Univariate Polynomial Solutions of Nonlinear Polynomial Difference Equations

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Abstract

We study real-polynomial solutions $P(x)$ of difference equations of the form $G(P(x-\tau_1), \ldots, P(x-\tau_s)) + G_0(x)=0$, where τ_i are real numbers, $G(x_1, \ldots, x_s)$ is a real polynomial of a total degree $D \geq 2$, and $G_0(x)$ is a polynomial in x. We consider the following problem: given τ_i, G and G_0, find an upper bound on the degree d of a real-polynomial solution $P(x)$, if exists.

We reduce this problem to finding a univariate polynomial for which d is a root. We formulate a sufficient condition under which such polynomial exists. Using this condition, we can give an effective bound on d, for instance, for all difference equations $G\big(P(x-1), P(x-2), P(x-3)\big) + G_0(x) = 0$ with quadratic G, and all difference equations $G\big(P(x), P(x-\tau)\big) + G_0(x) = 0$ with G of an arbitrary degree.

In the constructions we use Newton-Girard identities between elementary and power-sum symmetric polynomials.

Key words: difference equation, polynomial, elementary symmetric polynomials, power-sum symmetric polynomials, Newton-Girard identities, system of linear equations.

1. Introduction

We study polynomial solutions of difference equations of the form

$$G(P(x-\tau_1), \ldots, P(x-\tau_s)) + G_0(x) = 0 \tag{1}$$

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where $G(x_1, \ldots, x_s) \in \mathcal{R}[x_1, \ldots, x_s]$ is a real polynomial of a *total degree* $D \geq 2$ in s variables, $G_0(x) \in \mathcal{R}[x]$, and $\tau_i \in \mathcal{R}$, with $1 \leq i \leq s$, are pairwise different numbers. Our aim is to *bound the degree* d of a real-polynomial solution $P(x)$.

We call these equations *non-linear polynomial difference equations with constant coefficients*. We use the terminology "with constant coefficients" because we consider polynomials $G(x_1, \ldots, x_s)$ with constant, not-depending on x, coefficients. We believe that extending the proposed method for difference equations where the coefficients of $x_1^{i_1} \ldots, x_s^{i_s}$ depend on x, requires only technical adjustments. We leave it for the future work.

The approach in a nutshell and the outline of the paper

Let d denote the degree of a solution $P(x)$. We are looking for a univariate *degree polynomial*, that is a polynomial for which d is a root. A degree polynomial for a linear recurrence relation with polynomial coefficients is constructed in section 8.3 of the book (Petkovsek et al., 1996). It is easy to see, that a recurrence relation $P(n) = G(n, P(n-1), \ldots, P(n-s)) + G_0(n)$ is a specific case of a difference equation.

As one expects, our reasoning is based on equating the corresponding coefficients in the right- and left-hand-side of an identity of two polynomials in x. We apply this scheme not to the original equation 1, but to the equivalent one

$$G_D\big(P(x-\tau_1), \ldots, P(x-\tau_s)\big) = -G_{\leq D-1}\big(P(x-\tau_1), \ldots, P(x-\tau_s)\big) + G_0(x) \qquad (2)$$

where $G(x_1, \ldots, x_s)$ is presented as the sum $G_D(x_1, \ldots, x_s) + G_{\leq D-1}(x_1, \ldots, x_s)$ with G_D being the homogeneous part with total degree D and $G_{\leq D-1}$ contains the terms of G with total degrees $\leq D-1$.

Without lost of generality we assume that $d(D-1) > deg(G_0)$ (otherwise d is clearly bounded by $deg(G_0)/(D-1)$). Then the degree of x in the r.h.s. of equation 2 is at most $d(D-1)$. The degree of x of the left-hand side is at most dD. For all $0 \leq l \leq d-1$ the coefficients of x^{dD-l} on the left-hand side must vanish because $dD - l > d(D-1)$. In sections 2 and 3, we give necessary set-up and show that these coefficients are functions of the *power-sum symmetric polynomials* p_l of the roots $\mathbf{r} = (\rho_1, \ldots \rho_d)$ of $P(x)$, where $p_l(y_1, \ldots, y_n) := y_1^l + \ldots + y_n^l$ and $p_0(\mathbf{r}) := d$. Moreover, for real polynomials $P(x)$ the values $p_l(\mathbf{r})$ are always real, even if there are complex roots. We construct polynomials $S_l(u_0, u_1, \ldots, u_l)$ such that the coefficient of x^{dD-l} of the l.h.s. of equation 2 is equal to $S_l\big(p_0(\mathbf{r}), p_1(\mathbf{r}), \ldots, p_l(\mathbf{r})\big)$In general, S_l cannot be taken as degree polynomials, because they depend on $1 + l$ variables.

The main contribution of this work is that we analyse a case, when one can eliminate the variables $u_1, \ldots, u_l$ in such a way that we obtain 1-variate polynomial for which d is a root. We focus on the $(u_1, \ldots u_l)$-free term $S_l^*(u_0)$ of S_l. If there is l such that $S_l(u_0, u_1, \ldots, u_l) = S_l^*(u_0) \not\equiv 0$, and assuming $d > l$, then $S_l^*(u_0)$ can be taken as a degree polynomial. In the *framework lemma*, in section 2, we give a sufficient condition for such reduction to be possible. The kernel of constructions are the coefficients of those terms of S_l, in which at least one of $u_1, \ldots u_l$ occurs. We can eliminate such terms and bound degree d if these coefficients are presentable *as linear combinations* of the coefficients of u_0^μ of the polynomials $S_{l'}^*(u_0)$, where $l' < l, \mu \leq l'$. In sections 4 and 6, respectively, we study two independent cases for which the conditions of the framework lemma hold and therefore we can bound d:

- if $L := \min\{l | S_l^*(u_0) \not\equiv 0\} \leq 5$ then either $d \leq \max\{L, \deg(G_0)\}$, or d is a root of $S_L^*(u_0)$, (theorem 5 and an example in section 7),
- $d \leq \max\{D, \deg(G_0)/(D-1)\}$ for all difference equations $G\big(P(x), P(x-\tau)\big) + G_0(x) = 0$ (theorem 7).

In section 8 we sum up our results and outline the future work. Routine proofs and other technicalities can be found in the Appendix or technical report. The proofs are supported by calculations in Maple (worksheet `nonlineardifeq.mw`), available from the site `http://resourceanalysis.cs.ru.nl/index.html` under the item *Technical reports*.

1.1. Related Work

The bound $d \leq D$ for $G(P(x), P(x-\tau)) = 0$ (where $G_0 \equiv 0$) is similar to the result $d = D$ for ordinary differential equations of the form $G(P(x), P(x-1)) = 0$ where $G(x_1, x_2)$ is irreducible in rational field extension, see (Feng et al., 2008). First, note that in our case τ is an arbitrary real number, not necessarily 1. Second, we do not demand irreducibility of G, and if G is reducible then the inequation $d < D$ may hold. For instance, $P(x) = x$ solves $\big(P(x) - P(x-1)\big)^2 - 1 = 0$. Here $d = 1 < 2 = D$ and the polynomial $G(x_1, x_2) = (x_1 - x_2)^2 - 1$ is reducible: $G(x_1, x_2) = (x_1 - x_2 - 1)(x_1 - x_2 + 1)$.

Contrary to *linear difference equations*, there is no general theory for solving non-linear ones.

In paper (Tang et al., 2010) the authors investigate the global behavior of solutions of non-linear difference equation of the form $x_{n+1} = (\alpha + x_n)/(A + Bx_n + x_{n-k})$, where $n \geq 0$, the parameters are positive real numbers and initial conditions $x_{-k}, \ldots, x_0$ are non-negative real numbers, $k \geq 2$. One of the results is that every solution is bounded from above and from below by positive constants. In paper (Özkan Öcalan, 2009) one gives necessary and sufficient conditions for the oscillation of solutions x_n of nonlinear difference equations with advanced arguments. First, the author considers equations with constant coefficients of the form $x_{n+1} - x_n + \sum_{i=1}^{m} p_i f_i(x_{n-k_i}) = 0$ where $\lim_{u\to 0} \frac{f_i(u)}{u} = M_i$ with $0 < M < +\infty$. Then, the result is generalized to equations with non-constant coefficients, p_{in}.

A number of results had been obtain for recurrence relations. A bound on the degree of polynomial solutions to linear homogeneous recurrence relations with polynomial coefficients is obtained in Chapter 8.3 of the book *A=B* (Petkovsek et al., 1996). It is done via a degree polynomial. In the paper (Mezzarobba and Salvy, 2010) a similar problem is considered for complex polynomials, satisfying linear recurrence relations with rational-polynomial coefficients. Here the authors constructively define a real sequence that majorates the complex polynomial sequence. In (Borcea et al., 2011) one gives the asymptotic ratio $\lim_{n\to\infty} \frac{f(n+1)(\mathbf{x})}{f(n)(\mathbf{x})}$ for $f(n)(x)$ satisfying the linear recurrence equation of the form $f(n)(\mathbf{x}) + \alpha_1(n)(\mathbf{x})f(n-1)(\mathbf{x}) + \ldots + \alpha_1(n-s)(\mathbf{x})\alpha_s f(n-s)(\mathbf{x}) = 0$.
In (Rolanía and Lagomasino, 2007) the authors consider asymptotic behavior of the recurrence relation of the form $f(n)(z) = b(n)(z)f(n-1) + a^2(n)(z)f(n-2)(z)$, where $b(n)(z)$, $a(n)(z)$ are analytic in a certain complex domain.

Orthogonal polynomials on the real line always satisfy a three-term recurrence relation (see e.g. (van Assche and Foupouagnagni, 2003)):

$$P(n+1)(x) = (x - \beta(n))P(n)(x) - \gamma(n)P(n-1)(x)$$

It is still linear w.r.t. $P(n)(x)$ (but the polynomial solution is two-variate). However, the recurrence relations for the coefficients $\beta(n)$ and $\gamma(n)$ satisfy *non-linear recurrence relations* of a particular form or systems of such relations. An example of such system may be found in (van Assche and Foupouagnagni, 2003). It defines the coefficients $\beta(n)$ and $\gamma(n)$ such that $P(n)(x)$ is a generalized Charlier polynomial. The asymptotic behavior is such coefficients is studied as well.

In (Máté and Nevai, 1985) the authors study asymptotics for the recurrence relations of the form $H(f(n), f(n{+}1), \ldots, f(n{+}s), 1/n) = 0$, where H is a complex-valued function of $s+2$ real variables all of whose partial derivatives of order $\leq m$ are continuous in a neighborhood of the origin $\bar{0}$ and $\sum_{i=0}^{s} z^j \frac{\partial H}{\partial x_j}(\bar{0}) \neq 0$ for all complex number z with $|z| = 1$. The authors define numbers $c_1, \ldots, c_m$ such that $f(n) = \sum_{l=1}^{m} c_l n^{-l} + o(n^{-m})$. In the later publications the authors extend this result for *systems* of such recurrence relations. To our opinion this result cannot be applied to our problem (taken $\tau_i = i$). The arguments can be found in the technical report (Shkaravska and van Eekelen, 2010).

Indeed, recurrence relation (1) can be converted into the form above in two ways.

- Either we define $f(n) := (1/n^{d+1})P(n)$, where $d = \deg(P)$, and then $f(n)$ will be having the just mentioned form $f(n) = \sum_{l=1}^{m} c_l n^{-l} + o(n^{-m})$ and satisfy

  $$f(n) = \big(1/n^{d+1}\big)G\big((n-1)^{d+1}f(n-1), \ldots, (n-s)^{d+1}f(n-s)\big)$$

  However this conversion is not possible unless we know the degree d.
- Or we use the derived equation $P(x) = G(P(x-1), \ldots, P(x-s))$ that holds for all real numbers x (see Lemma 1). Then we have that

  $$P\big(1/n^t\big) = G\Big(P\big(1/n^t - 1\big), \ldots, P\big(1/n^t - s\big)\Big)$$

  for some $t \geq 2$. However the obvious in this case definition $f(n) := P(1/n^t)$ (that indeed has the mentioned above form) does not fit the recurrence scheme $f(n) = G(f(n-1), \ldots, f(n-s))$ since $P(1/n^t - s) \neq P\big(1/(n-s)^t\big) = f(n-s)$. It is not clear, which concrete recurrence schema, with all known coefficients, describes $f(n)$.

2. Special presentation of monomials $a_{i_1 \ldots i_s} x_1^{i_1} \ldots x_s^{i_s}$

2.1. Polynomial difference equations $G\big(P(x-\tau_1), \ldots, P(x-\tau_s)\big) = 0$

Before studying difference relations in detail, we note that a *recurrence relation* with a polynomial solution defined on natural numbers determines a difference equation with the same schema.

Lemma 1. Let a polynomial $P(x)$ satisfy $G(P(n-1), \ldots, P(n-s)) + G_0(n) = 0$ for all integer $n \geq n_0$, for some n_0. Then $G(P(x-1), \ldots, P(x-s)) + G_0(x)$ for all real $x \in \mathcal{R}$.

Proof. From the condition of the lemma it follows that the polynomial in x, namely $G(P(x-1), \ldots, P(x-s)) + G_0(x)$, is equal to zero in some $\deg(P) + 1$ pairwise different points. From this follows that it is zero for all $x \in \mathcal{R}$. □

This property makes the difference equation analysis applicable to analysis of recurrence relations.

Lemma 2. Let a function $f(x)$ (which is not necessarily a polynomial) satisfy $f(x) = G(f(x-1), \ldots, f(x-s))$ for all real x. Then any $g(x)$, such that $g(x) = f(x+y)$ for some real number y, satisfies the equation $g(x) = G(g(x-1), \ldots, g(x-s))$ as well.

Proof. By the definition of g one has $g(x) = f(x+y) = G(f(x+y-1), \ldots, f(x+y-s)) = G(f(x-1+y), \ldots, f(x-s+y) = G(g(x-1), \ldots, g(x-s))$.
□

To begin with, we recall the definition: the *total degree* D of a multivariate polynomial $G(x_1, \ldots, x_s) = \sum_{0 \leq i_1+\ldots+i_s} a_{i_1 \ldots i_s} x^{i_1} \ldots x^{i_s}$, where $i_1, \ldots, i_s$ are non-negative integers, is given by $D = \max\{i_1 + \ldots + i_s | a_{i_1 \ldots i_s} \neq 0\}$, and D must be finite.

Without lost of generality we assume that the translation set $T = \{\tau_1, \ldots, \tau_s\}$ is ordered according to the standard ordering of real numbers $\mathcal{R}$: $\tau_1 < \ldots < \tau_s$.

