The adjoint group of an Alexander quandle.

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To an abelian group $M$ equipped with an automorphism $T$ one can associate a quandle $A(M,T)$ called its Alexander quandle. It is given by the set $M$ together with the quandle operation $*$ defined by $y * x = Ty + x - Tx$. To any quandle $Q$ one can associate a group $\text{Adj}(Q)$ called the adjoint group of $Q$. It is defined as the abstract group with one generator $e_x$ for each $x \in Q$ and one relation $e_y e_x = e_x^{-1} e_y e_x$ for each $x, y \in Q$.

It is the purpose of this note to show that the adjoint group of an Alexander quandle $Q(M,T)$ has an elegant description in terms of $M$ and $T$, at least if the quandle is connected, which is the case if $1 - T$ is invertible. From this description one gets a similar description of the fundamental group of $Q(M,T)$ based at $0 \in M$. This note can be viewed as an exercise inspired by [2], to which we refer for motivation and definitions.

The adjoint group $A = \text{Adj}(A(M,T))$ acts from the right on $M$ by the formula $p \cdot e_x = p * x$. This defines a homomorphism $\rho$ from $A$ to the group $G$ of quandle automorphisms of $A(M,T))$. Thus $p \cdot e_0^{-1} = T^{-1} p$ and $p \cdot e_0^{-1} e_x = p + x - Tx$. From one sees that

$$p \cdot e_0^{-1} e_x e_0^{-1} e_y = p + x - Tx + y - Ty = p \cdot e_0^{-1} e_{x+y}$$

Therefore

$$e_0^{-1} e_{x+y} = \gamma(x,y) e_0^{-1} e_x e_0^{-1} e_y$$

for some $\gamma(x,y) \in \text{Adj}(Q)$ which acts trivially on $M$ and thus is an element of $K = \ker(\rho)$. The group $K$ is a central subgroup of $A$ as explained in [2].

From the definition of $\gamma(x,y)$ we see that $\gamma(0,y) = 1$ and $\gamma(x,0) = 1$ for all $x$ and $y$. Furthermore the formulas

$$e_0^{-1} e_{x+y+z} = \gamma(x,y+z) e_0^{-1} e_x e_0^{-1} e_{y+z}$$

$$= \gamma(x, y+z) e_0^{-1} e_x \gamma(y,z) e_0^{-1} e_y e_0^{-1} e_z$$

$$e_0^{-1} e_{x+y+z} = \gamma(x+y,z) e_0^{-1} e_x e_0^{-1} e_y e_0^{-1} e_z$$

$$= \gamma(x+y, z) e_0^{-1} \gamma(x,y) e_x e_0^{-1} e_y e_0^{-1} e_z$$

show that

$$\gamma(x, y+z) \gamma(y,z) = \gamma(x+y, z) \gamma(x,y)$$

for all $x, y, z \in M$
This shows that \( \gamma \) is a group 2-cocycle for the group \( M \) with values in \( K \). We will not use this: our purpose is not to show that \( \gamma \) is a coboundary, but to show that it vanishes to a certain degree, by exploiting its relation with \( T \). However if \( \gamma \) were a coboundary then in particular \( \gamma(x,y) \) would be symmetric in \( x \) and \( y \). This is one of the motivations to consider the map \( \lambda: M \times M \to K \) defined by

\[
\lambda(x, y) = \gamma(y, x)^{-1}\gamma(x, y) = [e_0^{-1}e_y, e_0^{-1}e_x] \quad (3)
\]

The defining relation for \( A \) shows that \( e_0e_xe_0^{-1} = e_{T^{-1}x} \) or equivalently \( e_xe_0^{-1} = e_0^{-1}e_{T^{-1}x} \) for \( x \in M \). So we can rewrite \( e_{x+y} = \gamma(x, y)e_xe_0^{-1}e_y \) as \( e_{x+y} = \gamma(x, y)e_0^{-1}e_{T^{-1}x}e_y \). In other words

\[
e_xe_v = \gamma(Tu, v)^{-1}e_0e_{Tu+v} \quad \text{for all } u, v \in M \quad (4)
\]

If we substitute this twice in the defining relation we find that

\[
\gamma(Tu, v)^{-1}e_0e_{Tu+v} = e_xe_v = e_0e_{Tu+v-Tv} \\
= \gamma(Tv, Tu + v -Tv)^{-1}e_0e_{Tv+Tu+v-Tv}
\]

