The nilpotence degree of torsion elements in lambda–rings

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April 7, 2010

1 Introduction.

If A is a λ-ring then for each prime number p one has a map $\theta^p\colon A \to A$ such that

- The map $\psi^p\colon A \to A$ defined by $\psi^p(a) = a^p - p\theta^p(a)$ is a ring homomorphism.
- The following formulas hold for all $a, b \in A$:

$$\begin{aligned}\theta^p(1) &= 0\\ \theta^p(a+b) &= \theta^p(a) + \theta^p(b) + \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^j b^{p-j}\\ \theta^p(ab) &= \theta^p(a)\psi^p(b) + a^p\theta^p(b)\\ \theta^p\psi^p(a) &= \psi^p\theta^p(a)\end{aligned}$$

Moreover the maps ψ^p and θ^r commute also for $r \neq p$. Conversely if a commutative ring A is equipped with maps ψ^p and θ^p satisfying the above identities then A has a unique structure of λ-ring such that the ψ^p are the associated Adams operations. We refer to [1] for generalities about λ-rings.

In [2] it is shown that a torsion element in a λ-ring is nilpotent. The purpose of this note is to give a sharp estimate of the nilpotence degree.

Let us call a ring A equipped with one operation θ^p satisfying the above conditions a θ^p-ring. We will prove that if A is a θ^p-ring and $a \in A$ satisfies $p^e a = 0$ then $a^{p^e+p^{e-1}} = 0$. In order to show that this estimate is sharp we exhibit a λ-ring A and an element $a \in A$ such that $p^e a = 0$ but $a^{p^e+p^{e-1}-1} \neq 0$.

If a is an element of a λ-ring A and $na = 0$ for some $n \in \mathbf{Z}$ then by the above theorem we have

$$(np^{-e})^{p^e+p^{e-1}} a^{p^e+p^{e-1}} = 0$$

where p^e is the highest power of p dividing n. Since the greatest common divisor of the numbers $(np^{-e})^{p^e+p^{e-1}}$ is 1 it follows that

$$a^E = 0,\ \text{where } E = \max\{p^e + p^{e-1}\ ;\ p^e \text{ divides } n\}$$

2 The nilpotency estimate.

In this section A is a λ-ring at p, and $a \in A$ satisfies $p^e a = 0$ for some $e > 0$. We will write ψ^{p^e} for the e-fold iterate of ψ^p.

Proposition 1. *For any $b \in A$ one has $\theta^p(pb) = p^{p-1}b^p - \psi^p(b)$.*

Proof. By definition of λ-ring at p one has $\theta^p(pb) = \theta^p(p)b^p + \psi^p(p)\theta^p(b)$. Now substitute $\psi^p(p) = p$, $\theta^p(p) = p^{p-1} - 1$ and $p\theta^p(b) = b^p - \psi^p(b)$. □

Proposition 2. *For $0 \le k \le e$ one has $p^{e-k}\psi^{p^k}(a) = 0$.*

Proof. By induction on k. For $k = 0$ the conclusion $p^e a = 0$ is given. Now assume the statement is true for $k = m$. Then by the last proposition one has

$$\begin{aligned} 0 = \theta^p(0) &= \theta^p(p \cdot p^{e-m-1}\psi^{p^m}(a)) \\ &= p^{p-1}(p^{e-m-1}\psi^{p^m}(a))^p - \psi^p(p^{e-m-1}\psi^{p^m}(a)) \\ &= -\psi^p(p^{e-m-1}\psi^{p^m}(a)) = p^{e-m-1}\psi^{p^{m+1}}(a) \end{aligned}$$

since $(p-1) + p(e-m-1) \ge e - m$ for $m < e$. □

Proposition 3. *For $0 \le k \le e-1$ one has $a^{p^e+p^{e-1}} = \psi^{p^k}(a^{p^{e-k}+p^{e-k-1}})$.*

Proof. By induction on k. For $k = 0$ both sides are identical. Assume that the statement is true for $k = m < e - 1$. Then one has

$$\begin{aligned} a^{p^e+p^{e-1}} &= (\psi^{p^m}(a)^p)^{p^{e-m-2}(p+1)} \\ &= (\psi^p\psi^{p^m}(a) + p\theta^p\psi^{p^m}(a))^{p^{e-m-2}(p+1)} \\ &= \sum_i \binom{p^{e-m-2}(p+1)}{i} (p\theta^p\psi^{p^m}(a))^i \psi^{p^{m+1}}(a)^{p^{e-m-2}(p+1)-i} \end{aligned}$$

If $0 < i < p^{e-m-2}(p+1)$ then the number of factors p in the corresponding term is

$$\begin{aligned} & i + v_p\binom{p^{e-m-2}(p+1)}{i} \ge i + v_p(p^{e-m-2}(p+1)) - v_p(i) \\ & \ge i + (e+m-2) - (i-1) = e + m - 1 \end{aligned}$$

and it also contains a factor $\psi^{p^{m+1}}(a)$. Hence by Proposition 2 it vanishes.

