde{M}^1) \otimes 1) \\ &= \phi((1 \otimes t_1)(\widetilde{M}^1 \otimes 1)). \end{aligned}$$

The remaining relations are verified in the same way. □

Our goal is now to see that the parameters corresponding to the subalgebra H_F can be removed and that there is a family of instantons parameterised by the commutative algebra $\mathcal{A}[\mathfrak{M}(k;\hbar)]$. This follows from the fact that there is a right coaction

$$\delta_R : \mathcal{A}[\mathfrak{M}(k;\hbar)] \rtimes H_F \to (A[\mathfrak{M}(k;\hbar)] \rtimes H_F) \otimes H_F, \qquad \delta_R := \mathrm{id} \otimes \Delta, \tag{5.16}$$

where $\Delta : H_F \to H_F \otimes H_F$ is the coproduct on the Hopf algebra H_F. This coaction is by 'gauge transformations', in the sense that the projections $\widetilde{\mathsf{P}} \otimes 1$ and $\delta_R(\widetilde{\mathsf{P}})$ are unitarily equivalent in the matrix algebra $\mathrm{M}_{2k+2}\left((\mathcal{A}[\mathfrak{M}(k;\hbar)]\rtimes H_F)\otimes H_F\right)$ and so they define gauge equivalent families of instantons [5]. This means that the parameters determined by the subalgebra H_F in $\mathcal{A}[\mathfrak{M}(k;\hbar)]\rtimes H_F$ are just gauge parameters and so they may be removed. Indeed, by passing to the subalgebra of $\mathcal{A}[\mathfrak{M}(k;\hbar)]\rtimes H_F$ consisting of coinvariant elements under the coaction (5.16), *viz.*

$$\mathcal{A}[\mathfrak{M}(k;\hbar)] \cong \{a \in \mathcal{A}[\mathfrak{M}(k;\hbar)]\rtimes H_F \mid \delta_R(a) = a\otimes 1\},$$

we obtain a projection $\mathsf{P}_{k;\hbar}$ with entries in $\mathcal{A}[\mathfrak{M}(k;\hbar)]\otimes\mathcal{A}[\mathbb{R}^4_\hbar]$. The precise construction of the projection $\mathsf{P}_{k;\hbar}$ goes exactly as in [5], as does the proof of the fact that the Grassmann family of connections $\nabla = \mathsf{P}_{k;\hbar}\circ(\mathrm{id}\otimes\mathrm{d})$ has anti-self-dual curvature and hence defines a family of instantons on $\mathbb{R}^4_\hbar$.

For each point $x\in\mathfrak{M}(k;\hbar)$ there is an evaluation map

$$\mathrm{ev}_x\otimes\mathrm{id} : \mathcal{A}[\mathfrak{M}(k;\hbar)]\otimes\mathcal{A}[\mathbb{C}^4_\hbar] \to \mathcal{A}[\mathbb{C}^4_\hbar],$$

which in turn defines a self-conjugate monad over the noncommutative space $\mathbb{C}^4_\hbar$. In analogy with Remark 4.3, the matrices $(\mathrm{ev}_x\otimes\mathrm{id})\tilde{\sigma}_z$ and $(\mathrm{ev}_x\otimes\mathrm{id})\tilde{\tau}_z$ determine a complex of free right $\mathcal{A}[\mathbb{C}^4_\hbar]$-modules

$$0\to\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4_\hbar]\xrightarrow{(\mathrm{ev}_x\otimes\mathrm{id})\tilde{\sigma}_z}\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4_\hbar]\xrightarrow{(\mathrm{ev}_x\otimes\mathrm{id})\tilde{\tau}_z}\mathcal{L}\otimes\mathcal{A}[\mathbb{C}^4_\hbar]\to 0. \tag{5.17}$$

The same evaluation map determines a projection $(\mathrm{ev}_x\otimes\mathrm{id})\mathsf{P}_{k;\hbar}$ and hence an instanton connection on $\mathbb{R}^4_\hbar$.

As described in Proposition 4.7, the gauge freedom in the classical ADHM construction is precisely the freedom determined by the choice of bases of the vector spaces $\mathcal{H},\mathcal{K},\mathcal{L}$. Clearly we also have this freedom in the noncommutative construction as well: we write $\sim$ for the equivalence relation induced on the space $\mathfrak{M}(k;\hbar)$ by such changes of basis (*cf.* Definition 4.8). This leads to the following explicit description of the parameter space $\mathfrak{M}(k;\hbar)$ (*cf.* [14]).

Theorem 5.17. *For each positive integer $k\in\mathbb{Z}$, the space $\mathfrak{M}(k;\hbar)/\sim$ of equivalence classes of self-conjugate monads over $\mathbb{C}^4_\hbar$ is the quotient of the set of complex matrices $B_1,B_2\in\mathrm{M}_k(\mathbb{C})$, $I\in\mathrm{M}_{2\times k}(\mathbb{C})$, $J\in\mathrm{M}_{k\times 2}(\mathbb{C})$ satisfying the equations*

(i) $[B_1,B_2]+IJ=0$,

(ii) $[B_1,B_1^*]+[B_2,B_2^*]+II^*-J^*J=-\mathrm{i}\hbar(\alpha+\beta)\mathbb{1}_k$

by the action of $\mathrm{U}(k)$ *given by*

$$B_1\mapsto gB_1g^{-1},\qquad B_2\mapsto gB_2g^{-1},\qquad I\mapsto gI,\qquad J\mapsto Jg^{-1}$$

for each $g\in\mathrm{U}(k)$.

Proof. Recall that we write the monad maps $\tilde{\sigma}_z$, $\tilde{\tau}_z$ as

$$\tilde{\sigma}_z=\widetilde{M}^1z_1+\widetilde{M}^2z_2+\widetilde{M}^3z_3+\widetilde{M}^4z_4,\qquad\tilde{\tau}_z=\widetilde{N}^1z_1+\widetilde{N}^2z_2+\widetilde{N}^3z_3+\widetilde{N}^4z_4$$

for constant matrices $\widetilde{M}^j$, $\widetilde{N}^l$, where $j,l=1,\ldots,4$. Upon expanding out the condition $\tilde{\tau}_z\circ\tilde{\sigma}_z=0$ and using the commutation relations in Lemma 5.1, we find the conditions

$$\widetilde{N}^j\widetilde{M}^l+\widetilde{N}^l\widetilde{M}^j+\mathrm{i}\hbar(\alpha+\beta)(\delta^{j1}\delta^{l2}-\delta^{j2}\delta^{l1})=0 \tag{5.18}$$

for $j,l=1,\ldots,4$. Recall from Lemma 2.2 that the typical fibre $\mathbb{CP}^1$ of the twistor fibration $\mathbb{R}^4\times\mathbb{CP}^1$ has homogeneous coordinates z_1, z_1^*, z_2, z_2^*; it follows that the 'line at infinity' ℓ_∞ is recovered by setting $z_1 = z_2 = 0$. On this line, the monad condition $\tilde{\tau}_z \circ \tilde{\sigma}_z = 0$ becomes

$$\widetilde{N}^3\widetilde{M}^4 + \widetilde{N}^4\widetilde{M}^3 = 0, \qquad \widetilde{N}^3\widetilde{M}^3 = 0, \qquad \widetilde{N}^4\widetilde{M}^4 = 0. \tag{5.19}$$

