THE L^p-FOURIER TRANSFORM ON LOCALLY COMPACT QUANTUM GROUPS

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ABSTRACT. Using interpolation properties of non-commutative L^p-spaces associated with an arbitrary von Neumann algebra, we define a L^p-Fourier transform $1 \leq p \leq 2$ on locally compact quantum groups. We show that the Fourier transform determines a distinguished choice for the interpolation parameter as introduced by Izumi. We define a convolution product in the L^p-setting and show that the Fourier transform turns the convolution product into a product.

1. INTRODUCTION

Locally compact (l.c.) quantum groups have been introduced by Kustermans and Vaes in their papers [15], [16], see also [13], [22], [24]. Their definition put quantum groups in a von Neumann algebraic context. It is known that many aspects of harmonic analysis and representation theory of l.c. groups have an analogue or generalization in the setting of l.c. quantum groups. In particular, Kustermans and Vaes have shown that the Pontrjagin duality theorem has a quantum group analogue which was one of the motivations for their definition. So to every l.c. quantum group (M, Δ), one can associate a dual quantum group $(\hat{M}, \hat{\Delta})$ such that $(M, \Delta) = (\hat{\hat{M}}, \hat{\hat{\Delta}})$.

Van Daele [25] shows that the L^2-Fourier transform has a quantum group interpretation on an algebraic level. Van Daele suggests that this transform has a suitable interpretation in the operator algebraic framework. One of the motivations of the present paper is to give this interpretation.

As worked out in Section 3, the result for the L^2-Fourier transform on the operator algebraic level is seemingly disappointing. The L^2-Fourier transform reduces to the identity operator when considered as an operator on the GNS-spaces of a quantum group and its dual. This is no surprise, since classically the L^2-Fourier transform is implicitly used in the construction of the dual quantum group [25, p. 25] or [15], [16]. More precisely, let G be a l.c. abelian group and let $(L^\infty(G), \Delta_G)$ be the usual l.c. quantum group associated to G. The dual quantum group is given by $(\mathcal{L}(G), \hat{\Delta}_G)$, where $\mathcal{L}(G)$ is the group von Neumann algebra. This quantum group is (spatially) isomorphic to $(L^\infty(\hat{G}), \Delta_{\hat{G}})$ by means of the L^2-Fourier transform.

Despite the fact that the L^2-Fourier transform is trivial on the GNS-space level, it turns out to be essential in finding a distinguished L^p-Fourier transform for $1 \leq p \leq 2$. This is worked out in the present paper.

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On the other hand, the present paper is related to a collection of papers studying module structures of L^p-spaces associated to the Fourier algebra of l.c. groups [4], [3], [6]. These papers are based on the theory of non-commutative L^p-spaces associated to arbitrary (not necessarily semi-finite) von Neumann algebras, see mainly [9], [21] and also [8], [12], [20]. In the final remark of [4] the question of what a proper action of the L^1-space on the L^p-space associated to a quantum group would be remains open. We address this question by defining a left action of the L^1-space on the L^p-space associated with a quantum group. Classically, this action corresponds to the convolution product. For an arbitrary quantum group, we show that this generalized convolution product is turned into a product under the Fourier transform.

After the introduction, this paper is divided into two sections. First, Section 2 contains the technical part on complex interpolation spaces. We discuss Izumi's L^p-spaces and we prove the necessary results in order to define a L^p-Fourier transform using interpolation theory.

Second, Section 3 contains the part on Fourier theory on l.c. quantum groups. This part is less technical and more conceptual. Izumi's L^p-spaces depend on a parameter $z \in \mathbb{C}$, which defines the intersections of the various L^p-spaces. To be more precise, (M, M_*) is turned into a compatible couple of Banach spaces for every $z \in \mathbb{C}$ and the L^p-spaces are defined as complex interpolation spaces with respect to this compatible couple. In Section 3 we discuss the L^2-Fourier transform and show that only for $z = -\frac{1}{2} + it, t \in \mathbb{R}$, (M, M_*) is a compatible couple of Banach spaces such that there is a well-defined L^1-Fourier transform. We also investigate the dependence of $t \in \mathbb{R}$. Then we use the complex interpolation method to define a L^p-Fourier transform. Finally, we introduce an action of the L^1-space on the L^p-space which classically coincides with the convolution product.

Although Section 2 forms an essential part of the theory, we refer the reader interested in the conceptual parts on Fourier theory to Section 3. Where necessary we give references to the technical results of Section 2.

1.1. **Notations and conventions.** Let φ be a weight on a von Neumann algebra M. Let $\mathfrak{n}_\varphi = \{x \in M \mid \varphi(x^*x) < \infty\}, \mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi$. We denote $\nabla_\varphi, J_\varphi$ for the modular operator and modular conjugation and we set $S_\varphi = J_\varphi \nabla_\varphi$. σ_t^φ denotes the modular automorphism group of φ. We denote $\mathcal{T}_\varphi$ for the Tomita algebra defined by

$$\mathcal{T}_\varphi = \left\{x \in M \mid x \text{ is analytic w.r.t. } \sigma_t^\varphi \text{ and } \forall z \in \mathbb{C} : \sigma_z^\varphi(x) \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*\right\}.$$

Let $(\mathcal{H}, \pi, \Lambda)$ be the GNS-representation of M with respect to φ. For $\Lambda(x) \in \Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ we denote $\pi_l(\Lambda(x))$ for the the bounded operator defined by $\pi_l(\Lambda(x))\Lambda(y) = \Lambda(xy), y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. If a vector $\Lambda(x)$ in the left Hilbert algebra $\Lambda(\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*)$ is right bounded, we denote $\pi_r(\Lambda(x))$ for the bounded operator defined by $\pi_r(\Lambda(x))\Lambda(y) = \Lambda(yx), y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

For a subset $A \subseteq M$, we denote A^+ for the positive elements in A. Similarly, M_*^+ denotes the space of positive normal functionals on M. For $\omega \in M_*$, we donote $\overline{\omega} \in M_*$ for the functional defined by $\overline{\omega}(x) = \overline{\omega(x^*)}, x \in M$. Suppose that M acts on a Hilbert space $\mathcal{H}$ and let $\xi, \eta \in \mathcal{H}$. We denote $\omega_{\xi,\eta}$ for the functional $\omega_{\xi,\eta}(x) = \langle x\xi, \eta\rangle$. We use $\cdot$ for the closure of the product of two unbounded operators and $[\,]$ for the closure of a preclosed operator. The character ι will always stand for the identity homomorphism.

1.1.1. *Interpolation.* We recall the complex interpolation method from [1]. Let E_0, E_1 be Banach spaces. The couple (E_0, E_1) is called a compatible couple of Banach spaces if E_0 and E_1 are continuously embedded into a Banach space E. In that case, we identify E_0 and E_1 as subspaces of E. Let $\mathcal{S} = \{z \in \mathbb{C} \mid 0 \leq \Re(z) \leq 1\}$ and let $\mathcal{S}^\circ$ denote its interior. Let $\mathcal{G}(E_0, E_1)$ be the set of functions $f : \mathcal{S} \to E_0 + E_1$ such that

(1) f is bounded and continuous on $\mathcal{S}$ and analytic on $\mathcal{S}^\circ$;
(2) For $t \in \mathbb{R}, j \in \{0, 1\}$, $f(it + j) \in E_j$ and $t \mapsto f(it + j)$ is continuous and bounded with respect to the norm on E_j;
(3) For $j \in \{0, 1\}$, $\|f(it + j)\|_{E_j} \to 0$ as $t \to \infty$.

Note that at this point our notation is different from [1] and [9], where $\mathcal{G}$ is denoted by $\mathcal{F}$, which we reserve for the Fourier transform. For $f \in \mathcal{G}(E_0, E_1)$, we define a norm $|\!|\!| f |\!|\!| = \max\{\sup \|f(it)\|_{E_0}, \sup \|f(it + 1)\|_{E_1}\}$. Let $\theta \in [0, 1]$. We define $(E_0, E_1)_{[\theta]} \subseteq E$ to be the space $\{f(\theta) \mid f \in \mathcal{G}(E_0, E_1)\}$ with norm $\|x\|_{[\theta]} = \inf\{|\!|\!| f |\!|\!| \mid f(\theta) = x, f \in \mathcal{G}(E_0, E_1)\}$. With this norm, $(E_0, E_1)_{[\theta]}$ is a Banach space. This construction is called the complex interpolation method.

The following Riesz-Thorin-like theorem plays a central role in the present paper. The theorem says that the complex interpolation method for parameter $\theta \in [0, 1]$ is an exact interpolation functor of exponent θ, see [1].

Theorem 1.1 ([1, Theorem 4.1.2]). *Let (E_0, E_1), (F_0, F_1) be compatible couples of Banach spaces and let E and F be the respective Banach spaces where these couples are embedded in. Let $T_0 : E_0 \to F_0$ and $T_1 : E_1 \to F_1$ be bounded maps such that they coincide on the intersection $E_0 \cap E_1$ (interpreted within E and F). Then, there exists a unique bounded map $T : (E_0, E_1)_{[\theta]} \to (F_0, F_1)_{[\theta]}$ such that T restricted to E_j coincides with T_j, $j = 0, 1$. Moreover, $\|T\| \leq \|T_0\|^{1-\theta} \|T_1\|^\theta$.*

1.1.2. *Non-commutative L^p-spaces.* We mainly use the construction of non-commutative L^p-spaces by Izumi [9]. Fix a von Neumann algebra M together with a normal, semi-finite, faithful weight φ. Set $S = S_\varphi, \nabla = \nabla_\varphi$ and $J = J_\varphi$. For $z \in \mathbb{C}$,

$$L_{(z)} = \left\{x \in M \mid \exists \varphi_x^{(z)} \in M_* \text{ s.t. } \forall a, b \in \mathcal{T}_\varphi : \varphi_x^{(z)}(a^* b) = \langle x J \nabla^{\bar{z}} \Lambda(a) \mid J \nabla^{-z} \Lambda(b)\rangle\right\}.$$

One can show that $\mathcal{T}_\varphi^2 \in L_{(z)}$ for all $z \in \mathbb{C}$. For $x \in L_{(z)}$, we define a norm $\|x\|_{L_{(z)}} = \max\{\|x\|, \|\varphi_x^{(z)}\|\}$. We define norm-decreasing injections

$$\mu_\infty^{(z)} : L_{(z)} \to M : x \mapsto x;$$
$$\mu_1^{(z)} : L_{(z)} \to M_* : x \mapsto \varphi_x^{(z)}.$$

Define $\nu_\infty^{(z)} : M \to L_{(-z)}^*$ as the dual map of $\mu_1^{(-z)}$ and $\nu_1^{(z)} : M_* \to L_{(-z)}^*$ as the restriction to M_* of the dual map of $\mu_\infty^{(-z)}$. It is proved in [9, Theorem 2.5] that the outer rectangle of Figure 1 (a) commutes. This gives a a compatible couple of Banach spaces $(M, M_*)_{(z)}$ for every $z \in \mathbb{C}$. Moreover [9, Corollary 2.13],

$$\nu_\infty^{(z)}(\mu_\infty^{(z)}(L_{(z)})) = \nu_1^{(z)}(M_*) \cap \nu_\infty^{(z)}(M). \tag{1.1}$$

For $p \in (1, \infty)$, Izumi [9] defines $L_{(z)}^p(M, \varphi) = (M, M_*)_{(z)[1/p]}$, i.e. the complex interpolation space at parameter $\theta = 1/p$ with respect to the couple $(M, M_*)_{(z)}$. We will denote the injection of $L_{(z)}$ into $L^p_{(z)}(M,\varphi)$ by $\mu_p^{(z)}$. We denote the norm-decreasing injection of $L^p_{(z)}(M,\varphi)$ into $L^*_{(-z)}$ by $\nu_p^{(z)}$, see Figure 1 (a).

Notation 1.2. Note that by defintion $L^p_{(z)}(M,\varphi)$ is a linear subspace of $L^*_{(-z)}$. Therefore, we will omit the map $\nu_p^{(z)}$ in the notation if the norms of these spaces do not play a role in the statement. Similarly, we suppress $\mu_\infty^{(z)}$ in the notation if this is more convenient.

