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# Unipotent group actions on affine varieties

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ABSTRACT. Algebraic actions of unipotent groups  $U$  on affine  $k$ -varieties  $X$  ( $k$  an algebraically closed field of characteristic 0) for which the algebraic quotient  $X//U$  has small dimension are considered. In case  $X$  is factorial,  $O(X)^* = k^*$ , and  $X//U$  is one-dimensional, it is shown that  $O(X)^U = k[f]$ , and if some point in  $X$  has trivial isotropy, then  $X$  is  $U$  equivariantly isomorphic to  $U \times A^1(k)$ . The main results are given distinct geometric and algebraic proofs. Links to the Abhyankar-Sathaye conjecture and a new equivalent formulation of the Sathaye conjecture are made.

## 1. Preliminaries and Introduction

Throughout,  $k$  will denote a field of characteristic zero,  $k^{[n]}$  the polynomial ring in  $n$  variables over  $k$ , and  $U$  a unipotent algebraic group over  $k$ . Our interest is in algebraic actions of such  $U$  on affine  $k$ -varieties  $X$  (equivalently on their coordinate rings  $\mathcal{O}(X)$ ). An algebraic action of the one dimensional unipotent group  $\mathcal{G}_a = (k, +)$  is conveniently described through the action of a locally nilpotent derivation  $D$  of  $\mathcal{O}(X)$ . Specifically, for  $u \in \mathcal{G}_a = k$ , we have the automorphism  $u^*$  acting on  $\mathcal{O}(X)$  and it is well-known (see for example [1] page 16-17) that there exists a unique locally nilpotent derivation  $D : \mathcal{O}(X) \rightarrow \mathcal{O}(X)$  such that  $u^* = \exp(uD)$ . (One can obtain  $D$  by taking  $D(f) = \frac{u^*f - f}{u}|_{u=0}$ .) Similarly, if  $\mathcal{G}_a^n$  acts on  $X$ , then we have for each component  $\mathcal{G}_a$ -action a locally nilpotent derivation  $D_i$ , and for each element  $u = (u_1, \dots, u_n) \in \mathcal{G}_a^n$  we have the derivation  $D := u_1D_1 + \dots + u_nD_n$ . If the action is faithful, there is a canonical isomorphism of  $\text{Lie}(\mathcal{G}_a^n)$  with  $kD_1 + \dots + kD_n$ . In this case, the  $D_i$  commute.

The situation is similar for a general unipotent group action  $U \times X \rightarrow X$ . Because the action is algebraic, each  $f \in \mathcal{O}(X)$  is contained in a finite dimensional  $U$  stable subspace  $V_f$  on which  $U$  acts by linear transformations. Since  $U$  is unipotent, for each  $u \in U$ ,  $u^* - id$  is nilpotent on  $V_f$ , so that

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$\ln(u)(g) = \sum_{j=1}^{\infty} \frac{(u^* - id)^j g}{j}$  is a finite sum for all  $g \in V_f$ . One checks that

$D_u \equiv \ln(u)$  is a (locally nilpotent) derivation of  $\mathcal{O}(X)$  and  $u^* = \exp(D_u)$ . If the action is faithful, i.e.  $U \rightarrow \text{Aut}(X)$  is injective, there is a canonical isomorphism of  $\text{Lie}(U)$  with  $\{D_u \mid u \in U\}$ . In fact,  $\text{Lie}(U) = kD_1 + \dots + kD_m$  ( $m = \dim(U)$ ) for some locally nilpotent derivations  $D_i$ . In general the  $D_i$  do not commute. In fact, all of them commute if and only if  $U = \mathcal{G}_a^m$ .

Two useful facts about unipotent group actions on quas affine varieties  $V$  can be immediately derived from these observations:

- (1) Because each  $u \in U$  acts via a locally nilpotent derivation of  $\mathcal{O}(V)$ , the ring of invariants  $\mathcal{O}(V)^U$  is the intersection of the kernels of locally nilpotent derivations.
- (2) Since kernels of locally nilpotent derivations are factorially closed, their intersection is too, i.e.  $\mathcal{O}(V)^U$  is factorially closed. In particular if  $\mathcal{O}(V)$  is a UFD so is  $\mathcal{O}(V)^U$ .

We will use the fact that  $U$  is a special group in the sense of Serre. This means that a  $U$  action which is locally trivial for the étale topology is locally trivial for the Zariski topology. If  $G$  is a group acting on a variety  $X$ , we denote by  $X//G$  the algebraic quotient  $X//G := \mathbf{Spec} \mathcal{O}(X)^G$  and by  $X/G$  the geometric quotient (when it exists). By a free action we mean an action for which the isotropy subgroup of each element consists only of the identity. (A free action is faithful.)