Moreover, when $z_1 = z_2 = 0$ we see from the relations (5.7)–(5.8) that the coordinates z_3, z_4 and their conjugates are mutually commuting, so that the line ℓ_∞ is classical. The self-conjugacy of the monad implies that the restricted bundle over ℓ_∞ is trivial; therefore we can argue as in [22] to show that the map $\widetilde{N}^3\widetilde{M}^4 = -\widetilde{N}^4\widetilde{M}^3$ is an isomorphism. Using these conditions we choose bases for $\mathcal{H}, \mathcal{K}, \mathcal{L}$ such that $\widetilde{N}^3\widetilde{M}^4 = \mathbb{1}_k$ and

$$\widetilde{M}^3 = \begin{pmatrix} \mathbb{1}_{k\times k} \\ 0_{k\times k} \\ 0_{2\times k} \end{pmatrix}, \qquad \widetilde{M}^4 = \begin{pmatrix} 0_{k\times k} \\ \mathbb{1}_{k\times k} \\ 0_{2\times k} \end{pmatrix}, \qquad \widetilde{N}^3 = \begin{pmatrix} 0_{k\times k} \\ \mathbb{1}_{k\times k} \\ 0_{k\times 2} \end{pmatrix}^{\mathrm{tr}}, \qquad \widetilde{N}^4 = \begin{pmatrix} -\mathbb{1}_{k\times k} \\ 0_{k\times k} \\ 0_{k\times 2} \end{pmatrix}^{\mathrm{tr}}.$$

Now invoking conditions (5.18) for $j = 3, 4$ and $l = 1, 2$, the remaining matrices are necessarily of the form

$$\widetilde{M}^1 = \begin{pmatrix} B_1 \\ B_2 \\ J \end{pmatrix}, \qquad \widetilde{M}^2 = \begin{pmatrix} B_1' \\ B_2' \\ J' \end{pmatrix}, \qquad \widetilde{N}^1 = \begin{pmatrix} -B_2 \\ B_1 \\ I \end{pmatrix}^{\mathrm{tr}}, \qquad \widetilde{N}^2 = \begin{pmatrix} -B_2' \\ B_1' \\ I' \end{pmatrix}^{\mathrm{tr}}.$$

Using the conditions $\tilde{\tau}^*_{J(z)} = -\tilde{\sigma}_z$ and $\tilde{\sigma}^*_{J(z)} = \tilde{\tau}_z$, which correspond to the requirement that the monad be self-conjugate, we find that

$$B_1' = -B_2^*, \qquad B_2' = B_1^*, \qquad J' = I^*, \qquad I' = -J^*.$$

Thus in order to satisfy the condition $\tilde{\tau}_z \circ \tilde{\sigma}_z = 0$ it remains only to impose the conditions (5.18) in the cases $j = l = 1$ and $j = 1$, $l = 2$. The first of these is condition (i) in the theorem; the second case is equivalent to requiring

$$[B_1, B_1^*] + [B_2, B_2^*] + II^* - J^*J + \mathrm{i}\hbar(\alpha + \beta) = 0,$$

giving condition (ii) in the theorem. Just as in the classical case [11], it is evident that the remaining gauge freedom in this calculation is given by the stated action of $\mathrm{U}(k)$, whence the result. □

Finally, we comment on a significant difference between the parameter space M_k of instantons on the classical space $\mathbb{R}^4$ and the parameter space $\mathfrak{M}(k;\hbar)$ of instantons on the Moyal-deformed version $\mathbb{R}^4_\hbar$. Recall that, in the classical ADHM construction of §4.2, we needed to assume that the algebra-valued matrix ρ^2 of Eq. (4.11) is invertible. Formally adjoining an inverse ρ^{-2} to the algebra $\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{A}[\mathsf{M}_k] \otimes \mathcal{A}[\mathbb{R}^4]$ resulted in the deletion of a collection of points from the parameter space M_k. In contrast, this noncommutative ADHM construction does not require this. Using the Moyal ADHM equations themselves one shows that the matrix ρ^2, which now having passed to the commutative parameter space has entries in the algebra $\mathcal{A}[\mathfrak{M}(k;\hbar)] \otimes \mathcal{A}[\mathbb{R}^4_\hbar]$, is automatically invertible (we refer to [21, 13] for a proof).

6. The Connes-Landi Noncommutative Plane $\mathbb{R}^4_\theta$

Next we turn to the construction of instantons on the noncommutative plane $\mathbb{R}^4_\theta$, which is an example of a toric noncommutative manifold (or isospectral deformation) in the sense of [9]. In particular, $\mathbb{R}^4_\theta$ is obtained as a localisation of the Connes-Landi quantum four-sphere S^4_θ, just as in Lemma 2.1 (*cf.* [17]), although here we shall obtain it directly from classical $\mathbb{R}^4$ by cocycle twisting.

6.1. **Toric deformation of the space of monads.** Whereas the Moyal space-time $\mathbb{R}^4_\hbar$ was obtained by cocycle twisting along an action of the group of translation sysmmetries of space-time, the noncommutative space-time $\mathbb{R}^4_\theta$ is constructed by deforming the classical coordinate algebra $\mathcal{A}[\mathbb{R}^4]$ along an action of a group of rotational symmetries.

Indeed, for the twisting Hopf algebra we take $H = \mathcal{A}[\mathbb{T}^2]$, the algebra of coordinate functions on the two-torus $\mathbb{T}^2$. It is the commutative unital algebra

$$\mathcal{A}[\mathbb{T}^2] := \mathcal{A}[s_j, s_j^{-1} \mid j = 1, 2]$$

equipped with the Hopf $*$-algebra structure

$$s_j^* = s_j^{-1}, \qquad \Delta(s_j) = s_j \otimes s_j, \qquad \epsilon(s_j) = 1, \qquad S(s_j) = s_j^{-1} \tag{6.1}$$

for $j = 1, 2$, with Δ, ϵ extended as $*$-algebra maps and S extended as a $*$-anti-algebra map.