Izumi [9, Theorem 3.8] proves that for $z, z' \in \mathbb{C}$, there is an isometric isomorphism

$$U_{p,(z',z)} : L^p_{(z)}(M,\varphi) \to L^p_{(z')}(M,\varphi), \qquad p \in (1,\infty),$$

such that for $a \in \mathcal{T}_\varphi^2$,

$$U_{p,(z',z)}(\nu_\infty^{(z)}(a)) = \nu_\infty^{(z')}(\sigma^\varphi_{i\frac{r'-r}{p}-(s'-s)}(a)), \tag{1.2}$$

where $z = r + is$ and $z' = r' + is'$, $r, r', s, s' \in \mathbb{R}$. Note that by the discussion so far indeed $\nu_\infty^{(z)}(\mathcal{T}_\varphi^2) \subseteq L^p_{(z)}(M,\varphi)$ and $\nu_\infty^{(z')}(\mathcal{T}_\varphi^2) \subseteq L^p_{(z')}(M,\varphi)$.

Let us briefly comment on the connection with the L^p-spaces studied in [21]. Suppose that M acts on a Hilbert space $\mathcal{K}$ and let ϕ be a weight on the commutant M'. Define $L^p(\phi)$ as the space of all closed, densely defined operators x on $\mathcal{K}$ such that if $x = u|x|$ is the polar decomposition, then $u \in M$ and there exists a $\omega \in M_*^+$ such that $|x|^p$ equals the spatial derivative $d\omega/d\phi$. This definition is known as the Connes/Hilsum definition [8], [20]. Terp [21] proves that $L^p(\phi)$ is isometrically isomorphic to $L^p_{(0)}(M,\varphi)$. More precisely, Terp obtains maps μ_p, ν_p that make the diagram in Figure 1 (b) commutative. She proves that the image of ν_p is contained in $L^p_{(0)}(M,\varphi)$ [21, Proposition 33] and that ν_p is an isometric isomorphism $\nu_p : L^p(\phi) \to L^p_{(0)}(M,\varphi)$, see [21, Theorem 36].

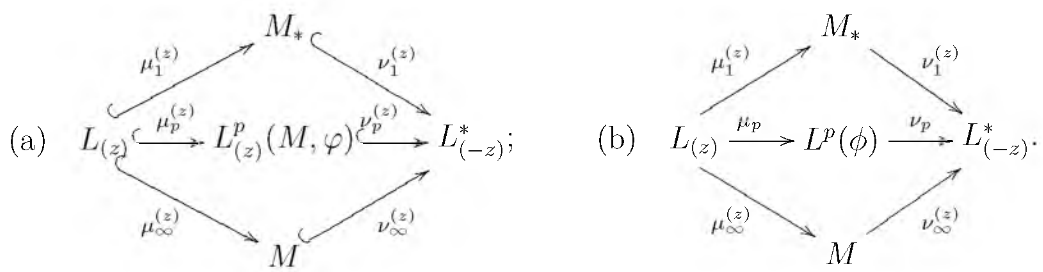

FIGURE 1. (a) Izumi's L^p-spaces. (b) Terp's L^p-spaces.

1.1.3. *Quantum groups.* We use the Kustermans-Vaes defintion of a locally compact quantum group [15], [16], see also [13], [22], [24].

Definition 1.3. A locally compact quantum group (M,Δ) consists of the following data:

(1) A von Neumann algebra M;
(2) A unital, normal $*$-homomorphism $\Delta : M \to M \otimes M$ satisfying the coassociativity relation $(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta$, where $\iota : M \to M$ is the identity;

(3) Two normal, semi-finite, faithful weights φ, ψ on M so that

$$\varphi\left((\omega\otimes\iota)\Delta(x)\right) = \varphi(x)\omega(1), \qquad \forall\, \omega\in M_*^+, \forall\, x\in\mathfrak{m}_\varphi^+ \qquad \text{(left invariance);}$$
$$\psi\left((\iota\otimes\omega)\Delta(x)\right) = \psi(x)\omega(1), \qquad \forall\, \omega\in M_*^+, \forall\, x\in\mathfrak{m}_\psi^+ \qquad \text{(right invariance).}$$

φ is the left Haar weight and ψ the right Haar weight.

Note that we suppress the Haar weights in the notation. The triple $(\mathcal{H},\pi,\Lambda)$ denotes the GNS-construction with respect to the left Haar weight φ. We may assume that M acts on the GNS-space $\mathcal{H}$.

There exists a unique unitary operator $W\in B(\mathcal{H}\otimes\mathcal{H})$ defined by

$$W^*\left(\Lambda(a)\otimes\Lambda(b)\right) = (\Lambda\otimes\Lambda)\left(\Delta(b)(a\otimes 1)\right), \qquad a,b\in\mathfrak{n}_\varphi.$$

W is known as the multiplicative unitary. It satisfies the pentagonal equation $W_{12}W_{13}W_{23} = W_{23}W_{12}$ in $B(\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H})$. Furthermore, $\Delta(x) = W^*(1\otimes x)W, x\in M$.

In [15], [16], it is proved that there exists a dual locally compact quantum group $(\hat{M},\hat{\Delta})$, so that $(\hat{\hat{M}},\hat{\hat{\Delta}}) = (M,\Delta)$. The dual left and right Haar weight are denoted by $\hat{\varphi}$ and $\hat{\psi}$. Similarly, all other dual objects will be denoted by a hat. By construction,

$$\hat{M} = \overline{\{(\omega\otimes\iota)(W)\mid \omega\in B(\mathcal{H})_*\}}^{\sigma-\text{strong}-*}.$$

Furthermore, $\hat{W} = \Sigma W^*\Sigma$, where Σ denotes the flip on $\mathcal{H}\otimes\mathcal{H}$. This implies that $W\in M\otimes\hat{M}$ and $M = \overline{\{(\iota\otimes\omega)(W)\mid\omega\in B(\mathcal{H})_*\}}^{\sigma-\text{strong}-*}$. For $\omega\in M_*$, we use the standard notation $\lambda(\omega) = (\omega\otimes\iota)(W)$. We denote $\mathcal{I}$ for the set of $\omega\in M_*$, such that $\Lambda(x)\mapsto\omega(x^*), x\in\mathfrak{n}_\varphi$ extends to a bounded functional on $\mathcal{H}$. By the Riesz theorem, for every $\omega\in\mathcal{I}$, there is a unique vector denoted by $\xi(\omega)\in\mathcal{H}$ such that $\omega(x^*) = \langle\Lambda(x),\xi(\omega)\rangle, x\in\mathfrak{n}_\varphi$. The dual left Haar weight $\hat{\varphi}$ is defined to be the unique normal, semi-finite, faithful weight on $\hat{M}$, with GNS-construction $(\mathcal{H},\iota,\hat{\Lambda})$ such that $\lambda(\mathcal{I})$ is a σ-strong-$*$/norm core for $\hat{\Lambda}$ and $\hat{\Lambda}(\lambda(\omega)) = \xi(\omega), \omega\in\mathcal{I}$.

2. Interpolation spaces

This section contains the preparatory work in order to study Fourier theory on l.c. quantum groups. The main result is that the non-commutative L^p-spaces as constructed by Izumi are not merely interpolation spaces between a von Neumann algebra and its predual. In case $p\in[1,2]$, Izumi's L^p-spaces turn out to be complex interpolation spaces between the GNS-space of a von Neumann algebra and its predual. In case $p\in[2,\infty]$, Izumi's L^p-spaces are complex interpolation spaces between a von Neumann algebra and its GNS-space. The result is proved in Theorem 2.11, only for the complex interpolation parameter $z = -\frac{1}{2}$. Its importance will become clear in Section 3. Most effort of proving this theorem is contained in checking the assumptions of the re-iteration theorem [1].

We warn the reader that this section contains mostly technical results. The part on Fourier theory on l.c. quantum groups is contained in Section 3 and we will give references to the present section where needed.

Fix a von Neumann algebra M together with a normal, semi-finite, faithful weight φ with GNS-construction $(\mathcal{H},\pi,\Lambda)$. If this is convenient, we will implicitly identify M and $\pi(M)$. The theory presented here works for general von Neumann algebras with a normal, semi-finite, faithful weight. In the next section M will be the von Neumann algebra of a l.c. quantum group and φ will be its left Haar weight.

The following proposition shows that $L_{(-1/2)}$ can be described by a condition that is in general more easy to check.

Proposition 2.1. (1) *Let* $L = \{x \in \mathfrak{n}_\varphi \mid \exists \omega_x \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_\varphi : \omega_x(y^*) = \langle \Lambda(x), \Lambda(y)\rangle\}$. *Then,* $L = L_{(-1/2)}$.

(2) *Similarly, let* $K = \{x \in \mathfrak{n}_\varphi^* \mid \exists \omega_x \in M_* \text{ s.t. } \forall y \in \mathfrak{n}_\varphi : \omega_x(y) = \langle \Lambda(y), \Lambda(x^*)\rangle\}$. *Then,* $K = L_{(1/2)}$.

Proof. We only give the proof of (1), since (2) can be proved similarly. We first prove $\subseteq$. For $x \in L$, $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned}\omega_x(a^*b) =&\langle \Lambda(x), \Lambda(b^*a)\rangle = \langle \Lambda(x), \pi_r(\Lambda(a))\Lambda(b^*)\rangle = \langle \Lambda(x), J\pi_l(J\Lambda(a))J\Lambda(b^*)\rangle \\ =&\langle \Lambda(x), J\pi(\sigma^\varphi_{i/2}(a)^*)J\Lambda(b^*)\rangle = \langle J\pi(\sigma^\varphi_{i/2}(a))J\Lambda(x), J\nabla^{1/2}\Lambda(b)\rangle \\ =&\langle \pi_r(J\nabla^{-1/2}\Lambda(a))\Lambda(x), J\nabla^{1/2}\Lambda(b)\rangle = \langle \pi(x)J\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle.\end{aligned}$$

Hence $x \in L_{(-1/2)}$ and $\omega_x = \varphi_x^{(-1/2)}$.

To prove $\supseteq$, we first prove that $M\mathcal{T}_\varphi^2 \subseteq L_{(-1/2)}$. Indeed, let $x \in M$ and let $c, d \in \mathcal{T}_\varphi$. The functional $M \ni y \mapsto \varphi(\sigma^\varphi_i(d)yxc)$ is normal. Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$\langle xcdJ\nabla^{-1/2}\Lambda(a), J\nabla^{1/2}\Lambda(b)\rangle = \langle \Lambda(xcd\sigma^\varphi_i(a)^*), \Lambda(b^*)\rangle = \varphi(bxcd\sigma^\varphi_i(a)^*) = \varphi(\sigma^\varphi_i(d)a^*bxc).$$

Hence, $xcd \in L_{(-1/2)}$.

Now, take $x \in L_{(-1/2)}$ and let $(e_j)_{j\in J}$ be a bounded net in $\mathcal{T}_\varphi$ such that $\sigma^\varphi_i(e_j)$ is bounded and such that $e_j \to 1$ σ-weakly, see [21, Lemma 9]. Then, $xe_j \to x$ σ-weakly. Furthermore,

$$\|\Lambda(xe_j)\|^2 = \varphi(e_j^*x^*xe_j) = \varphi^{(-1/2)}_{xe_j\sigma^\varphi_{-i}(e_j^*)}(x^*) \leq \|\varphi^{(-1/2)}_{xe_j\sigma^\varphi_{-i}(e_j^*)}\|\|x\|, \tag{2.1}$$

where the second equality is due to the previous paragraph. By [9, Proposition 2.6],

$$\varphi^{(-1/2)}_{xe_j\sigma^\varphi_{-i}(e_j^*)} = \varphi_x^{(-1/2)} \cdot \sigma^\varphi_i(e_j)e_j^*, \tag{2.2}$$

where for $\omega \in M_*, y \in M$, $\omega \cdot y$ is the normal functional defined by $(\omega \cdot y)(a) = \omega(ya), a \in M$. From (2.1) and (2.2) it follows that $(\Lambda(xe_j))_{j\in J}$ is a bounded net. Furthermore, for $a, b \in \mathcal{T}_\varphi$,

$$\langle \Lambda(xe_j), \Lambda(ab)\rangle = \varphi(b^*a^*xe_j) = \varphi(a^*xe_j\sigma^\varphi_{-i}(b^*)) \to \varphi(a^*x\sigma^\varphi_{-i}(b^*)).$$

Since $(\Lambda(xe_j))_{j\in J}$ is bounded, this proves that $(\Lambda(xe_j))_{j\in J}$ is weakly convergent. Since Λ is σ-weak/weak closed, this implies that $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$. So $L_{(-1/2)} \subseteq \mathfrak{n}_\varphi$.

