Transformations of one-dimensional Gibbs measures with infinite range interaction

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Abstract: We study single-site stochastic and deterministic transformations of one-dimensional Gibbs measures in the uniqueness regime with infinite-range interactions. We prove conservation of Gibbsianness and give quantitative estimates on the decay of the transformed potential. As examples, we consider exponentially decaying potentials, and potentials decaying as a power-law.

Key-words: Gibbs measures, potential, Koslov theorem, house-of-cards coupling, renormalization group transformation.

1 Introduction

Local transformations of Gibbs measures can be non-Gibbs. In [1], the mechanism behind the creation of non-Gibbsianness is explained as a hidden phase transition: conditioned on a certain configuration of the transformed spins, the original spins can exhibit a phase transition. Even if the untransformed system is not in a phase transition regime, by conditioning on the transformed configuration we can bring it into a regime of phase transition. In a regime of strong uniqueness, such as the Dobrushin uniqueness regime, or the complete analyticity regime, one expects that Gibbs measures turn into Gibbs measures under stochastic or deterministic disjoint-block transformations.

For one-dimensional systems in the uniqueness regime, one also expects that local transformations conserve the Gibbs property. Using disagreement percolation, this

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has been proved for finite-range potentials, [9]. The technique of disagreement percolation has however not been extended to the case of infinite range interactions, and in fact (at present) breaks down in that context. Further, it is also known that in the uniqueness regime in dimension one, decimating sufficiently many times brings the system into a regime where cluster expansion can be obtained, and hence the system becomes completely analytic [3]. Finally, in the context of dyamical systems, it has been shown recently [4] that a Gibbs measure with an exponentially decaying interaction transforms into a Gibbs measure with an interaction that decays at least as a stretched exponential under a transformation that "confuses" symbols (i.e., the transformed spin is determined by a partition of the untransformed spin).

In this paper we consider lattice spin systems in one dimension, with an interaction that is allowed to be of infinite range. We consider single-site stochastic and deterministic transformations. We prove that under a uniqueness condition (see 2.8 below), the transformed measure is Gibbs. We further prove that, if the initial interaction is exponentially decaying, then the transformed interaction decays exponentially as well. If the initial interaction decays (in some sense) as a power law with power α (which is chosen big enough to be in the uniqueness regime), then the tranformed interaction can be estimated with a (smaller) power as well.

The method of proof is based on two ingredients. One ingredient is classical: the single-site conditional probabilities of the transformed measure can be written as the expected value of a local function in a Gibbs measure that depends on the conditioning. The dependence on the conditioning, in the case of a single-site transformation is in the form of a spatially varying magnetic field. The second step is to control how the local function expectation depends on this magnetic field. This reduces to the problem of how well a local expectation is approximated by finite-volume Gibbs measure expectations (in a context which is not spatially homogeneous because of the presence of the magnetic field depending on the conditioning). In this second step we use coupling, in the spirit of [2]. As a consequence of this method, we obtain, besides Gibbsianness, estimates on the decay of the transformed potential (where we use the so-called Kozlov potential defined on lattice intervals).

Our paper is organized as follows: we start with basic definitions on Gibbs measures, potentials, and define the transformations that we consider. Section 2 is devoted to the case of stochastic single-site transformations. Section 3 contains the single-site deterministic case.

2 Gibbs measures and their transformations

2.1 One-dimensional Gibbs measures

We consider lattice spin systems, with configuration $\Omega = S^{\mathbb{Z}}$, where S, the single-site space, is a finite set. We equip Ω with the product topology. The set of all finite subsets of $\mathbb{Z}$ is denoted by $\mathcal{L}$. For $\Lambda \in \mathcal{L}$ and $\sigma \in \Omega$, we denote by σ_Λ the restriction of σ to Λ, while Ω_Λ denotes the set of all such restrictions.

A function $f : \Omega \to \mathbb{R}$ is called *local* if there exists a finite set $\Delta \subseteq \mathbb{Z}$ such that $f(\eta) = f(\sigma)$ for η and σ coinciding on Δ.

Continuity in the product topology coincides with quasi-locality, i.e., a function $f : \Omega \to \mathbb{R}$ is continuous if and only if it is a uniform limit of local functions, more precisely if

$$\lim_{\Lambda\uparrow\mathbb{Z}} \sup_{\xi,\zeta\in\Omega} |f(\omega_\Lambda\xi_{\Lambda^c}) - f(\omega_\Lambda\zeta_{\Lambda^c})| = 0, \tag{2.1}$$

Definition 2.1. *A function $\Phi : \mathcal{L} \times \Omega \to \mathbb{R}$ such that $\Phi(A, \sigma)$ depends only on $\sigma(x)$, $x \in A$ for $\forall A \in \mathcal{L}$, is called a* ***potential****. A potential is* ***uniformly absolutely convergent*** *if for all $x \in \mathbb{Z}$*

$$\sum_{A\ni x} \|\Phi(A, \sigma)\|_\infty < \infty, \tag{2.2}$$

where $\|\Phi(A, \sigma)\|_\infty = \sup_{\sigma\in\Omega} |\Phi(A, \sigma)|$.

For $\Phi \in \mathcal{B}$, $\zeta \in \Omega$, $\Lambda \in \mathcal{L}$, we define the finite-volume *Hamiltonian* with boundary condition ζ as

$$H_\Lambda^\zeta(\sigma) = \sum_{A\bigcap\Lambda\neq\varnothing} \Phi(A, \sigma_\Lambda\zeta_{\Lambda^c}). \tag{2.3}$$

Corresponding to this Hamiltonian we have the *finite-volume Gibbs measures* $\mu_\Lambda^{\Phi,\zeta}$, $\Lambda \in \mathcal{L}$, with boundary condition ζ, defined on Ω by

$$\int f(\xi)\mu_\Lambda^{\Phi,\zeta}(d\xi) = \sum_{\sigma_\Lambda\in\Omega_\Lambda} f(\sigma_\Lambda\zeta_{\Lambda^c}) \frac{\exp\left(-H_\Lambda^\zeta(\sigma)\right)}{Z_\Lambda^\zeta}, \tag{2.4}$$

where Z_Λ^ζ denotes the partition function normalizing $\mu_\Lambda^{\Phi,\zeta}$ to a probability measure and $f : \Omega \mapsto \mathbb{R}$ denotes any local function. For a probability measure μ on Ω, we denote by μ_Λ^ζ the condition probability distribution of $\sigma(x)$, $x \in \Lambda$, given $\sigma_{\Lambda^c} = \zeta_{\Lambda^c}$, which is of course only $\mu - a.s.$ defined.