This implies that \( \gamma(Tu, v) = \gamma(Tv, Tu + v -Tv) \) for \( u, v \in M \), in other words

\[
\gamma(x, y) = \gamma(Ty, x + y -Ty) \quad \text{for } x, y \in M \quad (5)
\]

and in particular

\[
\gamma(Ty, y -Ty) = 1 \quad \text{for } y \in M \quad (6)
\]

We now switch to additive notation for \( K \). From (5) and the cocycle relation we find

\[
\gamma(u, v) + \gamma(v -Tv, u) = \gamma(Tv, Tu + v -Tv + u) + \gamma(v -Tv, u) \\
= \gamma(Tv + v -Tv, u) + \gamma(Tv, v -Tv)
\]

and in particular

\[
\lambda(u, v) = \gamma(u, v) - \gamma(v, u) = -\gamma(v -Tv, u) \quad (7)
\]

Thus if \( \gamma \) were symmetric then \( \lambda \) would vanish, and so would \( \gamma \) since \( 1 - T \) is assumed to be invertible.

Now we look at the consequences for \( \lambda \) of the cocycle condition for \( \gamma \). If we substitute (7) in the cocycle condition for \( \gamma \) we find

\[
\lambda((1-T)^{-1}(x+y), z) + \lambda((1-T)^{-1}x, y) = \lambda((1-T)^{-1}x, y + z) + \lambda((1-T)^{-1}y, z)
\]

and putting \( x = u - Tu, y = v -Tv \) this yields

\[
\lambda(u + v, z) + \lambda(u, v -Tv) = \lambda(u, v -Tv + z) + \lambda(v, z) \quad (8)
\]
On the other hand subtracting two instances of the co-cycle condition for $\gamma$

$$\gamma(u, v + z) + \gamma(v, z) = \gamma(u + v, z) + \gamma(u, v)$$
$$\gamma(z, v + u) + \gamma(v, u) = \gamma(z + v, u) + \gamma(z, v)$$

we find

$$\lambda(u + v, z) + \lambda(u, v) = \lambda(u, v + z) + \lambda(v, z) \quad (9)$$

Subtracting $(9)$ from $(8)$ we find

$$\lambda(u, v - Tv) - \lambda(u, v) = \lambda(u, v - Tv + z) - \lambda(u, v + z) \quad (10)$$

This means that the right hand side of $(10)$ does not depend on $z$; in particular it has the same value for $z = -v$. Thus using the fact that $\lambda(u, 0) = 0$ we can rewrite $(10)$ as

$$\lambda(u, -Tv) = \lambda(u, v - Tv + z) - \lambda(u, v + z) \quad (11)$$

Substituting $a = v + z$ and $b = -Tv$ this yields

$$\lambda(u, b) = \lambda(u, a + b) - \lambda(u, a) \quad (12)$$

We have just proved that $\lambda$ is additive in its second coordinate. Since $\lambda$ is skew-symmetric it is in fact bi-additive. Thus (7) and the invertibility of $1 - T$ imply that $\gamma$ is bi-additive. Moreover using (6) we can simplify (5) to

$$\gamma(x, y) = \gamma(Ty, x) \text{ for all } x, y \quad (13)$$

This motivates the following definition and theorem.

**Definition 1.** Define $\tau : M \otimes M \to M \otimes M$ by the formula $\tau(x \otimes y) = Ty \otimes x$. Define $S(M, T)$ as $\operatorname{coker}(1 - \tau)$. Thus $\gamma$ can be viewed as a map from $S(T, M)$ to $K$. Finally define $F(M, T)$ as the set $\mathbb{Z} \times M \times S(M, T)$ with the multiplication given by

$$(k, x, \alpha)(m, y, \beta) = (k + m, Tmx + y, \alpha + \beta + [Tmx \otimes y])$$

**Theorem 1.** The groups $\text{Adj}(A(M, T))$ and $F(M, T)$ are isomorphic.

**Proof.** We define $\phi : \text{Adj}(A(M, T)) \to F(M, T)$ by setting $\phi(e_x) = (1, x, 0)$. To see that this is well defined we have to check the following:

$$\phi(e_x)\phi(e_{y+x}) = (1, x, 0)(1, Ty + x - Tx, 0)$$
$$= (2, Tx + (Ty + x - Tx), [Tx \otimes (Ty + x - Tx)])$$
$$= (2, Ty + x, [Ty \otimes x]) = (1, y, 0)(1, x, 0) = \phi(e_y)\phi(e_x)$$

which is the case since $[Tx \otimes Ty] = [Ty \otimes x]$ and $[Tx \otimes Tx] = [Tx \otimes x]$. We define $\psi : F(M, T) \to \text{Adj}(A(M, T))$ by setting $\psi(k, x, \alpha) = e_0^{k-1}e_x\gamma(\alpha)^{-1}$. 