Now, let again $x \in L_{(-1/2)}$ and let $y \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. We prove that $\varphi_x^{(-1/2)}(y^*) = \langle \Lambda(x), \Lambda(y)\rangle$. The proposition then follows by the easily established fact that $\mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ is a σ-weak/weak-core for Λ. Put

$$y_n = \frac{n}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(nt)^2}\sigma^\varphi_t(y)dt,$$

where the integral is taken in the σ-strong-$*$ sense. By standard techniques (c.f. the proof of [21, Lemma 9]), $y_n \in \mathcal{T}_\varphi$, y_n is bounded and y_n converges σ-weakly to y. Moreover, by [19, Lemma 2.4],

$$\Lambda(y_n) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(nt)^2} \nabla^{it} \Lambda(y) dt \to \Lambda(y) \qquad \text{weakly},$$

where the integral is a Bochner integral. Let $(e_j)_{j \in J}$ be a bounded net in $\mathcal{T}_\varphi$ converging σ-weakly to 1. The proposition follows from:

$$\begin{aligned} \langle \Lambda(x), \Lambda(y) \rangle &= \lim_{j \in J} \lim_n \langle \Lambda(x), e_j \Lambda(y_n) \rangle \\ &= \lim_{j \in J} \lim_n \langle x J \nabla^{-1/2} \Lambda(y_n), J \nabla^{1/2} \Lambda(e_j^*) \rangle = \lim_{j \in J} \lim_n \varphi_x^{(-1/2)}(y_n^* e_j) = \varphi_x^{(-1/2)}(y^*). \end{aligned}$$

□

In particular, let $y \in \mathfrak{n}_\varphi$. For $x \in L_{(-1/2)}$, $\varphi_x^{(-1/2)}(y^*) = \varphi(y^* x)$ and for $x \in L_{(-1/2)}$, $\varphi_x^{(1/2)}(y) = \varphi(xy)$.

Remark 2.2. Suppose that φ is a state. Then, $\mathfrak{n}_\varphi = L_{(-1/2)} = L_{(1/2)} = M$. Furthermore, $\varphi_x^{(-1/2)} = x \cdot \varphi$ and $\varphi_x^{(1/2)} = \varphi \cdot x$, where we use the notation $x \cdot \varphi \cdot y$ for the normal functional defined by $(x \cdot \varphi \cdot y)(z) = \varphi(yzx)$. Therefore, the map $\mu_1^{(-1/2)}$ is called the left injection and $\mu_1^{(1/2)}$ is called the right injection, see [12], [21] and [9].

Part of the next Corollary is already proved in [9]. Using the alternative descriptions of Proposition 2.1, it is easy to prove the remaining statements.

Corollary 2.3. *We have inclusions $M\mathcal{T}_\varphi^2 \subseteq L_{(-1/2)}$, $\mathcal{T}_\varphi^2 M \subseteq L_{(1/2)}$, $\mathcal{T}_\varphi^2 \subseteq L_{(-1/2)} \cap L_{(1/2)}$, $L_{(-1/2)}\mathcal{T}_\varphi \subseteq L_{(-1/2)}$, $\mathcal{T}_\varphi L_{(1/2)} \subseteq L_{(1/2)}$, $ML_{(-1/2)} \subseteq L_{(-1/2)}$ and $L_{(1/2)}M \subseteq L_{(1/2)}$. Moreover, $L_{(1/2)} = \{x^* \mid x \in L_{(-1/2)}\}$ and for $x \in L_{(-1/2)}$, $\varphi_{x^*}^{(1/2)} = \overline{\varphi_x^{(-1/2)}}$.*

Proof. The first inclusion was already proved in the proof of Proposition 2.1. Here we proved that for $x \in M, a, b \in \mathcal{T}_\varphi$, $\varphi_{xab}^{(-1/2)}(z) = \varphi(\sigma_i^\varphi(b)\, z\, xa), z \in M$. Similarly, one can prove that for $x, z \in M, a, b \in \mathcal{T}_\varphi, y_- \in L_{(-1/2)}, y_+ \in L_{(1/2)}$,

$$\begin{array}{lll} \varphi_{abx}^{(1/2)}(z) = \varphi(bx\, z\, \sigma_{-i}^\varphi(a)); & \varphi_{ab}^{(1/2)}(z) = \varphi(b\, z\, \sigma_{-i}^\varphi(a)); & \varphi_{ab}^{(-1/2)}(z) = \varphi(\sigma_i^\varphi(b)\, z\, a); \\ \varphi_{y_- a}^{(-1/2)}(z) = \varphi_{y_-}^{(-1/2)}(\sigma_i^\varphi(a) z); & \varphi_{ay_+}^{(1/2)}(z) = \varphi_{y_+}^{(1/2)}(z \sigma_{-i}^\varphi(a)); & \varphi_{xy_-}^{(-1/2)}(z) = \varphi_{y_-}^{(-1/2)}(zx); \\ \varphi_{y_+ x}^{(1/2)}(z) = \varphi_{y_+}^{(1/2)}(xz); & \varphi_{x^*}^{(1/2)} = \overline{\varphi_x^{(-1/2)}}. & \end{array}$$

□

In Section 3, we consider complex interpolation spaces between M_* and $L^2_{(-1/2)}(M, \varphi)$ and between $L^2_{(-1/2)}(M, \varphi)$ and M. This can be done by applying the re-iteration theorem [1, Theorem 4.6.1]. The following propositions serve as a preparation for this theorem.

Proposition 2.4. (1) *There is a unique unitary map $L^2_{(-1/2)}(M, \varphi) \to \mathcal{H}$ determined by $\nu_\infty^{(-1/2)}(a) \mapsto \Lambda(a)$, $a \in \mathcal{T}_\varphi^2$, see Notation 1.2.*

(2) *There is a unique unitary map $L^2_{(1/2)}(M, \varphi) \to \mathcal{H}^*$ determined by $\nu_\infty^{(1/2)}(a) \to \mid \Lambda(a^*) \rangle$, $a \in \mathcal{T}_\varphi^2$. Here, $\mid \Lambda(a^*) \rangle$ is the functional on $\mathcal{H}$ given by $\mid \Lambda(a^*) \rangle(\xi) = \langle \xi, \Lambda(a^*) \rangle$.*

Proof. (1) Fix a weight ϕ on M' and let $d = d\varphi/d\phi$ be the spatial derivative [19]. The suggested map is the composition of the isometric isomorphisms $U_{2,(0,-1/2)} : L^2_{(-1/2)}(M,\varphi) \to L^2_{(0)}(M,\varphi)$, $\nu_2^{-1} : L^2_{(0)}(M,\varphi) \simeq L^2(\phi)$, see Figure 1 (b), and the unitary $\mathcal{P}^{-1} : L^2(\phi) \to \mathcal{H}$ given in [21, Theorem 23]. Indeed, using Notation 1.2,

$$\begin{aligned}&\mathcal{P}^{-1}(\nu_2^{-1}(U_{2,(0,-1/2)}(\nu_\infty^{(-1/2)}(a)))) = \mathcal{P}^{-1}(\nu_2^{-1}(\nu_\infty^{(-1/2)}(\sigma^\varphi_{i/4}(a))))\\ =&\mathcal{P}^{-1}(\mu_2(\sigma^\varphi_{i/4}(a))) = \mathcal{P}^{-1}([d^{1/4}\sigma^\varphi_{i/4}(a)d^{1/4}]) = \mathcal{P}^{-1}([ad^{1/2}]) = \Lambda(a),\end{aligned}$$

where the first equality is due to (1.2), the second equality is the commutativity of Figure 1 (b) and the fact that $\mathcal{T}_\varphi^2 \subseteq L_{(-1/2)}$, the third equality is the explicit definition of μ_2 given in the proof of [21, Theorem 27] together with [8, Theorem 4], the fourth equality follows from [21, Lemma 22] togheter with [21, Theorem 26] and [8, Theorem 4], the last equality is [21, Theorem 23]. Since $\Lambda(\mathcal{T}_\varphi^2)$ is dense in $\mathcal{H}$, this determines a unitary map $L^2_{(-1/2)}(M,\varphi) \to \mathcal{H}$.

(2) Similarly, this map is given by the composition of the isometric isomorphisms $U_{2,(0,1/2)}$: $L^2_{(1/2)}(M,\varphi) \to L^2_{(0)}(M,\varphi)$, $\nu_2^{-1} : L^2_{(0)}(M,\varphi) \simeq L^2(\phi)$ as in [21, Theorem 36] and the isomorphism $L^2(\phi) \to \mathcal{H}^*$ determined by $d^{1/2}x^* \mapsto \mid \Lambda(x^*)\rangle, x \in \mathfrak{n}_\varphi$ (it follows from [21, Theorem 23] that this is an isomorphism). □

For convenience of notation, we will implicitly identify $L^2_{(-1/2)}(M,\varphi)$ and $\mathcal{H}$ as well as $L^2_{(1/2)}(M,\varphi)$ and $\mathcal{H}^*$ via the maps given in the previous proposition.

Proposition 2.5. *For* $\xi \in \mathcal{H}$, $y \in L_{(1/2)}$, $\langle \nu_2^{(-1/2)}(\xi), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \xi, \Lambda(y^*)\rangle$.

Proof. First assume that $\xi = \mu_2^{(-1/2)}(x), x \in L_{(-1/2)}$.

$$\begin{aligned}&\langle \nu_2^{(-1/2)}(\mu_2^{(-1/2)}(x)), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \nu_1^{(-1/2)}(\mu_1^{(-1/2)}(x)), y\rangle_{L^*_{(1/2)},L_{(1/2)}}\\ =&\langle \mu_1^{(-1/2)}(x), \mu_\infty^{(1/2)}(y)\rangle_{M_*,M} = \varphi_x^{(-1/2)}(y) = \varphi(yx) = \langle \Lambda(x), \Lambda(y^*)\rangle.\end{aligned}$$

The lemma follows by the identifications of Proposition 2.4 and the fact that $\mu_2^{(-1/2)}(L_{(-1/2)})$ is dense in $L^2_{(-1/2)}(M,\varphi)$, see [1, Theorem 4.2.2] and (1.1). □

Recall that $\mathcal{I} = \{\omega \in M_* \mid \Lambda(x) \mapsto \omega(x^*), x \in \mathfrak{n}_\varphi \text{ is bounded }\}$. For $\omega \in \mathcal{I}$, we have $\xi(\omega) \in \mathcal{H}$ and $\langle \xi(\omega), \Lambda(x)\rangle = \omega(x^*)$. For $\omega \in \mathcal{I}$ we define $\|\omega\|_\mathcal{I} = \max\{\|\omega\|, \|\xi(\omega)\|\}$. The following proposition determines the intersection of $L^2_{(-1/2)}(M,\varphi)$ and M_*.

Proposition 2.6. $\nu_1^{(-1/2)}(\mathcal{I}) = L^2_{(-1/2)}(M,\varphi) \cap \nu_1^{(-1/2)}(M_*)$, *where we have used Notation 1.2.*

Proof. We first prove $\supseteq$. Let $\xi \in \mathcal{H}$ and $\omega \in M_*$ be such that $\nu_2^{(-1/2)}(\xi) = \nu_1^{(-1/2)}(\omega)$. By Proposition 2.5, for $y \in L_{(1/2)}$,

$$\omega(y) = \langle \nu_1^{(-1/2)}(\omega), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \nu_2^{(-1/2)}(\xi), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \xi, \Lambda(y^*)\rangle.$$

Since $\Lambda(L_{(1/2)})$ contains $\Lambda(\mathcal{T}_\varphi^2)$ it is dense in $\mathcal{H}$, so $\omega \in \mathcal{I}$. To prove $\subseteq$, let $\omega \in \mathcal{I}$. By Proposition 2.5, for $y \in L_{(1/2)}$,

$$\begin{aligned}\langle \nu_2^{(-1/2)}(\xi(\omega)), y\rangle_{L^*_{(1/2)},L_{(1/2)}} &= \langle \xi(\omega), \Lambda(y^*)\rangle = \omega(y) \\ =&\langle \omega, \mu_\infty^{(1/2)}(y)\rangle_{M_*,M} = \langle \nu_1^{(-1/2)}(\omega), y\rangle_{L^*_{(1/2)},L_{(1/2)}}.\end{aligned}$$

Hence $\nu_2^{(-1/2)}(\xi(\omega)) = \nu_1^{-(1/2)}(\omega)$. ☐

The following lemma is a variant of [21, Lemma 9].