Now we re-index the coefficients $a_{i_1 \ldots i_s}$ of $G(x_1, \ldots, x_s)$. This is the reindexation I that maps, e.g. the term $a_{20}x_1^2$ to the term $\alpha_{11}x_1x_1$. Consider another example: let $D = 5$ and $s = 3$, so we take $G_5(x_1, x_2, x_3)$ of degree 5. Then the term $a_{2,3,0}x_1^2x_2^3$ may be written as $\alpha_{1,1,2,2,2}x_1x_1x_2x_2x_2$, where $\alpha_{k_1,k_1,k_2,k_2,k_2} = a_{2,3,0}$. In general, the reindexation I maps $(i_1 \ldots i_s)$ to $(k_1 \ldots k_D) = (1^{(i_1)}, 2^{(i_2)}, \ldots, s^{(i_s)})$, where $k^{(i)}$ denotes the i-dimensional vector $(k, \ldots, k)$. Thus, the we have defined map I from $\mathcal{I} := \{(i_1, \ldots, i_s) \,|\, i_1 + \ldots + i_s = D,\ i_j \in \mathcal{N}\}$ to $K := \{(k_1 \ldots k_D) \,|\, 1 \leq k_1 \leq \ldots \leq k_D \leq s\}$. It is a routine to show that I is a bijection.

Lemma 3. The reindexing I that sends $(i_1 \ldots i_s) \in \mathcal{I}$ to $(k_1 \ldots k_D) = (1^{(i_1)}, \ldots, s^{(i_s)})$, is a bijection.

Proof. By its definition, I is a map (i.e. is functional and everywhere defined). We have to prove that it is injective and surjective.

We prove injectivity by contradiction. Assume that there are two different indices, $(i_1 \ldots i_s)$ and $(i'_1 \ldots i'_s)$, that are mapped to the same $(k_1 \ldots k_D)$. Let $\ell = \min\{j | i_j \neq i'_j\}$. Therefore, $(i'_1 \ldots i'_s) = (i_1 \ldots i_{\ell-1}, i'_\ell, \ldots, i'_s)$. Now, $(k_1 \ldots k_D) = (1^{(i_1)}, \ldots, \ell^{(i_\ell)}, \ldots, s^{(i_s)}) \neq (1^{(i_1)}, \ldots, (\ell-1)^{(i_{\ell-1})}, \ell^{(i'_\ell)}, \ldots s^{(i'_s)}) = (k_1 \ldots k_D)$, which is a contradiction. So, the map I is an injection.

To prove surjectivity, we fix any $(k_1, \ldots, k_D) \in K$. It easy to see, that by the definition of K there exist $i_j \in \mathcal{N}$, such that $i_1 + \ldots + i_s = D$ and $(k_1, \ldots, k_D) = (1^{(i_1)}, \ldots, s^{(i_s)})$. We take $\mathbf{i} = (i_1, \ldots, i_s)$. Trivially, $I(\mathbf{i}) = (1^{(i_1)}, \ldots, s^{(i_s)}) = (k_1, \ldots, k_D)$. Therefore, the map I is a surjection. □

Using this reindexation, a polynomial $G_D(x_1, \ldots, x_s)$ is presented as $\Sigma_{\mathbf{k} \in K} \alpha_{\mathbf{k}} x_{k_1} \ldots x_{k_D}$, where $\mathbf{k} := (k_1, \ldots, k_D)$. For instance, for $D = 2, s = 3$ we have $K = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$, and for the polynomial $G_2(x_1, x_2, x_3) = x_1^2 - 2x_1x_2 + x_3^2$ we have $\alpha_{11} = 1$, $\alpha_{12} = -2$, $\alpha_{33} = 1$ and $\alpha_{13} = \alpha_{22} = \alpha_{23} = 0$. Consider another example: $G(x_1, \ldots, x_5) = a_{20000}x_1^2 - 2a_{10010}x_1x_4 + a_{00002}x_5^2$. The corresponding reindexed polynomial is $\alpha_{11}x_1^2 - 2\alpha_{14}x_1x_4 + \alpha_{55}x_5^2$. Here we have $D = 2$, $s = 5$, and all the possible indices are in the set

$$K = \{(1,1), (1,2), (1,3), (1,4), (1,5), (2,2), (2,3), (2,4), (2,5), (3,3), (3,4), (3,5), (4,4), (4,5)\}$$

Here we have that all α_{ij}, except α_{11}, α_{14}, α_{55}, vanish.

Further, a polynomial $G_D\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)$ is presented as $\Sigma_{\mathbf{k}\in K}\alpha_{\mathbf{k}}P(x-\tau_{k_1})\ldots P(x-\tau_{k_D})$. For the sake of convenience, we introduce the notations that reflects the corresponding reindexing of τ_i in $G\big(P(x-\tau_1),\ldots,P(x-\tau_s)\big)$:

$$\mathbf{t}(\mathbf{k}) := (\tau_{k_1},\ldots,\tau_{k_D}) \text{ for each } \mathbf{k}=(k_1,\ldots,k_D)\in K$$
$$\mathbf{t}=(t_1,\ldots,t_D) \text{ ranges over } \mathbf{t}(\mathbf{k}),\mathbf{k}\in K.$$

For example, for the polynomial $G_2\big(P(x-0.5),P(x-1),P(x-2.1)\big)=P(x-0.5)P(x-0.5)-2P(x-0.5)P(x-1)+P(x-2.1)P(x-2.1)$, we have $\tau_1=0.5,\tau_2=1,\tau_3=2.1$, and therefore, $\mathbf{t}(11)=(0.5,0.5)$, $\mathbf{t}(12)=(0.5,1)$ and $\mathbf{t}(33)=(2.1,2.1)$.

3. Coefficients of x in $G_D\big(P(x-\tau_1,\ldots,P(x-\tau_s)\big)$ as symmetric polynomials

Let a polynomial $P(x)$ be presented via its roots: $P(x)=A_d(x-\rho_1)\ldots(x-\rho_d)$. We want to see how the left-hand side of equation 2 looks if we substitute this presentation of $P(x)$ into it. Have a closer look at the term $P(x-t_1)\ldots P(x-t_D)$. Obviously, it is equal to $A_d^D\prod_{i=1}^D\prod_{j=1}^d(x-t_i-\rho_j)$. For this product, we want to find the coefficients $\varepsilon_l(\mathbf{t},\mathbf{r})$ of x^{Dd-l}, where $0\le l\le d-1$. The sums $(t_i+\rho_j)$, where $1\le i\le D$, $1\le j\le d$ are obviously the (only) roots of the polynomial $\prod_{i=1}^D\prod_{j=1}^d(x-t_i-\rho_j)$. Therefore, its coefficients $\varepsilon_l(\mathbf{t},\mathbf{r})$ are presented via *elementary symmetric polynomials* $e_l(y_1,\ldots,y_m):=\sum_{w\neq w'\to i_w\neq i_{w'}}y_{i_1}\ldots y_{i_l}$ and $e_0(y_1,\ldots,y_m)=1$ (Macdonald, 1979) in the standard way:

$$\varepsilon_l(\mathbf{t},\mathbf{r})=(-1)^l e_l(t_1+\rho_1,\ldots,t_i+\rho_j,\ldots,t_D+\rho_d) \qquad (3)$$

assuming that in the presentation one chooses $m=dD$. For the sake of convenience we denote the dD-dimensional vector $(t_1+\rho_1,\ldots,t_i+\rho_j,\ldots,t_D+\rho_d)$ via $(\mathbf{t}+\mathbf{r})$.

Now, we can proceed with the the coefficients of x^{dD-l} in the l.h.s. of equation 2. Let $N:=|K|$ and let the elements $\mathbf{k}$ of K be ordered in the obvious way: $\mathbf{k}_1=(1,\ldots,1),\mathbf{k}_2=(1,\ldots,1,2),\ldots,\mathbf{k}_N=(s,\ldots,s)$.

Lemma 4. If a polynomial P of a degree d solves equation 2 and $d>l$ for some $l>0$ then polynomial's roots $\mathbf{r}$ must solve the equation

$$\textstyle\sum_{\mathbf{k}\in K}\varepsilon_l(\mathbf{t}(\mathbf{k}),\mathbf{r})\alpha_{\mathbf{k}}=0 \qquad (4)$$

Proof. Due to $d>l$ we have that $dD-l>d(D-1)$. Since P solves equation 2, the coefficients $\sum_{\mathbf{k}\in K}\varepsilon_l(\mathbf{t}(\mathbf{k}),\mathbf{r})\alpha_{\mathbf{k}}$ at x^{dD-l} in the l.h.s. of equation 2 must vanish. □

Lemma 4 does not give direct information about d, since $\varepsilon_l(\mathbf{t},\mathbf{r})$ depends on d implicitly: d is the dimension of $\mathbf{r}$. To obtain from equation 4 an explicit equation over d we employ *power-sum symmetric polynomials* $p_l(y_1,\ldots,y_m):=y_1^l+\ldots+y_m^l$ (with $p_0(y_1,\ldots,y_m):=m$) and *Newton-Girard formula* (Macdonald, 1979):

$$e_l(\mathbf{y}) = (1/l)\textstyle\sum_{\kappa=1}^l(-1)^{\kappa-1}e_{l-\kappa}(\mathbf{y})p_\kappa(\mathbf{y})$$

For the sake of convenience, we introduce the following notations: $\bar{\alpha}$, $p_{l,\mathbf{t}}$ and $p_{l,\mathbf{r}}$ stay for $(\alpha_1, \dots, \alpha_N) = (\alpha_{\mathbf{k}})_{\mathbf{k}\in K}$, $p_l(\mathbf{t})$ and $p_l(\mathbf{r})$ respectively.

It is a routine to show, by the definition of p_κ and the binomial formula, that

$$p_\kappa(\mathbf{t}+\mathbf{r}) = \sum_{i=1}^{D}\sum_{j=1}^{d}(t_i+\rho_j)^\kappa = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda,\mathbf{t}}p_{\lambda,\mathbf{r}} \tag{5}$$

In more detail,

$$\begin{aligned} p_\kappa(\mathbf{t}+\mathbf{r}) = \quad & \textstyle\sum_{i=1}^{D}\sum_{j=1}^{d}(t_i+\rho_j)^\kappa = \sum_{i=1}^{D}\sum_{j=1}^{d}\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\rho_j^\lambda t_i^{\kappa-\lambda} = \\ & \textstyle\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}\sum_{j=1}^{d}\rho_j^\lambda t_i^{\kappa-\lambda} = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}t_i^{\kappa-\lambda}\sum_{j=1}^{d}\rho_j^\lambda = \\ & \textstyle\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\sum_{i=1}^{D}t_i^{\kappa-\lambda}p_{\lambda,\mathbf{r}} = \sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda,\mathbf{t}}p_{\lambda,\mathbf{r}} \end{aligned}$$

Let $\mathbf{v}_l$ and $\mathbf{u}_l$ denote the vectors of variables $(v_0, \dots, v_l)$ and $(u_0, \dots, u_l)$ respectively. To show how Newton-Girard formula is used to present $\varepsilon_l(\mathbf{t},\mathbf{r})$, we define inductively a family of functions $E_l(\mathbf{v}_l,\mathbf{u}_l)$. The definition mirrors Newton-Girard formula and identity 5:

Definition 1.

$$\begin{aligned} & E_0(\mathbf{v}_0,\mathbf{u}_0) := 1 \\ & E_l(\mathbf{v}_l,\mathbf{u}_l) := -(1/l)\textstyle\sum_{\kappa=1}^{l}E_{l-\kappa}(\mathbf{v}_{l-\kappa},\mathbf{u}_{l-\kappa})(\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}v_{\kappa-\lambda}u_\lambda) \end{aligned}$$

For instance, $E_1(\mathbf{v}_1,\mathbf{u}_1) = -v_1u_0 - v_0u_1$. Let $\mathbf{p}_{l,\mathbf{t}}$ and $\mathbf{p}_{l,\mathbf{r}}$ denote $(D, p_{1,\mathbf{t}}, \dots, p_{l,\mathbf{t}})$ and $(d, p_{1,\mathbf{r}}, \dots, p_{l,\mathbf{r}})$ respectively. It is a routine to establish the following connection between a coefficient $\varepsilon_l(\mathbf{t},\mathbf{r})$ and the function E_l:

Lemma 5. For all $l \geq 0$ the following identity holds:

$$\varepsilon_l(\mathbf{t},\mathbf{r}) = E_l(\mathbf{p}_{l,\mathbf{t}},\mathbf{p}_{l\mathbf{r}}) \tag{6}$$

Proof. Induction on l using Newton-Girard formula on the induction step.

For $l = 0$ immediately by the definitions one obtains $\varepsilon_0(\mathbf{t},\mathbf{r}) = 1 = E_0\big((D),(d)\big)$.

For $l > 0$ we apply Newton-Girard formula. Combining identity 3 with Newton-Girard identities, where $\mathbf{y} := \mathbf{t}+\mathbf{r} = (t_1+\rho_1, \dots, t_1+\rho_d, \dots, t_D+\rho_1, \dots, t_D+\rho_d)$, we obtain

$$(-1)^l\varepsilon_l(\mathbf{t},\mathbf{r}) = (1/l)\textstyle\sum_{\kappa=1}^{l}(-1)^{\kappa-1}(-1)^{l-\kappa}\varepsilon_{l-\kappa}(\mathbf{t},\mathbf{r})p_\kappa(\mathbf{t}+\mathbf{r})$$

From what follows that

$$\begin{aligned} \varepsilon_l(\mathbf{t},\mathbf{r}) = \; & -(1/l)\textstyle\sum_{\kappa=1}^{l}\varepsilon_{l-\kappa}(\mathbf{t},\mathbf{r})p_\kappa(\mathbf{t}+\mathbf{r}) =^{\text{identity 5}} \\ & -(1/l)\textstyle\sum_{\kappa=1}^{l}\varepsilon_{l-\kappa}(\mathbf{t},\mathbf{r})\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}p_{\kappa-\lambda,\mathbf{t}}p_{\lambda,\mathbf{r}} \end{aligned} \tag{7}$$

Using induction assumption for $\varepsilon_{l-\kappa}(\mathbf{t},\mathbf{r})$, we immediately obtain $\varepsilon_l(\mathbf{t},\mathbf{r}) = E_l(\mathbf{p}_{l,\mathbf{t}},\mathbf{p}_{l,\mathbf{r}})$. □

Using function E_l, we can symbolically compute $\varepsilon_l(\mathbf{t},\mathbf{r})$ for any $l > 0$. For instance, $\varepsilon_1(\mathbf{t},\mathbf{r}) = -E_1(\mathbf{p}_{1,\mathbf{t}},\mathbf{p}_{1,\mathbf{r}}) = -d\,p_{1,\mathbf{t}} - D\,p_{1,\mathbf{r}}$.