Definition 2.2. *For $\Phi \in \mathcal{B}$, we call μ a* ***Gibbs measure*** *with potential Φ if a version of its conditional probabilities coincides with the ones prescribed in (2.4), i.e., if*

$$\mu_\Lambda^{\Phi,\zeta} = \mu_\Lambda^\zeta \quad \mu - a.s. \quad \forall\Lambda \in \mathcal{L}, \zeta \in \Omega. \tag{2.5}$$

We assume that the potential Φ satisfies the following condition.

$$\sup_{i\in\mathbb{Z}} \sum_{A\ni i, diam(A)\geq K} \|\Phi(A, \sigma)\|_\infty = f(K) \tag{2.6}$$

where f satisfies

$$\sum_{n=0}^{\infty} f(n) < \infty \tag{2.7}$$

Under the condition 2.7, the potential Φ admits only one Gibbs measure $\mu = \mu_\Phi$, see [5], section 8.3. Condition 2.7 of course implies

$$\lim_{k\to\infty} \sum_{j\geq 0} f(j+k) = 0. \tag{2.8}$$

We abbreviate

$$F_k = \sum_{j\geq 0} 2f(j+k) \tag{2.9}$$

Remark that in the case of a translation invariant potential, the supremum in (2.6) can be omitted and then

$$f(K) = \sum_{A\ni 0, diam(A)\geq K} \|\Phi(A,\sigma)\|_\infty$$

Definition 2.3. *A version of conditional probabilities $\{\mu(\cdot|\zeta_{\Lambda^c}) : \zeta_{\Lambda^c} \in \Omega_{\Lambda^c}, \Lambda \in \mathcal{L}\}$ is called **uniformly non null** if for every $\Lambda \in \mathcal{L}$, there exists a constant $m_\Lambda > 0$ such that for every $\omega \in \Omega$*

$$\mu^\omega_\Lambda(\omega) \geq m_\Lambda. \tag{2.10}$$

The following theorem due to Kozlov [7] and Sullivan [11] gives a criterion to decide whether a given measure is Gibbsian.

Theorem 2.4. *A probability measure μ on $(\Omega, \mathcal{F})$ is a Gibbs measure with respect to a uniformly absolutely convergent potential iff there exists a version of its conditional probabilities that is continuous and uniformly non null.*

Remark 1. *Theorem 2.4 is constructive, i.e., the potential is constructed from the conditional probabilities. See section 3.1 for the explicit form. In our one-dimensional case, it is non-vanishing on lattice intervals only, i.e., sets of the form $[i,j] = \{i, i+1, \ldots, j\}$. Therefore, if we start from a Gibbs measure we can assume without loss of generality that the potential is non-zero only on lattice intervals.*

2.2 Transformations of Gibbs measures

We consider two types of transformation: single-site stochastic tranformations and single-site deterministic transformations.

We first consider single site stochastic transformation, i.e., for a given σ, the distribution of the image spin configuration is a product measure on $(S')^{\mathbb{Z}}$

$$T(\xi|\sigma) = \prod_{i\in\mathbb{Z}} P_i(\xi_i|\sigma_i). \tag{2.12}$$

Here, S' denotes the alphabet of the image-spin, and satisfies $|S'| \leq |S|$.

We assume that the transition kernel of a single site is strictly positive. That is, for $i \in \mathbb{Z}$,

$$\inf_{i\in\mathbb{Z},\xi_i,\sigma_i} P_i(\xi_i|\sigma_i) > 0. \tag{2.13}$$

The distribution of the image spin is then defined as

$$\mu \circ T(d\xi) = \int T(d\xi|\sigma)\mu(d\sigma). \tag{2.14}$$

The second case is a single-site deterministic transformation $T : \Omega \to \Omega'$ induced by a map $\varphi : S \to S'$ given by

$$(T(\sigma))_i =: \sigma'_i = \varphi(\sigma_i) \tag{2.15}$$

3 Stochastic single-site transformations

Theorem 3.1. *For single site stochastic transformations, if the potential Φ corresponding to the initial Gibbs measure μ satisfies condition (2.6),* (2.7)*, then the transformed measure $\mu \circ T$ is a Gibbs measure.*

Proof. First of all, $\{\mu \circ T(\cdot|\zeta_{\Lambda^c}) : \zeta_{\Lambda^c} \in \Omega_{\Lambda^c}, \Lambda \in \mathcal{L}\}$ is uniformly non null thanks to the positivity assumption of a single site's transformation kernel in (2.13). We then proceed with the proof in two steps.

First, we express the one-site conditional probabilities $\mu \circ T(\xi_0|\xi_{\mathbb{Z}\setminus\{0\}})$ as averages of a local observable over a Gibbs measure depending on the conditioning ξ. This is in the spirit of [8], but simpler since the transformation is stochastic, and hence the "constrained first layer model" of [8] is "not constrained" (given the image configuration, all configurations are possible as originals).

Second, we use a "house-of-cards" coupling technique (see (3.7)) in the spirit of [2] to prove the dependence of this local expectation on the conditioning ξ. We restrict to the conditional expectation of the transformed spin at the origin, given the transformed spins outside the origin. The same argument applies to conditional expectation of the spin at any other site.

