

The Witt group of non-degenerate braided fusion categories

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Abstract. We give a characterization of Drinfeld centers of fusion categories as non-degenerate braided fusion categories containing a Lagrangian algebra. Further we study the quotient of the monoid of non-degenerate braided fusion categories modulo the submonoid of the Drinfeld centers and show that its formal properties are similar to those of the classical Witt group.

1. Introduction

Tensor categories are ubiquitous in many areas of mathematics and it seems worthwhile to study them deeper. The simplest class of tensor categories is formed by so called fusion categories ([ENO1], see Section 2.1 below for a definition). It is known ([ENO1]) that over an algebraically closed field \mathbb{k} of characteristic zero there are only countably many equivalence classes of fusion categories and that the classification of these equivalence classes is essentially independent from the field \mathbb{k} (namely, an embedding of fields $\mathbb{k} \subset \mathbb{k}'$ induces a bijection between the sets of equivalence classes of fusion categories over \mathbb{k} and over \mathbb{k}'). Thus the classification of fusion categories seems to be a natural and interesting problem. This problem is very far from its solution at the moment.

An interesting additional structure that one might impose on a tensor category is a *braiding* ([JS2]). For a fusion category \mathcal{A} , its *Drinfeld center* $\mathcal{Z}(\mathcal{A})$ is a braided fusion category, see Section 2.3. Our first main result addresses the following question: when is a braided fusion category \mathcal{C} equivalent to the Drinfeld center of some fusion category? The answer we give is as follows: \mathcal{C} should be *non-degenerate* in the sense of [DGNO] and \mathcal{C} should contain a *Lagrangian algebra*, that is, a connected étale algebra of maximal possible size, see Section 4. More precisely, we show that the 2-groupoid of fusion categories is equivalent to the 2-groupoid of *quantum Manin pairs*, where a quantum Manin pair consists of a non-degenerate braided fusion category and a Lagrangian algebra in this

category. This result can be considered as (a step in) a reduction of the classification of all fusion categories to the classification of braided fusion categories.

The problem of classification of all braided fusion categories (even of non-degenerate ones) seems to be very interesting but is almost as inaccessible as a classification of all fusion categories. The second main result of this paper is an observation that there is an interesting algebraic structure in this classification. Namely, we prove that the quotient of the monoid of non-degenerate braided fusion categories by the submonoid of Drinfeld centers has formal properties similar to those of the classical Witt group of the quadratic forms over a field. Moreover, we show that the Witt group of finite abelian groups endowed with a non-degenerate quadratic form embeds naturally into this quotient. Thus we call it the *Witt group of non-degenerate braided fusion categories* and consider its computation as a fundamental problem in the study of fusion categories. Further we show that each Witt equivalence class contains a unique representative which is *completely anisotropic* (Theorem 5.13); this result is a counterpart of the statement that in the classical Witt group each Witt class contains a unique anisotropic quadratic form.

An interesting subgroup of the Witt group is the *unitary Witt group* (see Definition 5.24) consisting of the classes of *pseudounitary* braided fusion categories. A well-known source of examples of pseudounitary braided fusion categories is the representation theory of affine Lie algebras, see e.g. [BK], Chapter 7. Namely, for any simple finite dimensional Lie algebra \mathfrak{g} and a positive integer k one constructs a pseudounitary non-degenerate braided fusion category $\mathcal{C}(\mathfrak{g}, k)$ consisting of integrable highest weight modules of level k over the affinization of \mathfrak{g} . We do not know any elements of the unitary Witt group that are not in the subgroup generated by the classes $[\mathcal{C}(\mathfrak{g}, k)]$. It would be very interesting to find out whether such elements exist. The relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$ (or, more generally, between the classes of known braided fusion categories) are of great interest. By Corollary 5.9, any such relation produces at least one fusion category; one can hope to construct new examples of fusion categories in this way (see [CMS], Appendix, for an example of this kind). In Section 6 we give examples of such relations using the theory of *conformal embeddings* and *coset models* of central charge $c < 1$. It would be interesting to see whether other relations exist. At this moment even all relations between the classes $[\mathcal{C}(\mathfrak{sl}(2), k)]$ are not completely known (see Section 6.4).

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2. Preliminaries

Throughout this paper our base field \mathbb{k} is an algebraically closed field of characteristic zero.

2.1. Fusion categories. By definition (see [ENO1]), a *multi-fusion category* over \mathbb{k} is a \mathbb{k} -linear semisimple rigid tensor category with finitely many simple objects and finite

dimensional spaces of morphisms. A multi-fusion category is called a *fusion category* if its unit object $\mathbf{1}$ is simple. By a *fusion subcategory* of a fusion category we always mean a full tensor subcategory that is itself fusion (i.e., in particular rigid and semisimple.) Let Vec denote the fusion category of finite dimensional vector spaces over \mathbb{k} . Any fusion category \mathcal{A} contains a trivial fusion subcategory consisting of multiples of $\mathbf{1}$. We will identify this subcategory with Vec . A fusion category \mathcal{A} is called *simple* if Vec is the only proper fusion subcategory of \mathcal{A} .

A fusion category is called *pointed* if all its simple objects are invertible. For a fusion category \mathcal{A} we denote \mathcal{A}_{pt} the maximal pointed fusion subcategory of \mathcal{A} . We say that \mathcal{A} is *unpointed* if $\mathcal{A}_{\text{pt}} = \text{Vec}$.

We will denote by $\mathcal{A} \boxtimes \mathcal{B}$ the tensor product of fusion categories \mathcal{A} and \mathcal{B} . (Cf. [De], Section 5. Under the assumptions of this paper, where \mathbb{k} is algebraically closed and \mathcal{A}, \mathcal{B} semisimple, $\mathcal{A} \boxtimes \mathcal{B}$ can be obtained as the completion of the \mathbb{k} -linear direct product $\mathcal{A} \otimes_{\mathbb{k}} \mathcal{B}$ under direct sums and subobjects.)

For a fusion category \mathcal{A} we denote by $\mathcal{O}(\mathcal{A})$ the set of isomorphism classes of simple objects in \mathcal{A} .

Let \mathcal{A} be a fusion category and let $K(\mathcal{A})$ be its Grothendieck ring. There exists a unique ring homomorphism $\text{FPdim} : K(\mathcal{A}) \rightarrow \mathbb{R}$ such that $\text{FPdim}(X) > 0$ for any $0 \neq X \in \mathcal{A}$, see [ENO1], Section 8.1. (See also [ENO1], Section 9 for the observation that the results used below are independent of the ground field.) For a fusion category \mathcal{A} one defines (see [ENO1], Section 8.2) its *Frobenius–Perron dimension*:

$$(1) \quad \text{FPdim}(\mathcal{A}) = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)^2.$$

For any object X in \mathcal{A} let $[X]$ denote the corresponding element of the Grothendieck ring $K(\mathcal{A})$. One defines the (virtual) *regular object* of \mathcal{A} by

$$(2) \quad R_{\mathcal{A}} = \sum_{X \in \mathcal{O}(\mathcal{A})} \text{FPdim}(X)[X] \in K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R},$$

see e.g. [ENO1], Section 8.2. The regular object $R_{\mathcal{A}}$ has the following properties (see loc. cit.):

- (1) $\text{FPdim}(R_{\mathcal{A}}) = \text{FPdim}(\mathcal{A})$.
- (2) $[X]R_{\mathcal{A}} = \text{FPdim}(X)R_{\mathcal{A}}$ for any $X \in \mathcal{A}$.

(The first is obvious. The second is a restatement of the fact that the positive vector $(\text{FPdim}(X_i))_i$ is the (unique up to a scalar) common FP eigenvector, with respect to the canonical basis $[X_j]$, of the commuting operators $[X]$ acting on $K(\mathcal{A}) \otimes_{\mathbb{Z}} \mathbb{R}$ by multiplication. The proof only uses multiplicativity of the FP dimension. This also shows that $R_{\mathcal{A}}$ is actually characterized by the properties (1) and (2).)

Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories such that $\text{FPdim}(\mathcal{A}_1) = \text{FPdim}(\mathcal{A}_2)$. By [EO], Proposition 2.19, any fully faithful tensor functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is an equivalence.

There is another notion of dimension \mathcal{A} , the *categorical* (or *global*) dimension defined as follows (see [Mu2]). For each simple object X in \mathcal{A} pick an isomorphism $a_X : X \simeq X^{**}$ and set

$$(3) \quad \dim(\mathcal{A}) = \sum_{X \in \mathcal{U}(\mathcal{A})} |X|^2,$$

where $|X|^2 = \text{Tr}_X(a_X) \text{Tr}_{X^*}((a_X^{-1})^*)$. By [ENO1], Theorem 2.3, $\dim(\mathcal{A})$ is a non-zero element in \mathbb{k} .

A fusion category \mathcal{A} over $\mathbb{k} = \mathbb{C}$ is called *pseudo-unitary* if $\dim(\mathcal{A}) = \text{FPdim}(\mathcal{A})$, see [ENO1], Section 8.4. A pseudo-unitary fusion category \mathcal{A} has a unique spherical structure such that the categorical dimension $\dim(X)$ of any object X in \mathcal{A} equals $\text{FPdim}(X)$, see [ENO1], Proposition 8.23. It is easy to see that if \mathcal{A}_1 and \mathcal{A}_2 are pseudo-unitary, then so is $\mathcal{A}_1 \boxtimes \mathcal{A}_2$.

2.2. Braided fusion categories. A *braided* fusion category is a fusion category \mathcal{C} endowed with a braiding $c_{X,Y} : X \otimes Y \simeq Y \otimes X$, see [JS2]. For a braided fusion category its *reverse* \mathcal{C}^{rev} is the same fusion category with a new braiding $\tilde{c}_{X,Y} = c_{Y,X}^{-1}$. A braided fusion category is *symmetric* if $\tilde{c} = c$.

Recall from [Mu4] that objects X and Y of a braided fusion category \mathcal{C} are said to *centralize* each other if

$$(4) \quad c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y}.$$

The *centralizer* \mathcal{D}' of a fusion subcategory $\mathcal{D} \subset \mathcal{C}$ is defined to be the full subcategory of objects of \mathcal{C} that centralize each object of \mathcal{D} . It is easy to see that \mathcal{D}' is a fusion subcategory of \mathcal{C} . Clearly, \mathcal{D} is symmetric if and only if $\mathcal{D} \subset \mathcal{D}'$.

Definition 2.1 (see [DGNO], Definition 2.28 and Proposition 3.7). We will say that a braided fusion category \mathcal{C} is *non-degenerate* if $\mathcal{C}' = \text{Vec}$.

A non-degenerate braided fusion category $\mathcal{C} \neq \text{Vec}$ is *prime* if it has no proper non-degenerate braided fusion subcategories other than Vec . Clearly, a non-trivial simple braided fusion category is prime.

For a fusion subcategory \mathcal{D} of a non-degenerate braided fusion category \mathcal{C} one has the following properties, cf. [DGNO], Theorems 3.10, 3.14:

$$\begin{aligned} \mathcal{D}'' &= \mathcal{D}, \\ \text{FPdim}(\mathcal{D}) \text{FPdim}(\mathcal{D}') &= \text{FPdim}(\mathcal{C}). \end{aligned}$$

A *pre-modular* category is a braided fusion category equipped with a spherical structure. A pre-modular category \mathcal{C} is *modular* (i.e., its S -matrix is invertible) if and only if \mathcal{C} is non-degenerate [DGNO], Proposition 3.7. (Cf. also [Mu4].)

The following statement is well known. We include its proof for the reader's convenience.

Proposition 2.2. *Let $\mathcal{C} \neq \text{Vec}$ be a non-degenerate braided fusion category. Then*

$$(5) \quad \mathcal{C} = \mathcal{C}_1 \boxtimes \cdots \boxtimes \mathcal{C}_n,$$

where $\mathcal{C}_1, \dots, \mathcal{C}_n$ are prime non-degenerate subcategories of \mathcal{C} . Furthermore, if \mathcal{C} is unpointed, then its decomposition (5) into a tensor product of prime non-degenerate subcategories is unique up to a permutation of factors.

Proof. Existence of the decomposition (5) is established in [Mu4], Theorems 4.2, 4.5, for modular categories. Up to one argument that requires generalization, given by [DGNO], Theorem 3.13, the same proof works for non-degenerate fusion categories.

It remains to prove uniqueness. If $\mathcal{D} \subset \mathcal{C}$ is a fusion subcategory, let $\mathcal{D}_i \subset \mathcal{C}_i$ be the fusion subcategory generated by all simple objects $X_i \in \mathcal{C}_i$ such that there is a simple $X = X_1 \boxtimes \cdots \boxtimes X_i \boxtimes \cdots \boxtimes X_n \in \mathcal{D}$. Clearly we have $\mathcal{D} \subset \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, but the converse need not hold. If it does, we say that \mathcal{D} factorizes. Denoting by \mathcal{D}_{ad} the fusion subcategory of \mathcal{D} generated by $X \otimes X^*$, where X runs through simple objects of \mathcal{D} , the fact that $X \otimes X^* = (X_1 \otimes X_1^*) \boxtimes \cdots \boxtimes (X_n \otimes X_n^*)$ has $\mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1} \boxtimes (X_i \otimes X_i^*) \boxtimes \mathbf{1} \boxtimes \cdots \boxtimes \mathbf{1}$ as direct summand for each i implies that $\mathcal{D}_{\text{ad}} \supset (\mathcal{D}_{\text{ad}})_i$, thus \mathcal{D}_{ad} factorizes. Let $\mathcal{D} \subset \mathcal{C}$ be a non-degenerate fusion subcategory. Since \mathcal{C} is unpointed, i.e., $\mathcal{C}_{\text{pt}} = \text{Vec}$, \mathcal{D} is unpointed and by [DGNO], Corollary 3.27, we have $\mathcal{D}_{\text{ad}} = (\mathcal{D}_{\text{pt}})' \cap \mathcal{D} = \mathcal{D}$. Thus \mathcal{D} factorizes, i.e., $\mathcal{D} = \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, where each \mathcal{D}_i is non-degenerate. Since \mathcal{C}_i is prime, we must have either $\mathcal{D}_i = \text{Vec}$ or $\mathcal{D}_i = \mathcal{C}_i$ for each $i = 1, \dots, n$. In particular, every prime non-degenerate fusion subcategory $\mathcal{D} \subset \mathcal{C}$ coincides with some \mathcal{C}_i . Hence, (5) is unique up to a permutation of factors. \square

Remark 2.3. The proof actually also shows the following stronger result: If $\mathcal{D} \subset \mathcal{C}$ is an unpointed and non-degenerate fusion subcategory, then $\mathcal{D} = \mathcal{D}_1 \boxtimes \cdots \boxtimes \mathcal{D}_n$, where each \mathcal{D}_i is either $\mathcal{D}_i = \text{Vec}$ or $\mathcal{D}_i = \mathcal{C}_i$. This means that the prime factors \mathcal{C}_i that are unpointed appear in every prime factorization of \mathcal{C} , whether or not \mathcal{C} itself is unpointed.

2.3. Drinfeld center of a fusion category. For any fusion category \mathcal{A} its *Drinfeld center* $\mathcal{Z}(\mathcal{A})$ is defined as the category whose objects are pairs (X, γ_X) , where X is an object of \mathcal{A} and $\gamma_X : V \otimes X \simeq X \otimes V$, $V \in \mathcal{A}$, is a natural family of isomorphisms, satisfying a certain compatibility condition, see [JS1], Definition 3, or [K], Definition XIII.4.1. It is known that $\mathcal{Z}(\mathcal{A})$ is a non-degenerate braided fusion category and that

$$(6) \quad \dim(\mathcal{Z}(\mathcal{C})) = \dim(\mathcal{C})^2, \quad \text{FPdim}(\mathcal{Z}(\mathcal{C})) = \text{FPdim}(\mathcal{C})^2.$$

(See [Mu3], Theorems 3.16, 4.14, Proposition 5.10, for \mathcal{C} semisimple spherical and [ENO1], Theorem 2.15, Proposition 8.12, [DGNO], Corollary 3.9, for \mathcal{C} fusion.)

For a braided fusion category \mathcal{C} there are two braided functors

$$(7) \quad \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, c_{-,X}),$$

$$(8) \quad \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}) : X \mapsto (X, \tilde{c}_{-,X}).$$

These functors are fully faithful and so we can identify \mathcal{C} and \mathcal{C}^{rev} with their images in $\mathcal{Z}(\mathcal{C})$. These images centralize each other, i.e., $\mathcal{C}' = \mathcal{C}^{\text{rev}}$. (Cf. [Mu3], Proposition 7.3.) This allows us to define a braided tensor functor

$$(9) \quad G : \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \rightarrow \mathcal{Z}(\mathcal{C}).$$

It was shown in [Mu3], Theorem 7.10, and [DGNO], Proposition 3.7, that G is a braided equivalence if and only if \mathcal{C} is non-degenerate.

Let \mathcal{C} be a braided fusion category and let \mathcal{A} be a fusion category.

Definition 2.4. If $F : \mathcal{C} \rightarrow \mathcal{A}$ is a tensor functor, a structure of a *central functor* on F is a braided tensor functor $F' : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A})$ whose composition with the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ equals F .

Equivalently, a structure of central functor on F is a natural family of isomorphisms $Y \otimes F(X) \simeq F(X) \otimes Y$, $X \in \mathcal{C}$, $Y \in \mathcal{A}$, satisfying certain compatibility conditions, see [B], Section 2.1.

2.4. Separable algebras. Let \mathcal{A} be a fusion category. In this paper an *algebra* $A \in \mathcal{A}$ is an associative algebra with unit, see e.g. [O], Definition 3.1.

Definition 2.5. An algebra $A \in \mathcal{A}$ is said to be *separable* if the multiplication morphism $m : A \otimes A \rightarrow A$ splits as a morphism of A -bimodules.

Remark 2.6. (i) The morphism m is surjective (due to the existence of unit in A), so the definition makes sense.

(ii) Observe that if $F : \mathcal{A} \rightarrow \mathcal{B}$ is a tensor functor, then $F(A) \in \mathcal{B}$ is a separable algebra for a separable algebra $A \in \mathcal{A}$.

For an algebra $A \in \mathcal{A}$ let \mathcal{A}_A , ${}_A\mathcal{A}$, ${}_A\mathcal{A}_A$ denote, respectively, abelian categories of right A -modules, left A -modules, A -bimodules, see e.g. [O], Definition 3.1.

Proposition 2.7. For an algebra $A \in \mathcal{A}$ the following conditions are equivalent:

- (i) A is separable.
- (ii) The category \mathcal{A}_A is semisimple.
- (iii) The category ${}_A\mathcal{A}$ is semisimple.
- (iv) The category ${}_A\mathcal{A}_A$ is semisimple.

Proof. Assume that A is separable. Note that A considered as a bimodule over itself is a direct summand of the A -bimodule $A \otimes A$. Thus any $M = M \otimes_A A \in \mathcal{A}_A$ is a direct summand of $M \otimes_A A \otimes A = M \otimes A$. The object $M \otimes A \in \mathcal{A}_A$ is projective (see e.g. [O], Section 3.1). Thus any $M \in \mathcal{A}_A$ is projective and we have the implication (i) \Rightarrow (ii). The implication (i) \Rightarrow (iii) is proved similarly.

The implications (ii) \Rightarrow (iv) and (iii) \Rightarrow (iv) follow from [ENO1], Theorem 2.16, and [O], Remark 4.2. Finally, the implication (iv) \Rightarrow (i) is obvious. \square

Let \mathcal{C} be a braided fusion category. Recall that an algebra A in \mathcal{C} is called *commutative* if $m \circ c_{A,A} = m$, where $m : A \otimes A \rightarrow A$ is the multiplication of A , see e.g. [KiO], Definition 1.1.

Example 2.8. Let G be a finite group and let $\mathcal{A} = \text{Rep}(G)$ be the fusion category of finite dimensional representations of G . Let $A = \text{Fun}(G)$ be the algebra of \mathbb{k} -valued functions on G . The group G acts on A via left translations, so A can be considered as a commutative algebra in \mathcal{A} . The algebra A is called the *regular algebra* of the category $\mathcal{A} = \text{Rep}(G)$. Associating to $f \in A$ the function $\mu(f) : G \times G \rightarrow \mathbb{k}, (g, h) \mapsto \delta_{g,h} f(g)$, easy computations show that $\mu : A \rightarrow A \otimes A$ is a splitting of $m : A \otimes A \rightarrow A$ and a bimodule map. Thus A is separable. (Cf. [Br], p. 227, for a similar argument.)