Now we can reformulate 4, using the following definition.

Definition 2. Let $S_l(\mathbf{u}_l) := \sum_{\mathbf{k}\in K} E_l(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})}, \mathbf{u}_l)\alpha_{\mathbf{k}}$

This definition yields an explicit equation over $d, p_{1,\mathbf{r}}, \ldots, p_{l,\mathbf{r}}$, as stated in the next lemma.

Lemma 6. If a polynomial P of the degree d solves equation 2 and $d > l$ for some $l > 0$ then $S_l(\mathbf{p}_{l,\mathbf{r}}) = 0$.

Proof. By the lemma 5 and definition of S_l one straightforwardly has $\sum_{\mathbf{k}\in K} \varepsilon_l(\mathbf{t}(\mathbf{k}), \mathbf{r})\alpha_{\mathbf{k}} = S_l(\mathbf{p}_{l,\mathbf{r}})$. By lemma 4 we immediately obtain identity $S_l(\mathbf{p}_{l,\mathbf{r}}) = 0$. □

Yet, from the point of view of bounding the degree d, lemma 6 is too general. We are interested in cases when for some $L \geq 0$ the equations $S_l(u_0, u_1, \ldots, u_l) = 0$, with $0 \leq l \leq L$, yield a 1-variate equation $S^*_L(u_0) = 0$, where

Definition 3. $S^*_l(u_0)$ is a $u_1, \ldots, u_l$-free term of $S_l(u_0, u_1, \ldots, u_l)$.

To discover, when reduction to $S^*_L(u_0) = 0$ is possible for some L, we need to have a closer look at functions $E_l(\mathbf{v}_l, \mathbf{u}_l)$. These functions are obviously polynomials in $\mathbf{v}_l, \mathbf{u}_l$. The total degree of $v_0, \ldots, v_l$ and $u_0, \ldots, u_l$ is l, however one can prove more precise connection between the degrees of v- and u-variables:

Lemma 7. For any monomial of $v_0^{j_0} \ldots v_l^{j_l} u_0^{i_0} \ldots u_l^{i_l}$ that occurs in $E_l(\mathbf{v}_l, \mathbf{u}_l)$ the following inequation holds: $0 \cdot j_0 + j_1 + 2j_2 + \ldots + lj_l + 0 \cdot i_0 + i_1 + 2i_2 + \ldots + li_l \leq l$.

The proof is routine induction on l.

Proof. For $l = 0$ the statement trivially holds: $0j_0 + 0i_0 = 0$.

For $l = 1$ we have $E_1 = -v_1u_0 - v_0u_1$. Thus, (j_0, j_1, i_0, i_1) ranges for two monomials, $-v_1u_0$ and $-v_0u_1$, over $(0, 1, 1, 0)$ and $(1, 0, 0, 1)$, respectively. Trivially, we have $0 + j_1 + 0 + 0 = 1$ and $0 + 0 + 0 + i_1 = 1$.

Now, let the property hold for all $l' < l$. We use the recursive definition of $E_l(\mathbf{v}_l, \mathbf{u}_l)$. It is a sum of monomials of $E_{l-\kappa}(\mathbf{v}_{l-\kappa}, \mathbf{u}_{l-\kappa})$ multiplied by $\binom{\kappa}{\lambda} v_{\kappa-\lambda}u_\lambda$. Therefore, it is enough to consider an arbitrary product of this form. Let q be a monomial of $E_{l-\kappa}(\mathbf{v}_{l-\kappa}, \mathbf{u}_{l-\kappa})$. Let q correspond to the degrees $(j_0, \ldots, j_{l-\kappa}, i_0, \ldots i_{l-\kappa})$. By the induction assumption $\sigma_{l-\kappa} := 0j_0 + j_1 + 2j_2 + \ldots + (l-\kappa)j_{l-\kappa} + 0i_0 + i_1 + 2i_2 + \ldots + (l-\kappa)i_{l-\kappa} \leq l - \kappa$. Now, for the considered product $qv_{\kappa-\lambda}u_\lambda$ we have one $v_{\kappa-\lambda}$ more and one u_λ more w.r.t. q. Therefore, $\sigma_l = \sigma_{l-\kappa} + (\kappa - \lambda) + \lambda \leq l - \kappa + (\kappa - \lambda) + \lambda \leq l$. □

This property is used when one wants to give the complete list of all non-vanishing coefficients of the degrees of $(u_1, \ldots, u_l)$. Now, consider $E_l(\mathbf{v}_l, \mathbf{u}_l)$ as a polynomial in $(u_1, \ldots, u_l)$. Let $\mathbf{i}_l$ denote a "degree vector" $(i_1, \ldots, i_l)$.

Definition 4. Let $A_{l\mathbf{i}_l}(\mathbf{v}, u_0)$ denote the coefficient of $u_1^{i_1} \ldots u_l^{i_l}$ in $E_l(\mathbf{v}_l, \mathbf{u}_l)$, that is

$$E_l(\mathbf{v}_l, \mathbf{u}_l) = \sum_{\mathbf{i}_l} A_{l\mathbf{i}_l}(\mathbf{v}_l, u_0) \cdot u_1^{i_1} \ldots u_l^{i_l}$$

It is routine to check the following presentation of $S_l(u_0, u_1, \ldots, u_l)$ as a polynomial in $u_1, \ldots, u_l$:

$$\begin{array}{l} S_l(u_0, u_1, \ldots, u_l) = \sum_{\mathbf{i}_l} F_{l\mathbf{i}_l}(u_0) \cdot u_1^{i_1} \ldots u_l^{i_l} \text{ where} \\ F_{l\mathbf{i}_l}(u_0) = \sum_{\mathbf{k} \in K} A_{l\mathbf{i}_l}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})}, u_0)\alpha_{\mathbf{k}} \end{array} \tag{8}$$

Indeed,

$$\begin{array}{rl} S_l(u_0, u_1, \ldots, u_l) = & \sum_{\mathbf{k} \in K} E_l(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})}, \mathbf{u}_l)\alpha_{\mathbf{k}} = \\ & \sum_{\mathbf{k} \in K} (\sum_{\mathbf{i}_l} A_{l\mathbf{i}_l}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})}, u_0) u_1^{i_1} \ldots u_l^{i_l})\alpha_{\mathbf{k}} = \\ & \sum_{\mathbf{i}_l} u_1^{i_1} \ldots u_l^{i_l} \big(\sum_{\mathbf{k} \in K} A_{l\mathbf{i}_l}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})}, u_0)\alpha_{\mathbf{k}} \big) \end{array}$$

Recall, that $S_l^*(u_0)$ is the u_i-free monomial of S_l. It easy to see that

$$S_l^*(u_0) := F_{l\mathbf{0}_l}(u_0) \tag{9}$$

where $\mathbf{0}_l$ denotes l-dimensional null-vector.

In its turn, any of $A_{l\mathbf{i}_l}(\mathbf{v}_l, u_0)$ is a polynomial in u_0, with the corresponding coefficients $B_{l\mathbf{i}_l\mu}(\mathbf{v}_l)$ of u_0^{μ}. As we will see soon, the coefficients $B_{l\mathbf{0}_l\mu}$ play a special role. For the sake of readability, we abuse the notations and denote them via $B_{l\mu}$. This yield the coefficients $B_{l\mu}^*$ of $S_l^*(u_0)$ at u_0^{μ}:

$$B_{l\mu}^* = \sum_{\mathbf{k} \in K} B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} \tag{10}$$

Indeed,

$$\begin{array}{rl} S_l^*(u_0) & = \sum_{\mathbf{k} \in K} A_{l\mathbf{0}_l}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})}, u_0)\alpha_{\mathbf{k}} = \\ & \sum_{\mathbf{k} \in K} \big(\sum_{\mu=0}^{l} B_{l\mu}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})}) u_0^{\mu} \big)\alpha_{\mathbf{k}} \\ & \sum_{\mu=0}^{l} u_0^{\mu} \big(\sum_{\mathbf{k} \in K} B_{l\mu}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} \big) = \end{array}$$

Now we can formulate and prove an auxiliary lemma. It exploits the possibility to *present* $A_{l\mathbf{i}_l}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})}, u_0)$ *as a linear combination of* $B_{l'\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})$ *where the coefficients of this linear combination do not depend on* $\mathbf{k}$, and $0 \leq l' < l$.

Lemma 8. Let $L > 0$ be such that for all $0 \leq l \leq L-1$ identities $S_l^*(u_0) \equiv 0$ hold, and moreover, for $\mathbf{i}_L \neq \mathbf{0}_L$ and $\mu \leq l$ there exist functions $H^D_{L\mathbf{i}_l l\mu}(u_0)$, such that

$$A_{L\mathbf{i}_L}(\mathbf{p}_{L,\mathbf{t}(\mathbf{k})}, u_0) = \sum_{l=0}^{L-1} \sum_{\mu=0}^{l} H^D_{L\mathbf{i}_L l\mu}(u_0) B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})}) \tag{11}$$

Then $S_L(u_0, u_1, \ldots, u_l) = S_L^*(u_0)$.

Proof. The condition $S_l(u_0) \equiv 0$, where $0 \leq l < L$, means that all S_l's coefficients of u_0^{μ} vanish. That is, due to identity 10, we have

$$\sum_{\mathbf{k} \in K} B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0 \tag{12}$$

Now, plug-in identity 11 from the condition of the lemma into the definition of F in identity 8:

$$F_{L\mathbf{i}_l}(u_0) = \sum_{\mathbf{k}\in K}\big(\sum_{l=0}^{L-1}\sum_{\mu=0}^{l} H^D_{L\mathbf{i}_l l\mu}(u_0)B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} =$$
$$\sum_{l=0}^{L-1}\sum_{\mu=0}^{l} H^D_{L\mathbf{i}_l l\mu}(u_0)\big(\sum_{\mathbf{k}\in K} B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}}\big) =^{\text{identity 12}}$$
$$\sum_{l=0}^{L-1}\sum_{\mu=0}^{l} H^D_{L\mathbf{i}_l l\mu}(u_0)\cdot 0 = 0$$

We have obtained that $F_{l\mathbf{i}_l}(u_0) \equiv 0$ for all $\mathbf{i}_l \neq \mathbf{0}$, and therefore (recall identity 8) we have $S_l(u_0, u_1, \ldots, u_l) = F_{l,\mathbf{0}_l}(u_0) = S_l^*(u_0)$. □

We finish this section with the main working *framework lemma*.

Lemma 9 (Framework)**.** Let $L = \min\{l \mid S_l^*(u_0) \not\equiv 0\}$ and for all $\mathbf{i}_l \neq \mathbf{0}_l$ and $\mu \leq l < L$ there exist functions $H^D_{L\mathbf{i}_L l\mu}(u_0)$ such that identities 11 hold. Then either $d \leq \max\{L, \deg(G_0)/(D-1)\}$, or d is a root of $S_L^*(u_0)$.

Proof. Consider the "or"-case: $d > L$ and $d > \deg(G_0)/(D-1)$. Then $dD - L > d(D-1) > \deg(G_0)$ which means that the coefficients at x^{dD-L} in the l.h.s. of equation 2 must vanish. We apply lemma 6 to obtain $S_L(\mathbf{p}_{L,\mathbf{r}}) = 0$. Next, we apply lemma 8 and obtain $S_L(u_0, u_1, \ldots, u_l) = S_L^*(u_0)$. From this and the condition $S_L(\mathbf{p}_{L,\mathbf{r}}) = 0$, it follows that $S_L^*(d) = 0$. □

In the remaining sections we study two independent cases for which the conditions of the framework lemma hold and therefore, we can bound d.

4. Existence of a degree polynomial for $0 \leq l \leq 5$

We begin this section considering, when "linear-combination" identities like 11 hold for functions $E_l(\mathbf{v}_l, \mathbf{u}_l)$ in general.

Lemma 10. For all $1 \leq L \leq 5$ and $1 \leq \mu \leq l < L$, for all $\mathbf{i}_L \neq \mathbf{0}_L$ there exists functions $H_{L\mathbf{i}_L l\mu}(v_0, u_0)$ such that $A_{L\mathbf{i}_L}(\mathbf{v}_L, u_0) = \sum_{l=0}^{L-1}\sum_{\mu=0}^{l} H_{L\mathbf{i}_L l\mu}(v_0, u_0)B_{l\mu}(\mathbf{v}_l)$.

Proof.
The coefficients $A_{L\mathbf{i}_L}(\mathbf{v}_L, u_0)$, $B_{l\mu}(\mathbf{v}_l)$ and $H_{L\mathbf{i}_L l\mu}(v_0, u_0)$ are computed symbolically for all $0 \leq L \leq 5$, $0 \leq \mu \leq l < L$ in `coefficients_submission.mw`. We have used procedure `NormalForm` that implements division of the polynomial $A_{L\mathbf{i}_L}(\mathbf{v}_L, u_0)$ by the polynomials from the set $\{B_{l\mu}(\mathbf{v}_l)\}_{0\leq l<L,\ \mu\leq l}$. The coefficients $H_{L\mathbf{i}_L l\mu}(v_0, u_0)$ are fully given in Appendix, section 8.1.
□

To provide the reader with intuition behind our constructions, we consider the cases $l = 0, 1, 2$ in more detail. As we will see, the cases $l = 0, 1$ are degenerated. The case $l = 2$ is a good instance for the general schema. In section 7 we will consider an example for $l = 2$ as well.

4.1. l=0

Assume that $d > 0$. Then $Dd > d$, so we have to cancel n^{Dd} on the left-hand side of equation 2, that is $S_0(d) = 0$ by lemma 6. Using the definition $S_0(u_0) = \sum_{\mathbf{k}\in K} E_0(D, u_0)$ and the definition $E_0(\mathbf{v}_0, \mathbf{u}_0) = 1$ we obtain $S_0(u_0) \equiv \sum_{\mathbf{k}\in K} \alpha_{\mathbf{k}}$, and therefore

$$S_0^*(d) = \sum_{\mathbf{k}\in K} \alpha_{\mathbf{t}(\mathbf{k})} = 0 \tag{13}$$

If $\sum_{\mathbf{k}\in K} \alpha_{\mathbf{k}} \neq 0$ then the coefficient at n^{Dd} on the l.h.s. does not vanish and the assumption $d > 0$ cannot hold. Therefore, $d = 0$ and the polynomial solution of equation 2 can be only constant. For the sake of uniformity we take $S_0^*(u_0) \equiv \sum_{\mathbf{t}\in K} \alpha_{\mathbf{t}}$ as a degree polynomial, although in this case it is degenerated to a constant and does not have roots.

If $S_0^*(u_0) \equiv 0$ then continue to check for $d > 1$.