Step 1.

$$\begin{aligned}
\mu \circ T(\xi_0|\xi_{\mathbb{Z}\setminus\{0\}}) &= \lim_{\Lambda\uparrow\mathbb{Z}} \frac{\mu_\Lambda \circ T(\xi)}{\sum_{\widetilde{\xi}_0} \mu_\Lambda \circ T(\widetilde{\xi}_0\xi_{\Lambda\setminus\{0\}})} \\
&= \lim_{\Lambda\uparrow\mathbb{Z}} \frac{\sum_{\sigma_\Lambda} \mu_\Lambda \circ T(\xi|\sigma_\Lambda)\mu_\Lambda(\sigma_\Lambda)}{\sum_{\sigma_\Lambda}\sum_{\widetilde{\xi}_0} \mu_\Lambda \circ T(\widetilde{\xi}_0\xi_{\Lambda\setminus\{0\}}|\sigma_\Lambda)\mu_\Lambda(\sigma_\Lambda)} \\
&= \lim_{\Lambda\uparrow\mathbb{Z}} \frac{\sum_{\sigma_\Lambda}\prod_i P_i(\xi_i|\sigma_i)\mu_\Lambda(\sigma_\Lambda)}{\sum_{\sigma_\Lambda}\prod_{i\neq 0} P_i(\xi_i|\sigma_i)\mu_\Lambda(\sigma_\Lambda)} \\
&= \lim_{\Lambda\uparrow\mathbb{Z}} \frac{\sum_{\sigma_\Lambda}\prod_i P_i(\xi_i|\sigma_i)\mu_\Lambda(\sigma_\Lambda)}{\sum_{\sigma_\Lambda}\prod_i P_i(\xi_i|\sigma_i)\mu_\Lambda(\sigma_\Lambda)\frac{1}{P_0(\xi_0|\sigma_0)}} \\
&= \left(\mu^\xi\left(\frac{1}{P_0\left(\xi_0|\sigma_0\right)}\right)\right)^{-1}
\end{aligned} \tag{3.1}$$

where μ^ξ is a new Gibbs measure with potential

$$\Phi_A^\xi(\sigma) = \begin{cases} \Phi_A(\sigma) & \text{if } |A| > 1 \\ \Phi_A(\sigma) - \log P_i(\xi_i|\sigma_i) & \text{if } A = \{i\} \end{cases}. \tag{3.2}$$

and where the expectation

$$\left(\mu^\xi\left(\frac{1}{P_0\left(\xi_0|\sigma_0\right)}\right)\right)^{-1}$$

in (3.1) is w.r.t. σ_0, with fixed ξ. Remark that this Gibbs measure is uniquely defined, because it is a single-site modification of the original potential Φ, for which we have uniqueness by condition 2.8. The equalities in (3.1) are almost surely with respect to $\mu \circ T$. Therefore, it suffices to show that $\mu^\xi(P_0^{-1}(\xi_0|\sigma_0))$ is continuous as a function of ξ. Indeed, this then implies that $\mu \circ T(\xi_0|\xi_{\mathbb{Z}\setminus\{0\}})$ admits a version that is continuous as a function of ξ, which implies Gibbsianness, by Theorem 2.4. The problem boils down to proving (cf. (2.1))

$$\lim_{\Lambda\uparrow\mathbb{Z}} \left|\mu^{\xi_\Lambda\eta_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0)) - \mu^{\xi_\Lambda\eta'_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0))\right| = 0.$$

The form of the potential of $\mu^{\xi_\Lambda\eta_{\Lambda^c}}$, given in (3.2), implies that the Hamiltonian of the corresponding finite-volume Gibbs measure $\mu_{\Lambda,\zeta}^{\xi_\Lambda\eta_{\Lambda^c}}$ with boundary condition ζ has the following form

$$H_{\Lambda,\zeta}^{\xi_\Lambda\eta_{\Lambda^c}}(\sigma) = H_{\Lambda,\zeta}(\sigma) - \sum_{i\in\Lambda}\log P_i(\xi_i|\sigma_i).$$

Hence $\mu_{\Lambda,\zeta}^{\xi_\Lambda\eta_{\Lambda^c}}$ is independent of η and denoted as $\mu_{\Lambda,\zeta}^{\xi_\Lambda}$, which implies that

$$\left|\mu^{\xi_\Lambda\eta_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0)) - \mu^{\xi_\Lambda\eta'_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0))\right| \leq$$
$$\left|\mu^{\xi_\Lambda\eta_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0)) - \mu_{\Lambda,\zeta}^{\xi_\Lambda\eta_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0))\right| + \left|\mu_{\Lambda,\zeta}^{\xi_\Lambda\eta'_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0)) - \mu^{\xi_\Lambda\eta'_{\Lambda^c}}(P_0^{-1}(\xi_0|\sigma_0))\right|.$$

At this stage, it suffices to prove that, uniformly in ζ,

$$\mu^\xi(P_0^{-1}(\xi_0|\sigma_0)) = \lim_{\Lambda\uparrow\mathbb{Z}} \mu_{\Lambda,\zeta}^{\xi_\Lambda}(P_0^{-1}(\xi_0|\sigma_0)). \tag{3.3}$$

Step 2. It is sufficient for (3.3) if we can prove

$$\mu^{\widetilde{\Phi}}(\sigma_0) = \lim_{\Lambda\uparrow\mathbb{Z}} \mu_{\Lambda,\zeta}^{\widetilde{\Phi}}(\sigma_0),$$

where $\widetilde{\Phi}$ is a general potential satisfying condition (2.7). More precisely, we will prove that

$$\lim_{l\to\infty}\sup_{\zeta,\zeta'}\left|\mu_{[-l,l],\zeta}^{\widetilde{\Phi}}(\sigma_0) - \mu_{[-l,l],\zeta'}^{\widetilde{\Phi}}(\sigma_0)\right| = 0, \tag{3.4}$$

where $\mu_{[-l,l],\zeta}^{\widetilde{\Phi}}$ means the measure for configurations on $[-l,l]$ conditioned on the boundary $\zeta_{[-l,l]^c}$. For simplicity, we will omit the superscript $\widetilde{\Phi}$ hereafter. The speed of this convergence to zero (as a function of l) will determine the decay of the potential associated to the transformed measure (see later).