Lemma 2.7. *Let $\delta > 0$. There exists a net $(e_j)_{j\in J}$ in $\mathcal{T}_\varphi$ such that* (1) $\|\sigma_z^\varphi(e_j)\| \leq e^{\delta\Im(z)^2}$, (2) $e_j \to 1$ *strongly and* (3) $\sigma_{i/2}^\varphi(e_j) \to 1$ *σ-weakly.*

Proof. Let $(f_j)_{j\in J}$ and $(e_j)_{j\in J}$ be nets as in [21, Lemma 9]. This lemma proves already that $(e_j)_{j\in J}$ satisfies (1) and (2). Now, (3) follows, since for $\xi \in \mathcal{H}$,

$$\begin{aligned}\langle \sigma_{\frac{i}{2}}^\varphi(e_j)\xi, \xi\rangle =&\langle \omega_{\xi,\xi}, \sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2}\sigma_t^\varphi(f_j)dt\rangle = \langle \sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2}(\omega_{\xi,\xi}\circ\sigma_t^\varphi)dt, f_j\rangle \\ \to&\langle \sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2}(\omega_{\xi,\xi}\circ\sigma_t^\varphi)dt, 1\rangle = \sqrt{\frac{\delta}{\pi}}\int_{-\infty}^{\infty} e^{-\delta(t-\frac{i}{2})^2}dt\langle \xi,\xi\rangle = \langle \xi,\xi\rangle,\end{aligned}$$

where the last equality is obtained by means of the residue formula for meromorphic functions. So $\sigma_{\frac{i}{2}}^\varphi(e_k)$ is bounded and converges weakly, hence σ-weakly to 1. ☐

Proposition 2.8. *The map $k : L_{(-1/2)} \to \mathcal{I} : x \mapsto \varphi_x^{(-1/2)}$ is injective, norm-decreasing and has dense range.*

Proof. Suppose that $x \in L_{(-1/2)}$ and $\varphi_x^{(-1/2)} = 0$, then $0 = (\varphi_x^{(-1/2)})(x^*) = \varphi(x^*x)$. So $x = 0$ and hence k is injective. For $x \in L_{(-1/2)}$, $\|\varphi_x^{(-1/2)}\| \leq \|x\|_{L_{(-1/2)}}$ and

$$\|\xi(\varphi_x^{(-1/2)})\| = \|\Lambda(x)\| = \|\varphi_x^{(-1/2)}(x^*)\|^{1/2} \leq \|\varphi_x^{(-1/2)}\|^{1/2}\|x^*\|^{1/2} \leq \|x\|_{L_{(-1/2)}},$$

so that k is norm-decreasing. Now we prove that the range of k is dense in $\mathcal{I}$. We identify $\mathcal{I}$ with the subspace $\{(\omega, \xi(\omega)) \mid \omega \in \mathcal{I}\} \subseteq M_* \times \mathcal{H}$. We equip $M_* \times \mathcal{H}$ with the norm $\|(\omega,\xi)\|_{\max} = \max\{\|\omega\|, \|\xi\|\}$. The norm coincides with $\|\cdot\|_{\mathcal{I}}$ on $\mathcal{I}$. The dual of $(M_* \times \mathcal{H}, \|\cdot\|_{\max})$ can be identified with $(M \times \mathcal{H}, \|\cdot\|_{\mathrm{sum}})$, where $\|(x,\xi)\|_{\mathrm{sum}} = \|x\| + \|\xi\|$. Let $N \subseteq M \times \mathcal{H}$ be the space of all (y,η) such that $\langle(\omega,\xi(\omega)),(y,\eta)\rangle = 0$ for all $\omega \in \mathcal{I}$. The dual of $\mathcal{I}$ is given by $(M \times \mathcal{H})/N$ eqquiped with the quotient norm.

Now, let $(y,\eta) \in M \times \mathcal{H}$ be such that $\langle(\varphi_x^{(-1/2)}, \Lambda(x)), (y,\eta)\rangle_{M_*\times\mathcal{H}, M\times\mathcal{H}} = 0$ for all $x \in L_{(-1/2)}$. The proof is finished if we can show that $(y,\eta) \in N$. In order to do this, let $(e_j)_{j\in J}$ be a net as in Lemma 2.7. Put $a_j = \sigma_{-\frac{i}{2}}^\varphi(e_j)$. By the assumptions on (y,η), for $x \in \mathcal{T}_\varphi$,

$$(\varphi_{xa_j}^{(-1/2)})(y) = -\langle \Lambda(xa_j), \eta\rangle. \tag{2.3}$$

For the left hand side we find

$$\begin{aligned}(\varphi_{xa_j}^{(-1/2)})(y) &= \langle \pi(y)J\nabla^{\frac{1}{2}}\Lambda(x^*), J\nabla^{-\frac{1}{2}}\Lambda(a_j)\rangle \\ =&\langle \Lambda(x), \pi(y^*)S\nabla^{-1}\Lambda(a_j)\rangle = \langle \Lambda(x), \Lambda(y^*\sigma_i^\varphi(a_j)^*)\rangle,\end{aligned} \tag{2.4}$$

where the first equality follows from [9, Proposition 2.3]. For the right hand side of (2.3) we find

$$\langle \Lambda(xa_j), \eta\rangle = \langle J\pi(\sigma^{\varphi}_{-\frac{i}{2}}(a_j^*))J\Lambda(x), \eta\rangle = \langle \Lambda(x), J\pi(\sigma^{\varphi}_{\frac{i}{2}}(a_j))J\eta\rangle. \tag{2.5}$$

Hence (2.3) together with (2.4) and (2.5) yield

$$\Lambda(y^*\sigma^{\varphi}_{i}(a_j)^*) = -J\pi(\sigma^{\varphi}_{\frac{i}{2}}(a_j))J\eta.$$

Hence, since $\sigma^{\varphi}_{\frac{i}{2}}(a_j) = e_j \to 1$ strongly, $\Lambda(y^*\sigma^{\varphi}_{i}(a_j)^*) \to -\eta$ weakly. For $\omega \in \mathcal{I}$,

$$\langle \xi(\omega), \eta\rangle = -\lim_{j\in J}\langle \xi(\omega), \Lambda(y^*\sigma^{\varphi}_{i}(a_j)^*)\rangle = -\lim_{j\in J}\omega(\sigma^{\varphi}_{i}(a_j)y) = -\lim_{j\in J}\omega(\sigma^{\varphi}_{\frac{i}{2}}(e_j)y) = -\omega(y).$$

Thus $(y,\eta) \in N$. □

For $x \in \mathfrak{n}_\varphi$, we put $\|x\|_{\mathfrak{n}_\varphi} = \max\{\|x\|, \|\Lambda(x)\|\}$. It turns out that $\mathfrak{n}_\varphi$ is the intersection of M and $L^2_{(-1/2)}(M,\varphi)$.

Proposition 2.9. *Recall Notation 1.2. We have equalities*

$$\nu^{(-1/2)}_{\infty}(\mathfrak{n}_\varphi) = L^2_{(-1/2)}(M,\varphi) \cap \nu^{(-1/2)}_{\infty}(M) = L^2_{(-1/2)}(M,\varphi) \cap \nu^{(-1/2)}_{\infty}(\overline{\mu^{(-1/2)}_{\infty}(L_{(-1/2)})}).$$

Proof. First we prove that $\nu^{(-1/2)}_{\infty}(\mathfrak{n}_\varphi) = L^2_{(-1/2)}(M,\varphi) \cap \nu^{(-1/2)}_{\infty}(M)$. For $x \in \mathfrak{n}_\varphi, y \in L_{(1/2)}$, using Propositions 2.4 (a) and 2.5 for the first equality,

$$\begin{aligned}&\langle \nu^{(-1/2)}_{2}(\Lambda(x)), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \Lambda(x), \Lambda(y^*)\rangle \\ =&\varphi(yx) = \varphi^{(1/2)}_{y}(x) = \langle x, \mu^{(1/2)}_{1}(y)\rangle_{M,M_*} = \langle \nu^{(-1/2)}_{\infty}(x), y\rangle_{L^*_{(1/2)},L_{(1/2)}},\end{aligned}$$

so $\nu^{(-1/2)}_{2}(\Lambda(x)) = \nu^{(-1/2)}_{\infty}(x)$. Hence the inclusion $\subseteq$ follows.

Now let $x \in M$, $\xi \in \mathcal{H}$ be such that $\nu^{(-1/2)}_{\infty}(x) = \nu^{(-1/2)}_{2}(\xi)$, see Proposition 2.4. For $y \in L_{(1/2)}$,

$$\begin{aligned}&(\varphi^{(1/2)}_{y})(x) = \langle x, \mu^{(1/2)}_{1}(y)\rangle_{M,M_*} = \langle \nu^{(-1/2)}_{\infty}(x), y\rangle_{L^*_{(1/2)},L_{(1/2)}} \\ =&\langle \nu^{(-1/2)}_{2}(\xi), y\rangle_{L^*_{(1/2)},L_{(1/2)}} = \langle \xi, \Lambda(y^*)\rangle,\end{aligned}$$

by Proposition 2.5. For $a \in \mathcal{T}^2_\varphi$, $y \in \mathfrak{n}_\varphi$, using Corollary 2.3,

$$\langle \Lambda(xa), \Lambda(y)\rangle = \varphi(y^*xa) = \varphi^{(-1/2)}_{a}(y^*x) = \varphi^{(1/2)}_{\sigma^{\varphi}_{i}(a)}(y^*x) = \langle \xi, \Lambda(y\sigma^{\varphi}_{i}(a)^*)\rangle = \langle \pi_r(\Lambda(a))\xi, \Lambda(y)\rangle.$$

So for $a \in \mathcal{T}^2_\varphi$, $\Lambda(xa) = \pi_r(\Lambda(a))\xi$. Let $(e_j)_{j\in J}$ be a net as in Lemma 2.7. Put $a_j = e_j^2$. Then $xa_j \to x$ σ-weakly. Furthermore, $\pi_r(\Lambda(a_j))\xi = J\pi(\sigma^{\varphi}_{i/2}(a_j)^*)J\xi \to \xi$ weakly, hence $\Lambda(xa_j)$ converges weakly. Since Λ is σ-weak/weak closed, $x \in \mathrm{Dom}(\Lambda) = \mathfrak{n}_\varphi$ and $\xi = \Lambda(x)$. This proves $\supseteq$.