The paper is organized as follows: Section 2 contains some examples which illustrate the main results and clarify their hypotheses. The main results are proved in Section 3 from a geometric perspective, and Section 4 gives them an algebraic interpretation. (The algebraic and geometric viewpoint both have their merits: the geometric viewpoint lends itself to possible generalizations, while the algebraic proofs are constructive and can be more easily used in algorithms.) In section 5 we elaborate on some implications of the main results for the Sathaye conjecture, and on the motivation for studying this problem.

## 2. Examples

The following examples are valuable in various parts of the text.

**Example 1.** Let  $X = k^3$ , and  $U := \{u_{a,b,c} \mid a, b, c \in k\}$  where

$$u_{a,b,c} := \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

a unipotent group acting by  $u_{a,b,c}(x, y, z) = (x + a, y + az + b, z + c)$  (which indeed is an algebraic action). For each  $(a, b, c) \in k^3$  we thus have an automorphism, and its associated derivation on  $k[X, Y, Z]$  is  $D_{a,b,c} = a\partial_X + (az + b - \frac{ac}{2})\partial_Y + c\partial_Z$ . Set  $D_1 = \partial_Y, D_2 := \partial_X + Z\partial_Y, D_3 = \partial_Z$ . As a Lie algebra  $\text{Lie}(U)$  is generated by  $D_1, D_2, D_3$ . One checks that  $D_1$  commutes

with  $D_2, D_3$ , but  $[D_2, D_3] = D_1$ . However, restricted to  $k[X, Y, Z]^{D_1} = k[X, Z]$ ,  $D_2$  and  $D_3$  do commute, as they coincide with the derivations  $\partial_X$  and  $\partial_Z$ . Furthermore, as a  $k$  vector space  $\text{Lie}(U)$  has basis  $\partial_X, \partial_Y, \partial_Z$ .

**Example 2.** Let  $\mathcal{O}(X) = A = k[X, Y, Z]$ , and  $D_1 = Z\partial_X, D_2 = \partial_Y$ . These locally nilpotent derivations generate a  $U = (\mathcal{G}_a)^2$ -action on  $k^3$  given by  $(a, b) \cdot (x, y, z) \rightarrow (x + az, y + b, z)$ . Now  $k[Z] = A^{D_1, D_2} = \mathcal{O}(X/U)$ .  $D_1, D_2$  are linearly independent over  $k[Z]$ . When calculating modulo  $Z - \alpha$  where  $\alpha \in k$ , we notice that  $D_1 \bmod (Z - \alpha), D_2 \bmod (Z - \alpha)$  are linearly independent over  $A/(Z - \alpha)$  except when  $\alpha = 0$ . However, defining  $\mathcal{M} := (\text{Lie}(U) \otimes k(Z)) \cap \text{DER}(A) = (k(Z)D_1 + k(Z)D_2) \cap \text{DER}(A)$  we see that  $\mathcal{M} = k[Z]\partial_X + k[Z]\partial_Y$ . The derivations  $\partial_X, \partial_Y$  are linearly independent modulo each  $Z - \alpha$ . And for each  $\alpha \in k$ , we have  $A/(Z - \alpha) \cong k^{[2]}$ .

**Example 3.** Let  $P := X^2Y + X + Z^2 + T^3$ ,  $\mathcal{X} := \{(x, y, z, t) \mid P(x, y, z, t) = 0\}$ . Let  $A := k[x, y, z, t] := k[X, Y, Z, T]/(P) = \mathcal{O}(\mathcal{X})$ . The commuting locally nilpotent derivations  $2Z\partial_Y - X^2\partial_Z, 3T^2\partial_Y - X^2\partial_T$  on  $k[X, Y, Z, T]$  map  $P$  to zero, and hence induce derivations  $D_1, D_2$  on  $A$ . They are linearly independent over  $A^{D_1, D_2} = k[X]$  and since they commute, induce a  $(\mathcal{G}_a)^2$ -action on  $\mathcal{X}$ . Modulo  $X - \alpha$ ,  $D_1, D_2$  are linearly independent, except when  $\alpha = 0$ . Now defining  $\mathcal{M} := (\text{Lie}(U) \otimes k(X)) \cap \text{DER}(A) = k[X]D_1 + k[X]D_2 = \text{Lie}(U) \otimes k[X]$ , we see that  $\mathcal{M}$  modulo  $X - \alpha$  is a  $k$ -module of dimension 2 except when  $\alpha = 0$ , when it is of dimension 1. Also,  $A/(X - \alpha) \cong k^{[2]}$  except when  $\alpha = 0$ , when it is isomorphic to  $R[X]$  where  $R = k[Z, T]/(Z^2 + T^3)$ .