To prove (3.4), we couple the measures $\mu_{[-l,l],\zeta}(\sigma_{[-l,l]} = \cdot)$ and $\mu_{[-l,l],\zeta'}(\sigma_{[-l,l]} = \cdot)$, i.e., we construct a probability measure on pairs $(\sigma^1_{[-l,l]}, \sigma^2_{[-l,l]})$ with marginals $\mu_{[-l,l],\zeta}(\sigma_{[-l,l]} = \cdot)$ and $\mu_{[-l,l],\zeta'}(\sigma_{[-l,l]} = \cdot)$. The construction of the coupling follows an iterative procedure (inspired by [6], Section 7), where we generate in every stage a pair of two spins corresponding to the interior boundary spins at that stage. Initially, we generate $(\sigma^1_{-l}, \sigma^1_l)$ and $(\sigma^2_{-l}, \sigma^2_l)$ according to the maximal coupling[1] of $\mu_{[-l,l],\zeta}(\sigma_{-l} = \cdot, \sigma_l = \cdot)$ and $\mu_{[-l,l],\zeta'}(\sigma_{-l} = \cdot, \sigma_l = \cdot)$. Having generated $(\sigma^i_{-l}, \sigma^i_l)$,$(\sigma^i_{-l+1}, \sigma^i_{l-1})$ $\ldots$ $(\sigma^i_{-l+m}, \sigma^i_{l-m})$, for $i = 1, 2$, we generate $(\sigma^1_{-l+m+1}, \sigma^1_{l-m-1})$ and $(\sigma^2_{-l+m+1}, \sigma^2_{l-m-1})$ according to the maximal coupling of

$$\mu_{[-l+m+1,l-m-1],\zeta\sigma^1_{[-l+1,-l+m]\bigcup[l-1,l-m]}}(\sigma_{-l+m+1} = \cdot, \sigma_{l-m-1} = \cdot)$$

and

$$\mu_{[-l+m+1,l-m-1],\zeta'\sigma^2_{[-l+1,-l+m]\bigcup[l-1,l-m]}}(\sigma_{-l+m+1} = \cdot, \sigma_{l-m-1} = \cdot).$$

To estimate $|\mu_{[-l,l],\zeta}(\sigma_0) - \mu_{[-l,l],\zeta'}(\sigma_0)|$, we use the coupling just described, and proceed as in a "house-of-cards coupling" method of Bressaud-Fernández-Galves [2]. When we generate the symbols $\sigma_{-l+k}, \sigma_{l-k}$, we think of this as being at time instant k in the coupling. Suppose that for the last m time instants in the coupling, we had matches, then as in [2] we have to estimate the probability of a mismatch at time instant $m+1$. This is done in the next lemma.

Lemma 3.2. *For $-l < -n_2 < -n_1 \leq 0 \leq n_1 < n_2 < l$, $n_2 - n_1 = m$, let ξ and ζ be two configurations on the complement of $[-n_1, n_1]$ such that they agree on $\Delta_m = [-n_2, -n_1 - 1] \bigcup [n_1 + 1, n_2]$, then*

$$\sup_{\alpha,\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} |\mu_{[-n_1,n_1],\zeta}(\sigma_{-n_1} = \alpha, \sigma_{n_1} = \beta) - \mu_{[-n_1,n_1],\xi}(\sigma_{-n_1} = \alpha, \sigma_{n_1} = \beta)| \leq 2(e^{F_m} - 1),$$

where F_m is defined in (2.9).

Proof. Start with

$$\mu_{[-n_1,n_1],\zeta}(\sigma_{-n_1} = \alpha, \sigma_{n_1} = \beta) = \sum_{\sigma'} \frac{e^{-H^\zeta_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^\zeta_{[-n_1,n_1]}},$$

where we abbreviated $\alpha\sigma'\beta$ to be the configuration $\sigma_{[-n_1,n_1]}$ with $\sigma_{-n_1} = \alpha$, $\sigma_{n_1} = \beta$ and $\sigma_{[-n_1+1,n_1-1]} = \sigma'$, and where the sum runs over all configurations σ' on $[-n_1 +$

[1]For details of coupling and maximal coupling, we refer to [12].

$1, n_1-1]$. We then proceed as follows:

$$\begin{aligned}
&\sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left|\mu_{[-n_1,n_1],\zeta}(\sigma_{-n_1}=\alpha,\sigma_{n_1}=\beta)-\mu_{[-n_1,n_1],\xi}(\sigma_{-n_1}=\alpha,\sigma_{n_1}=\beta)\right| \\
&= \sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left|\sum_{\sigma'}\frac{e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\zeta}_{[-n_1,n_1]}}-\sum_{\sigma'}\frac{e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\xi}_{[-n_1,n_1]}}\right| \\
&\le \sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left\{\left|\frac{\sum_{\sigma'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\zeta}_{[-n_1,n_1]}}-\frac{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\zeta}_{[-n_1,n_1]}}\right|\right. \\
&\qquad\qquad \left.+\left|\frac{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\zeta}_{[-n_1,n_1]}}-\frac{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{Z^{\xi}_{[-n_1,n_1]}}\right|\right\} \\
&\le \sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left\{\left|\frac{\sum_{\sigma'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}-1\right|+\left|\frac{Z^{\xi}_{[-n_1,n_1]}}{Z^{\zeta}_{[-n_1,n_1]}}-1\right|\right\} \\
&= \sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left\{\left|\frac{\sum_{\sigma'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}-1\right|+\left|\frac{\sum_{\alpha'\sigma'\beta'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha'\sigma'\beta')}}{\sum_{\alpha'\sigma'\beta'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha'\sigma'\beta')}}-1\right|\right\},
\end{aligned}$$

where the sums in the second fraction run over all configuration $\alpha'\sigma'\beta'$ on $[-n_1,n_1]$. By using the elementary inequalities $\min_{i\in\{1,\dots,n\}}\frac{a_i}{b_i}\le\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i}\le\max_{i\in\{1,\dots,n\}}\frac{a_i}{b_i}$ and $|e^x-1|\le e^{|x|}-1$, we obtain