In order to deform the twistor fibration, we need to equip the various coordinate algebras with left H-comodule structures. There is a Hopf algebra projection from $\mathcal{A}[\mathrm{GL}^+(2, \mathbb{H})]$ onto H, defined on generators by

$$\pi : \mathcal{A}[\mathrm{GL}^+(2, \mathbb{H})] \to H, \qquad \begin{pmatrix} \alpha_1 & -\alpha_2^* & \beta_1 & -\beta_2^* \\ \alpha_2 & \alpha_1^* & \beta_2 & \beta_1^* \\ \gamma_1 & -\gamma_2^* & \delta_1 & -\delta_2^* \\ \gamma_2 & \gamma_1^* & \delta_2 & \delta_1^* \end{pmatrix} \mapsto \begin{pmatrix} s_1 & 0 & 0 & 0 \\ 0 & s_1^* & 0 & 0 \\ 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & s_2^* \end{pmatrix} \tag{6.2}$$

and extended as a $*$-algebra map. Using Eq. (2.16), this projection determines a left H-coaction $\Delta_\pi : \mathcal{A}[\mathbb{C}^4] \to H \otimes \mathcal{A}[\mathbb{C}^4]$ by

$$\mathcal{A}[\mathbb{C}^4] \to H \otimes \mathcal{A}[\mathbb{C}^4], \qquad z_j \mapsto \varsigma_j \otimes z_j, \tag{6.3}$$

extended as a $*$-algebra map, where we use the shorthand notation $(\varsigma_j) = (s_1, s_1^*, s_2, s_2^*)$ for the generators of H. Using the identification of generators in Eq. (2.10), this induces a coaction on the space-time algebra,

$$\mathcal{A}[\mathbb{R}^4] \to H \otimes \mathcal{A}[\mathbb{R}^4], \qquad \zeta_1 \mapsto \varsigma_1\varsigma_4 \otimes \zeta_1, \quad \zeta_2 \mapsto \varsigma_2\varsigma_4 \otimes \zeta_2, \tag{6.4}$$

and extended as a $*$-algebra map, making $\mathcal{A}[\mathbb{R}^4]$ into a left H-comodule $*$-algebra.

As a twisting cocycle on H, we take the linear map defined on generators by

$$F : H \otimes H \to \mathbb{C}, \qquad F(s_j, s_l) = \exp(\mathrm{i}\pi\Theta_{jl}) \tag{6.5}$$

and extended as a Hopf bicharacter in the sense of Eq.(1.3). Here the deformation matrix Θ is the 2×2 real anti-symmetric matrix

$$\Theta = (\Theta_{jl}) = \tfrac{1}{2} \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$$

for $0 < \theta < 1$ a real parameter. It is straightforward to check using the formulæ (1.9)–(1.10) and (1.12) that the product, antipode and $*$-structure on H are in fact undeformed by F, so that $H = H_F$ as a Hopf $*$-algebra. However, the effect of the twisting on the H-comodule algebras $\mathcal{A}[\mathbb{C}^4]$ and $\mathcal{A}[\mathbb{R}^4]$ is non-trivial. In what follows we write $\eta_{jl} := F^{-2}(\varsigma_j, \varsigma_l)$, namely

$$(6.6) \qquad (\eta_{jl}) = \begin{pmatrix} 1 & 1 & \mu & \bar{\mu} \\ 1 & 1 & \bar{\mu} & \mu \\ \bar{\mu} & \mu & 1 & 1 \\ \mu & \bar{\mu} & 1 & 1 \end{pmatrix}, \qquad \mu = e^{\mathrm{i}\pi\theta}.$$

Lemma 6.1. *The relations in the H-comodule algebra $\mathcal{A}[\mathbb{C}^4]$ are twisted into*

$$(6.7) \qquad z_j z_l = \eta_{lj} z_l z_j, \qquad z_j z_l^* = \eta_{jl} z_l^* z_j, \qquad z_j^* z_l = \eta_{jl} z_l z_j^*, \qquad z_j^* z_l^* = \eta_{lj} z_l^* z_j^*$$

for each $j, l = 1, \ldots, 4$.

Proof. The cocycle-twisted product on the H-comodule algebra $\mathcal{A}[\mathbb{C}^4]$ is defined by the formula (1.14). Just as in Lemma 5.1, the corresponding algebra relations can be expressed using the R-matrix (1.13): in this case one finds that the R-matrix takes the values

$$(6.8) \qquad \mathcal{R}(\varsigma_j, \varsigma_l) = F^{-2}(\varsigma_j, \varsigma_l) = \eta_{jl}, \qquad \mathcal{R}(\varsigma_j, \varsigma_l^*) = F^{-2}(\varsigma_j, \varsigma_l^*) = \eta_{lj}.$$

By explicitly computing Eqs. (5.9) (and omitting the product symbol $\cdot_F$), one obtains the relations stated in the lemma. We denote by $\mathcal{A}[\mathbb{C}^4_\theta]$ the algebra generated by $\{z_j, z_j^* \mid j = 1, \ldots, 4\}$ modulo the relations (6.7). In this way, we have that $\mathcal{A}[\mathbb{C}^4_\theta]$ is a left H_F-comodule $*$-algebra. □

Lemma 6.2. *The algebra relations in the H-comodule algebra $\mathcal{A}[\mathbb{R}^4]$ are twisted into*

$$(6.9) \qquad \zeta_1\zeta_2 = \lambda\zeta_2\zeta_1, \quad \zeta_1^*\zeta_2^* = \lambda\zeta_2^*\zeta_1^*, \quad \zeta_2^*\zeta_1 = \lambda\zeta_1\zeta_2^*, \quad \zeta_2\zeta_1^* = \lambda\zeta_1^*\zeta_2,$$

where the deformation parameter is $\lambda := \mu^2 = e^{2\pi\mathrm{i}\theta}$.

Proof. The product on $\mathcal{A}[\mathbb{R}^4]$ is once again twisted using the formula (1.14). Again omitting the product symbol $\cdot_F$, the relations are computed to be as stated. We denote by $\mathcal{A}[\mathbb{R}^4_\theta]$ the algebra generated by ζ_1, ζ_2 and their conjugates, subject to these relations. They make $\mathcal{A}[\mathbb{R}^4_\theta]$ into a left H_F-comodule $*$-algebra. □

Remark 6.3. Since the generators z_1, z_2 and their conjugates generate a commutative subalgebra of $\mathcal{A}[\mathbb{C}^4_\theta]$, it is easy to see using Lemma 2.2 that it is only the base space $\mathbb{R}^4_\theta$ of the localised twistor bundle that is deformed. The typical fibre $\mathbb{CP}^1$ remains classical and the localised twistor algebra is isomorphic to the tensor product $\mathcal{A}[\mathbb{R}^4_\theta] \otimes \mathcal{A}[\mathbb{CP}^1]$.