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To see that \( \psi \) is a homomorphism we have to check the following:

\[
\psi(k, x, \alpha)\psi(m, y, \beta) = e^{-1}e_{x}\gamma(\alpha)^{-1}e_{y}\gamma(\beta)^{-1} \\
= e^{-1}e_{x}e_{y}(\alpha)^{-1}e_{y}e_{y}(\beta)^{-1} = e^{-1}e_{x}e_{y}(\alpha + \beta)^{-1} \\
= e^{m+1}e_{T^{m}x+y}\gamma(T^{m}x+y)^{-1}(\alpha + \beta)^{-1} \\
= \psi(k + m, T^{m}x + y, \alpha + \beta + [T^{m}x \otimes y])
\]

which is the case \( e_{z}e_{y} = e_{z+y}[z \otimes y]^{-1} \) for \( z = T^{m}x \) by (1).

From \( \psi(\phi(x)) = \psi(1, x, 0) = e_{x} \) we see that \( \phi = 1 \). The other composition requires more work; first we compute

\[
\phi(\gamma[u \otimes v]^{-1}) = \phi(e_{u}e_{v}^{-1}e_{v}^{-1}e_{v}) = (1, u + v, 0)^{-1}(1, u, 0)(1, 0, 0)^{-1}(1, u, 0) \\
= (1, u, 0)(1, 0, 0)(1, u, 0) \\
= (1, u + v, 0)(1, u + v, [u \otimes v] = (0, 0, [u \otimes v])
\]

which shows that \( \phi(\gamma(\alpha)^{-1}) = (0, 0, \alpha) \) for all \( \alpha \). From this we get

\[
\phi(\psi(k, x, \alpha)) = \phi(e^{-1})\phi(e_{x})\phi(\gamma(\alpha)^{-1}) = (k - 1, 0, 0)(1, x, 0)(0, 0, \alpha) = (k, x, \alpha)
\]

so we find that \( \phi \psi = 1 \). \( \square \)

For any quandle \( Q \) there is a unique homomorphism \( \epsilon: \text{Adj}(Q) \rightarrow \mathbb{Z} \) such that \( \epsilon(e_{x}) = 1 \) for all \( x \in Q \); the kernel is denoted by \( \text{Adj}(Q)^{o} \). It is clear that \( \epsilon(\alpha) = 0 \) for all \( \alpha \), so \( \epsilon(\psi(k, x, \alpha)) = k \). Therefore under \( \psi \) the subgroup \( \text{Adj}(A(M, T))^{o} \) of \( \text{Adj}(A(M, T)) \) corresponds to the subgroup \( F(M, T)^{o} \) of \( F(M, T) \) consisting of the triples \( (0, x, \alpha) \). Note that on \( F(M, T)^{o} \) the multiplication simplifies to

\[
(0, x, \alpha)(0, y, \beta) = (0, x + y, \alpha + \beta + [x \otimes y])
\]

For any quandle the fundamental group based at \( q \in Q \) is defined as \( \pi_{1}(Q, q) = \{ g \in \text{Adj}(Q)^{o} \mid q \cdot g = q \} \). For these definitions we refer to [2]. In order to describe this in terms of \( (M, T) \) for the case \( Q = A(M, T) \) we need to translate the action of \( \text{Adj}(A(M, T)) \) on \( M \) into an action of \( F(M, T) \) on \( M \).

One can easily check that \( 0 \cdot \psi(k, x, \alpha) = x - Tx \) for all \( k, x \) and \( \alpha \). This implies that \( 0 \cdot (0, x, \alpha) = 0 \) if and only if \( x = 0 \), which means that \( \pi_{1}(A(M, T), 0) \) is isomorphic to \( S(M, T) \).

**Example 1.** Let \( F \) be a field, let \( M = F[t]/(t^{2} + at + b) \) and let \( T \) be multiplication by the class of \( t \). Then \( T \) is an automorphism if \( b \neq 0 \) and \( A(M, T) \) is connected if \( 1 + a + b \neq 0 \). In this case \( S(M, T) \) isomorphic to \( K/(b^{2} + ab - a - 1) \). Thus \( A(M, T) \) is simply connected if \( b^{2} + ab - a - 1 \neq 0 \). The entry for \( F = \mathbb{Z}/(3) \) and \( f(t) = t^{2} - t + 1 \) in the table on page 49 of [1] is not compatible with this, but it is a misprint.
References

   Also math.GT/9903135.