More generally we say that a braided fusion category \mathcal{E} is *Tannakian* ([De]) if there is a braided equivalence $F : \mathcal{E} \simeq \text{Rep}(G)$; in this case the algebra $F^{-1}(A)$ (with $A \in \text{Rep}(G)$ as above) is called a regular algebra $A_{\mathcal{E}}$ of \mathcal{E} . It is known that the algebra $A_{\mathcal{E}}$ is unique up to isomorphism. (Such an isomorphism is non-unique, in particular $\text{Aut } A_{\mathcal{E}} \cong G$.) See e.g. [DGNO], Section 2.13.

2.5. Equivariantization and de-equivariantization. Let \mathcal{A} be a fusion category with an action of a finite group G . In this case one can define the fusion category \mathcal{A}^G of G -equivariant objects in \mathcal{A} . Objects of this category are objects X of \mathcal{A} equipped with an isomorphism $u_g : g(X) \rightarrow X$ for all $g \in G$ such that

$$u_{gh} \circ \gamma_{g,h} = u_g \circ g(u_h),$$

where $\gamma_{g,h} : g(h(X)) \rightarrow gh(X)$ is the natural isomorphism associated to the action. Morphisms and tensor product of equivariant objects are defined in an obvious way. This category is called the *G-equivariantization* of \mathcal{A} . One has $\text{FPdim}(\mathcal{A}^G) = |G| \text{FPdim}(\mathcal{A})$. See [Br], [Mu5] and [DGNO], Section 4, for details.

Example 2.9. Let H be a normal subgroup of G . Then there is a natural action of G/H on \mathcal{A}^H and $(\mathcal{A}^H)^{G/H} \cong \mathcal{A}^G$.

There is a procedure opposite to equivariantization, called the *de-equivariantization*. Namely, let \mathcal{A} be a fusion category and let $\mathcal{E} = \text{Rep}(G) \subset \mathcal{Z}(\mathcal{A})$ be a Tannakian subcategory which embeds into \mathcal{A} via the forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$. Let $A = \text{Fun}(G)$ be the regular algebra of \mathcal{E} . It is a separable commutative algebra in $\mathcal{Z}(\mathcal{A})$ and so the category \mathcal{A}_G of left A -modules in \mathcal{A} is a fusion category with the tensor product \otimes_A , called *de-equivariantization* of \mathcal{A} . One has $\text{FPdim}(\mathcal{A}_G) = \text{FPdim}(\mathcal{A})/|G|$.

The above constructions are canonically inverse to each other, i.e., there are canonical equivalences $(\mathcal{A}_G)^G \cong \mathcal{A}$ and $(\mathcal{A}^G)_G \cong \mathcal{A}$, see [DGNO], Section 4.2.

2.6. Module categories over fusion categories. Let \mathcal{A} be a fusion category. A left \mathcal{A} -module category is a finite semisimple Abelian \mathbb{k} -linear category \mathcal{M} together with a

bifunctor $\otimes : \mathcal{A} \times \mathcal{M} \rightarrow \mathcal{M}$ and a natural family of isomorphisms

$$(X \otimes Y) \otimes M \simeq X \otimes (Y \otimes M) \quad \text{and} \quad \mathbf{1} \otimes M \simeq M$$

for $X, Y \in \mathcal{A}$, $M \in \mathcal{M}$, satisfying certain coherence conditions. See [O] for details and for the definitions of \mathcal{A} -module functors and their natural transformations. A typical example of a left \mathcal{A} -module category is the category \mathcal{A}_A of right modules over a separable algebra A in \mathcal{A} ([O]). An \mathcal{A} -module category is called *indecomposable* if it is not equivalent to a direct sum of two non-trivial \mathcal{A} -module categories.

The category of \mathcal{A} -module endofunctors of a right \mathcal{A} -module category \mathcal{M} will be denoted by $\mathcal{A}_{\mathcal{M}}^*$. It is known that $\mathcal{A}_{\mathcal{M}}^*$ is a multi-fusion category, see [ENO1], Theorem 2.18 (it is a fusion category if and only if \mathcal{M} is indecomposable).

Let \mathcal{M} be an indecomposable right \mathcal{A} -module category. We can regard \mathcal{M} as an $(\mathcal{A}_{\mathcal{M}}^*, \mathcal{A})$ -bimodule category. Its $(\mathcal{A}_{\mathcal{M}}^*, \mathcal{A})$ -bimodule endofunctors can be identified, on the one hand, with functors of left multiplication by objects of $\mathcal{Z}(\mathcal{A}_{\mathcal{M}}^*)$, and on the other hand, with functors of right multiplication by objects of $\mathcal{Z}(\mathcal{A})$. Combined, these identifications yield a canonical equivalence of braided categories

$$(10) \quad \mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{A}_{\mathcal{M}}^*).$$

This result is due to Schauenburg, see [S].

3. Étale algebras and central functors

3.1. Étale algebras in braided fusion categories.

Definition 3.1. An algebra $A \in \mathcal{C}$ is said to be *étale* if it is both commutative and separable. We say that an étale algebra $A \in \mathcal{C}$ is *connected* if $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$.

Remark 3.2. (i) The terminology of Definition 3.1 is justified by the fact that étale algebras in the usual sense can be characterized by the property from Definition 3.1.

(ii) Any étale algebra canonically decomposes as a direct sum of connected ones.

Example 3.3. (i) Let $\mathcal{C} \subset \mathcal{C}$ be a Tannakian subcategory. Then a regular algebra $A_{\mathcal{C}} \in \mathcal{C}$ (see Example 2.8) is connected étale.

(ii) Let \mathcal{C} be a pre-modular category. Let A be a commutative algebra in \mathcal{C} such that $\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1$, the pairing $A \otimes A \xrightarrow{m} A \rightarrow \mathbf{1}$ is non-degenerate, $\theta_A = \text{id}_A$, and $\dim(A) \neq 0$. It is proved in [KiO], Theorem 3.3, that such an A is connected étale.

Remark 3.4. In general if $A \in \mathcal{C}$ is a connected étale algebra and $A \rightarrow \mathbf{1}$ is a nonzero homomorphism (it is unique up to a scalar), then the pairing $A \otimes A \xrightarrow{m} A \rightarrow \mathbf{1}$ is non-degenerate. Indeed the kernel of this pairing would be a non-trivial ideal of A (= non-trivial subobject in the category \mathcal{C}_A); but the category \mathcal{C}_A is semisimple and

$$\dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}_A}(A, A) = \dim_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = 1.$$

In particular, this implies that any étale algebra is a self-dual object of \mathcal{C} (use Remark 3.2 (ii) for disconnected étale algebras).

3.2. From central functors to étale algebras.

Lemma 3.5. *Let \mathcal{C} be a braided fusion category, let \mathcal{A} be a fusion category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a central functor. Let $I : \mathcal{A} \rightarrow \mathcal{C}$ be the right adjoint functor of F . Then the object $A = I(\mathbf{1}) \in \mathcal{C}$ has a canonical structure of connected étale algebra.*

The category of right A -modules in \mathcal{C} is monoidally equivalent to the image of F , i.e., the smallest fusion subcategory of \mathcal{A} containing $F(\mathcal{C})$.

Proof. Let $\phi : \mathcal{C} \rightarrow \text{Vec}$ be the contravariant representable functor corresponding to A , that is, $\phi(X) = \text{Hom}_{\mathcal{C}}(X, A) \cong \text{Hom}_{\mathcal{A}}(F(X), \mathbf{1})$. The linear map

$$\begin{aligned} & \text{Hom}_{\mathcal{A}}(F(X_1), \mathbf{1}) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{A}}(F(X_2), \mathbf{1}) \\ & \rightarrow \text{Hom}_{\mathcal{A}}(F(X_1) \otimes F(X_2), \mathbf{1} \otimes \mathbf{1}) \cong \text{Hom}_{\mathcal{A}}(F(X_1 \otimes X_2), \mathbf{1}) \end{aligned}$$

defines a natural family

$$(11) \quad v_{X_1, X_2} : \phi(X_1) \otimes_{\mathbb{k}} \phi(X_2) \rightarrow \phi(X_1 \otimes X_2)$$

such that the compositions

$$(12) \quad \begin{aligned} & \phi(X_1) \otimes \phi(X_2) \otimes \phi(X_3) \rightarrow \phi(X_1 \otimes X_2) \otimes \phi(X_3) \rightarrow \phi(X_1 \otimes X_2 \otimes X_3), \\ & \phi(X_1) \otimes \phi(X_2) \otimes \phi(X_3) \rightarrow \phi(X_1) \otimes \phi(X_2 \otimes X_3) \rightarrow \phi(X_1 \otimes X_2 \otimes X_3) \end{aligned}$$

are equal. We claim that a morphism (11) is the same thing as an associative multiplication $m : A \otimes A \rightarrow A$. Namely, we define $m \in \text{Hom}(A \otimes A, A) = \phi(A \otimes A)$ by $m := v_{A, A}(\text{id}_A \otimes \text{id}_A)$, where id_A is considered as an element of $\phi(A)$. Now by naturality of v one has

$$v_{X_1, X_2}(f \otimes g) = m \circ (f \otimes g),$$

and associativity of m follows from (12).

By definition, $\text{Hom}_{\mathcal{C}}(\mathbf{1}, A) = \text{Hom}_{\mathcal{A}}(F(\mathbf{1}), \mathbf{1}) = \text{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1}) = \mathbb{k}$. It is easy to see that the image of $1 \in \mathbb{k}$ in $\text{Hom}_{\mathcal{C}}(\mathbf{1}, A)$ is a unit of the algebra A .

Next we want to prove the commutativity of m . By its definition, m is the image of a certain morphism $\tilde{m} \in \text{Hom}_{\mathcal{A}}(F(A \otimes A), \mathbf{1})$ under the bijection

$$\text{Hom}_{\mathcal{A}}(F(A \otimes A), \mathbf{1}) \cong \text{Hom}_{\mathcal{C}}(A \otimes A, A).$$

By naturality of the adjunction bijections, $m \circ c_{A, A}$ corresponds to

$$\tilde{m} \circ F(c_{A, A}) \in \text{Hom}_{\mathcal{A}}(F(A \otimes A), \mathbf{1}).$$

The equality $\tilde{m} = \tilde{m} \circ F(c_{A,A})$ follows from commutativity of the following diagram, where F' is the central structure, i.e., a braided tensor functor $F' : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{A})$ lifting $F : \mathcal{C} \rightarrow \mathcal{A}$:

$$\begin{array}{ccccccc}
 F'(A \otimes A) & \xrightarrow{\sim} & F'(A) \otimes F'(A) & \xrightarrow{l \otimes l} & \mathbf{1} \otimes \mathbf{1} & \xrightarrow{\sim} & \mathbf{1} \\
 \downarrow F'(c_{A,A}) & & \downarrow c_{F'(A), F'(A)} & & \downarrow c_{\mathbf{1}, \mathbf{1}} & & \downarrow \text{id}_{\mathbf{1}} \\
 F'(A \otimes A) & \xrightarrow{\sim} & F'(A) \otimes F'(A) & \xrightarrow{l \otimes l} & \mathbf{1} \otimes \mathbf{1} & \xrightarrow{\sim} & \mathbf{1}
 \end{array}$$

Here $l \in \text{Hom}_{\mathcal{C}}(F(A), \mathbf{1})$ is the image of id_A under $\text{Hom}_{\mathcal{C}}(A, A) \cong \text{Hom}_{\mathcal{A}}(F(A), \mathbf{1})$. The left square commutes since F' is a braided functor, and the right one since $c_{\mathbf{1}, \mathbf{1}} = \text{id}$. That the middle square commutes is more subtle, since $l : F(A) \rightarrow \mathbf{1}$ only is a morphism in \mathcal{A} but not in $\mathcal{Z}(\mathcal{A})$. It commutes nevertheless since the braiding of $\mathcal{Z}(\mathcal{A})$ is natural for such morphisms w.r.t. the second argument. (Since $c_{(X, e_X), (Y, e_Y)} = e_X(Y)$ and the half-braiding, $Y \mapsto e_X(Y)$ is natural w.r.t. all morphisms $Y \rightarrow Y'$ in \mathcal{A} .)

That the category of right A -modules in \mathcal{C} identifies with the image of F in \mathcal{A} follows from [EO], Theorem 3.17 (cf. also [O], Theorem 3.1). Thus \mathcal{C}_A is semisimple. By Proposition 2.7 semisimplicity of the category of A -modules implies the semisimplicity of the category of A -bimodules. In particular, the morphism of A -bimodules $m : A \otimes A \rightarrow A$, thus A is separable. \square

Example 3.6. (i) Let $\mathcal{C} = \text{Rep}(G)$ and let $F : \mathcal{C} \rightarrow \text{Vec}$ be the forgetful functor. Then the étale algebra A from Lemma 3.5 is the regular algebra, see Example 2.8.

(ii) Let Vec_G^ω be the fusion category of finite dimensional G -graded vector spaces with the associativity constraint twisted by a 3-cocycle $\omega \in Z^3(G, \mathbb{k}^\times)$. Let $\mathcal{C} = \mathcal{Z}(\text{Vec}_G^\omega)$ and let $F : \mathcal{C} \rightarrow \text{Vec}_G^\omega$ be the forgetful functor. Then the étale algebra A from Lemma 3.5 is the regular algebra of $\text{Rep}(G) \subset \mathcal{C}$.

(iii) Let $\mathcal{C} = \mathcal{Z}(\text{Rep}(G)) \cong \mathcal{Z}(\text{Vec}_G)$ and let $F : \mathcal{C} \rightarrow \text{Rep}(G)$ be the forgetful functor. Then the étale algebra A from Lemma 3.5 is the group algebra of G considered as an algebra in \mathcal{C} . Notice that in this case the algebra $F(A)$ in the symmetric tensor category $\text{Rep}(G)$ is non-commutative unless G is commutative.

Remark 3.7. Lemma 3.5 fails over fields of characteristic $p > 0$. Namely the algebra $A = I(\mathbf{1})$ is still commutative (with the same proof), but it can fail to be separable. Here is a counter-example. Let G be a finite abelian group of order divisible by p . Take $\mathcal{C} = \text{Vec}_G$, i.e., \mathcal{C} is the category of finite dimensional G -graded vector spaces with the obvious symmetric braided structure. Let $\mathcal{D} = \text{Vec}$ and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be the functor of forgetting the grading. Then A is the group algebra of G , which is not étale. In this example the category of A -bimodules identifies with $\text{Rep}(G)$ and is not semisimple.

3.3. The tensor category \mathcal{C}_A corresponding to an étale algebra A . Let \mathcal{C} be a braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra. Let \mathcal{C}_A be the category of right A -modules and let

$$(13) \quad F_A : \mathcal{C} \rightarrow \mathcal{C}_A : X \mapsto X \otimes A$$

be the free module functor. The category \mathcal{C}_A is semisimple by Proposition 2.7.

Using the braiding we can define two left A -module structures on a right A -module M by

$$(14) \quad A \otimes M \xrightarrow{c_{A,M}} M \otimes A \rightarrow M \quad \text{or by} \quad A \otimes M \xrightarrow{c_{M,A}^{-1}} M \otimes A \rightarrow M.$$

Both structures make M an A -bimodule, and we will denote the results by M_+ and M_- , respectively. Clearly, the functors $M \mapsto M_{\pm}$ are sections of the forgetful functor ${}_A\mathcal{C}_A \rightarrow \mathcal{C}_A$.

Since the category ${}_A\mathcal{C}_A$ of A -bimodules in \mathcal{C} is a tensor category, we obtain in this way two tensor structures \otimes_{\pm} on \mathcal{C}_A which are opposite to each other. For definiteness, when we consider \mathcal{C}_A as a tensor category, the tensor structure \otimes_- is understood. By definition, we have tensor functors $\mathcal{C}_A \rightarrow {}_A\mathcal{C}_A$ and $\mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A$.

Now the functor $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ has an obvious structure of tensor functor. The category \mathcal{C}_A is rigid since any object M in \mathcal{C}_A is a direct summand of the rigid object

$$F_A(M) = M \otimes A = M \otimes_A (A \otimes A).$$

The unit object of \mathcal{C}_A is $A = F_A(\mathbf{1})$ and the connectedness of A implies that $A \in \mathcal{C}_A$ is simple. Thus, \mathcal{C}_A is a fusion category. Alternatively, this follows from the fact that ${}_A\mathcal{C}_A$ is fusion, cf. e.g. [O], and the fact that the functors $M \mapsto M_{\pm}$ from \mathcal{C}_A and $\mathcal{C}_A^{\text{rev}}$ to ${}_A\mathcal{C}_A$ are tensor embeddings.

Example 3.8. Let \mathcal{C} be a braided fusion category and let $\mathcal{E} \subset \mathcal{C}$ be a Tannakian subcategory. Let $A \in \mathcal{E}$ be the regular algebra (which is connected étale by Example 3.3 (i)). In the terminology of [DGNO], Section 4.2, the fusion category \mathcal{C}_A introduced above is the de-equivariantization of \mathcal{C} (cf. Section 2.5) viewed as a fusion category over \mathcal{E} .

3.4. The central functor $\mathcal{C} \rightarrow \mathcal{C}_A$. Observe that the free module functor (13) admits a natural structure of a central functor, see Definition 2.4. Indeed, we have $F_A(X) = X \otimes A$, and, hence, $F_A(X) \otimes_A Y = X \otimes Y$. Similarly, $Y \otimes_A F_A(X) = Y \otimes X$. These two objects are isomorphic via the braiding of \mathcal{C} (using the commutativity of A , one can check that the braiding gives an isomorphism of A -modules) and, hence, F_A lifts to a braided tensor functor

$$(15) \quad F'_A : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$$

whose composition with the forgetful functor $\mathcal{Z}(\mathcal{C}_A) \rightarrow \mathcal{C}_A$ equals F_A . This construction is in a sense converse to Lemma 3.5:

Lemma 3.9. *Let $A \in \mathcal{C}$ be a connected étale algebra and let $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ be the central functor as above. Then the algebra object $A_{F_A} = I(\mathbf{1})$ obtained from F_A according to Lemma 3.5 is isomorphic to A .*

Proof. The adjoint of the functor $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ is given by the forgetful functor $I : \mathcal{C}_A \rightarrow \mathcal{C}$. The unit of \mathcal{C}_A being (A, m) , we have $I(\mathbf{1}_{\mathcal{C}_A}) = A$. It is straightforward to see

that the construction of the algebra structure on $A = I(\mathbf{1}_{\mathcal{C}_A})$ defined in (the proof of) Lemma 3.5 recovers the original algebra structure. \square

Let $\mathcal{A}_1, \mathcal{A}_2$ be fusion categories. We will say that a tensor functor $F : \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is *surjective* if any object in \mathcal{A}_2 is a subobject of some $F(X)$, $X \in \mathcal{A}_1$.

Remark 3.10. Some authors use the term *dominant* functor for what we call a surjective functor, see [Br], [BN].

Lemma 3.11. *For a connected étale algebra A in a braided fusion category \mathcal{C} we have*

$$(16) \quad \text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}.$$

Proof. The functor (13) is surjective. Considering the multiplicity of the unit object on both sides of the identity proven in [ENO1], Proposition 8.8, we obtain

$$\frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(\mathcal{C}_A)} = \sum_{X \in \mathcal{O}(\mathcal{C})} \text{FPdim}(X) [F_A(X) : \mathbf{1}] = \text{FPdim}(I(\mathbf{1})),$$

where $\mathcal{O}(\mathcal{C})$ denotes the set of simple objects of \mathcal{C} and I is the right adjoint of F_A . Since $A = I(\mathbf{1})$, the result follows. \square

3.5. Subcategory $\mathcal{C}_A^0 \subset \mathcal{C}_A$ of dyslectic modules. Let \mathcal{C} be a braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra and recall the discussion of the tensor functors $M \mapsto M_{\pm}$ from \mathcal{C}_A and $\mathcal{C}_A^{\text{rev}}$ to ${}_A\mathcal{C}_A$ in Section 3.3.