$$\begin{aligned}
&\sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left\{\left|\frac{\sum_{\sigma'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}{\sum_{\sigma'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)}}-1\right|+\left|\frac{\sum_{\alpha'\sigma'\beta'}e^{-H^{\zeta}_{[-n_1,n_1]}(\alpha'\sigma'\beta')}}{\sum_{\alpha'\sigma'\beta'}e^{-H^{\xi}_{[-n_1,n_1]}(\alpha'\sigma'\beta')}}-1\right|\right\} \\
&\le \sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left\{\left(e^{\sup_{\sigma'}\left|H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)\right|}-1\right)\right. \\
&\qquad\qquad \left.+\left(e^{\sup_{\alpha'\sigma'\beta'}\left|H^{\xi}_{[-n_1,n_1]}(\alpha'\sigma'\beta')-H^{\zeta}_{[-n_1,n_1]}(\alpha'\sigma'\beta')\right|}-1\right)\right\} \\
&= 2\left(e^{\sup_{\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m},\alpha\sigma'\beta}\left|H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)\right|}-1\right),
\end{aligned}$$

Now

$$\begin{aligned}
&\sup_{\alpha\sigma'\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \left|H^{\xi}_{[-n_1,n_1]}(\alpha\sigma'\beta)-H^{\zeta}_{[-n_1,n_1]}(\alpha\sigma'\beta)\right| \\
&\le \sum_{j=-n_1}^{n_1}\ \sum_{A\ni j,diam(A)\ge(n_2-j)\bigwedge(j-(-n_2))} 2\parallel\Phi(A,\sigma)\parallel_\infty \\
&\le \sum_{k=0}^{2n_1}\sup_{j\in\mathbb{Z}}\ \sum_{A\ni j,diam(A)\ge k+m} 2\parallel\Phi(A,\sigma)\parallel_\infty \\
&\le \sum_{k=0}^{\infty}2f(k+m)=F_m.
\end{aligned}$$

(Recall for the above inequalities that $m = n_2 - n_1$.) $\square$

As a consequence of the lemma, the probability of mismatch after m matches is dominated by

$$\gamma_m := 2(e^{F_m} - 1), \tag{3.5}$$

Then the probability that we are not coupled at time $k = l$ (i.e., the spins at the origin in the coupling are unequal) can be estimated by

$$\left|\mu_{[-l,l],\zeta}(\sigma_0) - \mu_{[-l,l],\zeta'}(\sigma_0)\right| = \left|\mathbb{E}_{\mathbb{P}_{12}}(\sigma_0^1 - \sigma_0^2)\right|$$

where $\mathbb{P}_{12}$ denotes the coupling of the measures $\mu_{[-l,l],\zeta}(\sigma_{[-l,l]} = \cdot)$ and $\mu_{[-l,l],\zeta'}(\sigma_{[-l,l]} = \cdot)$ just described.

Remark that by the non-nulness of Gibbs measures, we have that

$$\sup_{\alpha,\beta,\zeta,\xi,\zeta_{\Delta_m}=\xi_{\Delta_m}} \mu_{[-n_1,n_1],\zeta}(\sigma_{-n_1} = \alpha, \sigma_{n_1} = \beta) < 1 - \delta$$

for some $0 < \delta < 1$. As in [2], we then consider the auxiliary Markov chain S_n on $\{0, 1, 2, \cdots\}$ whose transition probabilities are

$$\begin{cases} \mathbb{P}(S_{n+1} = m + 1 | S_n = m) = 1 - \min\{\gamma_m, 1 - \delta\} \\ \mathbb{P}(S_{n+1} = 0 | S_n = m) = \min\{\gamma_m, 1 - \delta\}. \end{cases} \tag{3.6}$$

On the other hand, we have the process that counts the number of matches (the so-called "house-of-cards" process), defined by

$$\begin{cases} Z_0 = 0 \\ Z_{n+1} = \begin{cases} Z_n + 1 & \text{if } (\sigma^1_{-l+n}, \sigma^1_{l-n}) = (\sigma^2_{-l+n}, \sigma^2_{l-n}) \\ 0 & \text{otherwise} \end{cases} & \text{for } n = 0, 1, 2, \ldots \end{cases} \tag{3.7}$$

By Proposition 1 in [2], we have

$$\left|\mu_{[-l,l],\zeta}(\sigma_0) - \mu_{[-l,l],\zeta'}(\sigma_0)\right| = \left|\mathbb{E}_{\mathbb{P}_{12}}(\sigma_0^1 - \sigma_0^2)\right| = \mathbb{P}(Z_l = 0) \le \mathbb{P}(S_l = 0). \tag{3.8}$$

Finally condition (2.8) insures that $\gamma_n \to 0$ as $n \to +\infty$. Then by Proposition 2 in [2], we have $\mathbb{P}(S_l = 0) \to 0$ as $l \to +\infty$, which completes the proof. $\square$

3.1 The transformed potential

Definition 3.3. *If μ is a measure that admits a continuous version of the conditional probabilities $\mu(\xi_i|\xi_{\mathbb{Z}\setminus\{i\}})$, $i \in \mathbb{Z}$, then we call φ an estimate for the rate of continuity if*

$$\sup_{\xi,\zeta} \left|\mu(\xi_i|\xi_{[-n,n]\setminus\{i\}}\zeta_{[-n,n]^C}) - \mu(\xi_i|\xi_{\mathbb{Z}\setminus\{i\}})\right| \le \varphi(n). \tag{3.9}$$

In the previous section we showed that for our transformed Gibbs measure, $\mathbb{P}(S_n = 0)$ is an estimate for the rate of continuity. We now show the decay of the Kozlov potential associated to μ, when we have an estimate on the rate of continuity. We start from the following explicit form of the potential of theorem 2.4, see [7], [10]. We assume, without loss of generality, that the finite alphabet contains a distinguished symbol denoted by "+".