If $i = p^{e-m-2}(p+1)$ then the term is a multiple of $p^{e+1-m}\theta^p\psi^{p^m}(a)$, since $p^j(p+1) \ge j+3$ for all $j \ge 0$. But this expression equals

$$\begin{aligned} & p^{e+1-m}\psi^{p^m}\theta^p(a) = p^{e-m}\psi^{p^m}(a^p - \psi^p(a)) = \\ & p^{e-m}\psi^{p^m}(a)^p - \psi^p(p^{e-m}\psi^{p^m}(a)) = 0 - 0 = 0 \end{aligned}$$

by Proposition 2.

Thus only the term with $i = 0$ remains, which proves the statement for $k = m + 1$. □

Theorem 1. $a^{p^e+p^{e-1}} = 0$.

Proof. By the last Proposition we have

$$\begin{aligned} a^{p^e+p^{e-1}} &= \psi^{p^{e-1}}(a^{p+1}) = \psi^{p^{e-1}}(a)\psi^{p^{e-1}}(a)^p \\ &= \psi^{p^{e-1}}(a)(\psi^p\psi^{p^{e-1}}(a) + p\theta^p\psi^{p^{e-1}}(a)) \\ &= \psi^{p^{e-1}}(a)\psi^{p^e}(a) + p\psi^{p^{e-1}}(a)\theta^p\psi^{p^{e-1}}(a) \end{aligned}$$

which vanishes since $\psi^{p^e}(a) = 0$ and $p\psi^{p^{e-1}}(a) = 0$ by Proposition 2. □

3 The example ring.

We write K for the localization $\mathbf{Z}_{(p)}$ of the ring of ordinary integers at p, in which every prime $r \neq p$ is invertible. It caries a unique structure of λ-ring such that all Adams operations ψ^q coincide with the identity map.

We write R for the polynomial ring $K[x, y]$, and $\phi \colon R \to R$ for the homomorphism given by $\phi(x) = x^p - py$, $\phi(y) = y^p$.

Proposition 4. *For every $f \in R$ one has $\phi(f) = f^p \mod pR$.*

Proof. The statement is obviously true if $f \in K$ or $f = x$ or $f = y$. Moreover if it is true for f and g then it is also true for $f + g$ and fg. □

The Proposition tells us that we can define a θ^p -structure on R by putting $\psi^p(f) = \phi(f)$ and $\theta^p(f) = p^{-1}(f^p - \phi(f))$. One can extend this to a structure of λ-ring by declaring $\psi^r(x) = 0$ and $\psi^r(y) = 0$ for primes r different from p.

Definition 1. $F_n \in \mathbf{Z}[s, t]$ is defined recursively by:

$$F_0(s,t) = s, \qquad F_n(s,t) = F_{n-1}(s,t)^p - pF_{n-1}(t,0)$$

Lemma 1. *One has*

$$F_n(s,t) = F_{n-1}(s^p - pt, t^p)$$

Proof. By induction. For $n = 1$ both sides read $s^p - pt$. If the statement is true for n then $F_{n+1}(s,t) = F_n(s,t)^p - pF_n(t,0) = F_{n-1}(s^p - pt, t^p) - pF_{n-1}(t^p, 0) = F_n(s^p - pt, t^p)$. □

The point of this definition is that

$$\psi^p F_n(x,y) = F_n(\psi^p(x), \psi^p(y)) = F_n(x^p - py, y^p) = F_{n+1}(x,y)$$

Definition 2. The ideal J of R is the one generated by the $p^{e-n}F_n(x,y)$ for $0 \le n \le e$ and by y^{p^e}.

Proposition 5. *The ideal J is stable under θ^p.*

Proof. From the formulas for θ^p of a sum and θ^p of a product it follows that is sufficient to check that the generators of J are mapped to J. For $n < e$ we have

$$\begin{aligned}\theta^p(p^{e-n}F_n(x,y)) &= p^{-1}\left((p^{e-n}F_n(x,y))^p - \psi^p(p^{e-n}F_n(x,y))\right)\\ &= p^{(e-n)p-1}F_n(x,y) - p^{e-n-1}F_{n+1}(x,y)\end{aligned}$$

where the second term is obviously in J and the first term is in J since $(e-n)p-1 \geq e-n$. Furthermore

$$\begin{aligned}\theta^p(F_e(x,y)) &= p^{-1}\left(F_e(x,y)^p - F_e(x^p - py, y^p)\right)\\ &= p^{-1}\left(F_e(x,y)^p - F_{e+1}(x,y)\right)\\ &= p^{-1}(pF_e(y,0)) = f_e(y,0) = y^{p^e}\end{aligned}$$

and finally $\theta^p(y^{p^e}) = 0$ since $\psi^p(y) = y^p$. □

The Proposition tells us that the quotient ring $A = R/J$ inherits a structure of θ^p-ring, and in fact a structure of λ-ring. We will show now that the class a of x in R/J satisfies $a^{p^e+p^{e-1}-1} \neq 0$.