**Example 4.** The  $U = \mathcal{G}_a \times \mathcal{G}_a$  action on  $\mathbb{A}^2(k)$  given by

$$U \times \mathbb{A}^2 \ni ((s, t), (x, y)) \mapsto (x, y + t + sx) \in \mathbb{A}^2$$

is faithful and fixed point free. However every point in  $\mathbb{A}^2$  has a non-trivial isotropy subgroup. If  $x \neq 0$ , then  $((s, -sx), (x, y)) \mapsto (x, y)$  and  $((s, 0), (0, y)) \mapsto (0, y)$ .

### 3. Main Results

The following simple lemma is useful in a number of places.

**Lemma 1.** Let  $U$  be a unipotent algebraic group acting algebraically on a factorial quasiprojective variety  $X$  of dimension  $n$  satisfying  $\mathcal{O}(X)^* = k^*$ . If the action is not transitive and some point  $x \in X$  has orbit of dimension  $n - 1$ , then  $\mathcal{O}(X)^U = k[f]$  for some  $f \in \mathcal{O}(X)$

PROOF. Since  $n - 1$  is the maximum orbit dimension there is a Zariski open subset  $V$  of  $X$  for which the geometric quotient  $V/U$  exists as a variety. Then the transcendence degree of the quotient field  $K$  of  $\mathcal{O}(V/U)$  is equal to 1. Since  $K = qf(\mathcal{O}(X)^U)$  and

$$\mathcal{O}(X)^U = \mathcal{O}(X) \cap K,$$

is a ufd,  $\mathcal{O}(X)^U$  is finitely generated over  $k$ . From  $(\mathcal{O}(X)^U)^* = k^*$ , we conclude that  $\mathcal{O}(X)^U = k[f]$  for some  $f \in \mathcal{O}(X)$   $\square$

### 3.1. Unipotent actions having zero-dimensional quotient.

**Theorem 1.** *Let  $U$  be an  $n$ -dimensional unipotent group acting faithfully on an affine  $n$ -dimensional variety  $X$  satisfying  $\mathcal{O}(X)^* = k^*$ . If either*

- a) *Some  $x \in X$  has trivial isotropy subgroup or*
- b)  *$n = 2$ ,  $X$  is factorial, and  $U$  acts without fixed points,*

*then the action is transitive. In particular  $X \cong k^n$ .*

PROOF. In case a) there is an open affine subset  $V$  of  $X$  on which  $U$  acts without fixed points. Since  $U$  has the same dimension as  $V$ ,  $V//U$  is zero-dimensional, hence  $\mathcal{O}(V//U)$  is a field. This field contains  $k$ , and its units are contained in  $\mathcal{O}(X)^* = k^*$ , hence  $\mathcal{O}(V//U) = k$ . It follows that there exists an open set  $V'$  of  $X$  for which  $V'/U \cong \mathbf{Spec} k$ . Thus  $V' \cong U$  as a variety, and therefore  $V' \cong k^n$ . If  $v \in V'$ , then  $Uv = V'$ . Since  $U$  is unipotent, all orbits are closed, hence  $V'$  is closed in  $X$ . Since it is of dimension  $n$ , and  $X$  is irreducible of dimension  $n$ , we have that  $V' = X$ .

In case b)  $X$  is necessarily smooth since it is smooth in codimension 1 and every orbit is infinite. If  $X$  has a two dimensional (i.e. dense) orbit then the conclusion follows as in a). So we assume for each  $x \in X$  that the orbit  $Ux$  is one dimensional, given as  $\exp(uD)x$ , and therefore isomorphic to  $\mathbb{A}^1(k)$  by the discussion in the introduction. From Lemma 1 we conclude that  $\mathcal{O}(X)^U = k[f]$  for some  $f \in \mathcal{O}(X)$

Note that factorial closure of  $\mathcal{O}(X)^U$  implies that  $f - \lambda$  is irreducible for every  $\lambda \in k$ . The absence of nonconstant units implies that  $X \rightarrow \mathbf{Spec}(k[f])$  is surjective, and all fibers are  $U$  orbits. Smoothness of  $X$  implies in addition that this mapping is flat, hence an  $\mathbb{A}^1$  bundle over  $\mathbb{A}^1$ . But any such bundle is trivial, so we conclude again that  $X \cong \mathbb{A}^2$ .  $\square$

Example 4 of the previous section illustrates case b).