4.2. l=1

Assume that $d > 1$ then, again comparing the l.h.s. and the r.h.s. of equation 2 we have $Dd - 1 > Dd - d = (D-1)d \geq d$, so we have to cancel n^{Dd-1} as well. By lemma 6 this means that $S_1(d, p_{1,\mathbf{r}}) = 0$. Now we simplify this equation, using condition 13.

By the definition, $S_1(u_0, u_1) := \sum_{\mathbf{k}\in K} E_1(D, p_{1,\mathbf{t}(\mathbf{k})}, u_0, u_1)\alpha_{\mathbf{k}}$. By the definition, $E_1(v_0, v_1, u_0, u_1) = -v_0 u_1 - v_1 u_0$. From what follows that

$$\begin{aligned} S_1(u_0, u_1) &= -Du_1 \textstyle\sum_{\mathbf{k}\in K} \alpha_{\mathbf{k}} - u_0 \sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} \\ &=^{\text{equation 13}} -u_0 \textstyle\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} \\ &=^{\text{definition of } S_l^*} S_1^*(u_0) \end{aligned}$$

Therefore,

$$S_1^*(d) = -d \sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0 \tag{14}$$

Taking into account that $d > 1$ this implies $\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$. If $\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} \neq 0$ the coefficient at n^{Dd-1} on the l.h.s. does not vanish and the assumption $d > 1$ cannot hold. Therefore, $d = 0, 1$ and the polynomial solution of equation 2 can be only a constant or a linear function. We take $S_1^*(u_0) \equiv -u_0 \sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}$ as a degree polynomial. In this case it has only one solution $d = 0$, which does not make sense for $d > 1$.

If $\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$, which means that $S_1^*(u_0) \equiv 0$, we continue to check for $d > 2$.

4.3. l=2

Assume that $d > 2$ then, comparing the l.h.s. and the r.h.s. of equation 2 we have $Dd - 2 > Dd - d = (D-1)d \geq d$, so we have to cancel n^{Dd-2}. By lemma 6 this means that $S_2(d, p_{1,\mathbf{r}}, p_{2,\mathbf{r}}) = 0$. We are to simplify this equation, using conditions 13 and 14. By the definition, $S_2(u_0, u_1, u_2) = \sum_{\mathbf{k}\in K} E_2(D, p_{1,\mathbf{t}(\mathbf{k})}, p_{2,\mathbf{t}(\mathbf{k})}, u_0, u_1, u_2)\alpha_{\mathbf{k}}$. One unfolds the recursive definition of $E_2(v_0, v_1, v_2, u_0, u_1, u_2)$ and obtains the following coefficients $A_{2,i_1 i_2}(v_0, v_1, v_2, u_0)$ at $u_1^{i_1} u_2^{i_2}$, see Appendix and Maple script as well:

- $A_{200}(v_0, v_1, v_2, u_0) = (1/2)u_0^2 v_1^2 - (1/2)u_0 v_2$ is the u_1, u_2-free term of E_2,
- $A_{210}(v_0, v_1, v_2, u_0) = u_0 v_0 v_1 - v_1$,
- $A_{220}(v_0, v_1, v_2, u_0) = (1/2)v_0^2$,

• $A_{201}(v_0, v_1, v_2, u_0) = -(1/2)v_0$.
Apply definition 8 to obtain:
• $F_{2,00}(u_0) = (u_0/2)\sum_{\mathbf{k}\in K}(u_0 p^2_{1,\mathbf{t}(\mathbf{k})} - p_{2,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}}$,
• $F_{2,10}(u_0) = (u_0 D - 1)\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = (1/u_0 - D)S^*_1(u_0)$,
• $F_{2,20}(u_0) = (1/2)D^2\sum_{\mathbf{k}\in K}\alpha_{\mathbf{k}} = (1/2)D^2 S^*_0(u_0)$,
• $F_{2,01}(u_0) = -(1/2)DS^*_0(u_0)$,
• the coefficients for remaining degrees $u_1^{i_1}u_2^{i_2}$ of E_2 are zero.
Since $S^*_1(u_0) \equiv S^*_0(u_0) \equiv 0$, we immediately obtain that the coefficients $F_{2,i_1 i_2}(u_0)$ vanish for $(i_1 i_2) \neq (00)$, and therefore $S_2(u_0, u_1, u_2) = S^*_2(u_0)$. Thus,

$$S^*_2(d) = (d/2)\sum_{\mathbf{k}\in K}(dp^2_{1,\mathbf{t}(\mathbf{k})} - p_{2,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0 \tag{15}$$

Taking into account that $d > 2$ this implies $\sum_{\mathbf{k}\in K}(dp^2_{1,\mathbf{t}(\mathbf{k})} - p_{2,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0$. If this identity does not hold, then the coefficient at n^{Dd-2} on the l.h.s. does not vanish and the assumption $d > 2$ does not hold. Therefore, $d = 0, 1$ and the polynomial solution of equation 2 can be “at most” a quadratic function. We take $S^*_2(u_0)$ as a degree polynomial. In this case it has two solutions: $d = 0$, which does not make sense for $d > 2$, and

$$d = \frac{\sum_{\mathbf{k}\in K} p_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}}{\sum_{\mathbf{k}\in K} p^2_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}}$$

If $\sum_{\mathbf{k}\in K}(dp^2_{1,\mathbf{t}(\mathbf{k})} - p_{2,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} \equiv 0$, which means that $S^*_2(u_0) \equiv 0$, we continue to check for $d > 3$.

4.4. l=3

Assume that $d > 3$ then, comparing the l.h.s. and the r.h.s. of equation 2 we have $Dd - 3 > Dd - d = (D-1)d \geq d$, so we have to cancel n^{Dd-3}. By lemma 6 this means that $S_3(d, p_{1,\mathbf{r}}, p_{2,\mathbf{r}}, p_{3,\mathbf{r}}) = 0$. We are to simplify this equation, using conditions 13, 14 and 15. By the definition,

$$S_3(u_0, u_1, u_2, u_3) = \sum_{\mathbf{k}\in K} E_3\big((D, p_{1,\mathbf{t}(\mathbf{k})}, p_{2,\mathbf{t}(\mathbf{k})}, p_{3,\mathbf{t}(\mathbf{k})}), (u_0, u_1, u_2, u_3)\big)\alpha_{\mathbf{k}}$$

One unfolds the recursive definition of $E_3(\mathbf{v}_3, \mathbf{u}_3)$ and obtains the coefficients $A_{3\mathbf{i}_3}(\mathbf{v}_3, u_0)$ at $u_1^{i_1}u_2^{i_2}u_3^{i_3}$. The polynomial $A_{3\mathbf{0}_3}(\mathbf{v}_3, u_0)$ and the coefficients $H_{3\mathbf{i}_3 l\mu}(v_0, u_0)$ for the remaining $A_{3\mathbf{i}_3}(\mathbf{v}_3, u_0)$ are given in Appendix (subsection 8.1).

Further, using the conditions 13, 14 and 15 we obtain, that

$$S^*_3(\mathbf{u}_3) = \sum_{\mathbf{k}\in K}\Big(-(1/6)p^3_{1,\mathbf{t}(\mathbf{k})}u_0^3 + (1/2)p_{1,\mathbf{t}(\mathbf{k})}p_{2,\mathbf{t}(\mathbf{k})}u_0^2 - (1/3)p_{3,\mathbf{t}(\mathbf{k})}u_0\Big)\alpha_{\mathbf{k}}$$

If $S^*_0(u_0) \equiv S^*_1(u_1) \equiv S^*_2(u_0) \equiv 0$, then either $d \leq 3$ or $S^*_3(d) = 0$, and if $S^*_3(u_0) \not\equiv 0$ then d must be amongst its (natural) roots. If $S^*_3(u_0) \equiv 0$ as well, continue to check for $d > 3$.

Now we can formulate the main result, which gives us an effective bound on d in the case when there exists $0 \leq L \leq 5$ such that $S^*_L(u_0) \not\equiv 0$.

Theorem 5. *If $L := \min\{l | S^*_l(u_0) \not\equiv 0\} \leq 5$, then either $d \leq \max\{L, \deg(G_0)/(D-1)\}$ or d must be amongst the natural roots of $S^*_L(u_0)$.*

Proof. The condition $L \leq 5$ together with lemma 10 immediately yield the conditions of the framework lemma. Applying it straightforwardly gives us the desired conclusion. $\square$

In section 7 we will consider application of theorem 5 to a quadratic difference equation of the form $G\big(P(x-1), P(x-2), P(x-3)\big) = 0$. It turns out that the theorem can be applied for any such equation.

Corollary 1. For any difference equation 2 with $D = 2$ and $\tau_i = i$ with $i = 1, 2, 3$, there is $0 \leq L \leq 5$ such that $S^*_L(u_0) \not\equiv 0$. Therefore, the degree d of a polynomial solution P either does not exceed $\max\{L, \deg(G_0)/(D-1)\}$, or must be among the natural roots of polynomial $S^*_L(u_0)$.

Proof. Assume the opposite: $S^*_0(u_0) \equiv \ldots \equiv S^*_5(u_0) \equiv 0$. We will show that in this case G_D is reduced to a zero polynomial. With $D = 2$ and $\tau_i = i$, where $i = 1, 2, 3$, we have $T = K = \{(1,1), (1,2), (1,3), (2,2), (2,3), (3,3)\}$. Compute the concrete values of $B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})$ for all $\mathbf{k} \in K$, $1 \leq l \leq 5$, $1 \leq \mu \leq l$, and B_{00}. These values form he matrix of the following over-defined linear system of 16 equations w.r.t. 6 variables $\alpha_{\mathbf{k}}$ (see Appendix, section 8.2): $\sum_{\mathbf{k}\in K} B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0$. This system has only zero solution $\bar{\alpha} = \mathbf{0}_6$ which means that $G_D \equiv 0$ and the difference equation degenerates to a linear one with $D' = D - 1 = 1$. $\square$

In the same way one can prove

Corollary 2. For any difference equation 2 with $D = 3$ and $\tau_i = i$ with $i = 1, 2$, there is $0 \leq L \leq 5$ such that $S^*_L(u_0) \not\equiv 0$. Therefore, the degree d of a polynomial solution P either does not exceed $\max\{L, \deg(G_0)/(D-1)\}$, or must be among the natural roots of polynomial $S^*_L(u_0)$.

Proof. Assume the opposite: $S^*_0(u_0) \equiv \ldots \equiv S^*_5(u_0) \equiv 0$. We will show that in this case G_D is reduced to a zero polynomial. Take

$$T = K = \{(1,1,1), (1,1,2), (1,2,2), (2,2,2)\},$$

and compute $B_{l\mu}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})})$ for all $\mathbf{k} \in K$, $1 \leq l \leq 5$, $1 \leq \mu \leq l$, and B_{00}. Out of the conditions

$$\sum_{\mathbf{t}\in K} B_{l\mu}(D, p_{1,\mathbf{t}(\mathbf{k})}, \ldots, p_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0$$

obtain the over-defined system of 16 linear equations w.r.t. 4 variables $\alpha_{\mathbf{k}}$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -3 & -4 & -5 & -6 \\ -3/2 & -3 & -9/2 & -6 \\ 9/2 & 8 & 25/2 & 18 \\ -1 & -10/3 & -17/3 & -8 \\ 9/2 & 12 & 45/2 & 36 \\ -9/2 & -32/3 & -125/6 & -36 \\ -3/4 & -9/2 & -33/4 & -12 \\ 33/8 & 107/6 & 923/24 & 66 \\ -27/4 & -24 & -225/4 & -108 \\ 27/8 & 32/3 & 625/24 & 54 \\ -3/5 & -34/5 & -13 & -96/5 \\ 15/4 & 28 & 267/4 & 120 \\ -63/8 & -134/3 & -2915/24 & -252 \\ 27/4 & 32 & 375/4 & 216 \\ -81/40 & -128/15 & -625/24 & -324/5 \end{pmatrix} \bar{\alpha} = \bar{0}$$

The matrix of this system is computed and the system is solved by the generic script

$$\texttt{corollaries} := \texttt{proc}(\texttt{s} :: \texttt{posint}, \texttt{Dc} :: \texttt{posint})$$

that can be found in Maple worksheet `corollaries.mw`. The system has only the trivial solution, $\alpha_{\mathbf{k}} = 0$ for all $\mathbf{k} \in K$, $1 \leq l \leq 5$, $1 \leq \mu \leq l$ and $B_{00} = 1$, so the recurrence relation degenerates to a linear recurrence relation with $D' = D - 1$. $\square$

We complete this section by showing that for $l = 6$ there exist $\mathbf{i}_6$ such that $A_{6\mathbf{i}_l}$ is not a linear combination of $B_{l'\mu}$, where $0 \leq l' < 6$, and therefore we cannot apply the framework lemma. Thus, if there is no $S_l^*(u_0) \not\equiv 0$ for some $l \leq 5$ then our approach, in general, does not give a bound on d.

4.5. Case $l \geq 6$: linear dependency between the coefficients cannot be proven in general

In this section we show that in general $S_0^*(u_0) \equiv \ldots \equiv S_5^*(u_0) \equiv 0$ does not imply $S_6(\mathbf{u}_6) \equiv S_6^*(u_0)$.