Definition 3.12. A module $M \in \mathcal{C}_A$ is *dyslectic* (or *local*, in alternative terminology) if the identity map id_M is an isomorphism of A -bimodules $M_+ \simeq M_-$.

Equivalently, a module $M \in \mathcal{C}_A$ is dyslectic if the diagram

$$(17) \quad \begin{array}{ccc} M \otimes A & \xrightarrow{c_{A, M} \circ c_{M, A}} & M \otimes A \\ & \searrow \rho & \swarrow \rho \\ & M & \end{array}$$

commutes. Here $\rho : M \otimes A \rightarrow M$ denotes the action of A on M .

The notion of dyslectic module was introduced by Pareigis in [P]. See also [KiO].

Remark 3.13. Note that a simple $M \in \mathcal{C}_A$ is dyslectic if and only if $M_+ \simeq M_-$ as A -bimodules. Indeed, since the functors $M \mapsto M_{\pm}$ from \mathcal{C}_A to ${}_A\mathcal{C}_A$ are embeddings, for any simple $M \in \mathcal{C}_A$ any isomorphism between A -bimodules M_+ and M_- must be a multiple of id_M .

Dyslectic modules form a full subcategory of \mathcal{C}_A which will be denoted by \mathcal{C}_A^0 . It is known (see [P], Section 2, and [KiO]) that \mathcal{C}_A^0 is closed under \otimes_A and that the braiding in \mathcal{C} induces a natural braided structure in \mathcal{C}_A^0 . Thus, \mathcal{C}_A^0 is a braided fusion category.

Example 3.14. Let $\mathcal{E} \subset \mathcal{C}$ be a Tannakian subcategory and let $A \in \mathcal{E}$ be a regular algebra, see Example 2.8. Then [DGNO], Proposition 4.56 (i), says that \mathcal{C}_A^0 is equivalent to the de-equivariantization of \mathcal{E}' , cf. Section 2.5.

Lemma 3.15. *Let \mathcal{C} be a braided fusion category, let A be an étale algebra in \mathcal{C} , and let X be an object of \mathcal{C} . Then the free module $X \otimes A$ is dyslectic if and only if X centralizes A .*

Proof. Consider the following diagram, where we omit identity maps and associativity constraints:

$$(18) \quad \begin{array}{ccccc} & & A \otimes X \otimes A & & \\ & \nearrow c_{X \otimes A, A} & & \nwarrow c_{A, X \otimes A} & \\ X \otimes A \otimes A & \xrightarrow{c_{A, A}} & X \otimes A \otimes A & \xrightarrow{c_{A, A}} & X \otimes A \otimes A \\ & \searrow m_A & \nearrow m_A & \nwarrow m_A & \searrow m_A \\ & & X \otimes A & & \end{array}$$

The two upper triangles commute by the hexagon axioms and the two lower triangles commute since A is commutative. Therefore,

$$(\mathrm{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \mathrm{id}_A) = (\mathrm{id}_X \otimes m_A) \circ c_{A, X \otimes A} \circ c_{X \otimes A, A} \circ (\mathrm{id}_X \otimes c_{A, A}^{-1}),$$

which means that $X \otimes A$ is dyslectic if and only if

$$(19) \quad (\mathrm{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \mathrm{id}_A) = \mathrm{id}_X \otimes m_A.$$

In other words, commutativity of the perimeter of the above diagram is equivalent to commutativity of the diamond in the middle. Let $u_A : \mathbf{1} \rightarrow A$ denote the unit of A . Suppose that (19) holds. We have

$$\begin{aligned} c_{A, X} \circ c_{X, A} &= (\mathrm{id}_X \otimes m_A) \circ (\mathrm{id}_{X \otimes A} \otimes u_A) \circ c_{A, X} \circ c_{X, A} \\ &= (\mathrm{id}_X \otimes m_A) \circ (c_{A, X} \circ c_{X, A} \otimes \mathrm{id}_A) \circ (\mathrm{id}_{X \otimes A} \otimes u_A) \\ &= (\mathrm{id}_X \otimes m_A) \circ (\mathrm{id}_{X \otimes A} \otimes u_A) = \mathrm{id}_{X \otimes A}, \end{aligned}$$

where the third equality holds by (19). Thus, (19) is equivalent to $c_{A, X} \circ c_{X, A} = \mathrm{id}_{X \otimes A}$. Combining the above equivalences, we get the result. \square

3.6. Étale algebras in \mathcal{C}_A^0 and étale algebras over A . Let \mathcal{C} be a braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra. An algebra $B \in \mathcal{C}$ equipped with a unital homomorphism $f : A \rightarrow B$ is called *algebra over A* if the following diagram commutes:

$$\begin{array}{ccccc} A \otimes B & \xrightarrow{f \otimes \mathrm{id}} & B \otimes B & \xrightarrow{m_B} & B \\ \downarrow c_{B, A}^{-1} & & & \nearrow m_B & \\ B \otimes A & \xrightarrow{\mathrm{id} \otimes f} & B \otimes B & & \end{array}$$

In the language of [O], Section 5.4, we require that the morphism f lands in the right center of B ; in particular for a commutative algebra B this diagram commutes automatically. Notice that the morphism f is automatically injective since the algebra A has no nontrivial right ideals.

Observe that an algebra B over A has an obvious structure of right A -module, that is $B \in \mathcal{C}_A$. Moreover, any right B -module has an obvious structure of right A -module. The following statements are tautological:

(a) An algebra over A is the same as an algebra in \mathcal{C}_A .

(b) Let B be an algebra over A . Then right B -module in \mathcal{C}_A (with B considered as an algebra in \mathcal{C}_A) is the same as right B -module in \mathcal{C} . In particular, the categories $(\mathcal{C}_A)_B$ and \mathcal{C}_B are equivalent.

(c) Commutative algebra over A is the same as commutative algebra in $\mathcal{C}_A^0 \subset \mathcal{C}_A$.

Proposition 3.16 (cf. [FFRS], Lemma 4.13, and [D1], Proposition 2.3.3). *A commutative algebra over A is étale if and only if the corresponding algebra in \mathcal{C}_A^0 is étale. Under this bijection connected algebras correspond to connected ones.*

Proof. The first statement follows from the tautologies above combined with Proposition 2.7. The second statement is implied by the fact that a simple A -module M with $\text{Hom}_{\mathcal{C}}(\mathbf{1}, M) \neq 0$ is isomorphic to A , see e.g. [O], Lemma 3.2. \square

3.7. The category $\text{Rep}_{\mathcal{A}}(A)$ and its center. Let \mathcal{A} be a fusion category and let

$$F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$$

be the forgetful functor. Let $A \in \mathcal{Z}(\mathcal{A})$ be a connected étale algebra. Observe that any right $F(A)$ -module $M \in \mathcal{A}$ has a natural structure of left $F(A)$ -module defined as

$$F(A) \otimes M \simeq M \otimes F(A) \rightarrow M.$$

It is easy to verify that in this way M acquires a structure of $F(A)$ -bimodule.

Definition 3.17. The category $\text{Rep}_{\mathcal{A}}(A)$ is the tensor category of right $F(A)$ -modules in \mathcal{A} with tensor product $\otimes_{F(A)}$.

Remark 3.18. (i) Assume that \mathcal{C} is a braided fusion category and $A \in \mathcal{C}$ is a connected étale algebra. Then A can be considered as a connected étale algebra in $\mathcal{Z}(\mathcal{C})$ via the braided functor $\mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C})$ given in (7). In this case the categories \mathcal{C}_A and $\text{Rep}_{\mathcal{C}}(A)$ are identical. Nevertheless the tensor structures on \mathcal{C}_A and $\text{Rep}_{\mathcal{C}}(A)$ are *opposite* to each other.

(ii) The category $\text{Rep}_{\mathcal{C}}(A)$ is equivalent to the category of *left* $F(A)$ -modules.

Arguments similar to those in Section 3.3 show that $\text{Rep}_{\mathcal{A}}(A)$ is a semisimple rigid tensor category. Its unit object $F(A)$ may be reducible, so in general $\text{Rep}_{\mathcal{A}}(A)$ is not a

fusion category. In general $\text{Rep}_{\mathcal{A}}(A)$ is an example of a multi-fusion category, see Section 2.1.

Remark 3.19. Given an étale algebra $A \in \mathcal{Z}(\mathcal{A})$ there is a surjective tensor functor

$$\mathcal{A} \rightarrow \text{Rep}_{\mathcal{A}}(A) : X \mapsto X \otimes F(A).$$

Conversely, let $G : \mathcal{A} \rightarrow \mathcal{B}$ be a tensor functor and let $I : \mathcal{B} \rightarrow \mathcal{A}$ be its right adjoint. Then the object $I(\mathbf{1}) \in \mathcal{A}$ has a natural lift to $\mathcal{Z}(\mathcal{A})$. Moreover, it has a natural structure of an étale algebra in $\mathcal{Z}(\mathcal{A})$. The algebra $I(\mathbf{1}) \in \mathcal{Z}(\mathcal{A})$ is connected if and only if the functor G is not decomposable into a non-trivial direct sum of tensor functors. Similarly to Section 3.4 these two constructions are inverse to each other. See [BN] for details.

It is easy to see that the forgetful functor $\mathcal{Z}(\mathcal{A})_A^0 \hookrightarrow \mathcal{Z}(\mathcal{A})_A \rightarrow \text{Rep}_{\mathcal{A}}(A)$ has a canonical structure of central functor. Thus, it lifts to a braided tensor functor

$$(20) \quad \mathcal{Z}(\mathcal{A})_A^0 \rightarrow \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A)).$$

The following result was proved by Schauenburg (see [S], Corollary 4.5) under much weaker assumptions on the category \mathcal{A} and commutative algebra $A \in \mathcal{Z}(\mathcal{A})$ than ours.

Theorem 3.20. *The functor (20) is a braided equivalence $\mathcal{Z}(\mathcal{A})_A^0 \cong \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$.*

Sketch of proof. We just sketch a construction of an inverse functor. Let

$$M \in \mathcal{Z}(\text{Rep}_{\mathcal{A}}(A)).$$

For any $X \in \mathcal{A}$ consider $X \otimes F(A) \in \text{Rep}_{\mathcal{A}}(A)$. Then

$$(X \otimes F(A)) \otimes_{F(A)} M = X \otimes M \quad \text{and} \quad M \otimes_{F(A)} (X \otimes F(A)) = M \otimes X.$$

It is easy to see now that the central structure of M as $F(A)$ -module defines a central structure of M as an object of \mathcal{A} . Moreover one verifies directly that $F(A)$ -module structure on M gives A -module structure on this lift of M to $\mathcal{Z}(\mathcal{A})$; the resulting object of $\mathcal{Z}(\mathcal{A})_A$ lies in $\mathcal{Z}(\mathcal{A})_A^0$. Finally, this assignment has a natural structure of tensor functor. \square

Remark 3.21. Theorem 3.20 above implies that the unit object of the fusion category $\mathcal{Z}(\text{Rep}_{\mathcal{A}}(A))$ is indecomposable (recall that the algebra A is connected). It follows that the multi-fusion category $\text{Rep}_{\mathcal{A}}(A)$ is *indecomposable* in the sense of [ENO1], Section 2.4.

3.8. Properties of braided tensor functors.

Proposition 3.22. *Let \mathcal{C}, \mathcal{D} be braided fusion categories and let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a surjective braided tensor functor. Let $I : \mathcal{D} \rightarrow \mathcal{C}$ be the right adjoint functor of F and let $A := I(\mathbf{1})$ be the canonical connected étale algebra constructed in Lemma 3.5. Then $A \in \mathcal{C}^!$.*

Proof. Since F is a central functor, \mathcal{D} identifies with the category \mathcal{C}_A of A -modules in \mathcal{C} , cf. Section 3.4. We claim that every A -module is dyslectic, i.e., that $\mathcal{C}_A = \mathcal{C}_A^0$. Indeed,

the fusion category ${}_A\mathcal{C}_A$ identifies with the category of \mathcal{C} -module endofunctors of \mathcal{D} , see [O] (the action of \mathcal{C} on \mathcal{D} is defined via $F : \mathcal{C} \rightarrow \mathcal{D}$). Under this identification, for every simple object $M \in \mathcal{D}$ the bimodules M_{\pm} correspond to endofunctors of left and right multiplication by M . But these endofunctors are isomorphic via the braiding of \mathcal{D} , i.e., M is dyslectic.

In particular, for every $X \in \mathcal{C}$ the free A -module $X \otimes A$ is dyslectic. Hence, Lemma 3.15 implies that every $X \in \mathcal{C}$ centralizes A , i.e., $A \in \mathcal{C}'$. \square

Remark 3.23. Note that the étale algebra A from Proposition 3.22 is a commutative algebra in a symmetric fusion category \mathcal{C}' . Therefore, A belongs to the maximal Tannakian subcategory $\mathcal{E} = \text{Rep}(G) \subset \mathcal{C}'$. As is well known, every étale algebra $A \in \text{Rep}(G)$ is isomorphic to the algebra $\text{Fun}(G/H)$ of functions on G invariant under translations by elements of H for a uniquely determined subgroup $H \subset G$, the module category $\text{Rep}(G)_A$ is equivalent to $\text{Rep}(H)$ and the functor F_A identifies with the restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(H)$. In view of $A \in \mathcal{E}$, the restriction $F : \mathcal{E} \rightarrow F(\mathcal{E})$ of F to \mathcal{E} identifies with the restriction functor $\text{Rep}(G) \rightarrow \text{Rep}(H)$.

Corollary 3.24. *Let $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ be a surjective braided tensor functor between braided fusion categories. There exists a braided fusion category \mathcal{C} with an action of a finite group G , a subgroup $H \subset G$, and braided tensor equivalences $\mathcal{C}_1 \cong \mathcal{C}^G$, $\mathcal{C}_2 \cong \mathcal{C}^H$ such that the diagram*

$$(21) \quad \begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{C}^G & \xrightarrow{\text{Forg}} & \mathcal{C}^H \end{array}$$

commutes. Here $\text{Forg} : \mathcal{C}^G \rightarrow \mathcal{C}^H$ is the functor of “partially forgetting equivariance”.

Proof. By Proposition 3.22 there is an étale algebra A in \mathcal{C}'_1 such that $\mathcal{C}_2 \cong (\mathcal{C}_1)_A$. Let $\mathcal{E} = \text{Rep}(G)$ be the maximal Tannakian subcategory of \mathcal{C}'_1 and let $\mathcal{C} = (\mathcal{C}_1)_G$. Since equivariantization and de-equivariantization are mutually inverse constructions (see [DGNO], Theorem 4.4, and Section 2.5), we have $\mathcal{C}_1 \cong \mathcal{C}^G$.

By Remark 3.23 there is a subgroup $H \subset G$ such that $A = \text{Fun}(G/H)$. Note that a $\text{Fun}(G/H)$ -module in \mathcal{C}_1 is the same thing as an H -equivariant $\text{Fun}(G)$ -module, which implies $(\mathcal{C}_1)_A \cong ((\mathcal{C}_1)_G)^H = \mathcal{C}^H$. Furthermore, the forgetful functor $\mathcal{C}^G \rightarrow \mathcal{C}^H$ identifies with the given functor $F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \cong (\mathcal{C}_1)_A$ since both of them correspond to the same étale algebra $A = \text{Fun}(G/H)$. \square

Definition 3.25. A braided fusion category \mathcal{C} is called *almost non-degenerate* if the symmetric category \mathcal{C}' is either trivial or is equivalent to the category of super vector spaces.

In other words, \mathcal{C} is almost non-degenerate if \mathcal{C}' does not contain any non-trivial Tannakian subcategories.

Corollary 3.26. *Any braided tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ between braided fusion categories with \mathcal{C} almost non-degenerate is fully faithful.*

Remark 3.27. Using [EO], Theorem 2.5, and [De], Proposition 2.14, one can relax the assumptions of Corollary 3.26 on the category \mathcal{D} : it is enough to assume that \mathcal{D} is an abelian rigid braided tensor category with finite dimensional Hom spaces and finite lengths of all objects.

Let \mathcal{C} be a braided fusion category, $A \in \mathcal{C}$ be a connected étale algebra and $F_A : \mathcal{C} \rightarrow \mathcal{C}_A$ be the functor (13) with the central structure F'_A (15). The functor

$$(22) \quad T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A : M \boxtimes N \mapsto M_+ \otimes_A N_-$$

has a natural structure of tensor functor.

Corollary 3.28. *Assume \mathcal{C} is almost non-degenerate. Then the functor F'_A in (15) is fully faithful and the functor $T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A$ defined in (22) is surjective.*

Proof. The first assertion is Corollary 3.26. To prove the second assertion, observe that F'_A is dual to T_A (in the sense of [ENO1], Section 5.7) with respect to the module category \mathcal{C}_A . Indeed, an object $M \boxtimes N$ of $\mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}}$ corresponds to the $\mathcal{Z}(\mathcal{C}_A)$ -module endofunctor $M \otimes_A - \otimes_A N$ of \mathcal{C}_A . The functor dual to F'_A restricts this endofunctor to the \mathcal{C} -module endofunctor of \mathcal{C}_A by means of $F'_A : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$. This is precisely what $T_A(M \boxtimes N)$ does. So the result follows from [ENO1], Proposition 5.3. \square

3.9. Tensor complements. Let \mathcal{C} be a non-degenerate braided fusion category, see Definition 2.1. Let $A \in \mathcal{C}$ be a connected étale algebra. Then A can be considered as a connected étale algebra in \mathcal{C}^{rev} and in $\mathcal{Z}(\mathcal{C})$ via the embedding

$$\mathcal{C}^{\text{rev}} = \text{Vec} \boxtimes \mathcal{C}^{\text{rev}} \hookrightarrow \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}),$$

see (9).

Lemma 3.29. *Under the identification $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$ we have*

$$\mathcal{Z}(\mathcal{C})_A = \mathcal{C} \boxtimes \mathcal{C}_A^{\text{rev}} \quad \text{and} \quad \mathcal{Z}(\mathcal{C})_A^0 = \mathcal{C} \boxtimes (\mathcal{C}_A^{\text{rev}})_A^0.$$

Proof. The first statement is obvious and the second one is an immediate consequence. \square

Corollary 3.30. *For a non-degenerate \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ there is a braided equivalence $\mathcal{Z}(\mathcal{C}_A) \simeq \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\text{rev}}$. In particular the category \mathcal{C}_A^0 is non-degenerate.*

Proof. Combine Theorem 3.20 and Lemma 3.29. \square

Remark 3.31. (i) One can show that the embedding functor

$$\mathcal{C} = \mathcal{C} \boxtimes \text{Vec} \hookrightarrow \mathcal{C} \boxtimes (\mathcal{C}_A^0)^{\text{rev}} \cong \mathcal{Z}(\mathcal{C}_A)$$

is naturally isomorphic to the functor F'_A from (15), providing an alternative proof of the injectivity of that functor, as asserted in Corollary 3.28.

(ii) If we assume in addition that \mathcal{C} is modular and A is as in Example 3.3 (ii), then \mathcal{C}_A^0 has a natural spherical structure, see e.g. [KiO]. In this case Corollary 3.30 gives an alternative proof of [KiO], Theorem 4.5.

Corollary 3.32. *For a non-degenerate \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ we have*

$$(23) \quad \text{FPdim}(\mathcal{C}_A^0) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)^2}.$$

Proof. This follows immediately from Corollary 3.30 and equations (6) and (16). \square

4. Quantum Manin pairs

4.1. Definition of a quantum Manin pair. We start with the following consequence of Corollary 3.28.

Corollary 4.1. *Let \mathcal{C} be a non-degenerate braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra in \mathcal{C} . Assume that $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$. Then:*

- (i) *The functor $F'_A : \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{C}_A)$ defined in (15) is a braided tensor equivalence.*
- (ii) *The functor $T_A : \mathcal{C}_A \boxtimes \mathcal{C}_A^{\text{rev}} \rightarrow {}_A\mathcal{C}_A$ defined in (22) is a tensor equivalence.*

Proof. By Lemma 3.11,

$$\text{FPdim}(\mathcal{C}_A) = \frac{\text{FPdim}(\mathcal{C})}{\text{FPdim}(A)}.$$

Hence,

$$\text{FPdim}(\mathcal{Z}(\mathcal{C}_A)) = \frac{\text{FPdim}(\mathcal{C})^2}{\text{FPdim}(A)^2} = \text{FPdim}(\mathcal{C}),$$

see (6). Since by Corollary 3.28, F'_A is a fully faithful functor between categories of equal Frobenius–Perron dimension, it is necessarily an equivalence by [EO], Proposition 2.19. Hence the dual functor T_A is also an equivalence. \square

Definition 4.2. A *quantum Manin pair* (\mathcal{C}, A) consists of a non-degenerate braided fusion category \mathcal{C} and a connected étale algebra $A \in \mathcal{C}$ such that $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$.