**3.2. Unipotent actions having one-dimensional quotient.** The following theorem is the main result of this paper.

**Theorem 2 (Main theorem).** *Let  $U$  be a unipotent algebraic group of dimension  $n$ , acting on  $X$ , a factorial variety of dimension  $n + 1$  satisfying  $\mathcal{O}(X)^* = k^*$ .*

- (1) *If at least one  $x \in X$  has trivial stabilizer then  $\mathcal{O}(X)^U = \mathcal{O}(X//U) = k[f]$ . Furthermore,  $f^{-1}(\lambda) \cong k^n$  for all but finitely many  $\lambda \in k$ .*
- (2) *If  $U$  acts freely, then  $X$  is  $U$ -isomorphic to  $U \times k$ . In particular,  $X \simeq k^{n+1}$  and  $f$  is a coordinate.*

An important example to keep in mind is example 1, as this satisfies (1) but not (2). (There  $U = \mathcal{G}_a^2$ .)

PROOF OF THEOREM 2.

CLAIM 1:  $\mathcal{O}(X)^U = k[f]$ .

*Proof of claim 1:* This follows from lemma 1.

CLAIM 2:  $f : X \rightarrow k$  is surjective and has fibers isomorphic to  $U$ . The fibers are the  $U$ -orbits.

*Proof of claim 2:* The fibers  $f^{-1}(\lambda)$  are the zero loci of the irreducible  $f - \lambda$ , and are invariant under  $U$ . Since  $U$  acts freely on each fiber and orbits of unipotent group actions are closed, we see that the  $f$  fibers are exactly the  $U$  orbits in  $X$ . Thus  $f$  is a  $U$ -fibration (and, as the underlying variety of  $U$  is  $k^n$ , an  $\mathbb{A}^n$ -fibration).

CLAIM 3:  $X$  is smooth.

*Proof of claim 3:* The set  $X_{sing}$  is  $U$ -stable, hence it is a union of  $U$ -orbits. The  $U$ -orbits are the zero sets  $f - \lambda$ , hence of codimension 1. So  $X_{sing}$  is of codimension 1 or empty. But  $X$  is factorial, so in particular normal, which implies that the set of singular points of  $X$ , denoted by  $X_{sing}$ , is of codimension at least 2. This means that  $X_{sing}$  can only be empty.

CLAIM 4:  $f$  is smooth.

*Proof of claim 4:* All fibers of  $f$  are isomorphic to  $U$ , hence to  $k^n$ , by claim 2. Thus the fibers of  $f$  are geometrically regular of dimension  $n$ . Since  $X$  is smooth,  $f$  is flat, and proposition 10.2 of [2] yields that  $f$  is smooth.

CLAIM 5:  $X \times_f X$  is smooth.

*Proof of claim 5:*  $X \times_f X$  is smooth since it is a base extension of the smooth  $X$  by the smooth morphism  $f$ .

CLAIM 6:  $g : U \times X \rightarrow X \times_f X$  given by  $(u, x) \mapsto (x, ux)$  is an isomorphism.

*Proof of claim 6:* The map  $g$  restricted to  $U \times f^{-1}(\lambda)$  is a bijection onto  $\{(x, y) \mid f(x) = f(y) = \lambda\}$ . Taking the union over  $\lambda \in k$ , we get that  $g$  is a bijection. Since both  $U \times X$  and  $X \times_f X$  are smooth and  $g$  is a bijection, Zariski's Main Theorem implies that  $g$  is an open immersion if it is birational. If so then  $g$  must be an isomorphism since it is bijective.

From Rosenlicht's cross section theorem [6],  $X$  has a  $U$  stable open subset  $\tilde{X}$  on which the  $U$  action has a geometric quotient  $\tilde{X}/U$  and a  $U$  equivariant isomorphism  $\tilde{X} \cong U \times \tilde{X}/U$ . Restricting  $g$  to  $U \times \tilde{X} \rightarrow \tilde{X} \times_f \tilde{X}$  is clearly an isomorphism, so that  $g$  is birational.