The canonical differential calculi described in §3.1 are also deformed. The relations in the quantised calculi are given in the following lemmata.

Lemma 6.4. *The twisted differential calculus $\Omega(\mathbb{C}^4_\theta)$ is generated by the degree zero elements z_j, z_l^* and the degree one elements $\mathrm{d}z_j, \mathrm{d}z_l^*$ for $j, l = 1, \ldots, 4$, subject to the bimodule relations between functions and one-forms*

$$z_j \mathrm{d}z_l = \eta_{lj}(\mathrm{d}z_l)z_j, \qquad z_j \mathrm{d}z_l^* = \eta_{jl}(\mathrm{d}z_l^*)z_j$$

for $j,l=1,\dots,4$ *and the anti-commutation relations between one-forms*

$$\mathrm{d}z_j\wedge\mathrm{d}z_l+\eta_{lj}\mathrm{d}z_l\wedge\mathrm{d}z_j=0,\qquad \mathrm{d}z_j\wedge\mathrm{d}z_l^*+\eta_{jl}\mathrm{d}z_l^*\wedge\mathrm{d}z_j=0$$

for $j,l=1,\dots,4$.

Proof. One views the classical calculus $\Omega(\mathbb{C}^4)$ as a left H-comodule algebra and accordingly computes the deformed product using the twisting cocycle F. Since the exterior derivative d is H-equivariant, the (anti-) commutation relations in the deformed calculus $\Omega(\mathbb{C}^4_\theta)$ are exactly the same as the algebra relations in $\mathcal{A}[\mathbb{C}^4_\theta]$ but with d inserted appropriately. □

Lemma 6.5. *The twisted differential calculus* $\Omega(\mathbb{R}^4_\theta)$ *is generated by the degree zero elements* $\zeta_1,\zeta_1^*,\zeta_2,\zeta_2^*$ *and the degree one elements* $\mathrm{d}\zeta_1,\mathrm{d}\zeta_1^*,\mathrm{d}\zeta_2,\mathrm{d}\zeta_2^*$, *subject to the relations*

$$\zeta_1\,\mathrm{d}\zeta_2-\lambda\mathrm{d}\zeta_2\,\zeta_1=0,\qquad \zeta_2^*\mathrm{d}\zeta_1-\lambda\mathrm{d}\zeta_1\,\zeta_2^*=0,$$
$$\mathrm{d}\zeta_1\wedge\mathrm{d}\zeta_2+\lambda\mathrm{d}\zeta_2\wedge\mathrm{d}\zeta_1=0,\qquad \mathrm{d}\zeta_2^*\wedge\mathrm{d}\zeta_1+\lambda\mathrm{d}\zeta_1\wedge\mathrm{d}\zeta_2^*=0.$$

Proof. Once again, the classical calculus $\Omega(\mathbb{R}^4)$ is deformed as a twisted left H-comodule algebra, with the relations working out to be as stated. □

In particular, it is clear that the vector space $\Omega^2(\mathbb{R}^4_\theta)$ is the same as it is classically. The Hodge operator $*:\Omega^2(\mathbb{R}^4)\to\Omega^2(\mathbb{R}^4)$ commutes with the H-coaction in the sense that

$$\Delta_\pi(*\omega)=(\mathrm{id}\otimes *)\Delta_\pi(\omega),\qquad \omega\in\Omega^2(\mathbb{R}^4),$$

so that there is also a Hodge operator $*_\theta:\Omega^2(\mathbb{R}^4_\theta)\to\Omega^2(\mathbb{R}^4_\theta)$ defined by the same formula as it is classically. There is a decomposition of $\Omega^2(\mathbb{R}^4_\theta)$ into self-dual and anti-self-dual two-forms

$$\Omega^2(\mathbb{R}^4_\theta)=\Omega^2_+(\mathbb{R}^4_\theta)\oplus\Omega^2_-(\mathbb{R}^4_\theta)$$

which, at the level of vector spaces, is identical to the corresponding decomposition in the classical case.

We also apply the cocycle deformation the coordinate algebra $\mathcal{A}[\mathsf{M}_k]$ of the space of self-conjugate monads by viewing it as a left H-comodule algebra. We write $\mathcal{A}[\mathsf{M}_{k;\theta}]$ for the resulting cocycle-twisted left H_F-comodule algebra.

Proposition 6.6. *The noncommutative* $*$-*algebra* $\mathcal{A}[\mathsf{M}_{k;\theta}]$ *is generated by the matrix elements* M^j_{ab}, N^l_{dc} *for* $a,c=1,\dots,k$ *and* $b,d=1,\dots,2k+2$, *modulo the relations*

$$M^j_{ab}M^l_{cd}=\eta_{lj}M^l_{cd}M^j_{ab},\qquad N^j_{ba}N^l_{dc}=\eta_{lj}N^l_{dc}N^j_{ba},$$

together with the $*$-*structure* (4.6).

Proof. From Lemma 4.4 we read off the H-coaction on generators M^j, $j=1,\dots,4$, obtaining

$$M^j_{ab}\mapsto\varsigma_j^*\otimes M^j_{ab},\qquad N^l_{dc}\mapsto\varsigma_l^*\otimes N^l_{dc},$$

which we extend as a $*$-algebra map. The deformed relations follow immediately from an application of the twisting formula (1.14). The coaction of H on $\mathcal{A}[\mathsf{M}_k]$ does not depend on the matrix indices of the generators M^j, N^l, $j,l=1,\dots,4$, hence neither do the twisted commutation relations. In terms of the deformed product, the relations (4.4) are twisted into the relations

$$\sum_r \left(N^j_{dr} M^l_{rb} + \eta_{jl} N^l_{dr} M^j_{rb}\right) = 0$$

for all $j, l = 1, \dots, 4$ and $b, d = 1, \dots, k$. □

6.2. **The construction of instantons on $\mathbb{R}^4_\theta$.** Just as we did for the Moyal plane, we now use the noncommutative space of monads $\mathrm{M}_{k;\theta}$ to construct families of instantons on the Connes-Landi space-time $\mathbb{R}^4_\theta$.