Note that [1, Theorem 4.2.2] gives the first equality in

$$L^2_{(-1/2)}(M,\varphi) = (\overline{\mu^{(-1/2)}_{\infty}(L_{(-1/2)})}, M_*)_{(-1/2)[\frac{1}{2}]} \subseteq \nu^{(-1/2)}_{\infty}(\overline{\mu^{(-1/2)}_{\infty}(L_{(-1/2)})}) + \nu^{(-1/2)}_{1}(M_*),$$

and by (1.1),

$$\nu^{(-1/2)}_{\infty}(M) \cap \left(\nu^{(-1/2)}_{\infty}(\overline{\mu^{(-1/2)}_{\infty}(L_{(-1/2)})}) + \nu^{(-1/2)}_{1}(M_*)\right) = \nu^{(-1/2)}_{\infty}(\overline{\mu^{(-1/2)}_{\infty}(L_{(-1/2)})}).$$

Now,

$$
\begin{aligned}
&L^2_{(-1/2)}(M,\varphi)\cap \nu_\infty^{(-1/2)}(M)\\
=&L^2_{(-1/2)}(M,\varphi)\cap \nu_\infty^{(-1/2)}(M)\cap\left(\nu_\infty^{(-1/2)}(\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})})+\nu_1^{(-1/2)}(M_*)\right)\\
=&L^2_{(-1/2)}(M,\varphi)\cap \nu_\infty^{(-1/2)}(\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}).
\end{aligned}
$$

□

Proposition 2.10. *The map $k' : L_{(-1/2)} \to \mathfrak{n}_\varphi : x \mapsto x$ is injective, norm-decreasing and has dense range.*

Proof. The non-trivial part is that $k'(L_{(-1/2)})$ is dense in $\mathfrak{n}_\varphi$ with respect to $\|\cdot\|_{\mathfrak{n}_\varphi}$. To prove this, we identify $\mathfrak{n}_\varphi$ with the subspace $\{(x,\Lambda(x)) \mid x\in\mathfrak{n}_\varphi\}\subseteq M\times\mathcal{H}$. For $(x,\xi)\in M\times\mathcal{H}$, we set $\|(x,\xi)\|_{\max}=\max\{\|x\|,\|\xi\|\}$. So $\|\cdot\|_{\max}$ coincides with $\|\cdot\|_{\mathfrak{n}_\varphi}$ on $\mathfrak{n}_\varphi$. The dual of $(M\times\mathcal{H},\|\cdot\|_{\max})$ is given by $(M^*\times\mathcal{H},\|\cdot\|_{\mathrm{sum}})$, where $\|(\theta,\xi)\|_{\mathrm{sum}}=\|\theta\|+\|\xi\|$. Let $(\theta,\xi)\in M^*\times\mathcal{H}$ be such that for all $x\in L_{(-1/2)}$,

$$\theta(x)+\langle\Lambda(x),\xi\rangle=0. \tag{2.6}$$

We must prove that (2.6) holds for all $x\in\mathfrak{n}_\varphi$. We will first prove that (2.6) holds for $x\in\mathfrak{n}_\varphi\cap\mathfrak{n}_\varphi^*$.

We claim that there exists an $\omega\in M_*$ such that for $x\in L_{(-1/2)}\cap L_{(1/2)}$, $\omega(x)=\theta(x)$. From Corollary 2.3 it follows that $\overline{L_{(-1/2)}\cap L_{(1/2)}}$ is a C*-algebra. Let $(u_j)_{j\in J}$ be an approximate unit for the C*-algebra $\overline{L_{(-1/2)}\cap L_{(1/2)}}$ (here and in the rest of this proof the closure has to be interpreted within M). We may assume that $u_j\in(L_{(-1/2)}\cap L_{(1/2)})^+$. Set $\omega_j(x)=\theta(xu_j), x\in M$. Then $\omega_j\in M_*$. Indeed, let $(x_k)_{k\in K}$ be a bounded net in M converging σ-weakly to $x\in M$. $\omega_j(x_k)=\theta(x_ku_j)=-\langle\pi(x_k)\Lambda(u_j),\xi\rangle\to-\langle\pi(x)\Lambda(u_j),\xi\rangle=\theta(xu_j)=\omega_j(x)$. So ω_j is σ-weakly continuous on bounded sets, hence normal, c.f. [18, Theorem II.2.6].

Let ρ be a representation of $\overline{L_{(-1/2)}\cap L_{(1/2)}}$ on a Hilbert space $\mathcal{H}_\rho$ such that $\theta(x)=\langle\rho(x)\xi,\eta\rangle$ for certain vectors $\xi,\eta\in\mathcal{H}_\rho$. Then $\omega_j(x)=\langle\rho(x)\rho(u_j)\xi,\eta\rangle$. Since $\rho(u_j)\to1$ strongly, $\|\omega_j|_{\overline{L_{(-1/2)}\cap L_{(1/2)}}}-\theta|_{\overline{L_{(-1/2)}\cap L_{(1/2)}}}\|\to0$. $L_{(-1/2)}\cap L_{(1/2)}(\supseteq\mathcal{T}_\varphi^2)$ is σ-weakly, hence strongly dense in M so that by Kaplansky's density theorem $\|\omega_j\|=\|\omega_j|_{\overline{L_{(-1/2)}\cap L_{(1/2)}}}\|$. Hence ω_j is a Cauchy net in M_*. Let $\omega\in M_*$ be its limit. This proves the first claim.

Next, we claim that $L_{(-1/2)}\cap L_{(1/2)}$ is a σ-weak/weak core for Λ. Let $x\in\mathfrak{n}_\varphi=\mathrm{Dom}(\Lambda)$. Let $(e_j)_{j\in J}$ be as in Lemma 2.7. Then $L_{(-1/2)}\cap L_{(1/2)}\ni e_j^2xe_j^2\to x$ strongly, hence σ-weakly. Furthermore, $\Lambda(e_j^2xe_j^2)=\pi(e_j^2)J\pi(\sigma^\varphi_{\frac{i}{2}}(e_j^2)^*)J\Lambda(x)$ converges weakly to $\Lambda(x)$ which proves the claim.

Now, for $x\in L_{(-1/2)}\cap L_{(1/2)}$, $\omega(x)=\theta(x)=-\langle\Lambda(x),\xi\rangle$. Let $x\in\mathfrak{n}_\varphi$ and, by the previous paragraph, let $(x_i)_{i\in I}$ be a net in $L_{(-1/2)}\cap L_{(1/2)}$ converging σ-weakly to x such that $\Lambda(x_i)\to\Lambda(x)$ weakly. Then, we arrive at the following equation:

$$\omega(x)=\lim_{i\in I}\omega(x_i)=-\lim_{i\in I}\langle\Lambda(x_i),\xi\rangle=-\langle\Lambda(x),\xi\rangle. \tag{2.7}$$

Note that by Propositions 2.1 and 2.9, $\overline{L_{(-1/2)}\cap L_{(1/2)}}\subseteq\overline{\mathfrak{n}_\varphi\cap\mathfrak{n}_\varphi^*}\subseteq\overline{L_{(-1/2)}}\cap\overline{L_{(1/2)}}$. We claim that the inclusions are actually equalities. Indeed, let $x\in\overline{L_{(-1/2)}}\cap\overline{L_{(1/2)}}$ be positive.

Let x_n and y_n be sequences in $L_{(-1/2)}$, respectively $L_{(1/2)}$, converging in norm to x. Then, by Corollary 2.3, $y_n x_n \in L_{(-1/2)}L_{(1/2)} \subseteq \overline{L_{(-1/2)}} \cap \overline{L_{(1/2)}}$. $y_n x_n$ is norm convergent to x^2. So $x^2 \in \overline{L_{(-1/2)}} \cap \overline{L_{(1/2)}}$, hence $x \in \overline{L_{(-1/2)}} \cap \overline{L_{(1/2)}}$. From Corollary 2.3 it follows that $\overline{L_{(-1/2)}} \cap \overline{L_{(1/2)}}$ and $\overline{L_{(-1/2)} \cap L_{(1/2)}}$ are C*-algebras. Hence, $\overline{L_{(-1/2)} \cap L_{(1/2)}} = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^* = \overline{L_{(-1/2)}} \cap \overline{L_{(1/2)}}$.

Let $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$ and let $x_n \in L_{(-1/2)} \cap L_{(1/2)}$ be a sequence converging in norm to x. Then, by (2.7), $\theta(x) = \lim_{n\to\infty} \theta(x_n) = \lim_{n\to\infty} \omega(x_n) = \omega(x) = -\langle \Lambda(x), \xi\rangle$. Hence (2.6) follows for $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$.

Now, let $y \in \mathfrak{n}_\varphi$ and let $y = u|y|$ be its polar decomposition. Since (2.6) holds for $x \in L_{(-1/2)}$, we find for $x \in L_{(-1/2)}$,

$$(\theta \cdot u)(x) + \langle \Lambda(x), u^*\xi\rangle = 0, \tag{2.8}$$

where we define $(\theta \cdot u) \in M$ by $(\theta \cdot u)(a) = \theta(ua), a \in M$. If we apply the arguments in the previous paragraphs to the pair $(\theta \cdot u, u^*\xi)$, we see that actually (2.8) holds for all $x \in \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$. In particular, putting $x = |y|$, the required equation (2.6) follows. □

Now, we are prepared to apply the re-iteration theorem for the complex interpolation method.

Theorem 2.11. *Every interpolation space is understood with respect to Figure 1 (a) for the parameter $z = -1/2$, so $L^p(M,\varphi) = L^p_{(-1/2)}(M,\varphi), \ldots$*

1. *For $p \in (1,2]$, $(L^2(M,\varphi), M_*)_{[\frac{2}{p}-1]} = L^p(M,\varphi)$.*
2. *For $q \in [2,\infty)$, $(L^2(M,\varphi), M)_{[1-\frac{2}{q}]} = (M, L^2(M,\varphi))_{[\frac{2}{q}]} = L^q(M,\varphi)$.*

Proof. (1) By [1, Theorem 4.2.2 (b)] and (1.1),

$$(\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}, M_*)_{[\frac{1}{2}]} = \quad (M, M_*)_{[\frac{1}{2}]} = L^2(M,\varphi); \tag{2.9}$$

$$(\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}, M_*)_{[1]} = \quad (M, M_*)_{[1]} = L^1(M,\varphi); \tag{2.10}$$

here the closures are with respect to the norm in M. By [9, Proposition 2.4] and [1, Proposition 4.2.2], (2.10) equals $\nu_1^{(-1/2)}(M_*)$.

By [1, Theorem 4.2.2], (1.1) and [9, Proposition 2.4], $\mu_1^{(-1/2)}(L_{(-1/2)})$ is dense in M_*. Similarly, $\mu_\infty^{(-1/2)}(L_{(-1/2)})$ is dense in $\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}$. We find by Proposition 2.6, $\nu_1^{(-1/2)}(\mathcal{I}) = L^2_{(-1/2)}(M,\varphi) \cap \nu_1^{(-1/2)}(M_*)$. By Proposition 2.8, $\mu_1^{(-1/2)}(L_{(-1/2)})$ is $\|\cdot\|_{\mathcal{I}}$-dense in $\mathcal{I}$. Then we may apply the re-iteration theorem [1, Theorems 4.6.1],

$$\begin{aligned} L^p(M,\varphi) &= (M, M_*)_{[\frac{1}{p}]} = (\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}, M_*)_{[\frac{1}{p}]} \\ &= ((\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}, M_*)_{[\frac{1}{2}]}, (\overline{\mu_\infty^{(-1/2)}(L_{(-1/2)})}, M_*)_{[1]})_{[\frac{2}{p}-1]} = (L^2(M,\varphi), M_*)_{[\frac{2}{p}-1]}, \end{aligned}$$

where the second equality follows again by [1, Theorem 4.2.2].

(2) Using Propositions 2.9 and 2.10, one proves that $L^q(M,\varphi) = (L^2(M,\varphi), M)_{[1-\frac{2}{q}]}$, which in turn equals $(M, L^2(M,\varphi))_{[\frac{2}{q}]}$ by [1, Theorem 4.2.1]. □

We end this section by defining the embedding of $\mathcal{I}$ in $L^p_{(-1/2)}(M,\varphi)$ for $p \in [1,2]$.

Definition 2.12. Let $p \in [1,2]$. Consider the map

$$\varphi_x^{(-1/2)} \mapsto \mu_p^{(-1/2)}(x), \qquad x \in L_{(-1/2)}. \tag{2.11}$$

Note that by Proposition 2.8, $\mu_1^{(-1/2)}(L_{(-1/2)})$ is dense in $\mathcal{I}$ with respect to $\|\cdot\|_{\mathcal{I}}$. Furthermore, by Theorem 2.11, Proposition 2.6 and the definition of the norm on $L^p_{(-1/2)}(M,\varphi)$, it follows that (2.11) determines a norm-decreasing map $\rho_p : \mathcal{I} \to L^p_{(-1/2)}(M,\varphi)$.

Remark 2.13. ρ_p has dense range, see [1, Theorem 4.2.2 (a)]. Note that by Proposition 2.6, $\rho_2(\omega) = \xi(\omega), \omega \in \mathcal{I}$.