Lemma 11. If $S_l^*(u_0) \equiv 0$ holds for all $0 \leq l \leq 5$, then

$$S_6(\mathbf{u}_6) = S_6^*(u_0) - u_0 u_2 (1/8) \sum_{\mathbf{k} \in K} p_{2,\mathbf{t}(\mathbf{k})}^2 \alpha_{\mathbf{k}} + u_1^2 (1/8) \sum_{\mathbf{k} \in K} p_{2,\mathbf{t}(\mathbf{k})}^2 \alpha_{\mathbf{k}} \tag{16}$$

Proof. Compute all $A_{6\mathbf{i}_6}$ (see Appendix, section 8.1), and the corresponding coefficients $H_{6\mathbf{i}_6 l\mu}$ for the linear combinations over $B_{l\mu}(\mathbf{v}_l)$. One can directly see that all the coefficients, except for $A_{6,(200000)}$ and $A_{6,(010000)}$, do not depend on $v_1, \ldots, v_6$. Therefore, the corresponding sums $\sum_{\mathbf{k} \in K} A_{6\mathbf{i}_6}(\mathbf{p}_{6,\mathbf{t}(\mathbf{k})}, u_0)\alpha_{\mathbf{k}}$ vanish, since all $\sum_{\mathbf{k} \in K} B_{l\mu}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{k}} = 0$. Further, under the latter equations the coefficients $A_{6,(010000)}$ and $A_{6,(200000)}$ of u_2 and u_1^2, are reduced to $(1/4)u_0 v_2 B_{21}(v_1, v_2) = -(1/8)u_0 v_2^2$ and $-(1/4)v_2 B_{21}(v_1, v_2) = (1/8)v_2^2$ respectively. Desired identity 16 follows from these identities and the definition of $S_l(\mathbf{u}_l)$. $\square$

Now, the natural question is if it is possible at all, that there exists a difference equation for which $S_l^*(u_0) \equiv 0$ for all $0 \leq l \leq 5$ and therefore we cannot give a bound on the degree d of a possible polynomial solution. The answer is "yes" and an example of such a difference equation is $P(x-1)P(x-2)P(x-4) - 2P(x-1)P(x-3)P(x-3) + P(x-1)P(x-3)P(x-4) + P(x-2)P(x-2)P(x-3) - 2P(x-2)P(x-2)P(x-4) + P(x-2)P(x-3)P(x-3) = 0$. It is a routine to check that $S_l^*(u_0) \equiv 0$ for all $l = 0 \dots 5$. Moreover $\sum_{\mathbf{k}\in K} p_{2,\mathbf{k}}^2 \alpha_{\mathbf{k}} = 16$, which means by lemma 11 that $S_6(d, p_{1,\mathbf{r}}, p_{2,\mathbf{r}}) = 0$ is reduced to $S_6^*(d) - 2dp_{2,\mathbf{r}} + 2p_{1,\mathbf{r}}^2 = 0$ that is dependency on the solution's roots does not vanish. This example can be generalised by the following statement

Corollary 3. For any difference equation with $D = 3$ and $\tau_i = i$ with $i = 1..4$, the polynomials $S_l^*(x)$ are constant zeros for all $0 \leq l \leq 5$, if and only if the coefficients of G_D satisfy

$$\begin{aligned}
\alpha_{111} &= x_8 + x_2 + 6x_3 + x_4 + 6x_5 + 21x_6 + 56x_7 \\
\alpha_{112} &= -6x_8 - 6x_2 - 35x_3 - 6x_4 - 35x_5 - 120x_6 - 315x_7 \\
\alpha_{113} &= 6x_8 + 4x_2 + 24x_3 + 3x_4 + 20x_5 + 70x_6 + 180x_7 \\
\alpha_{114} &= -2x_8 - x_2 - 6x_3 - 3x_5 - 12x_6 - 30x_7 \\
\alpha_{122} &= 9x_8 + 11x_2 + 60x_3 + 12x_4 + 64x_5 + 210x_6 + 540x_7 \\
\alpha_{123} &= -18x_8 - 13x_2 - 72x_3 - 12x_4 - 66x_5 - 216x_6 - 540x_7 \\
\alpha_{124} &= 6x_8 + 6x_2 + 25x_3 + 23x_5 + 66x_6 + 153x_7 + x_1 + 6x_4 \\
\alpha_{133} &= 9x_8 - 2x_1 - 3x_2 - 9x_4 - 16x_5 - 27x_6 - 54x_7 \\
\alpha_{134} &= -6x_8 + 2x_2 + 3x_3 + 13x_5 + 28x_6 + 63x_7 + x_1 + 6x_4 \\
\alpha_{144} &= x_8 \\
\alpha_{222} &= -6x_2 - 27x_3 - 36x_5 - 108x_6 - 270x_7 - 8x_4 \\
\alpha_{223} &= 12x_2 + 45x_3 + 63x_5 + 171x_6 + 405x_7 + x_1 + 18x_4 \\
\alpha_{224} &= -8x_2 - 24x_3 - 34x_5 - 84x_6 - 189x_7 - 2x_1 - 12x_4 \\
\alpha_{233} &= x_1 \\
\alpha_{234} &= x_2 \\
\alpha_{244} &= x_3 \\
\alpha_{333} &= x_4 \\
\alpha_{334} &= x_5 \\
\alpha_{344} &= x_6 \\
\alpha_{444} &= x_7
\end{aligned} \tag{17}$$

for some real numbers $x_1, \dots, x_8$. In this case the sum $\sum_{\mathbf{k}\in K} p_{2,\mathbf{k}}^2 \alpha_{\mathbf{k}}$ is equal to $g(x_1, \dots, x_7) = 48x_2 + 144x_3 + 96x_4 + 240x_5 + 576x_6 + 1296x_7 + 16x_1$ and if $x_1, \dots, x_7$ are such that $g(x_1, \dots, x_7) \neq 0$, then $S_6(\mathbf{u}_6) \not\equiv S_6^*(u_0)$.

Proof. The proof is technically similar to the proof of Corollary 1. We construct a linear system w.r.t. $\bar{\alpha}$, which has solutions if and only if all $S_l^*(u_0) \equiv$ with $0 \leq l \leq 6$. The system is constructed and solved using Maple (run `corollaries(4,3)`). For $D = 3$ and $\tau_i = i$ with $i = 1..4$ we have that $T = K =$

$$\{(1,1,1),(1,1,2),(1,1,3),(1,1,4),(1,2,2),(1,2,3),(1,2,4),(1,3,3),(1,3,4),(1,4,4),$$
$$(2,2,2),(2,2,3),(2,2,4),(2,3,3),(2,3,4),(2,4,4),(3,3,3),(3,3,4),(3,4,4),(4,4,4)\}$$

Out of the conditions

$$\sum_{\mathbf{k}\in K} B_{l\mu}(D, p_{1,\mathbf{k}}, \ldots, p_{l,\mathbf{k}})\alpha_{\mathbf{k}} = 0$$

obtain the over-defined homogeneous system of 16 linear equations w.r.t. 10 variables $\alpha_{\mathbf{k}}$, with the matrix, given below in two parts (the first part gives columns 1-10, the second part gives columns 11-10):

1	1	1	1	1	1	1	1	1	1
−3	−4	−5	−6	−5	−6	−7	−7	−8	−9
−3/2	−3	−11/2	−9	−9/2	−7	−21/2	−19/2	−13	−33/2
9/2	8	25/2	18	25/2	18	49/2	49/2	32	81/2
−1	−10/3	−29/3	−22	−17/3	−12	−73/3	−55/3	−92/3	−43
9/2	12	55/2	54	45/2	42	147/2	133/2	104	297/2
−9/2	−32/3	−125/6	−36	−125/6	−36	−343/6	−343/6	−256/3	−243/2
−3/4	−9/2	−83/4	−129/2	−33/4	−49/2	−273/4	−163/4	−169/2	−513/4
33/8	107/6	1523/24	345/2	923/24	193/2	5411/24	4163/24	1979/6	4185/8
−27/4	−24	−275/4	−162	−225/4	−126	−1029/4	−931/4	−416	−2673/4
27/8	32/3	625/24	54	625/24	54	2401/24	2401/24	512/3	2187/8
−3/5	−34/5	−49	−1026/5	−13	−276/5	−1057/5	−487/5	−1268/5	−2049/5
15/4	28	1883/12	585	267/4	231	2933/4	5513/12	3224/3	7455/4
−63/8	−134/3	−4715/24	−639	−2915/24	−363	−23569/24	−18361/24	−4972/3	−23733/8
27/4	32	1375/12	324	375/4	252	2401/4	6517/12	3328/3	8019/4
−81/40	−128/15	−625/24	−324/5	−625/24	−324/5	−16807/120	−16807/120	−4096/15	−19683/40

1	1	1	1	1	1	1	1	1	1
−6	−7	−8	−8	−9	−10	−9	−10	−11	−12
−6	−17/2	−12	−11	−29/2	−18	−27/2	−17	−41/2	−24
18	49/2	32	32	81/2	50	81/2	50	121/2	72
−8	−43/3	−80/3	−62/3	−33	−136/3	−27	−118/3	−155/3	−64
36	119/2	96	88	261/2	180	243/2	170	451/2	288
−36	−343/6	−256/3	−256/3	−243/2	−500/3	−243/2	−500/3	−1331/6	−288
−12	−113/4	−72	−89/2	−353/4	−132	−243/4	−209/2	−593/4	−192
66	3275/24	856/3	1355/6	3217/8	1846/3	2673/8	3227/6	18683/24	1056
−108	−833/4	−384	−352	−2349/4	−900	−2187/4	−850	−4961/4	−1728
54	2401/24	512/3	512/3	2187/8	1250/3	2187/8	1250/3	14641/24	864
−96/5	−307/5	−1088/5	−518/5	−1299/5	−416	−729/5	−302	−2291/5	−3072/5
120	3835/12	896	1750/3	5091/4	2136	3645/4	5141/3	32279/12	3840
−252	−14497/24	−4288/3	−3436/3	−18261/8	−11660/3	−15309/8	−10235/3	−130493/24	−8064
216	5831/12	1024	2816/3	7047/4	3000	6561/4	8500/3	54571/12	6912
−324/5	−16807/120	−4096/15	−4096/15	−19683/40	−2500/3	−19683/40	−2500/3	−161051/120	−10368/5

This system has solutions of the form 17.

□

Contrary to cubic recurrence relations, for quadratic recurrence relations $S^*_l(u_0) \equiv 0$, where $0 \le l \le 5$, always implies that $S_6(\mathbf{u}_6) \equiv S^*_6(u_0)$.

Corollary 4. For all difference equations relations with $D = 2$, if $S^*_l(u_0) \equiv 0$, where $l = 0..5$, then $S_6(\mathbf{u}_6) \equiv S^*_6(u_0)$. From this follows that if $S^*_6(u_0) \not\equiv 0$, then either $d \le \max\{6, \deg(G_0)\}$, or d is one of the natural roots of $S^*_6(u_0)$ if they exist.

Proof. Recall lemma 11:

$$S_6(\mathbf{u}_6) = S^*_6(u_0) + u_2 \cdot (1/8)u_0 \sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} + u_1^2 \cdot (1/8) \sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}$$

It is easy to see that to prove the corollary one just need to prove $\sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$. We show that for $D = 2$ this follows from $S^*_4(u_0) \equiv 0$. For this we consider the coefficients B_{43}, B_{42} and B_{41} with $\mathbf{t} = (t_1, t_2)$:

- $B_{43} = -(1/4)p_{2,\mathbf{t}}p^2_{1,\mathbf{k}} =(-1/4)p_{2,\mathbf{t}}(t_1 + t_2)^2 = (-1/4)p_{2,\mathbf{t}}(t_1^2 + t_2^2 + 2t_1t_2) = (-1/4)p_{2,\mathbf{t}}(t_1^2 + t_2^2) - (1/4)p_{2,\mathbf{t}} \cdot 2t_1t_2 =(-1/4)p^2_{2,\mathbf{t}} - (1/2)(t_1^3t_2 + t_1t_2^3) =(-1/4)p^2_{2,\mathbf{t}} - (1/2)y_{\mathbf{t}}$, where $y_{\mathbf{t}}$ denotes $t_1^3t_2 + t_1t_2^3$;
- in $B_{42} = (1/3)p_{3,\mathbf{t}}p_{1,\mathbf{t}} + (1/8)p^2_{2,\mathbf{t}}$ we first pay attention to $p_{3,\mathbf{t}}p_{1,\mathbf{t}} =(t_1^3 + t_2^3)(t_1 + t_2) =t_1^4+t_2^4+t_1^3t_2+t_1t_2^3 = p_{4,\mathbf{t}}+y_{\mathbf{t}}$; second, we obtain $B_{42} = (1/3)(p_{4,\mathbf{t}}+y_{\mathbf{t}})+(1/8)p^2_{2,\mathbf{t}}$.

Since $S^*_4(u_0) \equiv 0$, we have

- (use B_{41}) $\sum_{\mathbf{k}\in K}(-1/4)p_{4,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$;
- (use B_{42}) $\sum_{\mathbf{k}\in K} \big((1/3)(p_{4,\mathbf{t}(\mathbf{k})}+y_{\mathbf{t}(\mathbf{k})})+(1/8)p^2_{2,\mathbf{t}(\mathbf{k})}\big)\alpha_{\mathbf{k}} = 0$, which due to the previous equation is reduced to $(1/3)\sum_{\mathbf{k}\in K} y_{\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} + (1/8)\sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$;
- (use B_{43}) $\sum_{\mathbf{k}\in K} \big((-1/4)p^2_{2,\mathbf{t}(\mathbf{k})}-(1/2)y_k\big)\alpha_{\mathbf{k}} = (-1/4)\sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}-(1/2)\sum_{\mathbf{k}\in K} y_{\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$.

Now, denote $\sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}$ as X and $\sum_{\mathbf{k}\in K} y_{\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}}$ as Y. From the equations above we obtain the following homogeneous linear system w.r.t. X, Y:

$$\begin{aligned} (1/8)X \quad &+(1/3)Y = 0 \\ (-1/4)X \quad &-(1/2)Y = 0 \end{aligned}$$

which has only zero solution $X = Y = 0$. Thus, we obtain $\sum_{\mathbf{k}\in K} p^2_{2,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{k}} = 0$ from what follows that $S_6(\mathbf{u}_6) = S^*_6(u_0)$. □

5. Example of difference equation solved by a polynomial of any degree (by any $(x-1)\dots(x-n)$)

This section consists of three parts. In the first part we give a difference equation such that it is solvable by any *Newton basis polynomial* $g_n(x) := (x - 1)\dots(x - n)$. In the second part we explain how we have constructed this equation, following an approach proposed in paper (van den Essen, 1992) to construct a differential equation for which any monomial x^n is a solution.

In the third part, we show why the conditions of the Framework lemma do not hold for the equation.

5.1. The equation

Let $\Delta(p)(x)$ and $\Delta^{(2)}(p)(x)$ denote differential operators $p(x)-p(x-1)$ and $\Delta(p)(x)-\Delta(p)(x-1)$ respectively. Let $H(x) := p(x)\cdot\Delta^{(2)}(p)(x)-\Delta^2(p)(x)$. It is a routine to show that the following lemma holds.