Theorem 3.4. *Let ν be a probability measure such that the conditional probabilities $\nu(\xi_i|\xi_{\mathbb{Z}\setminus\{i\}})$, $i \in \mathbb{Z}$, are non null and have a continuous version. Consider the potential, defined on lattice intervals (and vanishing on other subsets) by*

$$U([i,j],\xi) = \log \frac{\nu(\xi_i|\xi_{]i,j[}+)\nu(\xi_j|\xi_{]i,j[}+)}{\nu(\xi_i\xi_j|\xi_{]i,j[}+)}, \tag{3.10}$$

where the plus signs mean that conditioned sites outside the lattice interval $[i,j]$ all have the state $+$. If U is uniformly absolutely convergent, then ν is a Gibbs measure associated with the potential U.

We look now at this potential in our context, i.e., when ν is the transformed Gibbs measure $\mu \circ T$. By Theorem 3.1, $\mathbb{P}(S_n = 0)$ is an estimate for the rate of continuity of $\mu \circ T$. We can then estimate the potential: if φ is an estimate for the rate of continuity,

$$\begin{aligned} \nu(\xi_i|\xi_{]i,j[}+) &= \nu(\xi_i|\xi_{]i,j]}+)\nu(\xi_j|\xi_{]i,j[}+) + \sum_{\eta_j \neq \xi_j} \nu(\xi_i|\xi_{]i,j[}\eta_j+)\nu(\eta_j|\xi_{]i,j[}+) \\ &\leq \nu(\xi_i|\xi_{]i,j]}+)\left(1 + C\varphi(|j-i|)\right), \end{aligned} \tag{3.11}$$

where the constant C is bounded by the non-nullness assumption. Further, we have

$$\nu(\xi_i\xi_j|\xi_{]i,j[}+) = \nu(\xi_i|\xi_{]i,j]}+)\nu(\xi_j|\xi_{[i,j[}+).$$

So we have the estimate on Kozlov potential of the transformed measure

$$\left|U_{[i,j]}(\xi)\right| \leq \log(1 + C\varphi(|j-i|)) \leq C\varphi(|j-i|) \tag{3.12}$$

We now consider two relevant cases, according to behavior of F_m in (2.9).

1. If f in (2.6) decays *exponentially*, then F_N decays also exponentially as N increases. This implies that φ in (3.12) also decays exponentially, that is,

$$U_{[i,j]}(\xi) \leq e^{-\lambda|j-i|} \tag{3.13}$$

for some $\lambda > 0$.

2. In case that f decays as *a power law* i.e., for some $C > 0$,

$$f(k) \leq \frac{C}{k^\alpha}, \quad \text{for } \alpha > 1, \tag{3.14}$$

we have

$$F_N \leq \frac{C_1}{N^{\alpha-1}}, \tag{3.15}$$

where C_1 is a positive constant. This implies that φ in (3.9) decays as $\frac{C_1}{n^{\alpha-1}}$, which in turn implies that the transformed potential decays as

$$\|U_{[i,j]}\|_\infty \leq \frac{C_1}{(j-i)^{\alpha-1}}. \tag{3.16}$$

Hence, $\alpha > 2$ is sufficient to have uniform absolute summability of this potential (whereas $\alpha > 1$ is sufficient for Gibbsianness of the transformed measure)

E.g., if the original potential is a long-range Ising potential, i.e.,

$$\Phi(\{i,j\},\sigma) = \frac{\sigma_i \sigma_j}{|j-i|^\gamma}$$

then we need $\gamma > 2$ for the transformed measure to be Gibbsian, and $\gamma > 3$ for the transformed potential to be uniformly absolute convergent. Remark that for $\gamma < 2$ we do not have uniqueness of the associated Gibbs measure, so the transformed measure might be non-Gibbsian.

4 Deterministic single site transformations

As before, we consider the configuration space of the untransformed system $\Omega = S^{\mathbb{Z}}$, where S is a finite set, and the configuration space of the transformed system is $\Omega' = (S')^{\mathbb{Z}}$. The transformation $T : \Omega \to \Omega'$ now is induced by a map $\varphi : S \to S'$, via

$$(T(\sigma))_i =: \sigma'_i = \varphi(\sigma_i) \tag{4.1}$$

This is equivalent with defining the new spin σ_i via a partition of the single-site space S, which in the case of $S = \{1, \ldots, q\}$ and Φ the potential of the Potts-model has been called the fuzzy Potts model, see [9].

To deal with such transformations, we follow the approach of in [8]. This consists of writing the single-site conditional probabilities of the transformed measure in terms of a so-called *constrained restricted first layer measure*. The difference with stochastic transformations is that this measure does not necessarily have full support, i.e., given the second layer constraint $\xi \in \Omega'$, the first layer has to be such that its image coincides with ξ'.

As in the previous section, we start with a Gibbs measure μ on configurations $\sigma \in \Omega$. The potential Φ satisfies (2.8). We further abbreviate $\nu = \mu \circ T$ and $K(\eta_i|\sigma_i) = I(\varphi(\sigma_i) = \eta_i)$, where I denotes indicator, and for $\Lambda \subseteq \mathbb{Z}$ finite, $\Lambda_0 := \Lambda \setminus \{0\}$.