Now we are ready to prove the theorem. Using def. 0.10 p.16 of [5], and the fact (4) that  $f$  is smooth, together with (6), yields that  $f : X \rightarrow \mathbb{A}^1$  is an étale principal  $U$ -bundle and therefore a Zariski locally trivial principal  $U$  bundle as  $U$  is special. Such bundles are classified by the cohomology group  $H^1(U, k)$ , which is trivial because  $U$  is unipotent. Thus the bundle  $f : X \rightarrow k$  is trivial, which means that  $X \cong U \times k$ .  $\square$

**Remark 1.** (1) *To obtain  $\tilde{X}$  explicitly and avoid the use of Rosenlicht's theorem, recall that the action of  $U$  is generated by a finite set of  $\mathcal{G}_a$  actions each one given as the exponential of some locally nilpotent derivation  $D_i$  of  $\mathcal{O}(X)$ , indeed  $D_i \in \mathfrak{u}$ , the Lie algebra of*

- U. As such there is an open subset  $X_i$  of  $X$  on which  $D_i$  has a slice, and the corresponding  $\mathcal{G}_a$  acts by translation. Then  $\tilde{X} := \bigcap_{i=1}^s X_i$ .*
- (2) *One can avoid the use of the étale topology by applying a "Seshadri cover" [7]. One constructs a variety  $Z$  finite over  $X$ , necessarily affine, to which the  $U$  action extends so that*
- (a)  *$k(Z)/k(X)$  is Galois. Denote the Galois group by  $\Gamma$ .*
  - (b) *The  $\Gamma$  and  $U$  actions commute on  $Z$ .*
  - (c) *The  $U$  action on  $Z$  is Zariski locally trivial and, because the action on  $X$  is proper by claim 6,*
  - (d)  *$Y \equiv Z/U$  exists as a separated scheme of dimension 1, hence a curve, and affine because of the existence of nonconstant globally defined regular functions, namely  $\mathcal{O}(Z)^U$ .*
  - (e)  *$\mathcal{O}(X)^U \cong \mathcal{O}(Y)^\Gamma$  and  $X//U \cong X/U \cong Y/\Gamma$  shows that  $X \rightarrow X/U$  is Zariski locally trivial.*

#### 4. Algebraic Version

**4.1. Unipotent actions having zero dimensional kernel.** Let  $X$  be a quas affine variety, and  $U$  an algebraic group acting on  $X$ . We write  $A := \mathcal{O}(X)$  and denote by  $\mathfrak{u}$  the Lie algebra of  $U$ . In this section, we will make the following assumptions:

- (P) a)  $X$  and  $U$  are of dimension  $n$ .  
 b) There is a point  $x \in X$  such that  $\text{stab}(x) = \{e\}$ .  
 c)  $\mathcal{O}(X)^* = k^*$

**Definition 1.** *Assume (P). We say that  $D_1, \dots, D_n$  is a triangular basis of  $\mathfrak{u}$  (with respect to the action on  $X$ ) if*

- (1)  $\mathfrak{u} = kD_1 \oplus kD_2 \oplus \dots \oplus kD_n$  and
- (2) With subalgebras  $A_i$  of  $A$  given by  $A_1 := A$ ,  $A_i := A^{D_1} \cap \dots \cap A^{D_{i-1}}$ , the restriction of  $D_i$  to  $A_i$  commutes with the restrictions of  $D_{i+1}, \dots, D_n$ .

For a triangular basis, it is clear that  $D_j(A_j) \subseteq A_j$  for each  $j$ .

If  $U$  is unipotent then the existence of a triangular basis is a consequence of the Lie-Kolchin theorem. Indeed, the Lie algebra  $\mathfrak{u}$  of  $U$  is isomorphic to a Lie subalgebra of the full Lie algebra of upper triangular matrices over  $k$ . In particular  $\mathfrak{u}$  has a basis  $D_1, \dots, D_n$  satisfying  $[D_i, D_j] \in \text{span}\{D_1, \dots, D_{\min\{i,j\}-1}\}$ . By definition of the  $A_i$  this basis is triangular with respect to the action and  $D_1$  is in the center of  $\mathfrak{u}$ .