Lemma 12. Any Newton basis polynomial $g_n(x) := (x-1)\dots(x-n)$ solve the equation, w.r.t. p,

$$H(x-1)H(x)+\Delta(p)(x-1)\cdot p(x)\cdot H(x-1)-\Delta(p)(x-2)\cdot p(x-1)\cdot H(x)=0 \tag{18}$$

Proof. First, we compute $\Delta(p)(x),\Delta^{(2)}(p)(x)$ for $p=g_n(x)$:

$$\begin{aligned}
\Delta(g_n)(x) =\ & (x-1)(x-2)\dots(x-n)-(x-2)\dots(x-n)(x-n-1)= \\
& (x-2)\dots(x-n)(x-1-x+n+1)= \\
& n(x-2)\dots(x-n) \\
\Delta^{(2)}(g_n)(x) =\ & n(x-2)(x-3)\dots(x-n)-n(x-3)\dots(x-n)(x-n-1)= \\
& n(x-3)\dots(x-n)(x-2-x+n+1)= \\
& n(n-1)(x-3)\dots(x-n)
\end{aligned}$$

Second, compute $H(x) := g_n(x)\cdot\Delta^{(2)}(g_n)(x)-\Delta^2(g_n)(x)$:

$$\begin{aligned}
H(x) =\ & (x-1)\dots(x-n)\cdot n(n-1)(x-3)\dots(x-n)- \\
& n^2(x-2)^2\dots(x-n)^2= \\
& n(x-2)(x-3)^2\dots(x-n)^2\big((n-1)(x-1)-n(x-2)\big)= \\
& -n(x-2)(x-3)^2\dots(x-n)^2(x-n-1)
\end{aligned}$$

Third, compute all three summands in the l.h.s. of the equation:

$$\begin{aligned}
H_1(x) &= H(x-1)H(x)= \\
& -n(x-3)(x-4)^2\dots(x-n)^2(x-n-1)^2(x-n-2)\cdot \\
& -n(x-2)(x-3)^2\dots(x-n)^2(x-n-1)= \\
& n^2(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}$$

$$\begin{aligned}
H_2(x) &= \Delta(g_n)(x-1)\cdot g_n(x)\cdot H(x-1)= \\
& n(x-3)(x-4)\dots(x-n)(x-n-1)\cdot \\
& (x-1)(x-2)(x-3)(x-4)\dots(x-n)\cdot \\
& -n(x-3)(x-4)^2\dots(x-n)^2(x-n-1)^2(x-n-2)= \\
& -n^2(x-1)(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}$$

$$
\begin{aligned}
H_3(x) &= \Delta(g_n)(x-2) \cdot g_n(x-1) \cdot H(x) = \\
& n(x-4)\dots(x-n)(x-n-1)(x-n-2) \cdot \\
& (x-2)(x-3)(x-4)\dots(x-n)(x-n-1) \cdot \\
& -n(x-2)(x-3)^2(x-4)^2\dots(x-n)^2(x-n-1) = \\
& -n^2(x-2)^2(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)
\end{aligned}
$$

Now, compute $H_1(x) + H_2(x) - H_3(x) =$:

$$
\begin{aligned}
& n^2(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)\big(1-(x-1)+(x-2)\big) = \\
& n^2(x-2)(x-3)^3(x-4)^4\dots(x-n)^4(x-n-1)^3(x-n-2)\cdot 0 = 0
\end{aligned}
$$

Therefore, direct substitution shows that any $g_n(x)$ solves equation 18.
□

5.2. Construction

Constructing the difference equation 18 we followed an approach proposed in paper (van den Essen, 1992) to construct a differential equation for which any monomial x^n is a solution. Construction for difference equation is similar, except that we use *newton basis polynomials*, $g_n(x) = (x-1)\dots(x-n)$ for $n \geq 1$ instead of standard monomial basis x^n. This is not surprising since polynomials of this form are typically considered when one speaks about topics related to difference equations.

We recapitulate the necessary definitions

Definition 6.

$$
\begin{aligned}
g_n(x) &:= (x-1)\dots(x-n) \\
\Delta_n(x) &:= g_n(x) - g_n(x-1) \\
\Delta_n^{(2)}(x) &:= \Delta_n(x) - \Delta_n(x-1)
\end{aligned}
$$

It is a routine to check that

$$
\begin{aligned}
\Delta_n(x) = & (x-1)(x-2)\dots(x-n) - (x-2)\dots(x-n)(x-n-1) = \\
& (x-2)\dots(x-n)\big(x-1-(x-n-1)\big) = \\
& n(x-2)\dots(x-n)
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_n^{(2)}(x) = & n(x-2)\dots(x-n) - n(x-3)\dots(x-n)(x-n-1) = \\
& n(x-3)\dots(x-n)\big(x-2-(x-n-1)\big) = \\
& n(n-1)(x-3)\dots(x-n)
\end{aligned}
$$

Now,

$$
\begin{aligned}
g_n(x)\Delta_n^{(2)}(x) &= n(n-1)(x-1)(x-2)(x-3)^2\dots(x-n)^2\\
\Delta_n^2(x) &= n^2(x-2)^2(x-3)^2\dots(x-n)^2\\
g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x) &= n(x-2)(x-3)^2\dots(x-n)^2\big((n-1)(x-1)-n(x-2)\big)=\\
&n(x-2)(x-3)^2\dots(x-n)^2(nx-x-n+1-nx+2n)=\\
&-n(x-2)(x-3)^2\dots(x-n)^2(x-n-1)=\\
&-\Delta_n(x)\Delta_n(x-1)(1/n)
\end{aligned}
$$

Using $\dfrac{g_n(x)}{\Delta_n(x)}=\dfrac{x-1}{n}$, we obtain that

$$
g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)=-\frac{\Delta_n(x-1)g_n(x)}{x-1}
$$

and, therefore,

$$
x-1=-\frac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}
$$

Now we use the scheme of symbolic differentiation: for all functions $h_1(x)$ and $h_2(x)$, such that $h_1(x)=h_2(x)$ it follows that $h_1(x)-h_1(x-1)=h_2(x)-h_2(x-1)$. We take $h_1(x)=x-1$ and $h_2(x)=-\dfrac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}$ and obtain

$$
1=-\frac{\Delta_n(x-1)g_n(x)}{g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)}+\frac{\Delta_n(x-2)g_n(x-1)}{g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)}
$$

By standard transformations of the fractions we obtain

$$
\begin{aligned}
&\big(g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)\big)\big(g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)\big)+\\
&\Delta_n(x-1)g_n(x)\big(g_n(x-1)\Delta_n^{(2)}(x-1)-\Delta_n^2(x-1)\big)-\\
&\Delta_n(x-2)g_n(x-1)\big(g_n(x)\Delta_n^{(2)}(x)-\Delta_n^2(x)\big)=0
\end{aligned}
$$

5.3. The example in the context of our settings

After substitution of Δ and $\Delta^{(2)}$ in equation 18 by their definition and simplifications, equation 18 looks as follows:

$$
\begin{aligned}
&p(x-1)p(x-3)p(x)p(x-2)-2p(x-1)^3p(x-3)+p(x-2)^2p(x-1)^2+\\
&p(x)p(x-1)^2p(x-3)-2p(x)p(x-1)p(x-2)^2+p(x-2)p(x-1)^3 \qquad =0
\end{aligned}
\tag{19}
$$

It is easy to see, that the equation 19 is equivalent to

$$
\begin{aligned}
&p(x-3)p(x)p(x-2)-2p(x-1)^2p(x-3)+p(x-2)^2p(x-1)+\\
&p(x)p(x-1)p(x-3)-2p(x)p(x-2)^2+p(x-2)p(x-1)^2 \qquad =0
\end{aligned}
\tag{20}
$$

Indeed, if equation 20 is obtained from the previous one by division by $p(x-1)$, and $p(x) \equiv 0$ is not lost as a possible solution, because it solves equation 20 as well. Alternatively, one can substitute $g_n(x) = \prod_{i=1}^{n}(x-i)$ into the l.h.s. of the equation expressed in maple, and directly see that after evaluation the result of substitution is equal to zero. However, we want to show how this example fits our technique, that is we *directly* show that the coefficients at x^{3n-l} vanish for $0 \leq l \leq n-1$. Moreover, later in this section we show that equation 20 does not satisfy the conditions of the framework lemma.

To begin with, we need the following definitions and a technical lemma. Let $\bar{n} = (1, \ldots, n)$ be the vector of the roots of $g_n(x)$. Let $\varepsilon_{n,l}(\mathbf{t}(k), \bar{n})$ denote the coefficient at x^{3n-l} in $P_{n,k}(x) = g_n(x - t_{k,1})g_n(x - t_{k,2})_n(x - t_{k,3})$, where

k	$t_{k,1}$	$t_{k,2}$	$t_{k,3}$
1	0	2	3
2	1	1	3
3	1	2	2
4	0	1	3
5	0	2	2
6	1	1	2

Now we can establish a simple linear recurrence relation for $\varepsilon_{n,l}(\mathbf{t}(k), \mathbf{r})$, which we obtain not via Newton-Girard identities, but using the structure of $P_{n,k}(x)$.

Lemma 13. For any $n > 3$ and any $1 \leq k \leq 6$ one has

$$P_{n,k}(x) = (x-n)^3 P_{n-1,k}(x)$$

and, therefore, for any $3 \leq l \leq n$ one has that $\varepsilon_{n,l}(\mathbf{t}(k), \bar{n}) =$

$$\varepsilon_{n-1,l}(\mathbf{t}(k), \bar{n}) - 3n\varepsilon_{n-1,l+1}(\mathbf{t}(k), \bar{n}) + 3n^2\varepsilon_{n-1,l+2}(\mathbf{t}(k), \bar{n}) + n^3\varepsilon_{n-1,l+3}(\mathbf{t}(k), \bar{n})$$

Proof. We notice that for a fixed n and different k polynomials $P_{n,k}(x)$ have a common part $C_n(x) = (x-4)^3 \ldots (x-n)^3$, so that for any k it holds that $P_{n,k}(x) = C_n(x)R_{n,k}(x)$, where $R_{n,k}(x)$ are polynomials of the 8-th degree (which can be computed by division of $P_{n,k}(x)$ by $C_n(x)$, `maple` can do this operation):

$$\begin{aligned}
R_{n,1}(x) &= -(n-x+3)*(x-3)^2*(n+1-x)^2*(n-x+2)^2*(x-1)*(x-2) \\
R_{n,2}(x) &= -(n-x+2)*(n-x+3)*(x-2)^2*(x-3)^2*(n+1-x)^3 \\
R_{n,3}(x) &= -(n-x+2)^2*(x-2)*(n+1-x)^3*(x-3)^3 \\
R_{n,4}(x) &= (n-x+2)*(n-x+3)*(n+1-x)^2*(x-2)^2*(x-3)^2*(x-1) \\
R_{n,5}(x) &= (n+1-x)^2*(n-x+2)^2*(x-3)^3*(x-1)*(x-2) \\
R_{n,6}(x) &= (n-x+2)*(x-2)^2*(n+1-x)^3*(x-3)^3
\end{aligned}$$

Let $\delta_{n,k} = R_{n,k}(x) - R_{n-1,k}(x)$. It is easy to see that

$$P_{n,k}(x) = C_{n-1}(x)(x-n)^3\big(R_{n-1,k}(x) + \delta_{n,k}(x)\big) =$$
$$(x-n)^3 P_{n-1,k}(x) + (x-n)^3 C_{n-1}(x)\delta_{n,k}(x)$$

It is a routine to check (e.g. using maple) that $\sum_{k=1}^{6} \alpha_k \delta_{n,k}(x) \equiv 0$, from what follows that $\sum_{k=1}^{6} \alpha_k (x-n)^3 C_{n-1}(x)\delta_{n,k}(x) = (x-n)^3 C_{n-1}(x) \sum_{k=1}^{6} \alpha_k \delta_{n,k}(x) \equiv 0$. From this follows that

$$P_{n,k}(x) = (x-n)^3 P_{n-1,k}(x)$$

So the first statement of the lemma is proven. The second statement follows immediately, due to $(x-n)^3 = x^3 - 3nx^2 + 3n^2x - n^3$. The coefficient $c_{n,l}(\mathbf{t}(k), \bar{n})$ at x^i of $P_{n,k}(x)$ is equal to

$$c_{n-1,i-3}(\mathbf{t}(k), \bar{n}) - 3nc_{n-1,i-2}(\mathbf{t}(k), \bar{n}) + 3n^2 c_{n-1,i-1}(\mathbf{t}(k), \bar{n}) + n^3 c_{n-1,l}(\mathbf{t}(k), \bar{n})$$

From this follows that for $i = 3n-l$ we have $i-1 = 3(n-1)-(l+2)$, $i-2 = 3(n-1)-(l+1)$, $i-3 = 3(n-1)-l$ and $\varepsilon_{n-1,l}(\mathbf{t}(k), \bar{n})$ is equal to

$$\varepsilon_{n-1,l}(\mathbf{t}(k), \bar{n}) - 3n\varepsilon_{n-1,l+1}(\mathbf{t}(k), \bar{n}) + 3n^2 \varepsilon_{n-1,l+2}(\mathbf{t}(k), \bar{n}) + n^3 \varepsilon_{n-1,l+3}(\mathbf{t}(k), \bar{n})$$

which is what we want to prove. □

Now we can prove the following main, in this section, lemma.

Lemma 14. For all $n \geq 1$, one has $\sum_{k=1}^{6} \alpha_k P_{n,k}(x) \equiv 0$, and for all $0 \leq l \leq n-1$, one has $\sum_{k=1}^{6} \alpha_k \epsilon_{n,l}(\mathbf{t}(k), \bar{n}) \equiv 0$.

Proof. The proof is done induction on n.

For $n = 1, 2$ one can show that statements of the lemma hold by direct computations of $\sum_{k=1}^{6} \alpha_k P_{n,k}(x)$ and $\sum_{k=1}^{6} \alpha_k \epsilon_{n,l}(\mathbf{t}(k), \bar{n})$.

For $n \geq 3$ ... □

Now we show that the equation 20 doe snot satisfy the conditions of the framework lemma. It is done by direct computation of $S_l(u_0, u_1, \ldots, u_l)$ for $0 \leq l \leq 6$. We start with the following statement.

Lemma 15. For $0 \leq l \leq 5$ one has $S_l\big(u_0, \ldots, u_l\big) \equiv 0$. Moreover, $S_6(u_0, \ldots, u_6) = 2 * u_1^2 - (1/6) * u_0^2 - 2 * u_2 * u_0 + (1/6) * u_0^4 \not\equiv 0$.

Proof. These values are obtained by direct computations using a maple script that corresponds the following two-part pseudocode. In the first part we declare the arrays of variables and assign the parameters of the equation to the elements of these arrays:

```
L:=6 (*an upper bound for l*)
t:=array(1..L, 1..3) (*translations for s=3*)
α:=array(1..6) (*the coefficients of G_D with D=2*)
t_1:=[0, 2, 3], t_2:=[1, 1, 3], t_3:=[1,2, 2]
t_4:=[0,1, 3], t_5:=[0,2, 2], t_6:=[1,1, 2]
α_1:=1, α_2:=-2, α_3:=1
α_4:=1, α_4:=-2, α_6:=1
E':=array(1..L, 1..6) (*the values of E_l we compute for different l and t_k *)
S':=array(1..L) (*the values of S_l we compute for different l *)
```

The second part consist of the main procedure that computes S_lL

```
for l from 0 to L do
for k to 6 do
Ehelp := E_l(v_l, u_l);
for j from 0 to l do
Ehelp := eval(Ehelp, v_l = p_{l,t(k)})
end do;
E'_{l,k} := Ehelp;
end do;
S'_l := sum_{k=1}^{6} E'_{l,k} α_k;
end do;
```

The values $S_l\big(u_0, \dots, u_l\big) \equiv 0$, where $0 \leq l \leq 5$, and the symbolic expression for $S_6\big(u_0, \dots, u_6\big)$ are obtained by running this procedure. □

Lemma 16. For equation 20, $L := \min\{l | S^*_l(u_0) \not\equiv 0\} = 6$. Moreover, the linear-combination equations 11 does not hold for $A_{6\mathbf{i}_6}$.