The Pontryagin dual of the torus $\mathbb{T}^2$ is the discrete group $\widehat{\mathbb{T}}^2 \simeq \mathbb{Z}^2$. Given a pair of integers $(r_1, r_2) \in \mathbb{Z}^2$ we define unitary elements $\vec{u} = (u_1, u_2, u_3, u_4)$ of the algebra H_F by

$$\vec{u} = (u_1, u_2, u_3, u_4) = (\varsigma_1^{m_1}, \varsigma_2^{m_2}, \varsigma_3^{m_3}, \varsigma_4^{m_4}), \tag{6.10}$$

where $(m_j) = (r_1, r_1, r_2, r_2)$. It is clear that $u_1^* = u_2$ and $u_3^* = u_4$, and that each u_j is a group-like element of the Hopf algebra H_F, *i.e.* it transforms as $\Delta(u_j) = u_j \otimes u_j$ under the coproduct $\Delta : H_F \to H_F \otimes H_F$.

Lemma 6.7. *There is a canonical left action of H_F on the algebra $\mathcal{A}[\mathsf{M}_{k;\theta}]$ defined on generators by*

$$u_l \triangleright M^j_{ab} = \mathcal{R}(\varsigma_j^*, \varsigma_l^{m_l}) M^j_{ab} = \eta_{lj}^{m_l} M^j_{ab}, \qquad u_l \triangleright M^j_{ab}{}^* = \mathcal{R}(\varsigma_j, \varsigma_l^{m_l}) M^j_{ab}{}^* = \eta_{jl}^{m_l} M^j_{ab}{}^*$$

for $j, l = 1, \dots, 4$.

Proof. From Proposition 5.6 we know that $\mathcal{A}[\mathsf{M}_{k;\theta}]$ is a left H_F-comodule algebra; it is therefore also a left H_F-module algebra according to the formula (1.5), which works out to be as stated. □

This also gives us an action of the group $\mathbb{Z}^2$ on the algebra $\mathcal{A}[\mathsf{M}_{k;\theta}]$ by $*$-automorphisms,

$$\gamma : \mathbb{Z}^2 \to \mathrm{Aut}\, \mathcal{A}[\mathsf{M}_{k;\theta}]. \tag{6.11}$$

The smash product algebra $\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F$ corresponding the to H_F-action of Lemma 6.7 works out using the coproduct $\Delta(\varsigma_j) = \varsigma_j \otimes \varsigma_j$ on H_F and the formula (1.7) to have relations of the form

$$(M^j_{ab} \otimes u_l)(M^r_{cd} \otimes u_s) = \eta_{lr}^{m_l} \eta_{rj} \eta_{js}^{m_s} (M^r_{cd} \otimes u_s)(M^j_{ab} \otimes u_l)$$
$$(M^j_{ab} \otimes u_l)(M^r_{cd}{}^* \otimes u_s) = \eta_{rl}^{m_l} \eta_{rj} \eta_{sj}^{m_s} (M^r_{cd}{}^* \otimes u_s)(M^j_{ab} \otimes u_l)$$

for $j, l, r, s = 1, \dots, 4$, together with their conjugates. This is another special case of Example 1.1. We think of this smash product $\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F$ as an algebraic version the crossed product algebra $\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes_\gamma \mathbb{Z}^2$.

Lemma 6.8. *The algebra structure of the tensor product $\mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\theta]$ is determined by the relations in the respective subalgebras $\mathcal{A}[\mathsf{M}_{k;\theta}]$ and $\mathcal{A}[\mathbb{C}^4_\theta]$ given above, together with the cross-relations*

$$M^j z_l = \eta_{jl} z_l M^j, \qquad M^j z_l^* = \eta_{lj} z_l^* M^j, \qquad j, l = 1, \dots, 4,$$

as well as their conjugates.

Proof. The classical algebra $\mathcal{A}[\mathsf{M}_k] \otimes \mathcal{A}[\mathbb{C}^4]$ is equipped with the tensor product H_F-coaction of Eq. (1.4). We deform the product in this algebra using the formula (1.14). The cross-terms in the resulting algebra $\mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\theta]$ are computed to be as stated [5]. Once again, the symbol $\underline{\otimes}$ is to remind us that the algebra structure on the tensor product is not the standard one and has been twisted by the deformation procedure. □

Once again we have a pair of matrices σ_z and τ_z,

$$\sigma_z = \sum\nolimits_j M^j \otimes z_j, \qquad\qquad \tau_z = \sum\nolimits_j N^j \otimes z_j,$$

but whose entries live in the twisted algebra $\mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\theta]$. The matrix $\mathsf{V} := \begin{pmatrix} \sigma_z & \sigma_{J(z)} \end{pmatrix}$ is a $2k \times (2k+2)$ matrix with entries in $\mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\theta]$, using which we define $\rho^2 := \mathsf{V}^*\mathsf{V}$. From the projection $\mathsf{Q} := \mathsf{V}\rho^{-2}\mathsf{V}^*$ we construct the complementary matrix $\mathsf{P} := \mathbb{1}_{2k+2} - \mathsf{Q}$, which has entries in the algebra $\mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{R}^4_\theta]$.

It is clear that this matrix P is a self-adjoint idempotent, $\mathsf{P}^2 = \mathsf{P} = \mathsf{P}^*$. However, just as was the case for the Moyal plane, it does not define an honest family of projections in the sense of Definition 3.2, since it has values in the *twisted* tensor product algebra. We recover a genuine family of projections using the following lemma, in which we use the Sweedler notation $Z \mapsto Z^{(-1)} \otimes Z^{(0)}$ for the left coaction $\mathcal{A}[\mathbb{C}^4_\theta] \to H_F \otimes \mathcal{A}[\mathbb{C}^4_\theta]$ defined in Eq. (6.3).

Lemma 6.9. *There is a canonical $*$-algebra map*

$$\mu : \mathcal{A}[\mathsf{M}_{k;\theta}] \underline{\otimes} \mathcal{A}[\mathbb{C}^4_\theta] \to (\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\theta]$$

defined by $\mu(M \otimes Z) = M \otimes Z^{(-1)} \otimes Z^{(0)}$ for each $M \in \mathcal{A}[\mathsf{M}_{k;\theta}]$ and $Z \in \mathcal{A}[\mathbb{C}^4_\theta]$.