Remark 4.3. Observe that by (23) the condition $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$ is equivalent to the condition $\mathcal{C}_A^0 = \text{Vec}$.

Quantum Manin pairs form a 2-groupoid \mathfrak{QM} : a 1-morphism between two such pairs (\mathcal{C}_1, A_1) and (\mathcal{C}_2, A_2) is defined to be a pair (Φ, ϕ) , where $\Phi : \mathcal{C}_1 \simeq \mathcal{C}_2$ is a braided equivalence and $\phi : \Phi(A_1) \xrightarrow{\sim} A_2$ is an isomorphism of algebras; a 2-morphism between pairs

(Φ, ϕ) and (Φ', ϕ') is a natural isomorphism of tensor functors $\mu : \Phi \simeq \Phi'$ such that the following diagram commutes:

$$(24) \quad \begin{array}{ccc} \Phi(A_1) & \xrightarrow{\mu} & \Phi'(A_1) \\ \phi \searrow & & \swarrow \phi' \\ & A_2 & \end{array}$$

On the other hand, we have the 2-groupoid \mathfrak{FC} of fusion categories: objects are fusion categories, 1-morphisms are tensor equivalences, and 2-morphisms are isomorphisms of tensor functors. We have a 2-functor $\mathfrak{M} \rightarrow \mathfrak{FC}$ defined by $(\mathcal{C}, A) \mapsto \mathcal{C}_A$.

Proposition 4.4. *This 2-functor $\mathfrak{M} \rightarrow \mathfrak{FC}$ is a 2-equivalence.*

Proof. Let $\mathcal{A} \in \mathfrak{FC}$. The forgetful functor $F : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ has an obvious structure of central functor. Let $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ be its right adjoint. By Lemma 3.5, $I(\mathbf{1})$ is a connected étale algebra. It is known that $\text{FPdim}(I(\mathbf{1})) = \text{FPdim}(\mathcal{C})$, see e.g. [EO], Lemma 3.41. So (6) implies that $(\mathcal{Z}(\mathcal{A}), I(\mathbf{1})) \in \mathfrak{M}$. Thus we get a 2-functor $\mathfrak{FC} \rightarrow \mathfrak{M}$. Using Corollary 4.1 and the results from Section 3.4 we see that it is quasi-inverse to the 2-functor $\mathfrak{M} \rightarrow \mathfrak{FC}$. \square

Remark 4.5. Proposition 4.4 can be viewed as a categorical analogue of the following reconstruction of the double of a quasi-Lie bialgebra from a Manin pair (i.e., a pair consisting of a metric Lie algebra and its Lagrangian subalgebra) in the theory of quantum groups [Dr], Section 2:

Let \mathfrak{g} be a finite dimensional metric Lie algebra (i.e., a Lie algebra on which a non-degenerate invariant symmetric bilinear form is given). Let \mathfrak{l} be a Lagrangian subalgebra of \mathfrak{g} . Then \mathfrak{l} has a structure of a quasi-Lie bialgebra and there is an isomorphism between \mathfrak{g} and the double $\mathfrak{D}(\mathfrak{l})$ of \mathfrak{l} . The correspondence between Lagrangian subalgebras of \mathfrak{g} and doubles isomorphic to \mathfrak{g} is bijective, see [Dr], Section 2, for details.

4.2. Lagrangian algebras and module categories.

Definition 4.6. Let \mathcal{C} be a non-degenerate braided fusion category. A connected étale algebra in \mathcal{C} will be called *Lagrangian* if $\text{FPdim}(A)^2 = \text{FPdim}(\mathcal{C})$.

Thus, A is Lagrangian if and only if (\mathcal{C}, A) is a quantum Manin pair.

Remark 4.7. Let $\mathcal{E} \subset \mathcal{C}$ be a Lagrangian subcategory of \mathcal{C} , i.e., a Tannakian subcategory such that $\mathcal{E}' = \mathcal{E}$, see [DGNO], Definition 4.57. Then the regular algebra A of \mathcal{E} is a Lagrangian algebra in \mathcal{C} . Indeed, Example 3.14 says that $\mathcal{C}_A^0 = \text{Vec}$ and the statement follows from Remark 4.3.

Proposition 4.8. *Let \mathcal{A} be a fusion category and let $\mathcal{C} = \mathcal{Z}(\mathcal{A})$. There is a bijection between the sets of Lagrangian algebras in \mathcal{C} and indecomposable \mathcal{A} -module categories.*

Proof. By Corollary 4.1 every Lagrangian algebra $B \in \mathcal{C}$ determines a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{B})$, where $\mathcal{B} := \mathcal{C}_B$. Conversely, any braided equivalence between \mathcal{C} and $\mathcal{Z}(\mathcal{B})$ determines a surjective central functor $\mathcal{C} \rightarrow \mathcal{B}$ and, hence, a connected étale

algebra $A \in \mathcal{C}$, see Lemma 3.5. Combining Lemma 3.11 and equation (6), we see that the algebra A is Lagrangian. As we observed in Section 3.4 these two constructions are inverses of each other.

Thus it suffices to prove that the set of braided equivalences between $\mathcal{Z}(\mathcal{A})$ and centers of fusion categories is in bijection with indecomposable \mathcal{A} -module categories. This is done in [ENO3], Theorem 3.1, and [ENO2], Theorem 1.1. Namely, the bijection is provided by assigning to an \mathcal{A} -module category \mathcal{M} braided equivalence (10). \square

Remark 4.9. (i) It follows from the proof that the bijection in Proposition 4.8 has the following property: for a Lagrangian algebra $B \in \mathcal{C}$ the fusion category \mathcal{C}_B is equivalent to the dual category $\mathcal{A}_{\mathcal{M}}^*$ where \mathcal{M} is the module category corresponding to B .

(ii) Note that the bijection in Proposition 4.8 is given by the so-called full centre construction. In particular, $I(\mathbf{1})$ is the full centre of \mathcal{A} as a module category over itself. In the case when \mathcal{A} is modular, the statement of the proposition was verified in [KR], Theorem 3.22. Note also that in this case the bijection can be lifted to an equivalence of groupoids (module categories with module equivalences by one side and Lagrangian algebras and isomorphisms by the other) [DKR].

4.3. Lattice of subcategories. Let \mathcal{A} be a fusion category and let (\mathcal{C}, A) be the corresponding Manin pair. Here $\mathcal{C} = \mathcal{Z}(\mathcal{A})$ and $A = I(\mathbf{1})$, where $I : \mathcal{A} \rightarrow \mathcal{Z}(\mathcal{A})$ is the induction functor.

Let $\mathfrak{L}(\mathcal{A})$ denote the lattice of fusion subcategories of \mathcal{A} and let $L(A)$ denote the lattice of étale subalgebras of A .

Theorem 4.10. *There is a canonical anti-isomorphism of lattices $\mathfrak{L}(\mathcal{A}) \simeq L(A)$. If $B \subset A$ is the subalgebra corresponding to the subcategory $\mathcal{B} \subset \mathcal{A}$ under this anti-isomorphism, then $\text{FPdim}(B) \text{FPdim}(\mathcal{B}) = \text{FPdim}(\mathcal{A})$.*

Proof. We will construct mutually inverse order-reversing bijections

$$\alpha : \mathfrak{L}(\mathcal{A}) \rightarrow L(A) \quad \text{and} \quad \beta : L(A) \rightarrow \mathfrak{L}(\mathcal{A}).$$

Let $\mathcal{B} \subset \mathcal{A}$ be a fusion subcategory. Define the *relative center* $\mathcal{Z}_{\mathcal{B}}(\mathcal{A})$ to be the tensor category whose objects are pairs (X, γ_X) , where X is an object of \mathcal{A} and $\gamma_X : V \otimes X \simeq X \otimes V$, $V \in \mathcal{B}$ is a natural family of isomorphisms, satisfying the same compatibility condition as in the definition of $\mathcal{Z}(\mathcal{A})$. The forgetful functor $\mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$ has a factorization

$$\mathcal{Z}(\mathcal{A}) \xrightarrow{F_{\mathcal{B}}} \mathcal{Z}_{\mathcal{B}}(\mathcal{A}) \xrightarrow{\tilde{F}_{\mathcal{B}}} \mathcal{A}$$

where $F_{\mathcal{B}}$ and $\tilde{F}_{\mathcal{B}}$ are the obvious forgetful functors. Let $I_{\mathcal{B}}$ and $\tilde{I}_{\mathcal{B}}$ be the right adjoint functors of $F_{\mathcal{B}}$ and $\tilde{F}_{\mathcal{B}}$. The embedding $\mathbf{1} \subset \tilde{I}_{\mathcal{B}}(\mathbf{1})$ corresponding to the identity map under the isomorphism $\text{Hom}(\mathbf{1}, \tilde{I}_{\mathcal{B}}(\mathbf{1})) = \text{Hom}(\tilde{F}_{\mathcal{B}}(\mathbf{1}), \mathbf{1})$ induces an embedding of algebras $I_{\mathcal{B}}(\mathbf{1}) \subset I_{\mathcal{B}} \circ \tilde{I}_{\mathcal{B}}(\mathbf{1}) = I(\mathbf{1}) = A$. The algebra $I_{\mathcal{B}}(\mathbf{1})$ is separable (and hence étale), see Remark 3.19. We define

$$(25) \quad \alpha(\mathcal{B}) = I_{\mathcal{B}}(\mathbf{1}) \subset A.$$

An inclusion of subcategories $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \mathcal{A}$ induces a factorization

$$\mathcal{Z}(\mathcal{A}) \xrightarrow{F_{\mathcal{B}_2}} \mathcal{Z}_{\mathcal{B}_2}(\mathcal{A}) \rightarrow \mathcal{Z}_{\mathcal{B}_1}(\mathcal{A})$$

of the functor $F_{\mathcal{B}_1}$. This, in turn, yields an inclusion of subalgebras $I_{\mathcal{B}_2}(\mathbf{1}) \subset I_{\mathcal{B}_1}(\mathbf{1}) \subset A$. Thus the map α is order-reversing.

The functor $F_{\mathcal{B}}$ is surjective by [DGNO], Section 3.6. Hence we have

$$\mathrm{FPdim}(\alpha(\mathcal{B})) = \frac{\mathrm{FPdim}(\mathcal{A})}{\mathrm{FPdim}(\mathcal{B})}$$

by (the proof of) [ENO1], Corollary 8.11.

In the opposite direction, given an étale subalgebra $B \subset A$ we have a tensor functor $? \otimes_B A : \mathcal{C}_B \rightarrow \mathcal{C}_A$ inducing A -modules from B -modules. Let $\beta(B)$ be the full image in $\mathcal{C}_A = \mathcal{A}$ of the subcategory $\mathcal{C}_B^0 \subset \mathcal{C}_B$ under this functor. Observe that A considered as a B -module is dyslectic. It follows that the objects of $\beta(B)$ are precisely A -modules which are dyslectic as B -modules. This implies that the map β is order-reversing. Observe that the right adjoint functor of $? \otimes_B A$ is isomorphic to the forgetful functor $\mathcal{C}_A \rightarrow \mathcal{C}_B$ and sends the unit object of \mathcal{C}_A to $A \in \mathcal{C}_B^0 \subset \mathcal{C}_B$. Using again the proof of [ENO1], Corollary 8.11, we see that

$$(26) \quad \mathrm{FPdim}(\beta(B)) = \frac{\mathrm{FPdim}(A)}{\mathrm{FPdim}(B)}.$$

By construction, $\mathcal{C}_{\alpha(\mathcal{B})} = \mathcal{Z}_{\mathcal{B}}(\mathcal{A})$. We claim that the subcategory $\mathcal{C}_{\alpha(\mathcal{B})}^0 \subset \mathcal{C}_{\alpha(\mathcal{B})}$ identifies with $\mathcal{Z}(\mathcal{B}) \subset \mathcal{Z}_{\mathcal{B}}(\mathcal{A})$. Indeed, by Corollary 3.30 the category $(\mathcal{C}_{\alpha(\mathcal{B})}^0)^{\mathrm{rev}}$ identifies with the centralizer of \mathcal{C} in $\mathcal{Z}(\mathcal{C}_{\alpha(\mathcal{B})})$. On the other hand it is explained in [DGNO], Section 3.6, that $\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) = (\mathcal{A} \boxtimes \mathcal{B}^{\mathrm{op}})_{\mathcal{A}}^*$ (see Section 2.6 for the notation), so equation (10) implies $\mathcal{Z}(\mathcal{Z}_{\mathcal{B}}(\mathcal{A})) = \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{B})^{\mathrm{rev}}$. The central functor

$$\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \boxtimes \mathbf{1} \subset \mathcal{Z}(\mathcal{A}) \boxtimes \mathcal{Z}(\mathcal{B})^{\mathrm{rev}} = \mathcal{Z}(\mathcal{Z}_{\mathcal{B}}(\mathcal{A})) \rightarrow \mathcal{Z}_{\mathcal{B}}(\mathcal{A})$$

identifies with $F_{\mathcal{B}}$ with obvious central structure. Hence the subcategories

$$\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}) \boxtimes \mathbf{1} \subset \mathcal{Z}(\mathcal{Z}_{\mathcal{B}}(\mathcal{A})) \quad \text{and} \quad \mathcal{C} \subset \mathcal{Z}(\mathcal{C}_{\alpha(\mathcal{B})})$$

coincide and so do their centralizers in $\mathcal{Z}(\mathcal{Z}_{\mathcal{B}}(\mathcal{A}))$ and their images in $\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) = \mathcal{C}_{\alpha(\mathcal{B})}$. Our claim follows.

The induction functor

$$(27) \quad \mathcal{C}_{\alpha(\mathcal{B})} \rightarrow \mathcal{C}_A = \mathcal{A}$$

identifies with the forgetful functor $\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) \rightarrow \mathcal{A}$ and so maps surjectively $\mathcal{Z}(\mathcal{B}) = \mathcal{C}_{\alpha(\mathcal{B})}^0$ to \mathcal{B} . Thus, $\beta(\alpha(\mathcal{B})) = \mathcal{B}$.

Conversely, we claim that there is an equivalence $\mathcal{Z}_{\beta(B)}(\mathcal{A}) \simeq \mathcal{C}_B$ such that the forgetful functor $F_{\beta(B)} : \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{Z}_{\beta(B)}(\mathcal{A})$ identifies with the free module functor $\mathcal{C} \rightarrow \mathcal{C}_B$. This immediately implies that $\alpha(\beta(B)) = B$. To prove this claim, note that the braiding of \mathcal{C} allows to equip any A -module induced from \mathcal{C}_B with a morphism permuting it with the objects of $\beta(B) \subset \mathcal{C}_A$ (notice that for $M \in \mathcal{C}_B$ and $N \in \beta(B)$ we have $(M \otimes_B A) \otimes_A N = M \otimes_B N$ and $N \otimes_A (M \otimes_B A) = N \otimes_B M$). This gives rise to a tensor functor

$$(28) \quad F_A'' : \mathcal{C}_B \rightarrow \mathcal{Z}_{\beta(B)}(\mathcal{C}_A), \quad M \mapsto M \otimes_B A.$$

Recall the equivalence F_A' from Corollary 4.1 (i). It follows from the above definition that the diagram

$$(29) \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{F_A'} & \mathcal{Z}(\mathcal{C}_A) \\ \downarrow ? \otimes B & & \downarrow F_{\beta(B)} \\ \mathcal{C}_B & \xrightarrow{F_A''} & \mathcal{Z}_{\beta(B)}(\mathcal{C}_A) \end{array}$$

commutes. In particular, the induction functor (28) is surjective. Using [DGNO], equation (56), and equation (26), we get

$$\text{FPdim}(\mathcal{Z}_{\beta(B)}(\mathcal{A})) = \text{FPdim}(\beta(B)) \text{FPdim}(\mathcal{A}) = \frac{\text{FPdim}(A)^2}{\text{FPdim}(B)} = \text{FPdim}(\mathcal{C}_B).$$

Thus functor (28) is an equivalence by [EO], Proposition 2.20. This completes our proof. \square

Example 4.11. Let us illustrate Theorem 4.10. Let G be a finite group.

(i) Let $\mathcal{A} = \text{Rep}(G)$ be the fusion category of representations of G . Its fusion subcategories are of the form $\text{Rep}(G/N)$ where N ranges over the set of all normal subgroups of G . The étale algebra in $\mathcal{Z}(\text{Rep}(G))$ corresponding to the subcategory $\text{Rep}(G/N)$ is the group algebra $\mathbb{k}N$. As an object of $\mathcal{Z}(\text{Rep}(G))$ it has the following description. It is a G -graded algebra with non-zero graded components labelled by elements of N , the G -action on $\mathbb{k}N$ is the conjugation action (see [D1], where étale algebras in $\mathcal{Z}(\text{Rep}(G))$ were classified).

(ii) Let $\mathcal{A} = \text{Vec}_G^\omega$ be the fusion category of G -graded vector spaces with the associativity constraint twisted by a 3-cocycle $\omega \in Z^3(G, \mathbb{k}^\times)$. Fusion subcategories of \mathcal{A} correspond to subgroups $H \subset G$. A typical such subcategory is $\text{Vec}_H^{\omega|_H}$. The corresponding étale algebra in $\mathcal{Z}(\text{Vec}_G^\omega)$ is the algebra of \mathbb{k} -valued functions on G invariant under translations by elements of H .

Remark 4.12. Let \mathcal{C} be a non-degenerate braided fusion category and let $A \in \mathcal{C}$ be a connected étale algebra. Recall that $\mathcal{Z}(\mathcal{C}_A) \simeq \mathcal{C} \boxtimes (\mathcal{C}^{\text{rev}})_A^0$ (see Corollary 3.30) and the functor $\mathcal{C} = \mathcal{C} \boxtimes \mathbf{1} \subset \mathcal{Z}(\mathcal{C}_A) \rightarrow \mathcal{C}_A$ is isomorphic to the free module functor F_A , see Remark 3.31 (i). It follows that $A = A \boxtimes \mathbf{1} \in \mathcal{Z}(\mathcal{C}_A)$ is a subalgebra of the Lagrangian

algebra $I(1)$. It is easy to see that the corresponding subcategory of \mathcal{C}_A is precisely \mathcal{C}_A^0 . Thus Theorem 4.10 implies the following statement: the lattice of subalgebras of A is anti-isomorphic to the lattice of subcategories of \mathcal{C}_A containing \mathcal{C}_A^0 . Notice that Theorem 4.10 is a special case of this statement, see Remark 4.3.

4.4. Quantum Manin triples. Recall that a *Manin triple* consists of a metric Lie algebra \mathfrak{g} along with Lagrangian Lie subalgebras \mathfrak{g}_+ , \mathfrak{g}_- such that $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as a vector space. It was shown by Drinfeld in [Dr], Section 2, that Manin triples are in bijection with pairs of dual Lie bialgebras (cf. Remark 4.5).

Below we extend this result to the “quantum” setting.

Definition 4.13. A *quantum Manin triple* (\mathcal{C}, A, B) consists of a non-degenerate braided fusion category \mathcal{C} along with connected étale algebras A, B in \mathcal{C} such that both (\mathcal{C}, A) and (\mathcal{C}, B) are quantum Manin pairs and the category of (A, B) -bimodules in \mathcal{C} is equivalent to Vec .