**Proposition 1.** *Assume (P) and  $U$  unipotent. Then  $A \cong k[s_1, \dots, s_n] = k^{[n]}$  where  $D_i(s_i) = 1$ , and  $D_i(s_j) = 0$  if  $j > i$ .*

**PROOF.** We proceed by induction  $n = \dim \mathfrak{u}$ . If  $n = 1$ , then we have one nonzero LND on a dimension one  $k$ -algebra domain  $A$  satisfying  $A^* = k^*$ . It is well-known that this means that  $A \cong k[x]$  and the derivation is simply  $\partial_x$ . Suppose the theorem is proved for  $n - 1$ . Let  $D_1, D_2, \dots, D_n$  be a triangular

basis for  $\mathfrak{u}$ . Restricting to  $A^{D_1}$  and noting that  $D_1$  is in the center of  $\mathfrak{u}$ , we have an action of the Lie algebra  $\mathfrak{u}/kD_1$  which has the triangular basis  $k\overline{D_2} + \dots + k\overline{D_n}$  ( $\overline{D_i}$  denotes residue class modulo  $kD_1$ ). By construction  $\overline{D_i}(a) := D_i(a)$  is well defined, and by induction we find  $s_2, \dots, s_n \in A^{D_1}$  satisfying  $D_i(s_i) = 1, D_i(s_j) = 0$  if  $j > i \geq 2$ .  $D_i(s_j) = \delta_{ij}$ .

Next we consider a preslice  $p \in A$  such that  $D_1(p) = q, D_1(q) = 0$ , i.e.  $q = q(s_2, \dots, s_n)$ . We pick  $p$  in such a way that  $q$  is of lowest possible lexicographic degree w.r.t  $s_2 \gg s_3 \gg \dots \gg s_n$ . Now  $D_1(D_2(p)) = D_2D_1(p) = D_2(q)$ . Restricted to  $k[s_2, \dots, s_n]$ ,  $D_2 = \partial_{s_2}$ , so  $D_2(q)$  is of lower  $s_2$ -degree than  $q$ . Unless  $D_2(q) = 0$ , we get a contradiction with the degree requirements of  $q$ , as  $D_2(p)$  would be a ‘‘better’’ preslice having a lower degree derivative. Thus,  $q \in k[s_3, \dots, s_n]$ . Using the same argument for  $D_3, D_4$  etc. we get that  $q \in k^*$ . Hence,  $p$  is in fact a slice.  $\square$

**4.2. Unipotent actions having one-dimensional quotient.** With the same notations as in the previous section, we also denote the ring of  $U$  invariants in  $A$  by  $A^U$  and  $\mathbf{Spec} A^U$  by  $X//U$ . Note that  $A^U = \{a \in A \mid D(a) = 0 \text{ for all } D \in \mathfrak{u}\}$ . If  $U$  is unipotent and  $D_1, \dots, D_n$  is a triangular basis of  $\mathfrak{u}$ , we again write  $A_1 := A, A_{i+1} = A_i \cap A^{D_i}$ , noting that  $A^U = A_n$ . In this section we consider the conditions :

(Q1)  $U$  is a unipotent algebraic group of dimension  $n$  acting on an affine variety  $X$  of dimension  $n + 1$  with  $A^* = k^*$ .

and:

(Q)  $A^U = k[f]$  for some irreducible  $f \in A \setminus k$ .

**Remark 2.** According to Lemma 1, condition (Q1) along with the assumption that  $X$  is factorial and the existence of a point  $x \in X$  with  $\text{stab}(x) = \{e\}$ , implies that (Q) holds.

**Notation 1.** Assuming (Q), let  $\alpha \in k$ . Set  $\overline{A} := A/(f - \alpha)$  and write  $\overline{a}$  for the residue class of  $a$  in  $\overline{A}$  and  $\overline{D}$  for the derivation induced by  $D \in \mathfrak{u}$  on  $\overline{A}$ .

Our goal is to prove the following constructively:

**Theorem 3.** Assume (Q1) and (Q). Let  $D_1, \dots, D_n$  be a triangular basis of  $\mathfrak{u}$ .

- (1) For  $\alpha \in k$ ,
- (a) If  $\overline{D_1}, \dots, \overline{D_n}$  are independent over  $A/(f - \alpha)$ , then

$$A/(f - \alpha) \cong k^{[n]}.$$

- (b) There are only finitely many  $\alpha$  for which  $\overline{D_1}, \dots, \overline{D_n}$  are dependent over  $A/(f - \alpha)$ .

- (2) In the case that  $\overline{D_1}, \dots, \overline{D_n}$  are independent over  $A/(f - \alpha)$  for each  $\alpha \in k$ , then there are  $s_1, \dots, s_n \in A$  with  $A = k[s_1, \dots, s_n, f]$ , hence  $A$  is isomorphic to a polynomial ring in  $n + 1$  variables (and  $f$  is a coordinate).