Proof. The proof is identical to that of Lemma 5.10, save for the replacement of the coaction(5.4) by the coaction (6.3). □

As a consequence, we find that there are maps

$$\tilde{\sigma}_z : \mathcal{H} \otimes \mathcal{A}[\mathbb{C}^4_\theta] \to (\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4_\theta], \tag{6.12}$$

$$\tilde{\tau}_z : \mathcal{K} \otimes \mathcal{A}[\mathbb{C}^4_\theta] \to (\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{L} \otimes \mathcal{A}[\mathbb{C}^4_\theta] \tag{6.13}$$

defined by composing σ_z and τ_z with the map μ. With the coaction (6.3), they work out to be

$$\tilde{\sigma}_z := \sum\nolimits_r M^r \otimes \varsigma_r \otimes z_r, \qquad\qquad \tilde{\tau}_z := \sum\nolimits_r N^r \otimes \varsigma_r \otimes z_r,$$

which are respectively $k \times (2k+2)$ and $(2k+2) \times k$ matrices with entries in the noncommutative algebra $(\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\theta]$. With this in mind, we form the $(2k+2) \times 2k$ matrix $\widetilde{\mathsf{V}} := \begin{pmatrix} \tilde{\sigma}_z & \tilde{\sigma}_{J(z)} \end{pmatrix}$, this time yielding a $2k \times (2k+2)$ matrix with entries in $(\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\theta]$, and define $\widetilde{\rho}^2 := \widetilde{\mathsf{V}}^*\widetilde{\mathsf{V}}$. Just as in the classical case, in order to proceed we need to slightly enlarge the matrix algebra $\mathrm{M}_k(\mathbb{C}) \otimes \mathcal{A}[\mathsf{M}_{k;\theta}] \otimes \mathcal{A}[\mathbb{C}^4_\theta]$ by adjoining an inverse element $\widetilde{\rho}^{-2}$ for $\widetilde{\rho}^2$.

Proposition 6.10. *The $(2k+2) \times (2k+2)$ matrix $\widetilde{\mathsf{Q}} = \widetilde{\mathsf{V}}\widetilde{\rho}^{-2}\widetilde{\mathsf{V}}^*$ is a projection, $\widetilde{\mathsf{Q}}^2 = \widetilde{\mathsf{Q}} = \widetilde{\mathsf{Q}}^*$, with entries in the algebra $(\mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F) \otimes \mathcal{A}[\mathbb{C}^4_\theta]$ and trace equal to $2k$.*

Proof. The fact that $\widetilde{\mathsf{Q}}$ is a projection follows from the fact that Q is a projection and μ is a $*$-algebra map. By construction, the entries of the matrix $\widetilde{\rho}^2$ are central in the algebra $(\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F)\otimes\mathcal{A}[\mathbb{C}^4_\theta]$ (this follows from the fact that the corresponding classical matrix elements are coinvariant under the left H-coaction), from which it follows that the trace computation in Proposition 4.5 is valid in the noncommutative case as well [5]. □

From the projection $\widetilde{\mathsf{Q}}$ we construct the complementary projection $\widetilde{\mathsf{P}} := \mathbb{1}_{2k+2} - \widetilde{\mathsf{Q}}$; it has entries in the algebra $(\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F)\otimes\mathcal{A}[\mathbb{R}^4_\theta]$ and has trace equal to two. In analogy with Definition 3.2, the finitely-generated projective module

$$\mathcal{E} := \widetilde{\mathsf{P}}\left((\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F)\otimes\mathcal{A}[\mathbb{R}^4_\theta]\right)^{2k+2}$$

defines a family of rank two vector bundles over $\mathbb{R}^4_\theta$ parameterised by the noncommutative algebra $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$. We equip this family of vector bundles with the family of Grassmann connections associated to the projection $\widetilde{\mathsf{P}}$.

Proposition 6.11. *The curvature $F = \widetilde{\mathsf{P}}((\mathrm{id}\otimes \mathrm{d})\widetilde{\mathsf{P}})^2$ of the Grassmann family of connections $\nabla := (\mathrm{id}\otimes\mathrm{d})\circ\widetilde{\mathsf{P}}$ is anti-self-dual.*

Proof. From Lemma 5.5 we know that the space of two-forms $\Omega^2(\mathbb{R}^4_\theta)$ and the Hodge $*$-operator $*_\theta : \Omega^2(\mathbb{R}^4_\theta)\to\Omega^2(\mathbb{R}^4_\theta)$ are undeformed and equal to their classical counterparts; similarly for the decomposition $\Omega^2(\mathbb{R}^4_\theta) = \Omega^2_+(\mathbb{R}^4_\theta)\oplus\Omega^2_-(\mathbb{R}^4_\theta)$ into self-dual and anti-self-dual two-forms. This identification of the 'quantum' with the 'classical' spaces of two-forms survives the tensoring with the parameter space $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$, which yields that $(\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F)\otimes\Omega^2_\pm(\mathbb{R}^4_\theta)$ and $(\mathcal{A}[\mathsf{M}_k]\otimes H)\otimes\Omega^2_\pm(\mathbb{R}^4)$ are isomorphic as vector spaces. Computing the curvature F in exactly the same way as in Proposition 4.6, we see that it must be anti-self-dual, since the same is true in the classical case. □

6.3. **The toric ADHM equations.** The previous section produced a family of instantons parameterised by the noncommutative algebra $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$. Just as we did for the Moyal space-time, we would like to find a suitable commutative subalgebra of $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$ and hence a family of instantons parameterised by a classical space. In order to do this, we introduce elements of $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$ defined by

$$\widetilde{M^1_{ab}} := M^1_{ab}\otimes\varsigma_1, \qquad \widetilde{M^2_{ab}} := M^2\otimes\varsigma_2, \qquad \widetilde{M^3_{ab}} := M^3_{ab}\otimes 1, \qquad \widetilde{M^4_{ab}} := M^4_{ab}\otimes 1$$

for each $a = 1,\ldots,k$ and $b = 1,\ldots,2k+2$, together with their conjugates $\widetilde{M^j_{ab}}^*$, $j = 1,\ldots,4$.

Definition 6.12. We write $\mathcal{A}[\mathfrak{M}(k;\theta)]$ for the $*$-subalgebra of $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$ generated by the elements $\widetilde{M^j_{ab}}$, $\widetilde{M^l_{ab}}^*$.