Example 4.14. Let H be a semisimple Hopf algebra and let $\text{Rep}(H)$ denote the category of finite dimensional representations of H . Let $\mathcal{C} := \mathcal{Z}(\text{Rep}(H))$. It is well known that \mathcal{C} is equivalent, as a braided fusion category, to $\text{Rep}(D(H))$ where $D(H)$ is the Drinfeld double of H . There is a canonical Hopf algebra isomorphism $D(H) \cong D((H^*)^{\text{op}})$, where H^* denotes the dual Hopf algebra and op stands for the opposite multiplication. We thus have two central functors, to wit the forgetful functors,

$$\mathcal{C} \rightarrow \text{Rep}(H) \quad \text{and} \quad \mathcal{C} \rightarrow \text{Rep}((H^*)^{\text{op}}).$$

Let A and B denote the étale algebras in \mathcal{C} corresponding to these functors constructed as in Section 3.2.

We claim that (\mathcal{C}, A, B) is a quantum Manin triple. The only thing that needs to be checked is that the category of (A, B) -bimodules in \mathcal{C} is trivial. Note that $A = (H^*)^{\text{op}}$ and $B = H$ as $D(H)$ -module algebras (i.e., algebras in $\mathcal{C} = \text{Rep}(D(H))$). The category of $(H^*)^{\text{op}} \otimes H$ -bimodules in $\text{Rep}(D(H))$ is nothing but the category of $D(H)$ -Hopf modules which is equivalent to Vec by the Fundamental Theorem of Hopf modules (see [M] for the definition of a Hopf module and the Fundamental Theorem).

We explain now that any quantum Manin triple arises from the construction in Example 4.14. Let (\mathcal{C}, A, B) be a quantum Manin triple. Then Vec identified with (A, B) -bimodules has a structure of a \mathcal{C}_A -module category via \otimes_A . Equivalently, \mathcal{C}_A has a fiber functor, i.e., a tensor functor to Vec , see [O], Proposition 4.1. Thus $\mathcal{C}_A \cong \text{Rep}(H)$ for a semisimple Hopf algebra H , see [UI]. The dual category $(\mathcal{C}_A)_{\text{Vec}}^*$ is equivalent to \mathcal{C}_B (see Remark 4.9 (i)) and so $\mathcal{C}_B \cong \text{Rep}((H^*)^{\text{op}})$, see [O], Theorem 4.2.

Quantum Manin triples form a 2-groupoid \mathcal{G}_1 : a 1-morphism between triples $(\mathcal{C}_1, A_1, B_1)$ and $(\mathcal{C}_2, A_2, B_2)$ is defined to be a triple (Φ, ϕ, ψ) , where $\Phi: \mathcal{C}_1 \simeq \mathcal{C}_2$ is a braided equivalence and $\phi: \Phi(A_1) \xrightarrow{\sim} A_2$, $\psi: \Phi(B_1) \xrightarrow{\sim} B_2$ are isomorphisms of algebras; a 2-morphism between triples (Φ, ϕ, ψ) and (Φ', ϕ', ψ') is a natural isomorphism of tensor functors $\mu: \Phi \simeq \Phi'$ such that $\phi = \phi' \mu_{A_1}$ and $\psi = \psi' \mu_{B_1}$ (cf. diagram (24)).

Let \mathcal{G}_2 denote the 2-groupoid whose objects are pairs (\mathcal{A}, F) where \mathcal{A} is a fusion category and $F : \mathcal{A} \rightarrow \mathbf{Vec}$ is a fiber functor; 1-morphisms between (\mathcal{A}, F) and (\mathcal{A}', F') are pairs (ι, ν) where $\iota : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ is a tensor equivalence and $\nu : F \xrightarrow{\sim} F'\iota$ is an isomorphism of tensor functors; 2-morphisms between (ι_1, ν_1) and (ι_2, ν_2) are natural isomorphisms of tensor functors $m : \iota_1 \xrightarrow{\sim} \iota_2$ such that $\nu_2 = (F'm) \circ \nu_1$.

As we explained above a quantum Manin triple (\mathcal{C}, A, B) gives rise to a fusion category \mathcal{C}_A equipped with a fiber functor $F : \mathcal{C}_A \rightarrow \mathbf{Vec}$. This construction can be upgraded to a 2-functor $\mathcal{G}_1 \rightarrow \mathcal{G}_2$. Similarly, the construction from Example 4.14 can be upgraded to a 2-functor $\mathcal{G}_2 \rightarrow \mathcal{G}_1$ (we recall that by [UI] a pair $(\mathcal{A}, F) \in \mathcal{G}_2$ is isomorphic to the pair $(\mathbf{Rep}(H), F_H)$ where H is a semisimple Hopf algebra and $F_H : \mathbf{Rep}(H) \rightarrow \mathbf{Vec}$ is the forgetful functor).

Proposition 4.15. *The 2-functors above are mutually inverse 2-equivalences between \mathcal{G}_1 and \mathcal{G}_2 .*

The proof of Proposition 4.15 is similar to that of Proposition 4.4 and amounts to showing that the above constructions are inverses of each other. In fact, 2-groupoids \mathcal{G}_1 and \mathcal{G}_2 are also equivalent to the third 2-groupoid \mathcal{G}_3 which is defined in linear algebra terms: objects of \mathcal{G}_3 are semisimple Hopf algebras, 1-morphisms are twisted isomorphisms of Hopf algebras (defined in [D]), and 2-morphisms are gauge equivalences of twists. Details of these equivalences will be given elsewhere.

Finally, we give an easy criterion which allows us to recognize a quantum Manin triple. Let $R_{\mathcal{C}} \in K(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the regular object of \mathcal{C} , see Section 2.1.

Proposition 4.16. *Let \mathcal{C} be a non-degenerate braided fusion category and let (\mathcal{C}, A) , (\mathcal{C}, B) be quantum Manin pairs. The following conditions are equivalent:*

- (i) (\mathcal{C}, A, B) is a quantum Manin triple.
- (ii) $[A \otimes B] = R_{\mathcal{C}}$.
- (iii) $\dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B) = 1$.
- (iv) $\dim_{\mathbb{K}} \mathrm{Hom}_{\mathcal{C}}(A, B) = 1$.

Proof. Let us prove the implication (i) \Rightarrow (ii). The category of (A, B) -bimodules has a unique up to isomorphism simple object M . For any $X \in \mathcal{C}$, the object $A \otimes X \otimes B$ has an obvious structure of (A, B) -bimodule. Hence $[A \otimes X \otimes B] = r_X[M]$ for some positive integer r_X . Consequently

$$[A \otimes X \otimes B] = \frac{r_X}{r_1} [A \otimes B].$$

Computing the Frobenius–Perron dimension of both sides, we get

$$[A \otimes X \otimes B] = \mathrm{FPdim}(X)[A \otimes B].$$

Since the category \mathcal{C} is braided, we have

$$[X][A \otimes B] = [A \otimes X \otimes B] = \text{FPdim}(X)[A \otimes B].$$

Since $\text{FPdim}(A) = \text{FPdim}(B) = \sqrt{\text{FPdim}(\mathcal{C})}$, we have $\text{FPdim}(A \otimes B) = \text{FPdim}(\mathcal{C})$. Hence $[A \otimes B] = R_{\mathcal{C}}$, see Section 2.1.

The implication (ii) \Rightarrow (iii) is immediate and the equivalence (iii) \Leftrightarrow (iv) follows from Remark 3.4 since $\text{Hom}_{\mathcal{C}}(A, B) = \text{Hom}_{\mathcal{C}}(\mathbf{1}, {}^*A \otimes B) \simeq \text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B)$.

Let us prove the implication (iii) \Rightarrow (i). By Corollary 4.1 (i), the central functor

$$F_B : \mathcal{C} \rightarrow \mathcal{C}_B$$

is isomorphic to the forgetful functor $\mathcal{Z}(\mathcal{C}_B) \rightarrow \mathcal{C}_B$ (for a suitable choice of braided equivalence $\mathcal{C} \simeq \mathcal{Z}(\mathcal{C}_B)$). Consider the category $\text{Rep}_{\mathcal{C}_B}(A)$ (see Section 3.7). Notice that by Remark 3.18 (ii), this category coincides with the category of (A, B) -bimodules in \mathcal{C} . Thus, we need to prove that $\text{Rep}_{\mathcal{C}_B}(A) \simeq \text{Vec}$. Recall from Section 3.7 that the category $\text{Rep}_{\mathcal{C}_B}(A)$ has a structure of multi-fusion category. On the other hand the unit object $A \otimes B$ of this category is irreducible since $\text{Hom}_{A-B}(A \otimes B, A \otimes B) = \text{Hom}_{\mathcal{C}}(\mathbf{1}, A \otimes B)$. Thus, the multi-fusion category $\text{Rep}_{\mathcal{C}_B}(A)$ is in fact a fusion category. By Theorem 3.20 and Remark 4.3 we have $\mathcal{Z}(\text{Rep}_{\mathcal{C}_B}(A)) = \mathcal{C}_A^0 = \text{Vec}$. Thus (6) implies that $\text{FPdim}(\text{Rep}_{\mathcal{C}_B}(A)) = 1$, whence $\text{Rep}_{\mathcal{C}_B}(A) = \text{Vec}$. \square

5. Definition and properties of the Witt group

5.1. Definition of the Witt group.

Definition 5.1. Non-degenerate braided fusion categories \mathcal{C}_1 and \mathcal{C}_2 are *Witt equivalent* if there exists a braided equivalence $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$, where $\mathcal{A}_1, \mathcal{A}_2$ are fusion categories.

Remark 5.2. The equivalence relation in Definition 5.1 will not change if we allow \mathcal{A}_1 and \mathcal{A}_2 to be non-zero multi-fusion categories. Indeed, assume that

$$\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$$

where \mathcal{A}_1 and \mathcal{A}_2 are multi-fusion categories. We can assume that \mathcal{A}_1 and \mathcal{A}_2 are indecomposable in the sense of [ENO1], Section 2.4 (replace \mathcal{A}_1 and \mathcal{A}_2 by suitable summands otherwise). It follows from [EO], Lemma 3.24, Corollary 3.35, that for an indecomposable multi-fusion category \mathcal{A} there exists a fusion category \mathcal{A}' and a braided equivalence $\mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{A}')$. Our statement follows.

It is easy to see that Witt equivalence is indeed an equivalence relation. For example the transitivity holds since the conditions $\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$ and $\mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2') \simeq \mathcal{C}_3 \boxtimes \mathcal{Z}(\mathcal{A}_3)$ imply

$$\mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1 \boxtimes \mathcal{A}_2') \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2 \boxtimes \mathcal{A}_2') \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2' \boxtimes \mathcal{A}_2) \simeq \mathcal{C}_3 \boxtimes \mathcal{Z}(\mathcal{A}_3 \boxtimes \mathcal{A}_2).$$

We will denote the Witt equivalence class containing a category \mathcal{C} by $[\mathcal{C}]$. The set of Witt equivalence classes of non-degenerate braided fusion categories will be denoted \mathcal{W} . Clearly \mathcal{W} is a commutative monoid with respect to the operation \boxtimes . The unit of this monoid is $[\text{Vec}]$.

Lemma 5.3. *The monoid \mathcal{W} is a group.*

Proof. For a non-degenerate braided fusion category \mathcal{C} we have $\mathcal{Z}(\mathcal{C}) \simeq \mathcal{C} \boxtimes \mathcal{C}^{\text{rev}}$, see Section 2.3. Thus $[\mathcal{C}]^{-1} = [\mathcal{C}^{\text{rev}}]$. \square

Proposition 5.4. *Let $A \in \mathcal{C}$ be an étale connected algebra. Then $[\mathcal{C}_A^0] = [\mathcal{C}]$ in \mathcal{W} .*

Proof. This is immediate from Definition 5.1, Lemma 5.3 and Corollary 3.30. \square

Definition 5.5. The abelian group \mathcal{W} defined above is called *the Witt group of non-degenerate braided fusion categories*.

Remark 5.6. It is apparent from the definition that the group \mathcal{W} depends on the base field \mathbb{k} and should be denoted $\mathcal{W}(\mathbb{k})$. However it is known that any fusion category (or braided fusion category) is defined over the field of algebraic numbers $\overline{\mathbb{Q}}$, see [ENO1], Section 2.6. Thus an embedding $\overline{\mathbb{Q}} \subset \mathbb{k}$ induces an isomorphism $\mathcal{W}(\overline{\mathbb{Q}}) \simeq \mathcal{W}(\mathbb{k})$. In this sense we can talk about *the* Witt group of non-degenerate braided fusion categories (without mentioning the field \mathbb{k}). Of course this implies that the group \mathcal{W} carries a natural action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and should be considered together with this action.

Remark 5.7. It follows from [ENO1], Theorems 2.28, 2.31, and Remark 2.33 that there are countably many non-equivalent braided fusion categories. In particular, the group \mathcal{W} is at most countable. We will see later that \mathcal{W} is infinite.

Proposition 5.8. *Let \mathcal{C} be a non-degenerate braided fusion category. Then $\mathcal{C} \in [\text{Vec}]$ if and only if there exist a fusion category \mathcal{A} and a braided equivalence $\mathcal{C} \cong \mathcal{Z}(\mathcal{A})$.*

Proof. By definition, $\mathcal{C} \in [\text{Vec}]$ if and only if $\mathcal{C} \boxtimes \mathcal{Z}(\mathcal{B}_1) \simeq \mathcal{Z}(\mathcal{B}_2)$ with fusion categories \mathcal{B}_1 and \mathcal{B}_2 . By Proposition 4.4 there exists a connected étale algebra $A \in \mathcal{Z}(\mathcal{B}_1)$ such that $(\mathcal{Z}(\mathcal{B}_1), A)$ is a quantum Manin pair, see Definition 4.2. By abuse of notation we will denote by $A \in \mathcal{Z}(\mathcal{B}_2)$ the image of $\mathbf{1} \boxtimes A$ under the equivalence $\mathcal{C} \boxtimes \mathcal{Z}(\mathcal{B}_1) \simeq \mathcal{Z}(\mathcal{B}_2)$. Consider the multi-fusion category $\mathcal{A} = \text{Rep}_{\mathcal{B}_2}(A)$, see Section 3.7. By Theorem 3.20 we have $\mathcal{Z}(\mathcal{A}) \cong \mathcal{Z}(\mathcal{B}_2)_A^0$. On the other hand we have an obvious injective braided tensor functor

$$(30) \quad \mathcal{C} \rightarrow \mathcal{Z}(\mathcal{B}_2)_A^0 : X \mapsto (X \boxtimes \mathbf{1}) \otimes A.$$

We have

$$\text{FPdim}(\mathcal{C}) = \frac{\text{FPdim}(\mathcal{Z}(\mathcal{B}_2))}{\text{FPdim}(\mathcal{Z}(\mathcal{B}_1))} = \frac{\text{FPdim}(\mathcal{Z}(\mathcal{B}_2))}{\text{FPdim}(A)^2} = \text{FPdim}(\mathcal{Z}(\mathcal{B}_2)_A^0),$$

i.e., (30) is a fully faithful tensor functor between fusion categories of equal Frobenius–Perron dimension. Therefore, it is an equivalence by [EO], Proposition 2.19. The proposition follows, see Remarks 3.21 and 5.2. \square

Corollary 5.9. *We have $[\mathcal{C}] = [\mathcal{D}]$ if and only if there exists a fusion category \mathcal{A} and a braided equivalence $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A})$.*

5.2. Completely anisotropic categories.

Definition 5.10. We say that a non-degenerate braided fusion category is *completely anisotropic* if the only connected étale algebra $A \in \mathcal{C}$ is $A = \mathbf{1}$.

Remark 5.11. A completely anisotropic non-degenerate braided fusion category has no Tannakian subcategories other than Vec , i.e., it is *anisotropic* in the sense of [DGNO], Definition 5.16.

Lemma 5.12. *Let \mathcal{C} be a completely anisotropic category, let \mathcal{A} be a fusion category, and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a central functor. Then F is fully faithful.*

Proof. Let $I : \mathcal{A} \rightarrow \mathcal{C}$ be the right adjoint of F . Since \mathcal{C} is completely anisotropic, Lemma 3.5 implies that $I(\mathbf{1}) = \mathbf{1}$. Thus

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, Y) &\cong \text{Hom}_{\mathcal{C}}(X \otimes {}^*Y, \mathbf{1}) \\ &\cong \text{Hom}_{\mathcal{C}}(X \otimes {}^*Y, I(\mathbf{1})) \\ &\cong \text{Hom}_{\mathcal{A}}(F(X \otimes {}^*Y), \mathbf{1}) \\ &\cong \text{Hom}_{\mathcal{A}}(F(X) \otimes {}^*F(Y), \mathbf{1}) \\ &\cong \text{Hom}_{\mathcal{A}}(F(X), F(Y)). \end{aligned}$$

The result follows. \square

We will say that a connected étale algebra A in a braided fusion category \mathcal{C} is *maximal* if it is not a proper subalgebra of another such algebra. For any \mathcal{C} there exists at least one maximal connected étale algebra since by (16) the Frobenius–Perron dimensions of connected étale algebras are bounded by $\text{FPdim}(\mathcal{C})$.

Theorem 5.13. *Each Witt equivalence class in \mathcal{W} contains a completely anisotropic category that is unique up to braided equivalence.*

Proof. Let \mathcal{C} be a non-degenerate braided fusion category. Let $A \in \mathcal{C}$ be a maximal connected étale algebra. By Proposition 3.16 any connected étale algebra in \mathcal{C}_A^0 can be considered as a connected étale algebra in \mathcal{C} , so maximality of A is equivalent to \mathcal{C}_A^0 being completely anisotropic. Thus, Proposition 5.4 implies that any Witt equivalence class contains a completely anisotropic category.

Now let \mathcal{C} and \mathcal{D} be two completely anisotropic categories such that $[\mathcal{C}] = [\mathcal{D}]$. By Corollary 5.9 there exists a fusion category \mathcal{A} and a braided equivalence

$$\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A}).$$

In particular we have central functors $\mathcal{C} \rightarrow \mathcal{A}$ and $\mathcal{D}^{\text{rev}} \rightarrow \mathcal{A}$. By Lemma 5.12 these functors are fully faithful. Hence $\text{FPdim}(\mathcal{C}) \leq \text{FPdim}(\mathcal{A})$ and $\text{FPdim}(\mathcal{D}) \leq \text{FPdim}(\mathcal{A})$.

Combining this with (6) we see that $\text{FPdim}(\mathcal{C}) = \text{FPdim}(\mathcal{D}) = \text{FPdim}(\mathcal{A})$ and the functor $\mathcal{C} \rightarrow \mathcal{A}$ (and $\mathcal{D}^{\text{rev}} \rightarrow \mathcal{A}$) is an equivalence. In particular \mathcal{A} acquires a structure (in fact, two structures) of non-degenerate braided fusion category. Let \mathcal{C}' be the centralizer of \mathcal{C} in $\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A}) \simeq \mathcal{Z}(\mathcal{C})$. Then on one hand $\mathcal{C}' = \mathcal{D}^{\text{rev}}$ and on the other hand $\mathcal{C}' = \mathcal{C}^{\text{rev}}$, see Section 2.3. The result follows. \square

Corollary 5.14. *Let A and B be two maximal connected étale algebras in a non-degenerate braided fusion category \mathcal{C} . Then there exists a braided equivalence $\mathcal{C}_A^0 \simeq \mathcal{C}_B^0$. In particular $\text{FPdim}(A) = \text{FPdim}(B)$.*

Proof. The first statement is immediate from Theorem 5.13. The second one follows from (23). \square

The following result shows that Witt equivalence can also be understood without reference to the Drinfeld center:

Proposition 5.15. *Let $\mathcal{C}_1, \mathcal{C}_2$ be non-degenerate braided fusion categories. Then the following are equivalent:*

- (i) $[\mathcal{C}_1] = [\mathcal{C}_2]$, i.e., \mathcal{C}_1 and \mathcal{C}_2 are Witt equivalent.
- (ii) *There exist a non-degenerate braided fusion category \mathcal{C} , connected étale algebras $A_1, A_2 \in \mathcal{C}$ and braided equivalences $\mathcal{C}_1 \xrightarrow{\simeq} \mathcal{C}_{A_1}^0, \mathcal{C}_2 \xrightarrow{\simeq} \mathcal{C}_{A_2}^0$.*
- (iii) *There exist connected étale algebras $A_1 \in \mathcal{C}_1, A_2 \in \mathcal{C}_2$ and a braided equivalence $(\mathcal{C}_1)_{A_1}^0 \xrightarrow{\simeq} (\mathcal{C}_2)_{A_2}^0$.*

Proof. The implications (ii) \Rightarrow (i) and (iii) \Rightarrow (i) are immediate by Proposition 5.4.