**Definition 2.** Assume (Q1) and (Q), and a triangular basis  $D_1, \dots, D_n$  of  $u$ . Define

$$\mathcal{P}_i := \{p \in A \mid D_i(p) \in k[f], D_j(p) = 0 \text{ if } j < i\}$$

and

$$\mathcal{J}_i := D_i(\mathcal{P}_i) \subseteq k[f].$$

Thus  $\mathcal{P}_i$  is the set of "preslices" of  $D_i$  that are compatible with the triangular basis  $D_1, \dots, D_n$ .

**Lemma 2.** There exist  $p_i \in \mathcal{P}_i \setminus \{0\}$ ,  $p_i \in A_i$ , and  $q_i \in k^{[1]} \setminus \{0\}$  such that  $\mathcal{J}_i = q_i(f)k[f]$  and  $D_i(p_i) = q_i$ .

PROOF. First note that  $\mathcal{J}_i$  is not empty, as theorem 1 applied to  $A(f) := A \otimes k(f)$  gives an  $s_i \in A(f)$  which satisfies  $D_i(s_i) = 1$ ,  $D_j(s_i) = 0$  if  $j < i$ . Multiplying  $s_i$  by a suitable element of  $k[f]$  gives a nonzero element  $r(f)s_i$  of  $\mathcal{P}_i$ , and  $D_i(r(f)s_i) = r(f)$ . Because  $k[f] = \cap \ker(D_i)$ ,  $\mathcal{P}_i$  is a  $k[f]$ -module, and therefore  $\mathcal{J}_i$  is an ideal of  $k[f]$ . This means that  $\mathcal{J}_i$  is a principal ideal, and we take for  $q_i$  a generator (and  $p_i \in D_i^{-1}(q_i)$ ). Since  $D_j(p_i) = 0$  if  $j < i$ , we have  $p_i \in A_i$ .  $\square$

**Corollary 1.** The  $p_i$ ,  $1 \leq i \leq n$ , are algebraically independent over  $k$ .

PROOF. The  $s_i$  are certainly algebraically independent, and  $p_i \in k[f]s_i$ .  $\square$

**Lemma 3.** Assume (Q), and take  $p_i, q_i$  as in lemma 2. Then the  $D_i$  are linearly dependent modulo  $f - \alpha$  if and only if  $q_i(\alpha) = 0$  for some  $i$ .

PROOF. ( $\Rightarrow$ ): Suppose that  $0 \neq D := g_1 D_1 + \dots + g_n D_n$  satisfies  $\bar{D} = 0$  where  $g_i \in A$ , and not all  $\bar{g}_i = \bar{0}$ . Let  $i$  be the highest such that  $\bar{g}_i \neq \bar{0}$ . Then  $0 = \overline{D(p_i)} = \bar{g}_i \bar{D} \bar{p}_i = \bar{g}_i \overline{q_i(f)}$ . Since  $\bar{A}$  is a domain,  $q_i(\alpha) = \overline{q_i(f)} = 0$ . ( $\Leftarrow$ ): Assume  $f - \alpha$  divides  $q_i(f)$ . We need to show that the  $\bar{D}_i$  are linearly dependent over  $A/(f - \alpha)$ . Consider  $\bar{D}_i$  restricted to  $\bar{A}_i$ . If  $j > i$  then  $\bar{D}_i(p_j) = \overline{D_i(p_j)} = \bar{0}$ . Furthermore  $\bar{D}_i(\bar{p}_i) = \bar{q}_i(f) = q_i(\alpha) = 0$ . Hence,  $\bar{D}_i$  is zero if restricted to  $k[\bar{p}_i, \dots, \bar{p}_n]$ . But since this is of transcendence degree  $n$ , it follows that  $\bar{D}_i = 0$  on  $\bar{A}_i$ . Reversing the argument of ( $\Rightarrow$ ) yields the linear dependence of the  $\bar{D}_i$ .  $\square$

PROOF. (of theorem 3) Part 1: If  $\bar{D}_1, \dots, \bar{D}_n$  are independent, then Proposition 1 yields that  $\bar{A} \cong k^{[n]}$ . Lemma 3 states that for any point  $\alpha$  outside the zero set of  $q_1 q_2 \dots q_n$  we have  $A/(f - \alpha) \cong k^{[n]}$ . This zero set is either all of  $k$  or finite, yielding part 1.