Proposition 6.13. *The algebra $\mathcal{A}[\mathfrak{M}(k;\theta)]$ is a commutative $*$-subalgebra of the smash product $\mathcal{A}[\mathsf{M}_{k;\theta}]\bowtie H_F$.*

Proof. The generators $\widetilde{M^3_{ab}}$, $\widetilde{M^4_{ab}}$ and their conjugates are obviously central. The generators $\widetilde{M^1_{ab}}$, $\widetilde{M^2_{ab}}$ and their conjugates are also easily seen to commute amongst themselves. We check the case $j\in\{1,2\}$, $l\in\{3,4\}$, yielding

$$\begin{aligned}\widetilde{M^j_{ab}}\widetilde{M^l_{cd}} &= (M^j_{ab}\otimes\varsigma_j)(M^l_{ab}\otimes 1) = \eta_{jl}M^j_{ab}M^l_{cd}\otimes\varsigma_j = \eta_{jl}\eta_{lj}M^l_{cd}M^j_{ab}\otimes\varsigma_j\\ &= (M^l_{cd}\otimes 1)(M^j_{ab}\otimes\varsigma_j) = \widetilde{M^l_{cd}}\widetilde{M^j_{ab}}.\end{aligned}$$

All other pairs of generators are shown to commute using similar computations. □

Although we have changed our set of generators, we nevertheless combine them with the Hopf algebra H_F using a smash product construction. Let $\triangleright' : H_F \otimes \mathcal{A}[\mathfrak{M}(k;\theta)] \to \mathcal{A}[\mathfrak{M}(k;\theta)]$ be the left H_F-action defined by

$$\varsigma_l \triangleright' \widetilde{M^j_{ab}} = \eta_{lj} \widetilde{M^j_{ab}}, \qquad \varsigma_l \triangleright' \widetilde{M^j_{ab}}^* = \eta_{jl} \widetilde{M^j_{ab}}^*$$

for $j, l = 1, \ldots, 4$ and let $\mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F$ be the corresponding smash product algebra. The next proposition relates the parameter space $\mathcal{A}[\mathfrak{M}(k;\theta)]$ to the parameter space $\mathcal{A}[\mathsf{M}_{k;\theta}]$.

Theorem 6.14. *There is a $*$-algebra isomorphism $\phi : \mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F \to \mathcal{A}[\mathsf{M}_{k;\theta}] \rtimes H_F$ defined for each $h \in H_F$ by*

$$\widetilde{M^1_{ab}} \otimes h \mapsto M^1_{ab} \otimes \varsigma_1 h, \qquad \widetilde{M^2_{ab}} \otimes h \mapsto M^2_{ab} \otimes \varsigma_2 h,$$
$$\widetilde{M^3_{ab}} \otimes h \mapsto M^3_{ab} \otimes h, \qquad \widetilde{M^4_{ab}} \otimes h \mapsto M^4_{ab} \otimes h$$

and extended as a $*$-algebra map.

Proof. The given map is clearly a vector space isomorphism with inverse

$$M^1_{ab} \otimes h \mapsto \widetilde{M^1_{ab}} \otimes \varsigma_1^* h, \qquad M^2_{ab} \otimes h \mapsto \widetilde{M^2_{ab}} \otimes \varsigma_2^* h,$$
$$M^3_{ab} \otimes h \mapsto \widetilde{M^3_{ab}} \otimes h, \qquad M^4_{ab} \otimes h \mapsto \widetilde{M^4_{ab}} \otimes h,$$

extended as a $*$-algebra map. By definition, the map ϕ is a $*$-algebra homomorphism on the subalgebra $\mathcal{A}[\mathfrak{M}(k;\theta)]$, so it remains to check that it preserves the cross-relations with the subalgebra H_F. This is easy to verify: one has for example that

$$\begin{aligned}\phi(1 \otimes \varsigma_j)\phi(\widetilde{M^1} \otimes 1) &= (1 \otimes \varsigma_j)(M^1 \otimes \varsigma_1) = \eta_{j1} M^1 \otimes \varsigma_j \varsigma_1 \\ &= \eta_{j1}(M^1 \otimes \varsigma_1)(1 \otimes \varsigma_j) = \eta_{j1} \phi(\widetilde{M^1} \otimes 1)\phi(1 \otimes \varsigma_j).\end{aligned}$$

The remaining relations are checked in exactly the same way. □

Next we focus on the task of seeing how the parameters corresponding to the subalgebra H_F can be removed in order to leave a family of instantons parameterised by the commutative algebra $\mathcal{A}[\mathfrak{M}(k;\theta)]$. There is a right coaction

$$\delta_R : \mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F \to (\mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F) \otimes H_F, \qquad \delta_R := \mathrm{id} \otimes \Delta, \tag{6.14}$$

where $\Delta : H_F \to H_F \otimes H_F$ is the coproduct on the Hopf algebra H_F. This coaction is by gauge transformations, meaning that that the projections $\widetilde{\mathsf{P}} \otimes 1$ and $\delta_R(\widetilde{\mathsf{P}})$ are unitarily equivalent in the matrix algebra $\mathrm{M}_{2k+2}((\mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F) \otimes H_F)$ and so they define gauge equivalent families of instantons [5].

The parameters determined by the subalgebra H_F in $\mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F$ are therefore just 'gauge' parameters and so they may be removed by passing to the subalgebra of $\mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F$ consisting of coinvariant elements under the coaction (6.14), *viz.*

$$\mathcal{A}[\mathfrak{M}(k;\theta)] \cong \{a \in \mathcal{A}[\mathfrak{M}(k;\theta)] \rtimes H_F \mid \delta_R(a) = a \otimes 1\}.$$

In this way we obtain a projection $\mathsf{P}_{k;\theta}$ with entries in $\mathcal{A}[\mathfrak{M}(k;\theta)] \otimes \mathcal{A}[\mathbb{R}^4_\theta]$. The explicit details of the construction of the projection $\mathsf{P}_{k;\theta}$ are given in [5], together with a proof of the fact that the Grassmann family of connections $\nabla = \mathsf{P}_{k;\theta}\circ(\mathrm{id}\otimes \mathrm{d})$ also has anti-self-dual curvature and hence defines a family of instantons on $\mathbb{R}^4_\theta$.

Moreover, the commutative algebra $\mathcal{A}[\mathfrak{M}(k;\theta)]$ is the algebra of coordinate functions on a classical space of monads $\mathfrak{M}(k;\theta)$. For each point $x\in\mathfrak{M}(k;\theta)$ there is an evaluation map

$$\mathrm{ev}_x\otimes \mathrm{id} : \mathcal{A}[\mathfrak{M}(k;\theta)]\otimes\mathcal{A}[\mathbb{C}^4_\theta]\to\mathcal{A}[\mathbb{C}^4_\theta],$$

which in turn determines monad over the noncommutative space $\mathbb{C}^4_\theta$ in terms of the matrices $(\mathrm{ev}_x\otimes\mathrm{id})\widetilde{\sigma}_z$ and $(\mathrm{ev}_x\otimes\mathrm{id})\widetilde{\tau}_z$, *i.e.* a sequence of free right $\mathcal{A}[\mathbb{C}^4_\theta]$-modules

$$0\to\mathcal{H}\otimes\mathcal{A}[\mathbb{C}^4_\theta]\xrightarrow{(\mathrm{ev}_x\otimes\mathrm{id})\widetilde{\sigma}_z}\mathcal{K}\otimes\mathcal{A}[\mathbb{C}^4_\theta]\xrightarrow{(\mathrm{ev}_x\otimes\mathrm{id})\widetilde{\tau}_z}\mathcal{L}\otimes\mathcal{A}[\mathbb{C}^4_\theta]\to 0. \tag{6.15}$$

Recall from Proposition 4.7 that the gauge freedom in the classical ADHM construction is precisely the freedom to choose linear bases of the the vector spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$. Clearly we also have this freedom in the noncommutative construction and so we write $\sim$ for the resulting equivalence relation on the space $\mathfrak{M}(k;\theta)$ (*cf.* Definition 4.8). This yields the following description of the space $\mathfrak{M}(k;\theta)$ of classical parameters in the ADHM construction on $\mathbb{R}^4_\theta$.