(i) \Rightarrow (ii) By Definition 5.1, we have a braided equivalence

$$F : \mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1) \simeq \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2).$$

Thus we can define \mathcal{C} to be $\mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)$, the algebra A_1 to be $F(\mathbf{1} \boxtimes I_1(\mathbf{1}))$ and the algebra A_2 to be $\mathbf{1} \boxtimes I_2(\mathbf{1})$. Here $I_i : \mathcal{A}_i \rightarrow \mathcal{Z}(\mathcal{A}_i)$ are right adjoints to the forgetful functors $\mathcal{Z}(\mathcal{A}_i) \rightarrow \mathcal{A}_i$. Finally we define the braided equivalence $\mathcal{C}_1 \rightarrow \mathcal{C}_{A_1}^0$ as

$$\mathcal{C}_1 \rightarrow \mathcal{C}_1 \boxtimes \mathcal{Z}(\mathcal{A}_1)_{I_1(\mathbf{1})}^0 \xrightarrow{F} (\mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2))_{A_1}^0 = \mathcal{C}_{A_1}^0$$

and the braided equivalence $\mathcal{C}_2 \rightarrow \mathcal{C}_{A_2}^0$ as

$$\mathcal{C}_2 \rightarrow \mathcal{C}_2 \boxtimes \mathcal{Z}(\mathcal{A}_2)_{I_2(\mathbf{1})}^0 = \mathcal{C}_{A_2}^0.$$

(i) \Rightarrow (iii) Choose étale algebras $A_i \in \mathcal{C}_i$ such that the categories $(\mathcal{C}_i)_{A_i}^0$ are completely anisotropic. Now $[(\mathcal{C}_1)_{A_1}^0] = [\mathcal{C}_1] = [\mathcal{C}_2] = [(\mathcal{C}_2)_{A_2}^0]$ together with Theorem 5.13 implies the existence of a braided equivalence $(\mathcal{C}_1)_{A_1}^0 \xrightarrow{\simeq} (\mathcal{C}_2)_{A_2}^0$. \square

Remark 5.16. (1) The proposition implies that Witt equivalence is the equivalence relation \sim on non-degenerate braided fusion categories generated by ordinary braided equivalence \simeq and the relations $\mathcal{C} \sim \mathcal{C}_A^0$, where $A \in \mathcal{C}$ is an étale algebra. But the proposition is more precise in that it says that any two Witt equivalent categories can be joined by just two invocations of $\mathcal{C} \sim \mathcal{C}_A^0$ and either one (part (iii)) or two (part (ii)) braided equivalences.

(2) The proposition has applications to conformal field theory, cf. [Mu6].

5.3. The Witt group of metric groups and pointed categories. Recall that a *quadratic form* with values in \mathbb{k}^\times on a finite abelian group A is a function $q : A \rightarrow \mathbb{k}^\times$ such that $q(-x) = q(x)$ and $b(x, y) = \frac{q(x+y)}{q(x)q(y)}$ is bilinear, see e.g. [DGNO], Section 2.11.1. The pair (A, q) consisting of finite abelian group and quadratic form $q : A \rightarrow \mathbb{k}^\times$ is called a *pre-metric group*, see [DGNO], Section 2.11.2. A pre-metric group (A, q) is called *metric group* if the form q is non-degenerate (i.e., the associated bimultiplicative form $b(x, y)$ is non-degenerate).

To a pre-metric group (A, q) one assigns a unique up to a braided equivalence pointed braided fusion category $\mathcal{C}(A, q)$, where $q(a) \in \mathbb{k}^\times$ equals the braiding on the simple object $X_a \otimes X_a$ where X_a is a representative of an isomorphism class $a \in A$ (see e.g. [DGNO], Section 2.11.5). It was shown in [JS2] that this assignment is an equivalence between the 1-categorical truncation of the 2-category of pre-metric groups and that of the 2-category of pointed braided fusion categories.

The category $\mathcal{C}(A, q)$ is non-degenerate if and only if (A, q) is a metric group, see [DGNO], Sections 2.11.5 and 2.8.2.

Let (A, q) be a metric group and let $H \subset A$ be an *isotropic* subgroup (that is, $q|_H = 1$). Then $H \subset H^\perp$ where H^\perp is the orthogonal complement of H in A with respect to the bilinear form $b(x, y)$. Moreover, the restriction of q to H^\perp is the pull-back of a non-degenerate quadratic form $\tilde{q} : H^\perp/H \rightarrow \mathbb{k}^\times$. We say that $(H^\perp/H, \tilde{q})$ is an *m-subquotient* of (A, q) . Two metric groups are *Witt equivalent* if they have isomorphic m-subquotients (for some choice of isotropic subgroups in each of them), cf. [DGNO], Appendix A.7.1. The set of equivalence classes has a natural structure of abelian group (with addition induced by the orthogonal direct sum) and is called the *Witt group of metric groups*, see loc. cit. We will denote this group \mathcal{W}_{pt} .

Proposition 5.17. *The assignment*

$$(31) \quad \mathcal{W}_{\text{pt}} \rightarrow \mathcal{W} : (A, q) \mapsto [\mathcal{C}(A, q)]$$

induces a well-defined injective homomorphism $\mathcal{W}_{\text{pt}} \rightarrow \mathcal{W}$.

Proof. Let $H \subset A$ be an isotropic subgroup. Then the corresponding subcategory $\mathcal{C}(H, 1) \subset \mathcal{C}(A, q)$ is Tannakian, see e.g. [DGNO], Example 2.48. Let $B \in \mathcal{C}(H, 1)$ be the corresponding regular algebra, see Example 2.8. Then the category $\mathcal{C}(A, q)_B^0$ identifies with $\mathcal{C}(H^\perp/H, \tilde{q})$. In particular, $[\mathcal{C}(A, q)] = [\mathcal{C}(H^\perp/H, \tilde{q})]$. This implies that (31) is well defined.

It is known (see [DGNO], Section A.7.1) that each class in \mathcal{W}_{pt} has a representative (A, q) which is anisotropic, that is $q(x) \neq 1$ for $A \ni x \neq 1$. It is clear that the corresponding category $\mathcal{C}(A, q)$ is completely anisotropic. Thus, (31) is injective by Theorem 5.13. \square

In what follows we will identify the group \mathcal{W}_{pt} with its image in \mathcal{W} . The group \mathcal{W}_{pt} is explicitly known, see e.g. [DGNO], Appendix A.7. Namely,

$$\mathcal{W}_{\text{pt}} = \bigoplus_{p \text{ is prime}} \mathcal{W}_{\text{pt}}(p),$$

where $\mathcal{W}_{\text{pt}}(p) \subset \mathcal{W}_{\text{pt}}$ consists of the classes of metric p -groups.

The group $\mathcal{W}_{\text{pt}}(2)$ is isomorphic to $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by two classes

$$[\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_1)] \quad \text{and} \quad [\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q_2)],$$

where q_1, q_2 are any non-degenerate forms. For $p \equiv 3 \pmod{4}$ we have $\mathcal{W}_{\text{pt}}(p) \cong \mathbb{Z}/4\mathbb{Z}$ and the class $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q)]$ is a generator for any non-degenerate form q . For $p \equiv 1 \pmod{4}$ the group $\mathcal{W}_{\text{pt}}(p)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$; it is generated by the two classes $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q')]$ and $[\mathcal{C}(\mathbb{Z}/p\mathbb{Z}, q'')]$ with $q'(l) = \zeta^{l^2}$ and $q''(l) = \zeta^{nl^2}$, where ζ is a primitive p th root of unity in \mathbb{k} and n is any quadratic non-residue modulo p .

5.4. Property S. Let \mathcal{C} be a non-degenerate braided fusion category.

Definition 5.18. We say that \mathcal{C} has *property S* if the following conditions are satisfied:

(S1) \mathcal{C} is completely anisotropic.

(S2) \mathcal{C} is simple (that is, \mathcal{C} has no non-trivial fusion subcategories) and not pointed (so in particular $\mathcal{C} \not\cong \text{Vec}$).

We will also say that a class $w \in \mathcal{W}$ has property S if a completely anisotropic representative of w has property S. In Section 6.4 we will give infinitely many examples of non-degenerate braided fusion categories with property S.

Theorem 5.19. Let $\mathcal{D} = \bigboxtimes_{i \in I} \mathcal{C}_i$ where \mathcal{C}_i are braided fusion categories with property S. Assume that \mathcal{D} is a Drinfeld center of a fusion category. Then there is a fixed point free involution $a : I \rightarrow I$ such that $\mathcal{C}_{a(i)} \simeq \mathcal{C}_i^{\text{rev}}$

Proof. Assume that $\mathcal{D} = \mathcal{Z}(\mathcal{A})$ for some fusion category \mathcal{A} . Let

$$F : \mathcal{D} = \mathcal{Z}(\mathcal{A}) \rightarrow \mathcal{A}$$

be the forgetful functor. Choose a bijection $I = \{1, \dots, n\}$. For $1 \leq i \leq n$ let \mathcal{A}_i be the image of $\mathcal{C}_1 \boxtimes \mathcal{C}_2 \boxtimes \dots \boxtimes \mathcal{C}_i$ under F (so \mathcal{A}_i is a fusion subcategory of \mathcal{A}).

Claim. There is a subset $J_i \subset \{1, \dots, i\}$ such that F restricted to $\bigboxtimes_{j \in J_i} \mathcal{C}_j \subset \mathcal{D}$ is an equivalence $\bigboxtimes_{j \in J_i} \mathcal{C}_j \simeq \mathcal{A}_i$.

Proof of the Claim. We use induction on i . For $i = 1$ we set $J_1 = \{1\}$; in this case the claim follows from Lemma 5.12. Now consider the induction step. The subcategory \mathcal{A}_{i+1} is clearly generated by \mathcal{A}_i and (the image of) $\mathcal{C}_{i+1} \subset \mathcal{A}$ (recall that by Lemma 5.12, the functor F restricted to \mathcal{C}_{i+1} is fully faithful). There are two possibilities:

(a) The subcategories \mathcal{A}_i and \mathcal{C}_{i+1} intersect non-trivially in \mathcal{A} ; then \mathcal{A}_i contains \mathcal{C}_{i+1} since by (S2), \mathcal{C}_{i+1} has no non-trivial subcategories. In this case we set $J_{i+1} = J_i$.

(b) \mathcal{A}_i and \mathcal{C}_{i+1} intersect trivially. Then we set $J_{i+1} = J_i \cup \{i+1\}$. We claim that the forgetful functor $\bigotimes_{j \in J_{i+1}} \mathcal{C}_j \rightarrow \mathcal{A}$ is fully faithful. As in the proof of Lemma 5.12 it is sufficient to show that for any object $Z \in \bigotimes_{j \in J_{i+1}} \mathcal{C}_j$ we have $\text{Hom}_{\mathcal{A}}(F(Z), \mathbf{1}) = \text{Hom}_{\mathcal{D}}(Z, \mathbf{1})$. Clearly, we can restrict ourselves to the case when Z is simple. In this case $Z = X \boxtimes Y$ where $X \in \bigotimes_{j \in J_i} \mathcal{C}_j$ and $Y \in \mathcal{C}_{i+1}$ are simple. Then $F(Z) = F(X) \otimes F(Y)$ where $F(X) \in \mathcal{A}_i$ and $F(Y) \in F(\mathcal{C}_{i+1})$ are simple. Then $\text{Hom}_{\mathcal{A}}(F(Z), \mathbf{1}) = \text{Hom}_{\mathcal{A}}(F(X), F(Y)^*) = 0$ unless $X = \mathbf{1}$ and $Y = \mathbf{1}$. We are done in this case and the claim is proved. \square

We apply now the Claim with $i = n$; we see that $\mathcal{A} = \bigotimes_{j \in J_n} \mathcal{C}_j$. Thus

$$\mathcal{Z}(\mathcal{A}) = \bigotimes_{j \in J_n} (\mathcal{C}_j \boxtimes \mathcal{C}_j^{\text{rev}})$$

(see Section 2.3). The category \mathcal{D} does not contain non-trivial invertible objects. By Proposition 2.2 it has a unique decomposition into a product of simple categories. The result follows. \square

Corollary 5.20. *Let \mathcal{C} be a category with property S. Then $[\mathcal{C}] \in \mathcal{W}$ has order 2 if $\mathcal{C} \simeq \mathcal{C}^{\text{rev}}$ and otherwise $[\mathcal{C}] \in \mathcal{W}$ has infinite order.* \square

More precisely we have the following result. Let \mathcal{S} be the set of braided equivalence classes of categories with property S. Let $\mathcal{S}_2 \subset \mathcal{S}$ be the subset consisting of categories \mathcal{C} such that $\mathcal{C} \simeq \mathcal{C}^{\text{rev}}$ and let $\mathcal{S}' = \mathcal{S} \setminus \mathcal{S}_2$. It is clear that the set \mathcal{S} is at most countable, see Remark 5.7. It follows from (38) in Section 6.4 below that the set \mathcal{S}' (and hence \mathcal{S}) is infinite. Let $\mathcal{S}'_{\infty} \subset \mathcal{S}'$ be a maximal subset such that $\mathcal{C} \in \mathcal{S}'_{\infty}$ implies $\mathcal{C}^{\text{rev}} \notin \mathcal{S}'_{\infty}$.

Corollary 5.21. *Let $\mathcal{W}_{\mathcal{S}} \subset \mathcal{W}$ be the subgroup generated by the categories with property S. The map $(a_i)_{\mathcal{C}_i \in \mathcal{S}} \mapsto \prod_{\mathcal{C}_i \in \mathcal{S}} [\mathcal{C}_i]^{a_i}$ defines an isomorphism*

$$\bigoplus_{\mathcal{S}_2} \mathbb{Z}/2\mathbb{Z} \oplus \bigoplus_{\mathcal{S}'_{\infty}} \mathbb{Z} \simeq \mathcal{W}_{\mathcal{S}}. \quad \square$$

Remark 5.22. (1) It is clear that the set \mathcal{S}_2 is at most countable. However, we do not know whether it is empty and we do not know whether it is finite.

(2) The description of the group $\mathcal{W}_{\mathcal{S}}$ above is non-canonical due to the choice of the set \mathcal{S}'_{∞} . A better description is as follows: the set \mathcal{S} carries an involution σ which sends \mathcal{C} to \mathcal{C}^{rev} . We extend σ to the involution of the free abelian group $\mathbb{Z}[\mathcal{S}]$ generated by \mathcal{S} by linearity. Then $\mathcal{W}_{\mathcal{S}} \simeq \mathbb{Z}[\mathcal{S}]/\text{Image}(1 + \sigma)$.

(3) An argument similar to the proof of Theorem 5.19 shows that $\mathcal{W}_{\mathcal{S}} \cap \mathcal{W}_{\text{pt}} = \{1\}$. Thus the subgroup of \mathcal{W} generated by $\mathcal{W}_{\mathcal{S}}$ and \mathcal{W}_{pt} is isomorphic to $\mathcal{W}_{\mathcal{S}} \times \mathcal{W}_{\text{pt}}$.

(4) Assume that \mathcal{C}_i are braided fusion categories with property S and $\mathcal{C}_i^{\text{rev}} \not\cong \mathcal{C}_j$ for $j \neq i$. Corollary 5.21 implies that $\left[\bigboxtimes_{i \in I} \mathcal{C}_i \right] \neq 0$. A stronger statement is true: the category $\mathcal{D} = \bigboxtimes_{i \in I} \mathcal{C}_i$ is completely anisotropic. Indeed, by Lemma 3.9 it is sufficient to show that any surjective central functor $\mathcal{D} \rightarrow \mathcal{A}$ is an equivalence. This is proved by an argument parallel to the proof of Theorem 5.19; notice that the case (a) in the proof of the Claim never occurs since otherwise we would have a non-injective central functor $\mathcal{C}_i \boxtimes \mathcal{C}_j \rightarrow \mathcal{A}$; considering the image of this functor one shows that $\mathcal{C}_i^{\text{rev}} \simeq \mathcal{C}_j$ as in the proof of Theorem 5.13.

Corollary 5.23. *The \mathbb{Q} -vector space $\mathcal{W} \otimes_{\mathbb{Z}} \mathbb{Q}$ also has countable infinite dimension.*

Proof. Since \mathcal{S}_{∞} is infinite, the \mathbb{Q} -vector space $\mathcal{W}_{\mathcal{S}} \otimes_{\mathbb{Z}} \mathbb{Q}$ has countable infinite dimension. The result follows since the functor $? \otimes_{\mathbb{Z}} \mathbb{Q}$ is exact. \square

5.5. Central charge. From now on we will assume that $\mathbb{k} = \mathbb{C}$. Recall that any pseudo-unitary non-degenerate braided fusion category has a natural structure of modular tensor category (see e.g. [DGNO], Section 2.8.2).

Definition 5.24. Let $\mathcal{W}_{\text{un}} \subset \mathcal{W}$ be the subgroup consisting of Witt classes $[\mathcal{C}]$ of pseudo-unitary non-degenerate braided fusion categories \mathcal{C} .

Remark 5.25. Note that \mathcal{W}_{un} is *not* invariant under the Galois action from Remark 5.6 (for example the class $[\mathcal{C}(\text{sl}(2), 3)]_+ \in \mathcal{W}_{\text{un}}$ from Section 6.4 below has a Galois conjugate not lying in \mathcal{W}_{un}). In particular, $\mathcal{W}_{\text{un}} \subsetneq \mathcal{W}$.

Now recall that for a modular tensor category \mathcal{C} one defines the *multiplicative central charge* $\xi(\mathcal{C}) \in \mathbb{C}$, see [DGNO], Section 6.2. The following properties are well known, see e.g. [BK], Section 3.1.

Lemma 5.26. (i) $\xi(\mathcal{C})$ is a root of unity.

(ii) $\xi(\mathcal{C}_1 \boxtimes \mathcal{C}_2) = \xi(\mathcal{C}_1)\xi(\mathcal{C}_2)$.

(iii) $\xi(\mathcal{C}^{\text{rev}}) = \xi(\mathcal{C})^{-1}$. \square

The statement (i) (due to Anderson, Moore and Vafa, see [AM], [V]) allows us to consider the *additive central charge* $c = c(\mathcal{C}) \in \mathbb{Q}/8\mathbb{Z}$, which is related to $\xi(\mathcal{C})$ by $\xi(\mathcal{C}) = e^{2\pi ic/8}$.

Lemma 5.27. *Let \mathcal{C}_1 and \mathcal{C}_2 be two pseudo-unitary non-degenerate braided fusion categories considered as modular tensor categories. Assume that \mathcal{C}_1 and \mathcal{C}_2 are Witt equivalent. Then $\xi(\mathcal{C}_1) = \xi(\mathcal{C}_2)$.*

Proof. By Corollary 5.9, $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} \simeq \mathcal{Z}(\mathcal{A})$. Since the category $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}}$ is pseudo-unitary, so is \mathcal{A} (use (6)). Thus, the spherical structure on $\mathcal{C}_1 \boxtimes \mathcal{C}_2^{\text{rev}} = \mathcal{Z}(\mathcal{A})$ is induced by the spherical structure on \mathcal{A} . In this situation [Mu3], Theorem 1.2, says that $\xi(\mathcal{Z}(\mathcal{A})) = 1$. The result follows from Lemma 5.26. \square

Now for any class $w \in \mathcal{W}_{\text{un}}$ we define $\xi(w) = \xi(\mathcal{C})$ where \mathcal{C} is a pseudo-unitary representative of the class w ; according to Lemma 5.27 this is well defined. Similarly, we set $c(w) = c(\mathcal{C})$.