3. Fourier theory on quantum groups

In this section we first define a L^2-Fourier transform as motivated by [25]. It is a bounded operator between the L^2-spaces assocated with a quantum group and its dual. The L^2-Fourier transform trivializes once these L^2-spaces are identified with the GNS-spaces of the left (dual) Haar weight. This is exactly as one expects from the classical case, since the Fourier transform is implicitly used in the construction of the dual quantum group, see the remarks in the introduction. Once we have defined the L^2-Fourier transform, we show that there is a distinguished choice for a L^1-Fourier transform. Using the techniques of interpolation spaces we define a L^p-Fourier transform for $1 \leq p \leq 2$. Finally, we treat the convolution product and show that the Fourier transform turns convolution into multiplication.

In this section, let (M,Δ) be a locally compact quantum group and let $(\hat{M},\hat{\Delta})$ be its dual. We apply the results of Section 2 to M and $\hat{M}$. We add a hat in the notation to indicate that an object is defined with respect to the dual quantum group. So for $z \in \mathbb{C}$, $L^p_{(z)}(\hat{M},\hat{\varphi}), \hat{L}_{(z)}, \ldots$ are associated with $\hat{M}$. Recall that we denote $(\mathcal{H},\pi,\Lambda)$ for the GNS-construction with respect to the left Haar weight φ. Set $S = S_\varphi$, $\nabla = \nabla_\varphi$ and $J = J_\varphi$.

3.1. **The L^2-Fourier transform.** Here we define a Fourier transform between the L^2-space of a l.c. quantum group (M,Δ) and the L^2-space of its dual. The definition coincides with the algebraic definition suggested in [25].

Let (M,Δ) be a l.c. quantum group and let $(\hat{M},\hat{\Delta})$ be its dual. Recall from Proposition 2.1 that for $x \in L_{(-1/2)}$ and $y \in \mathfrak{n}_\varphi$, $\varphi_x^{-1/2}(y^*) = \varphi(y^*x)$. Hence,

$$\forall x \in L_{(-1/2)}, \varphi_x^{(-1/2)} \in \mathcal{I} \text{ and } \xi(\varphi_x^{(-1/2)}) = \Lambda(x). \tag{3.1}$$

Since $\mathcal{T}_\varphi^2$ is contained in $L_{(-1/2)}$, $\{\xi(\varphi_x^{(-1/2)}) \mid x \in L_{(-1/2)}\}$ is dense in $\mathcal{H}$. Consider the map

$$L_{(-1/2)} \to \mathfrak{n}_{\hat{\varphi}} : x \mapsto \lambda(\varphi_x^{(-1/2)}) = (\varphi_x^{(-1/2)} \otimes \iota)(W). \tag{3.2}$$

Identifying $\mathfrak{n}_\varphi$ and $\mathfrak{n}_{\hat{\varphi}}$ as subspaces of the GNS-space via Λ and $\hat{\Lambda}$, (3.1) and (3.2) yield an isometric map $\Lambda(x) \mapsto \hat{\Lambda}(\lambda(\varphi_x^{(-1/2)}))$. Using the identifications made in Proposition 2.4, we define the following L^2-Fourier transform.

Definition 3.1. We call the unitary extension of (3.2) the L^2-Fourier transform and denote it by $\mathcal{F}_2 : L^2_{(-1/2)}(M,\varphi) \to L^2_{(-1/2)}(\hat{M},\hat{\varphi})$.

In fact, under the indentifications of $L^2_{(-1/2)}(M, \varphi)$ and $L^2_{(-1/2)}(\hat{M}, \hat{\varphi})$ with their GNS-space $\mathcal{H}$, $\mathcal{F}_2$ becomes the identity map. This is what one would expect from the classical case of an abelian l.c. group. In this case, the Fourier transform is implicitly used in the construction of the dual quantum group and therefore trivializes, see the remarks in the introduction or [24, p. 25].

Remark 3.2. This transform coincides with the definition given in [25, Definition 1.3]. See [22] for algebraic quantum groups and their relations to locally compact quantum groups.

Remark 3.3. Kahng [10] defines an operator algebraic Fourier transform. However, [10, Definition 3] has to be given a more careful interpretation, since, if φ is not a state, the expression $(\varphi \otimes \iota)(W(a \otimes 1)), a \in \hat{\lambda}(\hat{\mathcal{I}})$, is in general undefined. In case φ is a state, Definition 3.1 equals Kahng's by (3.2) and Remark 2.2.

Remark 3.4. Using the isomorphisms $U_{2,(-1/2,z)} : L^2_{(z)}(M, \varphi) \to L^2_{(-1/2)}(M, \varphi)$ defined by (1.2), it is possible to define a Fourier transform $\mathcal{F}_{2,(z,w)} : L^2_{(z)}(M, \varphi) \to L^2_{(w)}(\hat{M}, \hat{\varphi}), z, w \in \mathbb{C}$. In the remainder of this paper we will see that only for the parameters $z = -1/2 + it, t \in \mathbb{R}$, Figure 1 (a) turns (M, M_*) into a compatible couple in such a way that (3.2) determines a bounded L^1-Fourier transform. Anticipating to this observation already, we define $\mathcal{F}_2$ to be the Fourier transform with respect to the interpolation spaces for parameters $z, w = -1/2$.

The Pontrjagin duality theorem [15], [16] for l.c. quantum groups states $(\hat{\hat{M}}, \hat{\hat{\Delta}}) = (M, \Delta)$. This immediately implies the dual analogue. Define $\hat{L}_{(-1/2)}$ as the set of all $x \in \mathfrak{n}_{\hat{\varphi}}$ such that there exists a $\hat{\varphi}_x^{(-1/2)} \in \hat{M}_*$ satisfying the condition

$$\forall y \in \mathfrak{n}_{\hat{\varphi}} : \quad \hat{\varphi}_x^{(-1/2)}(y^*) = \langle \hat{\Lambda}(x), \hat{\Lambda}(y) \rangle, \tag{3.3}$$

see Proposition 2.1. $\hat{L}_{(-1/2)}$ is dense in $L^2_{(-1/2)}(\hat{M}, \hat{\varphi})$ and we have the identity

$$\forall x \in \hat{L}_{(-1/2)}, \hat{\varphi}_x^{(-1/2)} \in \hat{\mathcal{I}} \text{ and } \Lambda(\hat{\lambda}(\hat{\varphi}_x^{(-1/2)})) = \hat{\Lambda}(x), \tag{3.4}$$

so that the map

$$\hat{L}_{(-1/2)} \to \mathfrak{n}_{\varphi} : x \mapsto \hat{\lambda}(\hat{\varphi}_x^{(-1/2)}) = (\hat{\varphi}_x^{(-1/2)} \otimes \iota)(\hat{W}) = (\iota \otimes \hat{\varphi}_x^{(-1/2)})(W^*), \tag{3.5}$$

extends to a unitary transformation $\hat{\mathcal{F}}_2 : L^2_{(-1/2)}(\hat{M}, \hat{\varphi}) \to L^2_{(-1/2)}(M, \varphi)$. Using Proposition 2.4 to identify $L^2_{(-1/2)}(M, \varphi)$ and $L^2_{(-1/2)}(\hat{M}, \hat{\varphi})$ with $\mathcal{H}$, (3.1) and (3.4) yield that $\mathcal{F}_2$, respectively $\hat{\mathcal{F}}_2$ act as the identity on $\mathcal{H}$. This proves the following result.

Proposition 3.5. *The inverse of $\mathcal{F}_2$ is given by $\hat{\mathcal{F}}_2$.*

3.2. **The L^1-Fourier transform.** Here we show that there is a distinguished choice for $z \in \mathbb{C}$ such that first of all Figure 1 (a) turns (M, M_*) into a compatible couple of Banach spaces and secondly, there exists a bounded L^1-Fourier transform $\mathcal{F}_1 : L^1_{(z)}(M, \varphi) \to L^\infty_{(z)}(\hat{M}, \hat{\varphi})$ that is compatible with the L^2-Fourier transform.

Fix $z = s + it \in \mathbb{C}$. We will implicitly identify $L^1_{(z)}(M, \varphi)$ with M_* and $L^\infty_{(z)}(\hat{M}, \hat{\varphi})$ as a subspace of $\hat{M}$. Note that $\mu_1^{(z)}(L_{(z)})$ is dense in M_* by [9, Proposition 2.3]. Therefore, by (3.2) the L^2-Fourier transform determines a densely defined map

$$\Phi^{(z)} : \mu_1^{(z)}(L_{(z)}) \to \hat{M} : \varphi_x^{(z)} \mapsto (\varphi_x^{(-1/2)} \otimes \iota)(W).$$

We will show that for $SU_q(2)$, the map $\Phi^{(z)}$ is bounded if and only if $z = -1/2 + it$, $t \in \mathbb{R}$. Then we define the L^1-Fourier transform as the bounded extension of $\Phi^{(-1/2)}$. Furthermore, we show how this Fourier transform is related to the bounded extensions of $\Phi^{(-1/2+it)}$.

Example 3.6. Let $(M, \Delta) = SU_q(2)$ and let φ be its Haar state [26], [27]. Let $\alpha, \gamma \in M$ denote the usual operators satisfying the relations

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \alpha\alpha^* + q^2\gamma\gamma^* = 1,\ \gamma\gamma^* = \gamma^*\gamma,\ q\gamma\alpha = \alpha\gamma,\ q\gamma^*\alpha = \alpha\gamma^*.$$

For $a, b \in \mathcal{T}_\varphi, x \in \mathcal{T}_\varphi^2$, $z = s + it \in \mathbb{C}$,

$$\begin{aligned}\varphi_x^{(z)}(a^*b) =& \langle xJ\nabla^{s-it}\Lambda(a), J\nabla^{-s-it}\Lambda(b)\rangle = \langle \Lambda(b), \pi_r(J\nabla^{s-it}\Lambda(x))\Lambda(a)\rangle \\ =& \langle \Lambda(b), \pi_r(\nabla^{-s+it+\frac{1}{2}}\Lambda(x^*))\Lambda(a)\rangle = \varphi(\sigma^\varphi_{-i(z-\frac{1}{2})}(x)a^*b) = \varphi(a^*b\sigma^\varphi_{-i(z+\frac{1}{2})}(x)).\end{aligned}$$

Note that $1, \alpha \in \mathcal{T}_\varphi$, so that for $n \in \mathbb{N}$, $\alpha^n \in \mathcal{T}_\varphi^2 \subseteq L_{(z)}$. For $x \in M$,

$$\varphi_{\alpha^n}^{(z)}(x) = \varphi(x\sigma^\varphi_{-i(z+\frac{1}{2})}(\alpha^n)) = q^{-2n(z+\frac{1}{2})}\varphi(x\alpha^n).$$

The von Neumann algebra M is isomorphic to $B(l^2(\mathbb{N})) \otimes L^\infty(\mathbb{T}) \simeq L^\infty(\mathbb{T}, B(l^2(\mathbb{N})))$, where $\mathbb{T}$ denotes the cirkel. Let $e_k, k \in \mathbb{N}$ be the canonical basis of $l^2(\mathbb{N})$. Let $x \in L^\infty(\mathbb{T}, B(l^2(\mathbb{N})))$ and write $x = x(t), t \in \mathbb{T}$. Then,

$$\varphi_{\alpha^n}^{(-1/2)}(x) = \varphi(x\alpha^n) = \frac{(1-q^2)}{2\pi}\int_{\mathbb{T}}\sum_{k=n}^{\infty} q^{2k}\sqrt{(q^{2k-2n+2}; q^2)_n}\langle x(t)e_{k-n}, e_k\rangle dt,$$

where $(a, q^2)_n = (1-a)(1-aq^2)\ldots(1-aq^{2n-2})$. Hence, $\|\varphi_{\alpha^n}^{(-1/2)}\| = \varphi_{\alpha^n}^{(-1/2)}(S^n \otimes 1)$, where $S : l^2(\mathbb{N}) \to l^2(\mathbb{N}) : e_k \mapsto e_{k+1}$ is the forward shift. By means of the Cauchy-Schwarz inequality and the q-binomial formula [7], for $z = s + it \in \mathbb{C}$, we get the estimate