Proof. Indeed, if $l \leq 5$, then $S^*_l(u_0) \equiv 0$, because $0 \equiv S_l(u_0, \dots, u_l) = S^*(u_0) + \sum_{\mathbf{i}_l \neq \mathbf{0}_l} F_{l\mathbf{i}_l}(u_0) u_1^{i_1} \dots u_l^{i_l}$ due to lemma 15. Due to the same lemma, $S^*_6(u_0) = -(1/6) * u_0^2 + (1/6) * u_0^4 \not\equiv 0$ (the $(u_1, \dots, u_6)$-free part of S_6), therefore $L = 6$.

Now, we recall the proof of lemma 11. There we compute all $A_{6\mathbf{i}_6}$ (see Appendix, section 8.1), and the corresponding coefficients $H_{6\mathbf{i}_6 l\mu}$ for the linear combinations over $B_{l\mu}(\mathbf{v}_l)$. We have directly see that all the coefficients, except for $A_{6,(200000)}$ and $A_{6,(010000)}$, do not depend on $v_1, \dots, v_6$. We see that $H_{6,(200000)} = -(1/4)v_2$ and $H_{6,(010000)} = (1/4)v_2 u_0$. However, it may be for a some particular choice of $\mathbf{t}(k)$, after substituting v_2 with any of $p_{2,\mathbf{t}(k)}$, all $H_{6,(200000)}(k)$ will be the same and all $H_{6,(010000)}(k)$ will be the same (so they actually will not be depending on k). Direct computations shows that this is not the case for the example:

k	$\mathbf{t}(k)$	$p_{2,\mathbf{t}(k)}$	$H_{6,(200000)}(k) = -(1/4)p_{2,\mathbf{t}(k)}$	$H_{6,(010000)}(k) = (1/4)u_0 p_{2,\mathbf{t}(k)}$
1	$(0,2,3)$	$2^2+3^2=13$	$-13/4$	$13/4u_0$
2	$(1,1,3)$	$1+1+3^2=11$	$-11/4$	$11/4u_0$
3	$(1,2,2)$	$1+2^2+2^2=9$	$-9/4$	$9/2u_0$
4	$(0,1,3)$	$1+3^2=10$	$-5/2$	$5/2u_0$
5	$(0,2,2)$	$2^2+2^2=8$	-2	$2u_0$
6	$(1,1,2)$	$1+1+2^2=6$	$-3/2$	$3/2u_0$

As we see, $H_{6,(200000)}(k)$ and $H_{6,(010000)}(k)$ are not constant as functions of $k = 1,\ldots,6$. Therefore, the condition of the framework lemma does not hold. $\square$

6. One-translation difference equations

In this section we study difference equations of the form

$$G(P(x), P(x-\tau)) + G_0(x) = 0 \tag{21}$$

For this equation we have $s = 2$ with $\tau_1 = 0$, $\tau_2 = \tau$. Further,

$$\mathrm{Nat}^D \supseteq K = \{\mathbf{k}_m = (1,\ldots,1,2,\ldots,2) \,|\, 2 \text{ occurs } 0 \le m \le D \text{ times}\}$$

$$\mathcal{R}^D \supseteq T = \{\mathbf{t}(\mathbf{k}_m) = (0,\ldots,0,\tau,\ldots,\tau) \,|\, \tau \text{ occurs } 0 \le m \le D \text{ times}\}$$

For the sake of convenience we denote $\mathbf{t}(\mathbf{k}_m)$ via $\mathbf{t}_m$, $\mathbf{p}_{l,\mathbf{t}(\mathbf{k}_m)}$ via $\mathbf{p}_{l,m}$ and $\alpha_{\mathbf{k}_m}$ via α_m. Our aim is to prove the following statement

Theorem 7. *The degree of a polynomial solution P of the equation 21, if exists, is $d \le \max\{D, \deg(G_0)/(D-1)\}$.*

by applying framework lemma. To show that the conditions of the lemma are satisfied we need to consider a few facts about $p_{l,m}$ and $S_l(\mathbf{u}_l)$ for equation 21. First of all, it is easy to see that $p_{l,m} = 0^l + \ldots + 0^l + \tau^l + \ldots + \tau^l = m\tau^l$. Second, it is a routine to check that $p_l(\mathbf{t}_m + \mathbf{r}) = m\sum_{\lambda=0}^{l}\binom{l}{\lambda}\tau^{l-\lambda}p_{\lambda,\mathbf{r}}$. Indeed, it follows straightforwardly from identity 5 and $p_{l,m} = m\tau^l$. Third, we obtain

$$E_l(\mathbf{p}_{l,m}, \mathbf{u}_l) = -(m/l)\sum_{\kappa=1}^{l} E_{l-\kappa}(\mathbf{p}_{l-\kappa,m}, \mathbf{u}_{l-\kappa})\sum_{\lambda=0}^{\kappa}\binom{\kappa}{\lambda}\tau^{k-\lambda}u_\lambda \tag{22}$$

From this one obtains recurrent formulae to compute $A_{l\mathbf{i}_l}(\mathbf{p}_{l,m}, u_0)$ and $B_{l\mathbf{i}_l\mu}(\mathbf{p}_{l,m})$ respectively

$$(-m/l)\big(\textstyle\sum_{\kappa=1}^{l}\sum_{\lambda=1}^{\kappa} A_{l-\kappa\ \mathbf{i}_l-1_\lambda}(\mathbf{p}_{l-\kappa,m}, u_0)\binom{\kappa}{\lambda}\tau^{\kappa-\lambda}\big) + \big(\sum_{\kappa=1}^{l} A_{l-\kappa\ \mathbf{i}_l}(\mathbf{p}_{l-\kappa,m}, u_0)\, u_0\, \tau^{\kappa}\big)\Big)$$

where $\mathbf{i}_l - 1_\lambda := (i_1,\ldots,i_{\lambda-1}, i_\lambda - 1, i_{\lambda+1},\ldots,i_l)$ and $A_{l-\kappa\ \mathbf{i}_l-1_\lambda} = 0$ if $i_\lambda = 0$

$$-(m/l)\Big(\big(\textstyle\sum_{\kappa=1}^{l-\mu}\sum_{\lambda=1}^{\kappa} B_{l-\kappa\ \mathbf{i}_l-1_\lambda,\mu}(\mathbf{p}_{l-\kappa,m})\binom{\kappa}{\lambda}\tau^{\kappa-\lambda}\big) + \big(\sum_{\kappa=1}^{l-\mu+1} B_{l-\kappa\ \mathbf{i}_l\ \mu-1}(\mathbf{p}_{l-\kappa,m})\tau^{\kappa}\big)\Big) \tag{23}$$

taking into account that $l-\kappa \geq \mu$ in the first summand for $B_{l\mu}$, and $l-\kappa \geq \mu-1$ in the second summand. Recall, that $B_{l\mu}$ is a shortcut for $B_{l\mathbf{0}_l\mu}$. Now we can prove the following lemma.

Lemma 17. For any l and $0 < \mu \leq l$ there exists a constant $C_{l\mu} > 0$ such that

$$B_{l,\mu}(\mathbf{p}_{l,m}) = (-1)^{\mu} C_{l\mu} \tau^l m^{\mu} \tag{24}$$

Proof. Induction on l and μ. For $l = 0$ we have $B_{00} = 1$ and therefore $C_{00} = 1$. For $l > 0$ we begin with $\mu = 0$: it is easy to see from identity 23, that all the monomials in $A_{l\mathbf{0}_l}$ contain u_0, that is $B_{l\mu} = 0$. For $\mu > 0$ we use the recurrent formula for $B_{l\mu}$:

$$B_{l\mu}(\mathbf{p}_{l,m}) = -(m/l) \textstyle\sum_{\kappa=1}^{l-\mu+1} B_{l-\kappa\ \mu-1}(\mathbf{p}_{l-\kappa,m})\tau^{\kappa} \tag{25}$$

For $\mu = 1$ we have $B_{l1} = (-m/l)B_{00}\tau^l$, with the only non-zero summand for $\kappa = l$, therefore $C_{l1} = (1/l)$. For $\mu > 1$, by induction assumption we straightforwardly obtain that $B_{l\mu}(\mathbf{p}_{l,m})$ is equal to

$$-(m/l) \sum_{\kappa=1}^{l-\mu+1} (-1)^{\mu-1} C_{l-\kappa,\mu-1} \tau^{l-\kappa} m^{\mu-1} \tau^{\kappa} = \tau^l m^{1+\mu-1} (-1)^{1+\mu-1} (1/l) \sum_{\kappa=1}^{l-\mu+1} C_{l-\kappa,\mu-1} \tag{26}$$

From this follows that for $\mu >$ one has $C_{l\mu} = (1/l) \sum_{\kappa=1}^{l-\mu+1} C_{l-\kappa,\mu-1} > 0$. □

Now we can prove theorem 7.

Proof. We show that the conditions of the framework lemma hold. First, we show that there exists $0 \leq L \leq D$ such that $S_L^*(u_0) \not\equiv 0$. Assume the opposite: $S_l^*(u_0) \equiv 0$ for all $0 \leq l \leq D$. It implies that the corresponding coefficients at u_0^l in $S_l^*(u_0)$ must be all zeros. This means, that by lemma 17, we obtain $\sum_{m=0}^{D} (-1)^l C_{ll} \tau^l m^l \alpha_m = 0$ which due to $\tau \neq 0$ and $C_{ll} > 0$ implies $\sum_{m=0}^{D} m^l \alpha_m = 0$ for all $0 \leq l \leq D$. That is we obtain a system with $D+1$ linear equations w.r.t. $D+1$ variables x_m. The matrix of this system has rank $D+1$ because its determinant is equal to Vandermonde determinant $D \times D$. Therefore, the system has only zero solution α_m which contradicts the fact that G is of degree D. Therefore there exists $S_L^*(u_0) \not\equiv 0$. W.l.o.g we assume that for all $0 \leq l \leq L-1$ identities $S_l^*(u_0) \equiv 0$ hold.

Second, a function $A_{l\mathbf{i}_l}(\mathbf{p}_{l,m}, u_0)$ can be seen as a polynomial in m with the coefficients $T_{l\mathbf{i}_l\mu}^{D,\tau}(u_0)$ of m^{μ}, where the values D and $\tau \in \mathcal{R}$ are given by the difference equation. Since $B_{l\mu}(\mathbf{p}_{l,m}) = (-1)^{\mu} C_{l\mu} \tau^l m^{\mu}$ by lemma 17, one can easily see that $A_{l\mathbf{i}_l}(\mathbf{p}_{l,m}, u_0)$ is a linear combination of $B_{\mu,\mu}(\mathbf{p}_{\mu,m})$, with the coefficients $H_{l\mathbf{i}_l\mu\mu}^{D,\tau}(u_0) = T_{l\mathbf{i}_l\mu}^{D,\tau}(u_0)/\big((-1)^{\mu} C_{\mu\mu} \tau^{\mu}\big)$.

Now, we prove the statement of the theorem.

Assume the opposite: $d > \max\{L, \deg(G_0)/(D-1)\}$. By the framework lemma we obtain that d is a root of $S_L^*(u_0)$. We note that $S_L^*(u_0) = \sum_{\mu=0}^{L} u_0^{\mu} \sum_{m=0}^{D} B_{L,\mu}(\mathbf{p}_{L,m})\alpha_m$ holds. Next, by lemma 17 we obtain that $S_L^*(u_0) = \sum_{\mu=0}^{L} u_0^{\mu} (-1)^{\mu} \tau^L C_{L,\mu} \sum_{m=0}^{D} m^{\mu} \alpha_m = 0$. Now, let us have a close look at sums $\sum_{m=0}^{D} m^{\mu} \alpha_m$ for $0 \leq \mu \leq L-1$. Since $S_{\mu}^*(u_0) \equiv 0$ for all $0 \leq \mu \leq L-1$ and applying lemma 17, we obtain $\sum_{m=0}^{D} (-1)^{\mu} C_{\mu\mu} \tau^{\mu} m^{\mu} \alpha_m = 0$, from what follows that $\sum_{m=0}^{D} m^{\mu}\alpha_m = 0$ for $0 \leq \mu \leq L-1$. Therefore, $S_L^*(u_0) = u_0^L(-1)^L\tau^L C_{L,L}\sum_{m=0}^{D} m^D\alpha_m = 0$. Since $S_L^*(u_0) \not\equiv 0$ we have $\sum_{m=0}^{D} m^D\alpha_m \neq 0$. From this follows that $S_L^*(d) = 0$ implies $d = 0$, which contradicts the assumption $d > D \geq 2$.