Theorem 6.15. *For $k\in\mathbb{Z}$ a positive integer, the space $\mathfrak{M}(k;\theta)/\sim$ of equivalence classes of self-conjugate monads over $\mathbb{C}^4_\theta$ is the quotient of the set of complex matrices $B_1, B_2\in\mathrm{M}_k(\mathbb{C})$, $I\in\mathrm{M}_{2\times k}(\mathbb{C})$, $J\in\mathrm{M}_{k\times 2}(\mathbb{C})$ satisfying the equations*

(i) $\bar{\mu}B_1B_2-\mu B_2B_1+IJ=0$,
(ii) $[B_1,B_1^*]+[B_2,B_2^*]+II^*-J^*J=0$

by the action of $\mathrm{U}(k)$ given by

$$B_1\mapsto gB_1g^{-1},\qquad B_2\mapsto gB_2g^{-1},\qquad I\mapsto gI,\qquad J\mapsto Jg^{-1}$$

for each $g\in\mathrm{U}(k)$.

Proof. We express the monad maps $\tilde{\sigma}_z$, $\tilde{\tau}_z$ as

$$\tilde{\sigma}_z=\widetilde{M}^1z_1+\widetilde{M}^2z_2+\widetilde{M}^3z_3+\widetilde{M}^4z_4,\qquad \tilde{\tau}_z=\widetilde{N}^1z_1+\widetilde{N}^2z_2+\widetilde{N}^3z_3+\widetilde{N}^4z_4$$

for constant matrices $\widetilde{M^j}$, $\widetilde{N^l}$, $j,l=1,\ldots,4$. Upon expanding out the condition $\tilde{\tau}_z\circ\tilde{\sigma}_z=0$ and using the commutation relations in Lemma 6.1, we find the conditions

$$\widetilde{N^j}\widetilde{M^l}+\eta_{jl}\widetilde{N^l}\widetilde{M^j}=0 \tag{6.16}$$

for $j,l=1,\ldots,4$. The typical fibre $\mathbb{CP}^1$ of the twistor fibration $\mathbb{R}^4\times\mathbb{CP}^1$ has homogeneous coordinates z_1, z_1^*, z_2, z_2^* and the 'line at infinity' ℓ_∞ is recovered by setting $z_1=z_2=0$. On this line, the monad condition $\tilde{\tau}_z\circ\tilde{\sigma}_z=0$ becomes

$$\widetilde{N}^3\widetilde{M}^4+\widetilde{N}^4\widetilde{M}^3=0,\qquad \widetilde{N}^3\widetilde{M}^3=0,\qquad \widetilde{N}^4\widetilde{M}^4=0. \tag{6.17}$$

Moreover, when $z_1=z_2=0$ we see from the relations (6.7) that the coordinates z_3, z_4 and their conjugates are mutually commuting, so that the line ℓ_∞ is classical. Self-conjugacy of the monad once again implies that the restricted bundle over ℓ_∞ is trivial, whence we can argue as in [22] to show that the map $\widetilde{N}^3\widetilde{M}^4 = -\widetilde{N}^4\widetilde{M}^3$ is an isomorphism. We choose bases for $\mathcal{H}, \mathcal{K}, \mathcal{L}$ such that $\widetilde{N}^3\widetilde{M}^4 = \mathbb{1}_k$ and

$$\widetilde{M}^3 = \begin{pmatrix} \mathbb{1}_{k\times k} \\ 0_{k\times k} \\ 0_{2\times k} \end{pmatrix}, \qquad \widetilde{M}^4 = \begin{pmatrix} 0_{k\times k} \\ \mathbb{1}_{k\times k} \\ 0_{2\times k} \end{pmatrix}, \qquad \widetilde{N}^3 = \begin{pmatrix} 0_{k\times k} \\ \mathbb{1}_{k\times k} \\ 0_{k\times 2} \end{pmatrix}^{\mathrm{tr}}, \qquad \widetilde{N}^4 = \begin{pmatrix} -\mathbb{1}_{k\times k} \\ 0_{k\times k} \\ 0_{k\times 2} \end{pmatrix}^{\mathrm{tr}}.$$

Using the conditions (6.16) for $j = 3, 4$ and $l = 1, 2$, the remaining matrices are necessarily of the form

$$\widetilde{M}^1 = \begin{pmatrix} B_1 \\ B_2 \\ J \end{pmatrix}, \qquad \widetilde{M}^2 = \begin{pmatrix} B_1' \\ B_2' \\ J' \end{pmatrix}, \qquad \widetilde{N}^1 = \begin{pmatrix} -\mu B_2 \\ \bar{\mu} B_1 \\ I \end{pmatrix}^{\mathrm{tr}}, \qquad \widetilde{N}^2 = \begin{pmatrix} -\bar{\mu} B_2' \\ \mu B_1' \\ I' \end{pmatrix}^{\mathrm{tr}}.$$

Invoking the relations $\tilde{\tau}^*_{J(z)} = -\tilde{\sigma}_z$ and $\tilde{\sigma}^*_{J(z)} = \tilde{\tau}_z$ corresponding to the fact that the monad is self-conjugate, we find that

$$B_1' = -\bar{\mu} B_2^*, \qquad B_2' = \mu B_1^*, \qquad J' = I^*, \qquad I' = -J^*.$$

Thus in order to fulfil the condition $\tilde{\tau}_z \circ \tilde{\sigma}_z = 0$ it remains to impose conditions (6.16) in the cases $j = l = 1$ and $j = 1, l = 2$. These are precisely conditions (i) and (ii) in the theorem. It is evident just as in the classical case [11] that the remaining freedom in this set-up is given by the stated action of $\mathrm{U}(k)$, whence the result. □

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