The main ingredient for proving this is the following elementary Proposition:

Proposition 6. *Let k be a commutative ring. Let $\rho\colon k[\xi,\eta] \to k[x,y]$ be the homomorphism given by $\rho(\xi) = x^p$, $\rho(\eta) = y^p$. Let M be an ideal of $k[\xi,\eta]$ with generators g_j, and let N be the ideal of $k[x,y]$ generated bu the $\rho(g_j)$. Then $k[x,y]/N$ is a free module over $k[\xi,\eta]/M$ via ρ with basis the classes of the $x^k y^m$ for $0 \leq k,m < p$.*

In particular if the class of ξ^n is nonzero in $k[\xi,\eta]/M$ then the class of x^{pn+p-1} is nonzero in $k[x,y]/N$.

Proof. If $b \in k[x,y]/N$ then it is the class $[f]$ of a certain $f \in k[x,y]$. Now $f \in k[x,y]$ can be written uniquely as $\sum_{0\leq k,m<p} x^k y^m \rho(f_{km})$ for certain $f_{km} \in k[\xi,\eta]$, and therefore $b = \sum_{k,m}[x^k y^m][\rho(f_{km})]$.

On the other hand if $\sum_{k,m}[x^k y^m][\rho(f_{km})] = 0$ then there must be $h_j \in k[x,y]$ such that $\sum_{k,m} x^k y^m \rho(f_{km}) = \sum h_j \rho(g_j)$. Moreover each h_j can be written as $\sum_{k,m} x^k y^m \rho(h_{jkm})$. Therefore

$$\sum_{k,m} x^k y^m \rho(f_{km}) = \sum_{k,m} x^k y^m \rho(\sum_j g_j h_{jkm})$$

By uniqueness this implies $f_{km} = \sum_j g_j h_{jkm}$. This means that $[f_{km}] = 0$. □

In order to apply this in an induction to prove the main theorem we need another property of the F_n:

Lemma 2. *If $n > 0$ then $F_n(s,t) = F_{n-1}(s^p,t^p) \mod p^n$.*

Proof. If $n = 1$ this reads $s^p - pt = s^p \mod p$. If the statement is true for n then

$$\begin{aligned} F_{n+1}(s,t) &= F_n(s,t)^p - pF_n(t,0) \\ &= (F_{n-1}(s^p,t^p) \mod p^n)^p - p(F_{n-1}(t^p,0) \mod p^n) \\ &= F_{n-1}(s^p,t^p)^p - pF_{n-1}(t^p,0) \mod p^{n+1} \\ &= F_n(s^p,t^p) \end{aligned}$$

Here we used the fact that $u = v \mod p^n$ implies $u^p = v^p \mod p^{n+1}$. □

In order to show that $a^{p^e+p^{e-1}-1}$ is nonzero in $A = K[x,y]/J$ we prove that is even nonzero in A/p^{e+1} (which is of course not a θ^p-ring):

Theorem 2. *The class of $x^{p^e+p^{e-1}-1}$ is nonzero in $\mathbf{Z}/p^{e+1}[x,y]$ modulo the ideal generated by the $p^{e-n}F_n(x,y)$ for $0 \leq n \leq e$ and by y^{p^e}*

Proof. We use induction in e. For $e = 1$ the statement says that $x^p \neq 0$ in $\mathbf{Z}/p^2[x,y]/(px, x^p - py, y^p)$. This is obvious by inspection.

If the statement is true for certain e then by Proposition 6 the class of $x^{p(p^e+p^{e-1}-1)+p-1} = x^{p^{e+1}+p^e-1}$ is nonzero in $\mathbf{Z}/p^{e+1}[x,y]$ modulo the ideal generated by the $p^{e-n}F_n(x^p,y^p)$ for $0 \leq n \leq e$ and by $y^{p^{e+1}}$. By Lemma 2 this ideal coincides with the ideal generated by the $p^{e-n}F_{n+1}(x,y)$ for $0 \leq n \leq e$ and by $y^{p^{e+1}}$.

A fortiori the class of $x^{p^{e+1}+p^e-1}$ is nonzero in $\mathbf{Z}/p^{e+2}[x,y]$ modulo the ideal generated by the same polynomials and $p^{e+1}F_0(x,y)$. But this is just the statement for $e+1$. □

References

[1] F.J.-B.J. Clauwens, The K-groups of λ–rings. Part I. Construction of the logarithmic invariant, Compos. Math. 61 (1987), 295–328.

[2] A.W.M. Dress, Induction and Structure Theorems for Orthogonal Representations of Finite Groups, Ann. of Math. 102 (1975), 291–325.