For clarity, we first repeat the main steps of [8] to rewrite the single-site conditional probabilities of ν in terms of a constrained restricted first layer measure.

$$\begin{aligned}\nu(\eta_0|\eta_{\Lambda_0}) &= \frac{\sum_{\sigma_\Lambda}\mu(\sigma_\Lambda)\prod_{i\in\Lambda}K(\eta_i|\sigma_i)}{\sum_{\sigma_\Lambda}\mu(\sigma_\Lambda)\prod_{i\in\Lambda_0}K(\eta_i|\sigma_i)} \\ &= \frac{\int\mu(d\zeta)\sum_{\sigma_\Lambda}\mu_{\Lambda,\zeta}(\sigma_\Lambda)\prod_{i\in\Lambda}K(\eta_i|\sigma_i)}{\int\mu(d\zeta)\sum_{\sigma_\Lambda}\mu_{\Lambda,\zeta}(\sigma_\Lambda)\prod_{i\in\Lambda_0}K(\eta_i|\sigma_i)}\end{aligned} \tag{4.2}$$

Now we consider the following auxiliary measure on the state space $\Omega_0 := S^{\Lambda_0}$.

$$\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(\sigma_{\Lambda_0}) := \frac{1}{N^\eta_{\Lambda,\zeta}}\exp\left(-\mathcal{H}^\zeta_{\Lambda_0}(\sigma_{\Lambda_0})\right)\prod_{i\in\Lambda_0}K(\eta_i|\sigma_i) \tag{4.3}$$

where $N^\eta_{\Lambda,\zeta}$ denotes the normalizing constant, and

$$\mathcal{H}^\zeta_{\Lambda_0}(\sigma_{\Lambda_0}) = \sum_{A\cap\Lambda_0\neq\varnothing, A\not\ni 0}\Phi(A,\sigma_\Lambda\zeta_{\Lambda^c}) \tag{4.4}$$

These measures concentrate on configurations $\sigma_{\Lambda_0}\in S^{\Lambda_0}$ compatible with η_{Λ_0}, i.e., such that $K(\eta_i|\sigma_i)\neq 0$ for all $i\in\Lambda_0$. For $\eta\in\Omega'$ fixed, they form a η-dependent specification on the configuration space $S^{\mathbb{Z}_0}$, i.e.,

a) $\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(\sigma_{\Lambda_0})$ is a probability measure on S^{Λ_0}

b) $\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(\sigma_{\Lambda_0})$ depends only in ζ on $\mathbb{Z}_0\setminus\Lambda_0$

c) Consistency: if we denote

$$\left(\gamma^\eta_{\Lambda_0}(g)\right)(\zeta) := \int\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(d\sigma_{\Lambda_0})g(\sigma_{\Lambda_0}\zeta_{\Lambda^c}) \tag{4.5}$$

then these η-dependent kernels γ^η_Λ satisfy

$$\gamma^\eta_{\Lambda_0}(\gamma^\eta_{\Lambda'_0}(g)) = \gamma^\eta_{\Lambda_0}(g) \tag{4.6}$$

for all $\Lambda\supset\Lambda'$ and all local functions g.

In terms of these measures, we can rewrite the conditional probability $\nu(\eta_0|\eta_{\Lambda_0})$ as follows.

$$\nu(\eta_0|\eta_{\Lambda_0}) = \frac{\int\mu(d\zeta)\frac{N^\eta_{\Lambda,\zeta}}{Z^\zeta_\Lambda}\int\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(d\sigma_{\Lambda_0})\psi^\zeta_{0,\Lambda}(\eta_0,\sigma_{\Lambda_0})}{\int\mu(d\zeta)\frac{N^\eta_{\Lambda,\zeta}}{Z^\zeta_\Lambda}\int\mu^{\eta_{\Lambda_0}}_{\Lambda_0,\zeta}(d\sigma_{\Lambda_0})\varphi^\zeta_{0,\Lambda}(\sigma_{\Lambda_0})} \tag{4.7}$$

where

$$\begin{aligned}\psi^\zeta_{0,\Lambda}(\eta_0,\sigma_{\Lambda_0}) &= \sum_{\sigma_0}e^{-h_0(\sigma_0\sigma_{\Lambda_0}\zeta_{\Lambda^c})}K(\eta_0|\sigma_0) \\ \varphi^\zeta_{0,\Lambda}(\sigma_{\Lambda_0}) &= \sum_{\sigma_0}e^{-h_0(\sigma_0\sigma_{\Lambda_0})\zeta_{\Lambda^c})}\end{aligned} \tag{4.8}$$

with

$$h_0(\sigma) = \sum_{A \ni 0} \Phi(A, \sigma) \tag{4.9}$$

and Z_Λ^ζ is the finite-volume partition function with boundary condition ζ, i.e.,

$$Z_\Lambda^\zeta = \sum_{\sigma_\Lambda} e^{-H_\Lambda^\zeta(\sigma_\Lambda)}$$

Notice that $\psi_{0,\Lambda}^\zeta(\eta_0, \sigma_{\Lambda_0})$ and $\varphi_{0,\Lambda}^\zeta(\sigma_{\Lambda_0})$ converge uniformly (in η_0, σ, ζ), as $\Lambda \uparrow \mathbb{Z}$ to

$$\begin{aligned} \psi_0(\eta_0, \sigma_{\mathbb{Z}\setminus\{0\}}) &= \sum_{\sigma_0} e^{-h_0(\sigma)} K(\eta_0|\sigma_0) \\ \varphi_0(\sigma_{\mathbb{Z}\setminus\{0\}}) &= \sum_{\sigma_0} e^{-h_0(\sigma)} \end{aligned} \tag{4.10}$$

4.1 Exponentially decaying potential

Let us now first look at the case where Φ decays exponentially. As a consequence, the decay to zero in (2.8) is exponential in k. We will prove here, that, as in the stochastic case, the transformed measure ν has an exponentially decaying interaction as well. In this case, for $\Lambda = [-n, n]$ there exist $C_1, c_1 > 0$ such that for all ζ, σ, η,

$$|\psi_{0,\Lambda}^\zeta(\eta_0, \sigma_{\Lambda_0}) - \psi_0(\eta_0, \sigma_{\mathbb{Z}\setminus\{0\}})| \leq C_1 e^{-c_1 n}$$

and similarly for φ_0. Our aim is then to show that there exist $C_2, c_2 > 0$ such that for all η, $n, m > n$,

$$|\nu(\eta_0|\eta_{[-n,n]_0}) - \nu(\eta_0|\eta_{[-m.m]_0})| \leq C_2 e^{-c_2 n}$$

The idea is once more to couple the measures $\mu_{\Lambda,\zeta}^{\eta_{\Lambda_0}}$ and $\mu_{\Lambda,\zeta'}^{\eta_{\Lambda_0}}$ for different boundary conditions, such that in the coupling the probability that $\sigma_i^1 \neq \sigma_i^2$ is bounded by $e^{-\alpha|n-i|\wedge|-n-i|}$ for some $\alpha > 0$. This coupling follows the same iterative procedure as in the stochastic case, and the estimates are identical. Next, we need to compare expectations of the functions ψ_0, φ_0 (instead of a function that only depends on σ_0 in the stochastic case). These functions ψ_0, φ_0 can however be exponentially well approximated by local functions. We spell out these steps in three lemmas.