Part 2: Lemma 3 tells us directly that for each  $1 \leq i \leq n$  and  $\alpha \in k$ , we have  $q_i(\alpha) \neq 0$ . But this means that the  $q_i \in k^*$ , so the  $p_i$  can be taken to be actual slices ( $s_i = p_i$ ). Using the fact that  $s_i \in A_i$  we obtain that  $A = A_1 = A_2[s_1] = A_3[s_2, s_1] = \dots = A_{n+1}[s_1, \dots, s_n] = k[s_1, \dots, s_n, f]$  as claimed.  $\square$

## 5. Consequences of the main theorems

This paper is originally motivated by the following result of [4]:

**Theorem 4.** *Let  $A = k[x, y, z]$  and  $D_1, D_2$  two commuting locally nilpotent derivations on  $A$  which are linearly independent over  $A$ . Then  $A^{D_1, D_2} = k[f]$  and  $f$  is a coordinate.*

Here the notation  $A^{D_1, D_2}$  means  $A^{D_1} \cap A^{D_2}$  the intersections of the kernels of  $D_1$  and  $D_2$ , which is the set of elements vanishing under  $D_1$  resp.  $D_2$ . (Note that for the  $\mathcal{G}_a$  action associated to  $D$ , this notation means  $\mathcal{O}(X/\mathcal{G}_a) = \mathcal{O}(X)^{\mathcal{G}_a} = \mathcal{O}(X)^D$ ). By a **coordinate** is meant an element  $f$  in  $k^{[n]}$  for which there exist  $f_2, \dots, f_n$  with  $k[f, f_2, \dots, f_n] = k^{[n]}$ . Equivalently,  $(f, f_2, \dots, f_n) : k^{[n]} \rightarrow k^{[n]}$  is an automorphism. The most important ingredient of this theorem is Kaliman's theorem [3].

In [4] it is conjectured that this result is true also in higher dimensions, i.e. having  $n$  commuting linearly independent locally nilpotent derivations on  $k^{[n+1]}$  should yield that their common kernel is generated by a coordinate. However, it seems that this conjecture is very hard, on a par with the well-known Sathaye conjecture:

**$SC(n)$  Sathaye-conjecture:** Let  $f \in A := k^{[n]}$  such that  $A/(f - \lambda) \cong k^{[n-1]}$ . Then  $f$  is a coordinate.

The Sathaye conjecture is proved for  $n \leq 3$  by the aforementioned Kaliman's theorem. Therefore, the original motivation was to find additional requirements in higher dimensions to achieve the result that  $f$  is a coordinate. The results in this paper give one such requirement, namely that  $k^{[n]}/(f - \lambda) \cong k^{[n-1]}$  for all constants  $\lambda$ .

Another consequence of the result of this paper is that the Sathaye conjecture is equivalent to

**$MSC(n)$  Modified Sathaye Conjecture:** Let  $A := k^{[n]}$ , and let  $f \in A$  be such that  $A/(f - \alpha) \cong k^{[n-1]}$  for all  $\alpha \in k$ . Then there exist  $n-1$  commuting locally nilpotent derivations  $D_1, \dots, D_{n-1}$  on  $A$  such that  $A^{D_1, \dots, D_{n-1}} = k[f]$  and the  $D_i$  are linearly independent modulo  $(f - \alpha)$  for each  $\alpha \in k$ .

**PROOF OF EQUIVALENCE OF  $SC(n)$  AND  $MSC(n)$ .** Suppose we have proven the  $MSC(n)$ . Then for any  $f$  satisfying “ $A/(f - \alpha) \cong k^{[n-1]}$  for all  $\alpha \in k$ ” we can find commuting LNDs  $D_1, \dots, D_{n-1}$  on  $A$  giving rise to a  $\mathcal{G}_a^{n-1}$  action satisfying the hypotheses of Theorem 2. Applying this theorem, we obtain that  $f$  is a coordinate in  $A$ . So the  $SC(n)$  is true in that case.

Now suppose we have proven the  $SC(n)$ . Let  $f$  satisfy the requirements of the  $MSC(n)$ , that is, “ $A/(f - \alpha) \cong k^{[n-1]}$  for all  $\alpha \in k$ ”. Since  $f$  satisfies the requirements of  $SC(n)$ ,  $f$  then must be a coordinate. So it has  $n-1$  so-called mates:  $k[f, f_2, \dots, f_n] = k^{[n]}$ . But then the partial derivative with respect to each of these  $n$  polynomials  $f, f_2, \dots, f_n$  defines a locally nilpotent

derivation. All of them commute, and the intersection of the kernels of the last  $n - 1$  derivations is  $k[f]$ ; so the MSC holds.  $\square$

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