Corollary 5.28. *The assignment $w \mapsto c(w)$ is a homomorphism $\mathcal{W}_{\text{un}} \rightarrow \mathbb{Q}/8\mathbb{Z}$.*

Proof. This is immediate from Lemma 5.26. \square

Remark 5.29. A non-degenerate pointed category $\mathcal{C}(A, q)$ has a canonical pseudo-unitary structure (characterized by the condition that dimensions of all simple objects are 1). The ribbon twist of the corresponding modular structure on $\mathcal{C}(A, q)$ is $\theta_{X_a} = q(a)1_{X_a}$, where X_a is a simple object corresponding to $a \in A$. The multiplicative central charge of $\mathcal{C}(A, q)$ is given by [DGNO], Section 6.1,

$$\xi(\mathcal{C}(A, q)) = \frac{1}{\sqrt{|A|}} \sum_{a \in A} q(a).$$

In particular, for a metric cyclic group of order 2 with the value of the quadratic form on the generator $q(1) = i \in k$ (with $i^2 = -1$) we have

$$\xi(\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)) = \frac{1+i}{\sqrt{2}}$$

so that the additive central charge is

$$(32) \quad c(\mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q)) = 1 \in \mathbb{Q}/8\mathbb{Z}.$$

6. Finite extensions of vertex algebras

6.1. Extensions of VOAs. Let V be a rational vertex algebra, that is, a vertex algebra satisfying conditions 1–3 from [H], Section 1. It is proved in loc. cit. that the category $\text{Rep}(V)$ of V -modules of finite length has a natural structure of modular tensor category; in particular $\text{Rep}(V)$ is a non-degenerate braided fusion category.

Note that a rational vertex algebra has to be *simple* (i.e., has no non-trivial ideals). This, in particular, means that VOA maps between rational vertex algebras are monomorphisms.

The category of modules $\text{Rep}(V \otimes U)$ of the tensor product of two (rational) vertex algebras is ribbon equivalent to the tensor product $\text{Rep}(V) \boxtimes \text{Rep}(U)$ of the categories of modules (see, for example [FHL]).

The following relation between the central charge c_V of a (unitary) rational VOA V and the central charge of the category of its modules $\text{Rep}(V)$ is well known to specialists (although we could not find a reference)¹⁾:

$$\xi(\text{Rep}(V)) = e^{\frac{2\pi i c_V}{8}}.$$

¹⁾ This relation can be verified directly for all the examples we consider later.

Now consider a finite extension of vertex algebras $V \subset W$, that is, V is a vertex subalgebra of W (with the same Virasoro vector) and W viewed as a V -module decomposes into a finite direct sum of irreducible V -modules²⁾. Then W considered as an object $A \in \text{Rep}(V)$ has a natural structure of commutative algebra; moreover this algebra satisfies the conditions from Example 3.3(ii) and hence is étale, see [KiO], Theorem 5.2³⁾. Furthermore, the restriction functor $\text{Rep}(W) \rightarrow \text{Rep}(V)$ induces a braided tensor equivalence $\text{Rep}(W) \simeq \text{Rep}(V)_A^0$. Thus, Proposition 5.4 implies that in this situation we have $[\text{Rep}(V)] = [\text{Rep}(W)]$. We can use this in order to construct examples of interesting relations in the group \mathcal{W} .

Example 6.1 (Chiral orbifolds). Let G be a finite group of automorphisms of a rational vertex algebra V . The sub-VOA of invariants V^G is called the *chiral orbifold* of V . In the case when the vertex subalgebra of invariants V^G is rational, we have a Witt equivalence between categories of modules $\text{Rep}(V)$, $\text{Rep}(V^G)$.

6.2. Affine Lie algebras and conformal embeddings. Let \mathfrak{g} be a finite dimensional simple Lie algebra and let $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. For any $k \in \mathbb{Z}_{>0}$ let $\mathcal{C}(\mathfrak{g}, k)$ be the category of highest weight integrable $\hat{\mathfrak{g}}$ -modules of level k , see e.g. [BK], Section 7.1, where this category is denoted $\mathcal{O}_k^{\text{int}}$. The category $\mathcal{C}(\mathfrak{g}, k)$ can be identified with the category $\text{Rep}(V(\mathfrak{g}, k))$ where $V(\mathfrak{g}, k)$ is the simple vertex algebra associated with the vacuum $\hat{\mathfrak{g}}$ -module of level k . In particular the category $\mathcal{C}(\mathfrak{g}, k)$ has a structure of modular tensor category, see [HL], [BK], Chapter 7.

Example 6.2. The category $\mathcal{C}(\mathfrak{sl}(n), 1)$ is pointed. It identifies with $\mathcal{C}(\mathbb{Z}/n\mathbb{Z}, q)$, where $q(l) = e^{\pi i l^2 \frac{n-1}{n}}$, $l \in \mathbb{Z}/n\mathbb{Z}$. More generally, $\mathcal{C}(\mathfrak{g}, 1)$ (with \mathfrak{g} simply laced) is pointed [FK].

It is known ([BK]) that the categories $\mathcal{C}(\mathfrak{g}, k)$ are pseudo-unitary. In particular, we have Witt classes $[\mathcal{C}(\mathfrak{g}, k)] \in \mathcal{W}_{\text{un}} \subset \mathcal{W}$. The following formula for the central charge is very useful, see e.g. [BK], 7.4.5:

$$(33) \quad c(\mathcal{C}(\mathfrak{g}, k)) = \frac{k \dim \mathfrak{g}}{k + h^\vee},$$

where h^\vee is the dual Coxeter number of the Lie algebra \mathfrak{g} .

One can construct examples of relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$ using the theory of *conformal embeddings*, see [BB], [SW], [KW]. Let $\bigoplus_i \mathfrak{g}^i \subset \mathfrak{g}'$ be an embedding (here \mathfrak{g}^i and \mathfrak{g}' are finite dimensional simple Lie algebras). We will symbolically write

²⁾ Note that finiteness is automatic if we assume that L_0 -eigenspaces are finite dimensional (which is standard and true e.g. for affine VOAs). Indeed, as a module over a rational vertex algebra V , W is completely reducible, i.e., is a sum of simple V -modules. Since V has only a finite number of non-isomorphic simple modules the only way for W not to be finite is to have infinite multiplicities (in decomposition into simple V -modules). That will contradict finite dimensionality of L_0 -eigenspaces.

³⁾ The proof of this result in [KiO] is not complete. However for examples we are going to consider in this section the arguments from [KiO], Section 5.5, are sufficient.

$\bigoplus_i (\mathfrak{g}^i)_{k_i} \subset \mathfrak{g}'_{k'}$ if the restriction of a $\hat{\mathfrak{g}}'$ -module of level k' to $\hat{\mathfrak{g}}^i$ has level k_i (in this case the numbers k_i are multiples of k'). Such an embedding defines an embedding of vertex algebras $\bigotimes_i V(\mathfrak{g}^i, k_i) \subset V(\mathfrak{g}', k')$; but in general this embedding does not preserve the Virasoro vector. In the case when it does the embedding $\bigoplus_i (\mathfrak{g}^i)_{k_i} \subset \mathfrak{g}'_{k'}$ is called conformal embedding; it is known that in this case the extension of vertex algebras $\bigotimes_i V(\mathfrak{g}^i, k_i) \subset V(\mathfrak{g}', k')$ is finite⁴⁾. Thus in view of Section 6.1, we get a relation

$$(34) \quad \prod_i [\mathcal{C}(\mathfrak{g}^i, k_i)] = [\mathcal{C}(\mathfrak{g}', k')].$$

The complete classification of the conformal embeddings was done in [BB], [SW] (see also [KW]) and is reproduced in the Appendix.

6.3. Cosets. Let $U \subseteq V$ be an embedding of rational vertex algebras, which does not preserve conformal vectors ω_U, ω_V (only operator products are preserved). The *centralizer* $C_V(U)$ is a vertex algebra with the conformal vector $\omega_V - \omega_U$, see [GKO]. Moreover the tensor product $U \otimes C_V(U)$ is mapped naturally to V and this map is a map of vertex algebras. In the case when V, U and $C_V(U)$ are rational we have a Witt equivalence of categories of modules

$$\text{Rep}(U) \boxtimes \text{Rep}(C_V(U)) \simeq \text{Rep}(U \otimes C_V(U))$$

and $\text{Rep}(V)$.

Let $\bigoplus_i (\mathfrak{h}^i)_{k_i} \subset \bigoplus_j (\mathfrak{g}^j)_{k'_j}$ be an embedding of vertex algebras non necessarily preserving the Virasoro vector as in Section 6.2. Let $\bigotimes_i V(\mathfrak{h}^i, k_i) \subset \bigotimes_j V(\mathfrak{g}^j, k'_j)$ be the corresponding embedding of the vertex algebras. The centralizer

$$C_{\bigotimes_j V(\mathfrak{g}^j, k'_j)} \left(\bigotimes_i V(\mathfrak{h}^i, k_i) \right)$$

is called the *coset model* and is denoted $\frac{\times_j (\mathfrak{g}^j)_{k'_j}}{\times_i (\mathfrak{h}^i)_{k_i}}$.

Sometimes coset models defined by different embeddings of semisimple Lie algebras are isomorphic. An example of such isomorphism was found by Goddard, Kent and Olive [GKO]. They observed that the coset models⁵⁾

$$\frac{A_{1,m} \times A_{1,1}}{A_{1,m+1}}, \quad \frac{C_{m+1,1}}{C_{m,1} \times C_{1,1}}$$

⁴⁾ This follows from the fact that L_0 -eigenspaces of $V(\mathfrak{g}, k)$ are finite dimensional.

⁵⁾ Here and in the Appendix the notation $X_{i,k}$ refers to the Lie algebra of type X_i at level k .

are isomorphic, since they are both isomorphic to the same rational Virasoro vertex algebra Vir_{c_m} with the central charge

$$(35) \quad c_m = 1 - \frac{6}{(m+2)(m+3)}.$$

We can use coset models in order to construct new relations in the Witt group as follows. Assume that the central charge c of a coset model vertex algebra $\frac{\times_j (\mathfrak{g}^j)_{k'_j}}{\times_i (\mathfrak{h}^i)_{k_i}}$ is positive⁶⁾ but less than 1⁷⁾. It is known that in this case $c = c_m$ for some positive integer m and the vertex algebra in question contains a rational vertex subalgebra Vir_{c_m} , see [GKO]. This implies that the rational vertex algebra $\bigotimes_j V(\mathfrak{g}^j, k'_j)$ is a finite extension of rational vertex algebra $\bigotimes_i V(\mathfrak{h}^i, k_i) \otimes \text{Vir}_{c_m}$. Thus according to the results of Section 6.1 we get a relation in the Witt group

$$(36) \quad \left(\prod_i [\mathcal{C}(\mathfrak{h}^i, k_i)] \right) \cdot [\text{Vir}_{c_m}] = \prod_j [\mathcal{C}(\mathfrak{g}^j, k'_j)].$$

A special case of this relation corresponding to the coset model $\frac{A_{1,m} \times A_{1,1}}{A_{1,m+1}}$ reads

$$(37) \quad [\text{Vir}_{c_m}] = [\mathcal{C}(\mathfrak{sl}(2), m)] [\mathcal{C}(\mathfrak{sl}(2), 1)] [\mathcal{C}(\mathfrak{sl}(2), m+1)]^{-1}.$$

Thus combining (36) and (37), we obtain relations between the classes $[\mathcal{C}(\mathfrak{g}, k)]$.

6.4. Examples for $\mathfrak{g} = \mathfrak{sl}(2)$. We give here some examples of relations (or absence thereof) between the classes $[\mathcal{C}(\mathfrak{sl}(2), k)]$. We refer the reader to [KiO], Section 6, for more details on the categories $\mathcal{C}(\mathfrak{sl}(2), k)$. Note that all étale algebras in these categories were classified in [KiO], Theorem 6.1.

(1) The category $\mathcal{C}(\mathfrak{sl}(2), 1)$ is pointed, moreover $\mathcal{C}(\mathfrak{sl}(2), 1) \simeq \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_+)$ where $q_+(1) = i$. In particular, the class $[\mathcal{C}(\mathfrak{sl}(2), 1)] \in \mathcal{W}$ has order 8.

(2) For any odd k , we have $\mathcal{C}(\mathfrak{sl}(2), k) \simeq \mathcal{C}(\mathfrak{sl}(2), k)_+ \boxtimes \mathcal{C}(\mathbb{Z}/2\mathbb{Z}, q_\pm)$ where $\mathcal{C}(\mathfrak{sl}(2), k)_+$ is the subcategory of “integer spin” representations and $q_\pm(1) = \pm i$ (see e.g. [KiO], Lemma 6.6). The category $\mathcal{C}(\mathfrak{sl}(2), k)_+$ for an odd $k \geq 3$ has property S. Using (33) and (32), we get

$$c(\mathcal{C}(\mathfrak{sl}(2), k)_+) = \frac{3k}{k+2} + (-1)^{(k+1)/2}.$$

In particular, $2c(\mathcal{C}(\mathfrak{sl}(2), k)_+) \neq 0 \in \mathbb{Q}/8\mathbb{Z}$, so

$$(38) \quad \mathcal{C}(\mathfrak{sl}(2), k)_+ \not\cong \mathcal{C}(\mathfrak{sl}(2), k)_+^{\text{rev}}.$$

This shows that the set \mathcal{S}_∞ from Section 5.4 is infinite.

⁶⁾ It is known (see [GKO]) that $c \geq 0$. The case $c = 0$ corresponds exactly to the conformal embeddings discussed in Section 6.2.

⁷⁾ The list of cosets with such central charge was given in [BG] and is reproduced in the Appendix.

Consider the category $\mathcal{C}(\mathfrak{sl}(2), 3)_+$. The class $[\mathcal{C}(\mathfrak{sl}(2), 3)_+] \in \mathcal{W}$ is a simplest example of element of \mathcal{W} of infinite order. We will say that a braided fusion category \mathcal{C} is a *Fibonacci category* if the Grothendieck ring $K(\mathcal{C})$ is isomorphic to $K(\mathcal{C}(\mathfrak{sl}(2), 3)_+)$ as a based ring. It is known that a pseudo-unitary Fibonacci category is equivalent to either $\mathcal{C}(\mathfrak{sl}(2), 3)_+$ or $\mathcal{C}(\mathfrak{sl}(2), 3)_+^{\text{rev}}$.

(3) The category $\mathcal{C}(\mathfrak{sl}(2), 2)$ is an example of Ising braided category, see [DGNO], Appendix B. In particular, it follows from [DGNO], Lemma B.24, that

$$[\mathcal{C}(\mathfrak{sl}(2), 2)]^2 = [\mathcal{C}(\mathbb{Z}/4\mathbb{Z}, q)], \quad \text{where } q(l) = e^{3\pi i l^2/4}.$$

Thus, the order of $[\mathcal{C}(\mathfrak{sl}(2), 2)] \in \mathcal{W}$ is 16.

Using [DGNO], Lemma B.24, it is easy to see that for an odd l we have

$$[\mathcal{C}(\mathfrak{sl}(2), 2)]^l = [\mathcal{C}],$$

where \mathcal{C} is an Ising braided category. Since there are precisely eight equivalence classes of Ising braided categories (see [DGNO], Corollary B.16), we get that for any Ising braided category \mathcal{C} there is a unique odd number l , $1 \leq l \leq 15$, such that $[\mathcal{C}] = [\mathcal{C}(\mathfrak{sl}(2), 2)]^l$. The number l is easy to compute from $c(\mathcal{C})$ using $c(\mathcal{C}(\mathfrak{sl}(2), 2)) = \frac{3}{2}$.

(4) There exists a conformal embedding $\mathfrak{sl}(2)_4 \subset \mathfrak{sl}(3)_1$. Thus

$$[\mathcal{C}(\mathfrak{sl}(2), 4)] = [\mathcal{C}(\mathfrak{sl}(3), 1)] = [\mathcal{C}(\mathbb{Z}/3\mathbb{Z}, q)], \quad \text{where } q(l) = e^{2\pi i l^2/3}.$$

In particular, the order of $[\mathcal{C}(\mathfrak{sl}(2), 4)] \in \mathcal{W}$ is 4.

(5) There exists a conformal embedding $\mathfrak{sl}(2)_6 \oplus \mathfrak{sl}(2)_6 \subset \mathfrak{so}(9)_1$. Thus

$$[\mathcal{C}(\mathfrak{sl}(2), 6)]^2 = [\mathcal{C}(\mathfrak{so}(9), 1)].$$

Notice that $\mathcal{C}(\mathfrak{so}(9), 1)$ is also an example of Ising braided category. Using the central charge one computes that

$$[\mathcal{C}(\mathfrak{sl}(2), 6)]^2 = [\mathcal{C}(\mathfrak{sl}(2), 2)]^3.$$

In particular, $[\mathcal{C}(\mathfrak{sl}(2), 6)] \in \mathcal{W}$ has order 32.

(6) The category $\mathcal{C}(\mathfrak{sl}(2), 8)$ is known to contain an étale algebra A such that $\mathcal{C}(\mathfrak{sl}(2), 8)_A^0$ is equivalent to the product of two Fibonacci categories, see e.g. [MPS], Theorem 4.1. Using the central charge one computes that

$$[\mathcal{C}(\mathfrak{sl}(2), 8)] = [\mathcal{C}(\mathfrak{sl}(2), 3)_+]^{-2}.$$

(7) There exists a conformal embedding $\mathfrak{sl}(2)_{10} \subset \mathfrak{sp}(4)_1$. Thus,

$$[\mathcal{C}(\mathfrak{sl}(2), 10)] = [\mathcal{C}(\mathfrak{sp}(4), 1)].$$

The category $\mathcal{C}(\mathfrak{sp}(4), 1)$ is an Ising braided category. Using the central charge one computes that $[\mathcal{C}(\mathfrak{sl}(2), 10)] = [\mathcal{C}(\mathfrak{sl}(2), 2)]^7$.

(8) Let $\mathfrak{g}(G_2)$ be a Lie algebra of type G_2 . There exists a conformal embedding

$$\mathfrak{sl}(2)_{28} \subset \mathfrak{g}(G_2)_1.$$

Thus,

$$[\mathcal{C}(\mathfrak{sl}(2), 28)] = [\mathcal{C}(\mathfrak{g}(G_2), 1)].$$

The category $\mathcal{C}(\mathfrak{g}(G_2), 1)$ is a Fibonacci category. Using the central charge one computes that

$$[\mathcal{C}(\mathfrak{sl}(2), 28)] = [\mathcal{C}(\mathfrak{sl}(2), 3)]_+.$$

(9) The category $\mathcal{C}(\mathfrak{sl}(2), k)$ with k divisible by 4 is known to contain an étale algebra A of dimension 2, see [KiO], Theorem 6.1. It is also known that in this case for $k \neq 4, 8, 28$ the category $\mathcal{C}(\mathfrak{sl}(2), k)_A^0$ has property S and is not equivalent to any category $\mathcal{C}(\mathfrak{sl}(2), k_1)_+$ with odd k_1 . Thus we get infinitely many more elements of the set \mathcal{S}_∞ . For example we see that $[\mathcal{C}(\mathfrak{sl}(2), 12)] \in \mathcal{W}$ has infinite order.

6.5. Holomorphic vertex algebras with $c = 24$. We recall that a rational vertex algebra V is called *holomorphic* if $\text{Rep}(V) = \text{Vec}$, that is the only simple V -module is V itself, see e.g. [DM]. In [Sc] Schellekens gives a conjectural list of 71 holomorphic vertex algebras with central charge $c = 24$, see also [DM]. Out of this list, 69 algebras are extensions of vertex algebras associated with affine Lie algebras as in Section 6.2. Thus in view of the discussion in Section 6.1, each of these algebras should give a conjectural relation between the classes $[\mathcal{C}(\mathfrak{g}, k)]$. Some of these relations can be deduced from the relations in Sections 6.2 and 6.3, but some others are genuinely new. For example entry No. 14 from the Schellekens list gives a conjectural relation $[\mathcal{C}(F_4, 6)] = [\mathcal{C}(\mathfrak{sl}(3), 2)]^{-1}$ which cannot be deduced from the results above.

6.6. Open questions. In this section we collect some open questions about the Witt group \mathcal{W} .

Question 6.3. Is it true that \mathcal{W} is a direct sum of cyclic groups? Is there an inclusion $\mathbb{Q} \subset \mathcal{W}$?

Question 6.4. Is \mathcal{W}_{un} generated by classes $[\mathcal{C}(\mathfrak{g}, k)]$?