(3.6)

$$\begin{aligned}q^{2n(s+\frac{1}{2})}\|\varphi_{\alpha^n}^{(z)}\| = \|\varphi_{\alpha^n}^{(-1/2)}\| = \varphi_{\alpha^n}^{(-1/2)}(S^n \otimes 1) = (1-q^2)\sum_{k=n}^{\infty} q^{2k}\sqrt{(q^{2k-2n+2}; q^2)_n} \\ \leq (1-q^2)\left(\sum_{k=n}^{\infty} q^{2k}\right)^{1/2}\left(\sum_{k=n}^{\infty} q^{2k}(q^{2k-2n+2}; q^2)_n\right)^{1/2} = (1-q^2)\left(\frac{q^{2n}}{1-q^2}\right)^{1/2}\left(\frac{q^{2n}}{1-q^{2n+2}}\right)^{1/2}.\end{aligned}$$

By a similar calculation involving the q-binomial formula,

$$\begin{aligned}\|\Phi^{(z)}(\varphi_{\alpha^n}^{(z)})\| =& \|((\varphi_{\alpha^n}^{(-1/2)}) \otimes \iota)(W)\| = \varphi((\alpha^n)^*\alpha^n) \\ =& (1-q^2)\sum_{k=n}^{\infty} q^{2k}(q^{2k-2n+2}; q^2)_n = \frac{q^{2n}(1-q^2)}{1-q^{2n+2}},\end{aligned} \tag{3.7}$$

where the second equation follows by the Peter-Weyl decomposition of W and the fact that $(\alpha^n)^*$ appears as the matrix coefficient $t^{(n/2)}_{-n/2,-n/2}$, see [11, Chapter 4] for details and the definition of the corepresentation matrix $t^{(l)}_{i,j}, l \in \frac{1}{2}\mathbb{N}, i,j \in \{-l, -l+1, \ldots, l\}$. From (3.6) and (3.7) we see that for all $n \in \mathbb{N}$,

$$\sup\left\{ \|\Phi^{(z)}(\omega)\| \mid \omega \in M_*, \|\omega\| = 1 \right\} \geq q^{2n(s+\frac{1}{2})} \left(\frac{1-q^2}{1-q^{2n+2}} \right)^{\frac{1}{2}},$$

so that $\Phi^{(z)}$ is unbounded in case $s < -\frac{1}{2}$. Similarly,

$$q^{-4n(s+\frac{1}{2})} \|\varphi^{(z)}_{(\alpha^n)^*}\|^2 \leq \|\Lambda((\alpha^n)^*)\|^2 = (1-q^2)\sum_{k=0}^{\infty} q^{2k}(q^{2k+2};q^2)_n = \frac{(1-q^2)}{1-q^{2n+2}}, \tag{3.8}$$

and

$$\|\Phi^{(z)}(\varphi^{(z)}_{(\alpha^n)^*})\| = \|((\varphi^{(-1/2)}_{(\alpha^n)^*}) \otimes \iota)(W)\| = \varphi(\alpha^n(\alpha^n)^*) = (1-q^2)\sum_{k=n}^{\infty} q^{2k}(q^{2k+2};q^2)_n = \frac{(1-q^2)}{1-q^{2n+2}}. \tag{3.9}$$

It follows from (3.8) and (3.9) that $\Phi^{(z)}$ is unbounded in case $s > -\frac{1}{2}$.

Remark 3.7. The fact that $\Phi^{(z)}$ is unbounded in case the real part of z is not equal to $-\frac{1}{2}$ is a consequence of the fact the left Haar weight is not a trace, so that σ^{φ}_t is non-trivial.

The discussion so far shows that in order to define a proper L^1-Fourier transform, the complex parameter z in Figure 1 (a) must be of the form $z = -\frac{1}{2} + it$, $t \in \mathbb{R}$. Here we choose $t = 0$.

Definition 3.8. For a l.c. quantum group (M, Δ), we define the L^1-Fourier transform as

$$\mathcal{F}_1 : M_* \to \hat{M} : \omega \to \lambda(\omega) = (\omega \otimes \iota)(W).$$

The map is a norm-decreasing map between Banach spaces. Similarly, we set

$$\hat{\mathcal{F}}_1 : \hat{M}_* \to M : \omega \mapsto \hat{\lambda}(\omega) = (\omega \otimes \iota)(\hat{W}) = (\iota \otimes \omega)(W^*).$$

We comment on the dependence of $\Phi^{(z)}$ on z. Recall from [9, Proposition 2.6] that for $z = -\frac{1}{2} + it$, $t, \in \mathbb{R}$, $L_{(z)}$ is invariant under σ^{φ}_r for $r \in \mathbb{R}$. Moreover, $L_{(z)} = L_{(-1/2)}$ and for $x \in L_{(z)}$,

$$\varphi^{(z)}_x = \varphi^{(-1/2)}_{\sigma^{\varphi}_t(x)} = \varphi^{(-1/2)}_x \circ \sigma^{\varphi}_{-t}.$$

Hence, the map $\Phi^{(z)} : M_* \to \hat{M}$ is bounded, we denote its bounded extension by $\mathcal{F}^{(z)}_1$ and for $\omega \in M_*$, $\mathcal{F}^{(z)}_1(\omega) = \mathcal{F}^{(-1/2)}_1(\omega \circ \sigma^{\varphi}_t) = \mathcal{F}_1(\omega \circ \sigma^{\varphi}_t)$.

Remark 3.9. The Fourier transform gives a canonical choice for the parameter $z \in \mathbb{C}$ that turns (M, M_*) into a compatible pair of Banach spaces as in Figure 1 (a). In [4], [3] and [6], the choice of this parameter does not seem distinguished. In particular, in the final remark of [4] it is questioned what would be the right choice. In [3] the choice $z = 0$ is considered, where [6] considers $z = -1/2$. The Fourier transform seems to give a natural answer to the final remarks of [4]. However, we did not try to put the results of [4] in a general L^p-setting.

3.3. **The L^p-Fourier transform,** $p \in [1,2]$**.** In this section we apply the complex interpolation method to obtain a L^p-Fourier transform in case $1 \leq p \leq 2$.

Theorem 3.10. *Every interpolation space is understood with respect to Figure 1 (a) for the parameter $z = -1/2$. Let $p \in [1,2]$ and set q by $\frac{1}{p} + \frac{1}{q} = 1$. There exists a unique bounded linear map $\mathcal{F}_p : L^p(M,\varphi) \to L^q(\hat{M},\hat{\varphi})$ such that $\hat{\nu}_q^{(-1/2)} \circ \mathcal{F}_p \circ \mu_p^{(-1/2)} = \hat{\nu}_\infty^{(-1/2)} \circ \mathcal{F}_1 \circ \mu_1^{(-1/2)} = \hat{\nu}_2^{(-1/2)} \circ \mathcal{F}_2 \circ \mu_2^{(-1/2)}$. Moreover, $\|\mathcal{F}_p\| \leq 1$.*

Proof. For $x \in L_{(-1/2)}$, $y \in \hat{L}_{(1/2)}$,

$$\begin{aligned}&\langle \hat{\nu}_\infty^{(-1/2)}(\mathcal{F}_1(\mu_1^{(-1/2)}(x))), y\rangle_{\hat{L}^*_{(1/2)}, \hat{L}_{(1/2)}} = \langle \mathcal{F}_1(\mu_1^{(-1/2)}(x)), \hat{\mu}_1^{(1/2)}(y)\rangle_{M,M_*}\\ =&\hat{\varphi}_y^{(1/2)}\left((\varphi_x^{(-1/2)} \otimes \iota)(W)\right) = \langle \hat{\Lambda}((\varphi_x^{(-1/2)} \otimes \iota)(W)), \hat{\Lambda}(y^*)\rangle = \langle \Lambda(x), \hat{\Lambda}(y^*)\rangle\\ =&\langle \hat{\nu}_2^{(-1/2)}(\mathcal{F}_2(\mu_2^{(-1/2)}(x))), y\rangle_{\hat{L}^*_{(1/2)}, \hat{L}_{(1/2)}},\end{aligned}$$

where the third equality is due to Proposition 2.1 and the last equality is due to Proposition 2.5. Hence, the theorem is true for $p = 1$ and $p = 2$. For arbitrary $p \in [1,2]$, the theorem follows from the fact that the complex interpolation functor is an exact interpolation functor [1] (or Theorem 1.1) and the fact that by Theorem 2.11, $(L^2(M,\varphi), M_*)_{[\frac{2}{p}-1]} = L^p(M,\varphi)$ and $(L^2(M,\varphi), M)_{[1-\frac{2}{q}]} = L^q(M,\varphi)$. □

Similarly, there is a dual Fourier transform, which we denote by $\hat{\mathcal{F}}_p$. In the following example we relate the present theory to the theory of L^p-spaces asssociated to semi-finite von Neumann algebras. We show that in this case the Fourier transform can be given by an explicit formula. The example is set up in the dual quantum group setting, in order to make it easier to relate it to special cases that have already been studied, see Remarks 3.12 and 3.14.

Example 3.11. We recall the definition of L^p-spaces associated with a semi-finite von Neumann algebra M with normal, faithful, semi-finite trace tr, see [20]. First, we extend tr to all positive, self-adjoint operators x affiliated with M, by setting $\mathrm{tr}(x) = \sup_{n\in\mathbb{N}} \mathrm{tr}\left(\int_0^n \lambda dE_\lambda\right)$, where $x = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition. For $p \in [1,\infty)$, set

$$L^p(M) = \{a \,\eta\, M \mid a \text{ is closed, densely defined}, \mathrm{tr}(|a|^p) < \infty\}, \tag{3.10}$$

where η means affiliated.

Let tr' be the trace on M' such that $d\mathrm{tr}/d\mathrm{tr}' = 1$, see [19] for the definition of the definition of the spatial derivative. In [20, p. 95-96] it is explained that we have an equality of Banach spaces $L^p(\mathrm{tr}') = L^p(M)$. Let φ be a normal, faithful, semi-finite weight on M. We define an isometric isomorphism $\pi_p : L^p_{(-1/2)}(M,\varphi) \simeq L^p(M)$ by $\pi_p = \nu_p^{-1} \circ U_{p,(0,-1/2)}$. Let d be the spatial derivative $d\varphi/d\mathrm{tr}'$. For $a, b \in \mathcal{T}_\varphi$,

$$\begin{aligned}&\pi_p(\nu_\infty^{(-1/2)}(ab)) = \nu_p^{-1}(U_{p,(0,-1/2)}(\nu_\infty^{(-1/2)}(ab))) = \nu_p^{-1}(\nu_\infty^{(0)}(\sigma^\varphi_{i/2p}(ab)))\\ =&\mu_p(\sigma^\varphi_{i/2p}(ab)) = d^{1/2p}\sigma^\varphi_{i/2p}(a) \cdot [\sigma^\varphi_{i/2p}(b)d^{1/2p}],\end{aligned} \tag{3.11}$$

where the equalities follow respectively from the definition of π_p, (1.2), the commutativity of Figure 1 (b) and the explicit formula for μ_p given in (50) of [21]. Note that by Notation 1.2, indeed $\nu_\infty^{(-1/2)}(ab) \in L^p_{(-1/2)}(M,\varphi)$. Using [21, Lemma 22], we find

$$\left(abd^{1/p}\right)^* \supseteq \left(d^{1/2p}\sigma^\varphi_{i/2p}(a)\cdot[\sigma^\varphi_{i/2p}(b)d^{1/2p}]\right)^* \in L^p(M). \tag{3.12}$$

It follows from [20, Part I, Corollary 15; Part IV, Corollaries 6 and 7] that (3.12) is actually an equality. (This can also be seen by following the proof of [8, Theorem 4 (1)], which contains a similar argument). Then, using Notation 1.2, (3.11) gives,

$$\pi_p(\mu_p^{(-1/2)}(ab)) = \pi_p(\nu_\infty^{(-1/2)}(ab)) = [abd^{1/p}]. \tag{3.13}$$