Lemma 4.1. *Let μ_1, μ_2 be two probability measures on S^{Λ_0} and $\mathbb{P}$ a coupling of them. Then for all functions $g : S^{\Lambda_0} \to \mathbb{R}$ we have*

$$\left| \int g \, d\mu_1 - \int g \, d\mu_2 \right| \leq \sum_{i \in \Lambda_0} \mathbb{P}(\sigma_i^1 \neq \sigma_i^2) \delta_i g \tag{4.11}$$

where $\delta_i g(\sigma) = \sup\{g(\sigma) - g(\sigma') : \sigma_j = \sigma'_j \ \forall\ j \neq i\}$

Proof. This is elementary and left to the reader. $\square$

Lemma 4.2. *For* $\Lambda = [-n, n]$ *there exists a coupling* $\mathbb{P}$ *of* $\mu^{\eta_{\Lambda_0}}_{\Lambda,\zeta}$ *and* $\mu^{\eta_{\Lambda_0}}_{\Lambda,\zeta'}$ *such that*

$$\mathbb{P}(\sigma^1_i \neq \sigma^2_i) \leq C_3 e^{-c_3 |n-i| \wedge |-n-i|} \tag{4.12}$$

where $C_3, c_3 > 0$ *do not depend on* ζ, ζ', n.

Proof. The coupling follows the iterative procedure as in the stochastic case, and the estimates in terms of the function f in (2.6) are identical. □

As a consequence of these lemmas we have the existence of a unique Gibbs measure μ^η on $S^{\mathbb{Z}_0}$ consistent with the specification $\mu^{\eta_{\Lambda_0}}_{\Lambda,\zeta}$, and for any local function g (with dependence set in Λ) we have the estimate

$$\sup_\xi \left| \int g d\mu^\eta - \gamma^\eta_\Lambda(\xi)(g) \right| \leq C_3 \sum_i \delta_i(g) e^{-c_3 |n-i| \wedge |-n-i|} \tag{4.13}$$

where we used the notation (4.5) and where $\Lambda = [-n, n]$

Lemma 4.3. *Suppose that* $g : S^{\mathbb{Z}_0} \to \mathbb{R}$ *is continuous and such that there exist* g_k *depending only on* $\sigma_i, i \in [-k, k]_0$ *such that*

$$\|g_k - g\|_\infty < C_4 e^{-c_4 k} \tag{4.14}$$

for some $C_4, c_4 > 0$. *Then there exists* $C_5, c_5 > 0$ *such that for* $\Lambda = [-n, n]$

$$\sup_\eta \left| \gamma^\eta_\Lambda(g) - \int g \; d\mu^\eta \right| \leq C_5 e^{-c_5 n} \tag{4.15}$$

Proof. Choose $\Lambda = [-n, n]$, and choose g_k as in (4.14). Write

$$|\gamma^\eta_\Lambda(g)(\zeta) - \gamma^\eta_\Lambda(g)(\xi)| \leq A + B + C \tag{4.16}$$

where

$$A := |\gamma^\eta_\Lambda(g)(\zeta) - \gamma^\eta_\Lambda(g_k)(\zeta)| \leq \|g - g_k\|_\infty \tag{4.17}$$

$$C := |\gamma^\eta_\Lambda(g_k)(\xi) - \gamma^\eta_\Lambda(g)(\xi)| \leq \|g - g_k\|_\infty \tag{4.18}$$

$$B := |\gamma^\eta_\Lambda(g_k)(\zeta) - \gamma^\eta_\Lambda(g_k)(\xi)| \leq 2 \sup_\xi \left| \gamma^\eta_\Lambda(g_k)(\xi) - \int g_k d\mu^\eta \right| \tag{4.19}$$

Now use (4.13), and the obvious inequality $\delta_i(g) \leq 2\|g\|_\infty$ to obtain

$$|\gamma^\eta_\Lambda(g)(\zeta) - \gamma^\eta_\Lambda(g)(\xi)| \leq 2C_4 e^{-c_4 k} + 4 \sup_k \|g_k\|_\infty \sum_{j=0}^{k} C_3 e^{-c_3 (n-j)} \tag{4.20}$$

Finally, choose $k = n/2$. □

4.2 Power law decaying potential

For the case where Φ decays according to a power law, more precisely, if

$$f(K) \leq Ck^{-\alpha} \tag{4.21}$$

where f is the function associated to the potential Φ as in (2.6), and $\alpha > 2$. Then we have the analogue of (4.12) (cf. the two cases considered after Theorem 3.4)

$$\mathbb{P}(\sigma_i^1 \neq \sigma_i^2) \leq C_3\left((n-i) \wedge (-n-i)\right)^{\alpha-1} \tag{4.22}$$

Next, the local approximations of the functions ψ_0 and φ_0 converge now only at power-law speed, i.e., the local approximations ψ_0^k, φ_0^k with dependence set $[-k,k]$ satisfy

$$\|\psi_0 - \psi_0^k\|_\infty < Ck^{-\alpha},\ \ \|\varphi_0 - \varphi_0^k\|_\infty < Ck^{-\alpha}$$

Therefore, in that case we find, using the same steps as in the exponential case, for all η, $n, m > n$,

$$|\nu(\eta_0|\eta_{[-n,n]_0}) - \nu(\eta_0|\eta_{[-m.m]_0})| \leq C_2 n^{-(\alpha-2)}$$

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