Remark 6.5. Notice that \mathcal{W}_{pt} is contained in the subgroup generated by $[\mathcal{C}(\mathfrak{g}, k)]$. Namely, the subgroup of \mathcal{W} generated by $[\mathcal{C}(\mathfrak{sl}(2), 1)]$ and $[\mathcal{C}(\mathfrak{sl}(2), 2)]$ contains $\mathcal{W}_{\text{pt}}(2)$. For a prime $p = 4k + 3$, the subgroup $\mathcal{W}_{\text{pt}}(p)$ is generated by $[\mathcal{C}(\mathfrak{sl}(p), 1)]$. Finally for a prime $p = 4k + 1$ choose a prime number $q < p$ which is a quadratic non-residue modulo p (it is easy to see that such a prime does exist). Then $\mathcal{W}_{\text{pt}}(p)$ is contained in the subgroup of \mathcal{W} generated by $[\mathcal{C}(\mathfrak{sl}(p), 1)]$ and $[\mathcal{C}(\mathfrak{sl}(pq), 1)]$ and $\mathcal{W}_{\text{pt}}(q)$. Thus we are done by induction.

Remark 6.6. Since the end of the eighties there is a common belief among physicists that all rational conformal field theories come from lattice and WZW models via coset and orbifold (and perhaps chiral extension) constructions (see [MS]). An analogous statement for modular categories would imply that the unitary Witt group is generated by classes of affine categories $\mathcal{C}(\mathfrak{g}, k)$.

Question 6.7. What are the relations in the subgroup of \mathcal{W} generated by $[\mathcal{C}(\mathfrak{g}, k)]$? Is it true that all relations in the subgroup generated by $[\mathcal{C}(\mathfrak{sl}(2), k)]$ are described in Section 6.4? Is it possible to express some nonzero power of $[\mathcal{C}(\mathfrak{sl}(2), 12)] \in \mathcal{W}$ in terms of $[\mathcal{C}(\mathfrak{sl}(2), k)]$, $k \neq 12$? What is the order of $[\mathcal{C}(\mathfrak{sl}(2), 14)] \in \mathcal{W}$?

Question 6.8. Is there a class $w \in \mathcal{W}_{\mathcal{S}}$ of order 2? Equivalently does exist a non-degenerate braided fusion category \mathcal{C} with property S and such that $\mathcal{C}^{\text{rev}} \simeq \mathcal{C}$?

Question 6.9. Is it true that torsion in \mathcal{W} is 2-primary? Is there an element of order 3 in \mathcal{W} ?

Question 6.10. What is the biggest finite order of an element of \mathcal{W} ? For example, are there elements of \mathcal{W} of order 64?

Appendix. Conformal embeddings and cosets with $c < 1$

Here we reproduce (from [BB], [SW]) the list of maximal embeddings starting with serial embeddings (rank-level dualities, (anti-)symmetric and regular embeddings and their variants) and followed up by sporadic embeddings. For the sake of compactness we use matrix algebra notations (instead of Dynkin symbols) for the rank-level embeddings (the first four):

$$\mathfrak{su}(m)_n \times \mathfrak{su}(n)_m \subseteq \mathfrak{su}(mn)_1, \quad \mathfrak{sp}(2m)_n \times \mathfrak{sp}(2n)_m \subseteq \mathfrak{so}(4mn)_1,$$

$$\mathfrak{so}(m)_n \times \mathfrak{so}(n)_m \subseteq \mathfrak{so}(mn)_1, \quad \mathfrak{so}(m)_4 \times \mathfrak{su}(2)_m \subseteq \mathfrak{sp}(2m)_1,$$

$$A_{n,n-1} \subseteq A_{\frac{(n-1)(n+2)}{2},1}, \quad A_{n,n+3} \subseteq A_{\frac{n(n+3)}{2},1},$$

$$A_{2n+1,2n+2} \subseteq B_{2n^2+4n+1,1}, \quad A_{2n,2n+1} \subseteq D_{2n(n+1),1},$$

$$B_{2n+1,4n+1} \subseteq B_{(4n+1)(n+1),1}, \quad A_{2n+1,4n+5} \subseteq B_{4n^2+7n+2,1},$$

$$B_{2n,4n-1} \subseteq D_{n(4n+1),1}, \quad B_{2n,4n+3} \subseteq D_{n(4n+3),1},$$

$$C_{2n,2n-1} \subseteq B_{4n^2-n-1,1}, \quad C_{2n+1,2n+2} \subseteq B_{(4n+1)(n+1),1},$$

$$C_{2n,2n+1} \subseteq D_{n(4n+1),1}, \quad C_{2n+1,2n} \subseteq D_{n(4n+3),1},$$

$$D_{2n,4n+2} \subseteq B_{4n^2+n-1,1}, \quad D_{2n+1,4n} \subseteq B_{n(4n+3),1},$$

$$D_{2n,4n-2} \subseteq D_{n(4n-1),1}, \quad D_{2n+1,4n+4} \subseteq D_{(n+1)(4n+1),1},$$

$$B_{n,2} \subseteq A_{2n,1}, \quad D_{n,2} \subseteq A_{2n-1,1},$$

$$\begin{aligned}
D_{1,1} \times A_{i,1} \times A_{n-i-1} &\subseteq A_{n,1}, & 1 \leq i \leq n-2, & & D_{1,1} \times A_{n-1,1} &\subseteq A_{n,1}, \\
D_{1,1} \times D_{n-1,1} &\subseteq D_{n,1}, & & & D_{1,1} \times A_{n-1,1} &\subseteq D_{n,1}, \\
A_{1,1} \times A_{1,1} \times D_{n-2,1} &\subseteq D_{n,1}, & & & D_{i,1} \times D_{n-i,1} &\subseteq D_{n,1}, & 3 \leq i \leq n-3, \\
A_{1,1} \times A_{1,1} \times B_{n-2,1} &\subseteq B_{n,1}, & & & D_{1,1} \times B_{n-1,1} &\subseteq B_{n,1}, \\
D_{i,1} \times B_{n-1,1} &\subseteq B_{n,1}, & 3 \leq i \leq n-2, & & D_{i,1} \times B_{n-i,1} &\subseteq B_{n,1}, \\
A_{1,2} \times D_{n-1,1} &\subseteq B_{n,1}, & & & D_{1,1} \times A_{n-1,2} &\subseteq C_{n,1}, \\
D_{1,1} \times D_{5,1} &\subseteq E_{6,1}, & & & A_{1,1} \times A_{5,1} &\subseteq E_{6,1}, \\
A_{2,1} \times A_{2,1} \times A_{2,1} &\subseteq E_{6,1}, & & & D_{1,1} \times E_{6,1} &\subseteq E_{7,1}, \\
A_{1,1} \times D_{6,1} &\subseteq E_{7,1}, & & & A_{7,1} &\subseteq E_{7,1}, \\
A_{2,1} \times A_{5,1} &\subseteq E_{7,1}, & & & D_{8,1} &\subseteq E_{8,1}, \\
A_{4,1} \times A_{4,1} &\subseteq E_{8,1}, & & & A_{2,1} \times E_{6,1} &\subseteq E_{8,1}, \\
A_{1,1} \times E_{7,1} &\subseteq E_{8,1}, & & & A_{8,1} &\subseteq E_{8,1}, \\
A_{1,1} \times C_{3,1} &\subseteq F_{4,1}, & & & G_{2,1} \times A_{1,8} &\subseteq F_{4,1}, \\
A_{1,3} \times A_{1,1} &\subset G_{2,1}, & & & A_{2,2} \times A_{2,1} &\subset F_{4,1}, \\
G_{2,1} \times A_{2,2} &\subseteq E_{6,1}, & & & A_{1,7} \times G_{2,2} &\subseteq E_{7,1}, \\
A_{1,3} \times F_{4,1} &\subseteq E_{7,1}, & & & G_{2,1} \times C_{3,1} &\subseteq E_{7,1}, \\
A_{2,6} \times A_{1,16} &\subseteq E_{8,1}, & & & G_{2,1} \times F_{4,1} &\subseteq E_{8,1}, \\
\\
A_{1,10} &\subseteq B_{2,1}, & A_{1,28} &\subseteq G_{2,1}, & A_{2,9} &\subseteq E_{6,1}, \\
A_{2,21} &\subseteq E_{7,1}, & A_{5,6} &\subseteq C_{10,1}, & A_{7,10} &\subseteq D_{35,1}, \\
B_{2,12} &\subseteq E_{8,1}, & B_{4,2} &\subseteq D_{8,1}, & C_{3,5} &\subseteq C_{7,1}, \\
C_{4,1} &\subseteq E_{6,1}, & C_{4,7} &\subseteq D_{21,1}, & D_{5,4} &\subseteq A_{15,1}, \\
D_{6,8} &\subseteq C_{16,1}, & D_{8,16} &\subseteq D_{64,1}, & E_{6,6} &\subseteq A_{26,1}, \\
E_{6,12} &\subseteq D_{39,1}, & E_{7,12} &\subseteq C_{28,1}, & E_{7,18} &\subseteq B_{66,1}, \\
E_{8,30} &\subseteq D_{124,1}, & F_{4,3} &\subseteq D_{13,1}, & F_{4,9} &\subseteq D_{26,1}, \\
G_{2,3} &\subseteq E_{6,1}, & G_{2,4} &\subseteq D_{7,1}.
\end{aligned}$$

Next we reproduce the list of cosets with central charge $0 < c < 1$ given in [BG]:

$$\begin{aligned}
\text{Vir}_{c_n} &= \frac{A_{1,1} \times A_{1,n}}{A_{1,n+1}}, & \text{Vir}_{c_n} &\subseteq \frac{A_{n+1,2}}{A_{n,2} \times u(1)}, \\
\text{Vir}_{c_n} &\subseteq \frac{C_{m+1,1}}{C_{m,1} \times C_{1,1}}, & \text{Vir}_{c_1} &\subseteq \frac{\text{so}(n)_1}{\text{so}(n-1)_1},
\end{aligned}$$

$$\begin{array}{lll}
 \mathrm{Vir}_{c_1} \cong \frac{A_{1,2}}{u(1)}, & \mathrm{Vir}_{c_2} \cong \frac{E_{7,2}}{A_{7,2}}, & \mathrm{Vir}_{c_3} \cong \frac{A_{2,1} \times A_{2,1}}{A_{2,2}}, \\
 \mathrm{Vir}_{c_3} \cong \frac{A_{1,3}}{u(1)}, & \mathrm{Vir}_{c_3} \cong \frac{E_{7,2}}{A_{1,2} \times D_{6,2}}, & \mathrm{Vir}_{c_4} \cong \frac{E_{6,1} \times E_{6,1}}{E_{6,2}}, \\
 \mathrm{Vir}_{c_3} \cong \frac{A_{2,2}}{A_{1,8}}, & \mathrm{Vir}_{c_1} \cong \frac{E_{8,2}}{D_{8,2}}, & \mathrm{Vir}_{c_2} \cong \frac{E_{7,1} \times E_{7,1}}{E_{7,2}}, \\
 \mathrm{Vir}_{c_3} \cong \frac{A_{5,1}}{C_{3,1}}, & \mathrm{Vir}_{c_2} \cong \frac{E_{8,2}}{A_{1,2} \times E_{7,1}}, & \mathrm{Vir}_{c_1} \cong \frac{E_{8,1} \times E_{8,1}}{E_{8,2}}, \\
 \mathrm{Vir}_{c_2} \cong \frac{B_{3,1}}{G_{2,1}}, & \mathrm{Vir}_{c_9} \cong \frac{E_{8,2}}{A_{8,2}}, & \mathrm{Vir}_{c_9} \cong \frac{E_{8,1} \times E_{8,2}}{E_{8,3}}, \\
 \mathrm{Vir}_{c_3} \cong \frac{E_{6,1}}{F_{4,1}}, & \mathrm{Vir}_{c_9} \cong \frac{F_{4,1}}{C_{3,2} \times A_{1,2}}, & \mathrm{Vir}_{c_8} \cong \frac{F_{4,1} \times F_{4,1}}{F_{4,2}}, \\
 \mathrm{Vir}_{c_4} \cong \frac{E_{6,2}}{C_{4,2}}, & \mathrm{Vir}_{c_2} \cong \frac{F_{4,1}}{B_{4,1}}, & \mathrm{Vir}_{c_7} \cong \frac{G_{2,1} \times G_{2,1}}{G_{2,2}}, \\
 \mathrm{Vir}_{c_5} \cong \frac{E_{6,2}}{A_{1,2} \times A_{5,2}}, & \mathrm{Vir}_{c_3} \cong \frac{G_{2,1}}{A_{2,1}}, & \mathrm{Vir}_{c_6} \cong \frac{G_{2,2}}{A_{1,2} \times A_{1,6}}.
 \end{array}$$

References

- [AM] *G. Anderson and G. Moore*, Rationality in conformal field theory, *Comm. Math. Phys.* **117** (1988), no. 3, 441–450.
- [BB] *A. Bais and P. Bouwknegt*, A classification of subgroup truncations of the bosonic string, *Nuclear Phys. B* **279** (1987), 561–570.
- [BK] *B. Bakalov and A. Kirillov Jr.*, *Lectures on Tensor categories and modular functors*, Amer. Math. Soc., Providence, RI, 2001.
- [B] *R. Bezrukavnikov*, On tensor categories attached to cells in affine Weyl groups, in: *Representation theory of algebraic groups and quantum groups*, *Adv. Stud. Pure Math.* **40**, Math. Soc. Japan, Tokyo (2004), 69–90.
- [BG] *P. Bowcock and P. Goddard*, Coset construction and extended algebras, *Nuclear Phys. B* **305** (1988), 685.
- [Br] *A. Bruguières*, Catégories prémodulaires, modularization et invariants des variétés de dimension 3, *Math. Ann.* **316** (2000), no. 2, 215–236.
- [BN] *A. Bruguières and S. Natale*, Exact sequences of tensor categories, preprint 2010, <http://arxiv.org/abs/1006.0569>.
- [CMS] *F. Calegari, S. Morrison and N. Snyder*, Cyclotomic integers, fusion categories, and subfactors, *Comm. Math. Phys.* **303** (2011), 845–896.
- [D] *A. Davydov*, Twisted automorphisms of Hopf algebras, preprint 2007, <http://arxiv.org/abs/0708.2757>.
- [D1] *A. Davydov*, Modular invariants for group-theoretic modular data I, *J. Algebra* **323** (2010), 1321–1348.
- [DB] *A. Davydov and T. Booker*, Commutative algebras in Fibonacci categories, preprint 2010, <http://arxiv.org/abs/1103.3537>.
- [DKR] *A. Davydov, L. Kong and I. Runkel*, Invertible defects and isomorphisms of rational CFTs, preprint 2010, <http://arxiv.org/abs/1004.4725>.
- [De] *P. Deligne*, Catégories tannakiennes, in: *The Grothendieck Festschrift, Volume II*, *Progr. Math.* **87**, Birkhäuser Boston, Boston, MA, (1990), 111–195.
- [DM] *C. Dong and G. Mason*, Holomorphic vertex operator algebras of small central charge, *Pacific J. Math.* **213** (2004), no. 2, 253–266.

- [DZ] C. Dong and W. Zhang, On classification of rational vertex operator algebras with central charges less than 1, *J. Algebra* **320** (2008), no. 1, 86–93.
- [Dr] V. G. Drinfeld, Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990), no. 6, 1419–1457.
- [DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik, On braided fusion categories I, *Selecta Math. (N.S.)* **16** (2010), 1–119.
- [ENO1] P. Etingof, D. Nikshych and V. Ostrik, On fusion categories, *Ann. of Math. (2)* **162** (2005), 581–642.
- [ENO2] P. Etingof, D. Nikshych and V. Ostrik, Fusion categories and homotopy theory, *Quantum Topol.* **1** (2010), 209–273.
- [ENO3] P. Etingof, D. Nikshych and V. Ostrik, Weakly group-theoretical and solvable fusion categories, *Adv. Math.* **226** (2011), 176–205.
- [EO] P. Etingof and V. Ostrik, Finite tensor categories, *Moscow Math. J.* **4** (2004), 627–654.
- [FHL] I. Frenkel, Y. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, *Mem. Amer. Math. Soc.* **104** (1993).
- [FK] I. Frenkel and V. Kac, Basic representations of affine Lie algebras and dual resonance models, *Invent. Math.* **62** (1980), 23–66.
- [FFRS] J. Frohlich, J. Fuchs, I. Runkel and C. Schweigert, Correspondences of ribbon categories, *Adv. Math.* **199** (2006), no. 1, 192–329.
- [GKO] P. Goddard, A. Kent and D. Olive, Unitary representations of the Virasoro algebra and super-Virasoro algebras, *Comm. Math. Phys.* **103** (1986), 105–119.
- [H] Y.-Z. Huang, Rigidity and modularity of vertex tensor categories, *Commun. Contemp. Math.* **10** (2008), suppl. 1, 871–911.
- [HL] Y.-Z. Huang and J. Lepowski, A theory of tensor products for module categories for a vertex operator algebra, I, II, *Selecta Math. (N.S.)* **1** (1995), no. 4, 699–756, 757–786.
- [JS1] A. Joyal and R. Street, Tortile Yang-Baxter operators in braided tensor categories, *J. Pure Appl. Algebra* **71** (1991), 43–45.
- [JS2] A. Joyal and R. Street, Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.
- [KW] V. Kac and M. Wakimoto, Modular and conformal invariance constraints in representation theory of affine algebras, *Adv. Math.* **70** (1988), no. 2, 156–236.
- [K] C. Kassel, *Quantum groups*, Grad. Texts in Math. **155**, Springer-Verlag, New York 1995.
- [KiO] A. Kirillov Jr. and V. Ostrik, On q -analog of McKay correspondence and ADE classification of $\widehat{\mathfrak{sl}}_2$ conformal field theories, *Adv. Math.* **171** (2002), no. 2, 183–227.
- [KR] L. Kong and I. Runkel, Cardy algebras and sewing constraints, I, *Comm. Math. Phys.* **292** (2009), 871–912.
- [M] S. Montgomery, *Hopf algebras and their actions on rings*, CBMS Reg. Conf. Ser. Math. **82**, Amer. Math. Soc., Providence, RI, 1993.
- [MS] G. Moore and N. Seiberg, Lectures on RCFT, in: *Superstrings '89* (Trieste 1989), World Sci. Publ., Singapore (1990), 1–129.
- [MPS] S. Morrison, E. Peters and N. Snyder, Knot polynomial identities and quantum group coincidences, *Quantum Topol.* **2** (2011), 101–156.
- [Mu1] M. Müger, Galois theory for braided tensor categories and the modular closure, *Adv. Math.* **150** (2000), no. 2, 151–201.
- [Mu2] M. Müger, From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories, *J. Pure Appl. Algebra* **180** (2003), no. 1–2, 81–157.
- [Mu3] M. Müger, From subfactors to categories and topology. II. The quantum double of tensor categories and subfactors, *J. Pure Appl. Algebra* **180** (2003), no. 1–2, 159–219.
- [Mu4] M. Müger, On the structure of modular categories, *Proc. Lond. Math. Soc.* **87** (2003), 291–308.
- [Mu5] M. Müger, Galois extensions of braided tensor categories and braided crossed G -categories, *J. Algebra* **277** (2004), no. 1, 256–281.
- [Mu6] M. Müger, On superselection theory of quantum fields in low dimensions, *Proceedings of the XVIth International Congress on Mathematical Physics* (Prague August 3–8, 2009), World Sci. Publ., Singapore (2010), 496–503.
- [O] V. Ostrik, Module categories, weak Hopf algebras and modular invariants, *Transform. Groups* **8** (2003), 177–206.
- [P] B. Pareigis, On braiding and dyslexia, *J. Algebra* **171** (1995), no. 2, 413–425.
- [S] P. Schauenburg, The monoidal center construction and bimodules, *J. Pure Appl. Algebra* **158** (2001), no. 2–3, 325–346.
- [Sc] A. N. Schellekens, Meromorphic $c = 24$ conformal field theories, *Comm. Math. Phys.* **153** (1993), 159–185.

- [SW] *A. Schellekens and N. Warner*, Conformal subalgebras of Kac–Moody algebras, *Phys. Rev. D* **34** (1986), no. 10, 3092–3096.
- [Ul] *K.-H. Ulbrich*, On Hopf algebras and rigid monoidal categories, *Israel J. Math.* **72** (1990), no. 1–2, 252–256.
- [V] *C. Vafa*, Toward classification of conformal theories, *Phys. Lett. B.* **199** (1987), 195–202.

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