Now, let (M,Δ) be a l.c. quantum group such that $\hat{M}$ is semi-finite. Let $p \in [1,2]$ and define $q \in [2,\infty]$ by $1/p + 1/q = 1$. Fix a trace tr on $\hat{M}$. We identify $L^p(\hat{M})$ with $L^p_{(-1/2)}(\hat{M},\hat{\varphi})$ via $\hat{\pi}_p$, i.e. the dual version of (3.13). Then, the L^p-Fourier transform becomes a map $\hat{\mathcal{F}}_p \circ \hat{\pi}_p^{-1} : L^p(\hat{M}) \to L^q_{(-1/2)}(M,\varphi)$. We describe this transform explicitly. Let D^{-2} be the Radon-Nikodym derivative of $\hat{\varphi}$ with respect to tr, so using the notation of [23] or [19, Section VIII.2], $\hat{\varphi} = \mathrm{tr}_{D^{-2}}$. Let tr$'$ be the trace on $\hat{M}'$ such that $d\mathrm{tr}/d\mathrm{tr}' = 1$. By [19, Theorem IX.3.8 (iii)], D^{-2} is the spatial derivative $d\varphi/d\mathrm{tr}'$. Let $a, b \in \mathcal{T}_{\hat{\varphi}}$. Put $y = [abD^{-2/p}] = \hat{\pi}_p(\hat{\mu}_p^{(-1/2)}(ab)) \in L^p(\hat{M})$. We find the following explicit formula for the Fourier transform:

$$\left(\nu_\infty^{(-1/2)}\right)^{-1}\left(\nu_q^{(-1/2)}\left(\left(\hat{\mathcal{F}}_p\circ\hat{\pi}_p^{-1}\right)(y)\right)\right) = \hat{\mathcal{F}}_1\left(\hat{\mu}_1^{(-1/2)}(ab)\right) = (\iota\otimes\varphi^{(-1/2)}_{[yD^{2/p}]})(W^*). \tag{3.14}$$

In particular, the formula of the Fourier transform depends on p. This is a consequence of the fact that the construction captured in Figure 1 (a) defines the intersection of $L^1(\hat{M})$ and $L^\infty(\hat{M})$ to be $\hat{L}_{(-1/2)}$, see (1.1). This is generally different from the naieve intersection $\hat{M} \cap L^1(\hat{M}) = \mathfrak{m}_{\mathrm{tr}}$. This intersection is deformed by means of the operator D^{-2}.

Remark 3.12. In the classical case of an abelian l.c. group $D^{-2} = 1$, since in this case the Plancherel weight is tracial [19, Section VII.3].

As a special case of Example 3.11 we study the case in which (M,Δ) satisfies the assumptions of the quantum Plancherel theorem [5].

Example 3.13. Let (M,Δ) be a l.c. quantum group such that $\hat{M}$ is type I and such that the universal dual C*-algebra $\hat{M}_u$ (see [14]) is separable. By the quantum Plancherel theorem [5], we have decompositions

$$\hat{M} \simeq \int^\oplus_{\mathrm{IC}(M)} B(\mathcal{H}_U)d\mu(U), \qquad W = \int^\oplus_{\mathrm{IC}(M)} \dim(U)\cdot U d\mu(U),$$

where IC(M) is the set of equivalence classes of irreducible, unitary corepresentations of M, $\mathcal{H}_U$ is the corepresentation space of $U \in \mathrm{IC}(M)$ and μ is the Plancherel measure, which is a standard measure, c.f. (proof of) [5, Theorem 3.4.1]. Let $\mathrm{tr}(x) = \int_{\mathrm{IC}(M)} \mathrm{tr}_{B(\mathcal{H}_U)}(x_U)d\mu(U)$, $x = (x_U)_{U\in\mathrm{IC}(M)} \in \hat{M}$, be the canonical trace on $\hat{M}$. Suppose that

(1) $\xi = (\xi_U)_{U\in\mathrm{IC}(M)}$ and $\eta = (\eta_U)_{U\in\mathrm{IC}(M)}$ are μ-measurable, essentially bounded fields of vectors with compact support;

(2) there exists a $n \in \mathbb{N}$, such that $\xi, \eta \in \left(\int_0^\infty \chi_{[0,n]}(\lambda) dE_\lambda\right)\mathcal{H}$. Here, $D^{-2} = \int_0^\infty \lambda dE_\lambda$ is the spectral decomposition and $\chi_{[0,n]}$ is the characteristic function of $[0,n]$.

Write θ_{ξ_U,η_U} for the Hilbert-Schmidt operator given by $\theta_{\xi_U,\eta_U} v = \langle v, \eta_U\rangle \xi_U$, $v \in \mathcal{H}_U$. Let $y = \int_{\mathrm{IC}(M)}^\oplus \theta_{\xi_U,\eta_U} d\mu(U)$ Then,

(1) $y \in L^p(\hat{M})$;
(2) $y \in \mathcal{T}_{\hat{\varphi}}$, since $\sigma_t^{\hat{\varphi}}$ is implemented by D^{-2}. Moreover, let $A = \mathrm{supp}(\xi) \cap \mathrm{supp}(\eta)$, where supp means support. Since $\int_{\mathrm{IC}(M)}^\oplus \chi_A(U) d\mu(U) \in \mathcal{T}_{\hat{\varphi}}$, also $y \in \mathcal{T}_{\hat{\varphi}}^2$;
(3) $\int_{\mathrm{IC}(M)}^\oplus \theta_{\xi_U, D_U^{2/p-2}\eta_U} d\mu(U) \in \mathfrak{m}_{\mathrm{tr}}$.

From [19, Lemma VIII.2.8] and Proposition 2.1 it follows that $\varphi^{(-1/2)}_{[yD^{2/p}]}(x) = \mathrm{tr}(x[yD^{2/p-2}]), x \in \hat{M}$. Hence, (3.14) yields,

$$
\begin{aligned}
&\left(\nu_\infty^{(-1/2)}\right)^{-1}\left(\nu_q^{(-1/2)}\left(\hat{\mathcal{F}}_p\left(\int_{\mathrm{IC}(M)}^\oplus \theta_{\xi_U,\eta_U} d\mu(U)\right)\right)\right) \\
&= (\iota \otimes \varphi^{(-1/2)}_{[yD^{2/p}]})(W^*) = \int_{\mathrm{IC}(M)} (\iota \otimes \omega_{\xi_U, D_U^{2/p-2}\eta_U})(U^*) d\mu(U),
\end{aligned} \tag{3.15}
$$

where the integral on the right hand side is with respect to the σ-weak topology.

Remark 3.14. Note that the special case $p = 2$ is already considered in [2, Lemma 3.3], see in particular the remark after [2, Remark 3.4]. Furthermore, note that again the dependence of the parameter p is apparent.

3.4. **Convolution product.** First, we define the product of elements of the L^∞-space and the L^p-space associated with M. The product extends the product within M.

Definition 3.15. Let $x \in M$. The map $m_x^\infty : M \to M : y \mapsto xy$ determines a bounded map $m_x^1 : M_* \to M_*$ such that $\nu_\infty^{(-1/2)} \circ m_x^\infty \circ \mu_\infty^{(-1/2)} = \nu_1^{(-1/2)} \circ m_x^1 \circ \mu_1^{(-1/2)}$. Explicitly, m_x^1 is the map $m_x^1(\omega) = x \cdot \omega, \omega \in M_*$, where $(x \cdot \omega)(z) = \omega(zx), z \in M$. By Theorem 1.1, for $p \in (1, \infty)$, there is a unique bounded map $m_x^p : L^p_{(-1/2)}(M, \varphi) \to L^p_{(-1/2)}(M, \varphi)$ such that $\nu_\infty^{(-1/2)} \circ m_x^\infty \circ \mu_\infty^{(-1/2)} = \nu_p^{(-1/2)} \circ m_x^p \circ \mu_p^{(-1/2)}$. Let $x \in M$ and let $y \in L^p_{(-1/2)}(M, \varphi)$. We will write xy for $m_x^p(y)$.

For $\omega_1, \omega_2 \in M_*$, we define the convolution product $\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta$. This product is well-known in the theory of l.c. quantum groups. We show that it is possible to extend it to the L^p-setting for $p \in [1, 2]$. Moreover, the convolution product is turned into the product of Definition 3.15 by the Fourier transfrom. Recall the map ρ_p from Definition 2.12, see also Remark 2.13.

Theorem 3.16. (1) *Let $x \in L_{(-1/2)}$ and let $\omega \in M_*$. Then $\omega * \varphi_x^{(-1/2)} \in \mathcal{I}$ and $\xi(\omega * \varphi_x^{(-1/2)}) = (\omega \otimes \iota)(W)\Lambda(x)$. In particular,*

$$
\|\rho_2(\omega * \varphi_x^{(-1/2)})\|_{L^2_{(-1/2)}(M,\varphi)} = \|\xi(\omega * \varphi_x^{(-1/2)})\| \leq \|\omega\|\|\Lambda(x)\| = \|\omega\|\|\mu_2^{(-1/2)}(x)\|_{L^2_{(-1/2)}(M,\varphi)}. \tag{3.16}
$$

(2) *Let* $p \in [1,2]$ *and* $\omega \in M_*$. *There is a bounded operator* $\omega *^p : L^p_{(-1/2)}(M,\varphi) \to L^p_{(-1/2)}(M,\varphi)$ *which is uniquely determined by* $\omega *^p \mu_p^{(-1/2)}(x) = \rho_p\left(\omega * \varphi_x^{(-1/2)}\right)$, *where* $x \in L_{(-1/2)}$. *Moreover,* $\|\omega *^p\| \leq \|\omega\|$.

(3) *For* $\omega \in M_*$, $a \in L^p_{(-1/2)}(M,\varphi)$, $\mathcal{F}_1(\omega)\mathcal{F}_p(a) = \mathcal{F}_p(\omega *^p a)$.

Proof. Let $\theta \in \hat{\mathcal{I}}$ and put $y = (\iota \otimes \theta)(W^*)$. Now (1) follows from,

$$\begin{aligned}&\overline{(\omega * \varphi_x^{(-1/2)})(y^*) = (\omega \otimes \varphi_x^{(-1/2)})\Delta((\iota \otimes \theta)(W^*)^*)} = (\omega \otimes \varphi_x^{(-1/2)} \otimes \overline{\theta})(W_{13}W_{23})\\ =&\theta\left(\left((\omega \otimes \iota)(W)(\varphi_x^{(-1/2)} \otimes \iota)(W) \right)^*\right) = \langle \hat{\Lambda}\left((\omega \otimes \iota)(W)(\varphi_x^{(-1/2)} \otimes \iota)(W)\right), \hat{\xi}(\theta)\rangle\\ =&\langle(\omega \otimes \iota)(W)\Lambda(x), \Lambda(y)\rangle,\end{aligned}$$

and the fact that $\{\Lambda((\iota \otimes \theta)(W^*)) \mid \theta \in \hat{\mathcal{I}}\}$ is dense in $\mathcal{H}$. To prove (2), note that for $\omega_1, \omega_2 \in M_*$, $\|\omega_1 * \omega_2\| \leq \|\omega_1\|\|\omega_2\|$. Hence (2) follows for $p = 1$ and also for $p = 2$ by (3.16). The morphisms are compatible with respect to the compatible couple Figure 1 (a). Hence the fact that the complex interpolation functor is exact, see Theorem 1.1, together with Theorem 2.11 yields (2). For $\omega_1, \omega_2 \in M_*$, note that

$$\mathcal{F}_1(\omega_1 * \omega_2) = (\omega_1 \otimes \omega_2 \otimes \iota)(\Delta \otimes \iota)(W) = (\omega_1 \otimes \omega_2 \otimes \iota)W_{13}W_{23} = (\omega_1 \otimes \iota)(W)(\omega_2 \otimes \iota)(W).$$

For $x \in L_{(-1/2)}, \omega \in M_*$,

$$\begin{aligned}&\hat{\nu}_q^{(-1/2)}(\mathcal{F}_p(\omega *^p \mu_p^{(-1/2)}(x))) = \hat{\nu}_\infty^{(-1/2)}(\mathcal{F}_1(\omega * \varphi_x^{(-1/2)}))\\ =&\hat{\nu}_\infty^{(-1/2)}(\mathcal{F}_1(\omega)\mathcal{F}_1(\varphi_x^{(-1/2)})) = \hat{\nu}_q^{(-1/2)}(\mathcal{F}_1(\omega)\mathcal{F}_p(\mu_p^{(-1/2)}(x))).\end{aligned}$$

Since the range of $\mu_p^{(-1/2)}$ is dense in $L^p_{(-1/2)}(M,\varphi)$ and $\hat{\nu}_q^{(-1/2)}$ is injective, (3) follows. □

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