Therefore, $d \leq \max\{L, \deg(G_0)/(D-1)\} \leq \max\{D, \deg(G_0)/(D-1)\}$. □

7. Example

The following difference equation gives an example of a polynomial solution with $d > 2$ be a root of the degree polynomial.

$$\begin{aligned} P(x) = {} & P(x-1)\cdot P(x-1) - 2\cdot P(x-1)\cdot P(x-2) + \\ & 3\cdot P(x-1)\cdot P(x-3) - 2\cdot P(x-2)\cdot P(x-2) - \\ & 17\cdot P(x-1) + 29\cdot x^2 - 45\cdot x + 51 \end{aligned}$$

To analyse this equation, it will be enough to calculate $S_0^*(u_0)$, $S_1^*(u_0)$ and $S_2^*(u_0)$. To do this, we use identity $S_l^*(u_0) = \sum_{\mathbf{k}\in K} A_{l,\mathbf{0}_l}(\mathbf{p}_{l,\mathbf{t}(\mathbf{k})}, u_0)\alpha_{\mathbf{k}}$. It is a routine to check that $A_{0,()}(\mathbf{v}_0, u_0) = 1$, $A_{1,(0)}(\mathbf{v}_1, u_0) = -v_1u_0$ and $A_{2,(00)}(\mathbf{v}_2, u_0) = -(1/2)v_1^2u_0^2 - (1/2)v_2u_0$ (see Appendix as well). Now we compute the values $p_{l,\mathbf{t}(\mathbf{k})}$ (for non-vanishing $\alpha_{k_1k_2}$):

(k_1,k_2)	$\mathbf{t}(k_1,k_2)$	$p_{1,\mathbf{t}(k_1,k_2)}$	$p^2_{1,\mathbf{t}(k_1,k_2)}$	$p_{2,\mathbf{t}(k_1,k_2)}$
$(1,1)$	$(1,1)$	$1+1=2$	4	$1^2+1^2=2$
$(1,2)$	$(1,2)$	$1+2=3$	9	$1^2+2^2=5$
$(1,3)$	$(1,3)$	$1+3=4$	16	$1^2+3^2=10$
$(2,2)$	$(2,2)$	$2+2=4$	16	$2^2+2^2=8$

Now, by substitutions $v_l := p_{l,\mathbf{t}(k_1,k_2)}$ we obtain

$$\begin{aligned} S_0^*(u_0) = {} & \textstyle\sum_{\mathbf{k}\in K}\alpha_{\mathbf{t}(\mathbf{k})} = 1-2+3-2 \equiv 0 \\ S_1^*(u_0) = {} & u_0\textstyle\sum_{\mathbf{k}\in K} p_{1,\mathbf{t}(\mathbf{k})}\alpha_{\mathbf{t}(\mathbf{k})} = u_0(1\cdot 2 - 2\cdot 3 + 3\cdot 4 - 2\cdot 4) \equiv 0 \\ S_2^*(u_0) = {} & u_0\big(\textstyle\sum_{\mathbf{k}\in K}(u_0p^2_{1,\mathbf{t}(\mathbf{k})} - p_{2,\mathbf{t}(\mathbf{k})})\alpha_{\mathbf{t}(\mathbf{k})}\big) = \\ & u_0\big(u_0\cdot(1\cdot 4 - 2\cdot 9 + 3\cdot 16 - 2\cdot 16) - \\ & (1\cdot 2 - 2\cdot 5 + 3\cdot 10 - 2\cdot 8)\big) = u_0(2u_0 - 6) \end{aligned}$$

From this follows, that if the difference equation has a polynomial solution of the degree $d > 2$ than for this degree must hold $2d - 6 = 0$, that is $d = 3$. It is a routine to check that $P(x) = x^3 + x^2 + x + 1$ solves the equation.

8. Conclusions and Outlook

We have considered polynomial solutions $P(x)$ of difference equations of the form $G(P(x-\tau_1),\ldots,P(x-\tau_s)) + G_0(x) = 0$, where $G(x_1,\ldots,x_s)$ is a known polynomial of a degree $D \geq 2$ and G_0 is a known polynomial in x. We study the cases when one can bound the degree d of a polynomial P if exists. For the difference equation we construct the family of polynomials $S_l^*(u_0)$, $l \geq 0$. We have shown that if $L := \min\{l | S_l^*(x) \not\equiv 0\} \leq 5$ then $d \leq \max\{L, \deg(G_0)/(D-1)\}$ or d must be amongst the natural roots of $S_L^*(u_0)$

(theorem 5). We have shown that in this way we can bound d for all quadratic difference equations with $\tau_i = i$, where $i = 1, 2, 3$ and all cubic difference equations with $\tau_i = 1, 2$ where $i = 1, 2$. In general, we cannot bound the degree of solutions of difference equations for which $S_l^*(u_0)$ are constant zeros for all $0 \leq l \leq 5$. However, we have proven that $d \leq \max\{D, \deg(G_0)\}$ for equations with $s = 2$, $\tau_1 = 0$ and $\tau_2 = \tau$, theorem 7.

An obvious direction of the future research is applying our technique to polynomial difference equations with polynomial non-constant coefficients. More challenging problem is to check if there a connections between the obtained results and Galois theory.

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Appendix

8.1. $B_{l\mu}$ *for* $0 \leq l' \leq l \leq 6$, $0 \leq \mu \leq l'$ *and the coefficients* $H_{l\mathbf{i}_l l'\mu}$

8.1.1. $E_0(\mathbf{v}_0, \mathbf{u}_0)$

immediately by the definitions we get $B_{00} = A_{0()} = 1$.

8.1.2. $E_1(\mathbf{v}_1, \mathbf{u}_1)$

Recall that $E_1(\mathbf{v}_1, \mathbf{u}_1) = -v_0 u_1 - v_1 u_0$. Therefore, the following identities hold.

- The coefficient of u_1^0 is $A_{10} = -u_0 v_1$. From this follows that:
  - · the coefficient of u_0^1 for A_{10} is $B_{11} = -v_1$.
  - · the coefficient of u_0^0 for A_{10} is $B_{10} = 0$.
- The coefficient of u_1^1 is $A_{11} = -v_0 = -v_0 \cdot B_{00}$, so $H_{1100} = -v_0$.

8.1.3. $E_2(\mathbf{v}_2, \mathbf{u}_2)$

We have

$$E_2(\mathbf{v}_2, \mathbf{u}_2) = (1/2)u_0^2 v_1^2 + (1/2)v_0^2 u_1^2 + v_1 u_0 v_0 u_1 - (1/2)v_2 u_0 - v_1 u_1 - (1/2)v_0 u_2$$

- Thus, the coefficient of $u_1^0 u_2^0$ is $A_{2\ 00} = (1/2)v_1^2 u_0^2 - (1/2)v_2 u_0$, that is
  - · $B_{22} = (1/2)v_1^2$,
  - · $B_{21} = -(1/2)v_2$,
  - · $B_{20} = 0$
- The coefficient of $u_1^1 u_2^0$ is $A_{2\ 10} = v_0 u_0 v_1 - v_1$. It is easy to see that $A_{2\ 10} = (-u_0 v_0 + 1)B_{11} + v_0^2 B_{00}$.
- $A_{2\ 20} = (1/2)v_0^2$. It is easy to see that $A_{2\ 20} = v_0^2 B_{00}$.
- The coefficient of $u_1^0 u_2^1$ is $A_{201} = -(1/2)v_0 = -(1/2)v_0 B_{00}$.

8.1.4. $E_3(\mathbf{v}_3, \mathbf{u}_3)$

The coefficient of $u_1^0 u_2^0 u_3^0$ is $A_{3\ 000} = (1/3)\big((1/2)v_1^2 u_0^2 - (1/2)v_2 u_0\big)v_1 u_0 + (1/3)v_1 u_0^2 v_2 - (1/3)v_3 u_0$, that is

- $B_{33} = -(1/6)v_1^3$,
- $B_{32} = (1/2)v_1 v_2$,
- $B_{31} = -(1/3)v_3$,
- $B_{30} = 0$

The presentation of $A_{3\mathbf{i}_3}$ as a linear combination of the form $H_{3\mathbf{i}_3 00}B_{00} + \sum_{1\le\mu\le l\le 2} H_{3\mathbf{i}_3 l\mu}B_{l\mu}$ is considered in detail in the table below.

$\mathbf{i}_3$	$H_{3\mathbf{i}_3 22}$	$H_{3\mathbf{i}_3 21}$	$H_{3\mathbf{i}_3 11}$	$H_{3\mathbf{i}_3 00}$
100	$-u_0 v_0^2 + 2u_0$	$-u_0 v_0 + 2$	0	0
200	0	0	$\frac{1}{2}v_0^2 u_0 - v_0$	0
300	0	0	0	$-\frac{1}{6}v_0^3$
010	0	0	$-\frac{1}{2}u_0 v_0 + 1$	0
110	0	0	0	$\frac{1}{2}v_0^2$
001	0	0	0	$-\frac{1}{3}v_0$

8.1.5. $E_4(\mathbf{v}_4, \mathbf{u}_4)$

The coefficient of $u_1^0 u_2^0 u_3^0 v_4^0$ is $A_{40000} = B_{44}u_0^4 + B_{43}u_0^3 + B_{42}u_0^2 + B_{41}u_0$ where

- $B_{44} = (1/24)v_1^4$,
- $B_{43} = -(1/4)v_2 v_1^2$,
- $B_{42} = (1/3)v_3 v_1 + (1/8)v_2^2$,
- $B_{41} = -(1/4)v_4$,
- $B_{40} = 0$.

The presentation of $A_{4\mathbf{i}_4}$ as a linear combination

$$H_{4\mathbf{i}_4 00} B_{00} + \sum_{1 \le \mu \le l \le 3} H_{4\mathbf{i}_4 l\mu} B_{l\mu}$$

is considered in detail in the table below.

$\mathbf{i}_4$	$H_{4\mathbf{i}_4 33}$	$H_{4\mathbf{i}_4 32}$	$H_{4\mathbf{i}_4 31}$	$H_{4\mathbf{i}_4 22}$	$H_{4\mathbf{i}_4 21}$	$H_{4\mathbf{i}_4 11}$	$H_{4\mathbf{i}_4 00}$
1000	$-v_0 u_0^3 + 3u_0^2$	$3u_0 - v_0 u_0^2$	$-u_0 v_0 + 3$	0	0	0	0
2000	0	0	0	$\frac{1}{2} v_0^2 u_0^2 - 2v_0 u_0 + 1$	$\frac{1}{2} v_0^2 u_0 - 2v_0$	0	0
3000	0	0	0	0	0	$-\frac{1}{6} v_0^3 u_0 + \frac{1}{2} v_0^2$	0
4000	0	0	0	0	0	0	$\frac{1}{24} v_0^4$
0100	0	0	0	$-\frac{1}{2} v_0 u_0^2 + 2u_0$	$-\frac{1}{2} v_0 u_0 + 3$	0	0
0200	0	0	0	0	0	0	$\frac{1}{8} v_0^2$
0010	0	0	0	0	0	$-\frac{1}{3} u_0 v_0 + 1$	0
0001	0	0	0	0	0	0	$-\frac{1}{4} v_0$
1100	0	0	0	0	0	$\frac{1}{2} v_0^2 u_0 - \frac{3}{2} v_0$	0
2100	0	0	0	0	0	0	$-\frac{1}{4} v_0^3$
1010	0	0	0	0	0	0	$\frac{1}{3} v_0^2$

8.1.6. $E_5(\mathbf{v}_5, \mathbf{u}_5)$

The coefficient of $u_1^0 u_2^0 u_3^0 u_4^0 u_5^0$ is $A_{50000} = B_{55} u_0^5 + B_{54} u_0^4 + B_{53} u_0^3 + B_{52} u_0^2 + B_{51} u_0$ where

- $B_{55} = -(1/120) v_1^5$,
- $B_{54} = (1/12) v_2 v_1^3$,
- $B_{53} = -(1/6) v_1^2 v_3 - (1/8) v_2^2 v_1$,
- $B_{52} = (1/4) v_4 v_1 + (1/6) v_3 v_2$,
- $B_{51} = -(1/5) v_5$,
- $B_{50} = 0$.

The presentation of $A_{5\mathbf{i}_5}$ as a linear combination

$$H_{5\mathbf{i}_5 00} B_{00} + \sum_{1 \le \mu \le l \le 4} H_{5\mathbf{i}_5 l\mu} B_{l\mu}$$

is considered in detail in the tables below.

$\mathbf{i}_5$	$H_{5\mathbf{i}_5 44}$	$H_{5\mathbf{i}_5 43}$	$H_{5\mathbf{i}_5 42}$	$H_{5\mathbf{i}_5 41}$	$H_{5\mathbf{i}_5 33}$	$H_{5\mathbf{i}_5 32}$
10000	$-v_0u_0^4+4u_0^3$	$4u_0^2-v_0u_0^3$	$-v_0u_0^2+4u_0$	$-u_0v_0+4$	0	0
20000	0	0	0	0	$\frac{1}{2}v_0^2u_0^3-3v_0u_0^2+3u_0$	$\frac{1}{2}v_0^2u_0^2-3u_0v_0+2$
30000	0	0	0	0	0	0
40000	0	0	0	0	0	0
50000	0	0	0	0	0	0
01000	0	0	0	0	$-\frac{1}{2}v_0u_0^3+3u_0^2$	$-\frac{1}{2}v_0u_0^2+4u_0$
02000	0	0	0	0	0	0
11000	0	0	0	0	0	0
12000	0	0	0	0	0	0
00100	0	0	0	0	0	0
10100	0	0	0	0	0	0
21000	0	0	0	0	0	0
20100	0	0	0	0	0	0
31000	0	0	0	0	0	0
01100	0	0	0	0	0	0
00010	0	0	0	0	0	0
10010	0	0	0	0	0	0
00001	0	0	0	0	0	0

$\mathbf{i}_5$	$H_{5\mathbf{i}_5 31}$	$H_{5\mathbf{i}_5 22}$	$H_{5\mathbf{i}_5 21}$	$H_{5\mathbf{i}_5 11}$	$H_{5\mathbf{i}_5 00}$
10000	0	0	0	0	0
20000	$-3v_0+\frac{1}{2}v_0^2u_0$	0	0	0	0
30000	0	$-\frac{1}{6}v_0^3u_0^2+v_0^2u_0-v_0$	$-\frac{1}{6}v_0^3u_0+v_0^2$	0	0
40000	0	0	0	$\frac{1}{24}v_0^4u_0-\frac{1}{6}v_0^3$	0
50000	0	0	0	0	$-\frac{1}{120}v_0^5$
01000	$-\frac{1}{2}u_0v_0+6$	0	0	0	0
02000	0	0	0	$\frac{1}{8}v_0^2u_0-\frac{1}{2}v_0$	0
11000	0	$\frac{1}{2}v_0^2u_0^2-3v_0u_0+2$	$\frac{1}{2}v_0^2u_0-4v_0$	0	0
12000	0	0	0	0	$-\frac{1}{8}v_0^3$
00100	0	$-\frac{1}{3}v_0u_0^2+2u_0$	$-\frac{1}{3}u_0v_0+4$	0	0
10100	0	0	0	$-\frac{1}{3}v_0^2u_0-\frac{4}{3}v_0$	0
21000	0	0	0	$-\frac{1}{4}v_0^3u_0+v_0^2$	0
20100	0	0	0	0	$-\frac{1}{6}u_0^3$
31000	0	0	0	0	$\frac{1}{12}v_0^4$
01100	0	0	0	0	$\frac{1}{6}v_0^2$
00010	0	0	0	$-\frac{1}{4}u_0v_0+1$	0
10010	0	0	0	0	$\frac{1}{4}v_0^2$
00001	0	0	0	0	$-\frac{1}{5}v_0$

8.1.7. $E_6(\mathbf{v}_5,\mathbf{u}_6)$

The coefficients $H_{6\mathbf{i}_6 l\mu}$ are given in the following tables:

$\mathbf{i}_6$	$H_{6\mathbf{i}_6 55}$	$H_{6\mathbf{i}_6 54}$	$H_{6\mathbf{i}_6 53}$	$H_{6\mathbf{i}_6 52}$	$H_{6\mathbf{i}_6 51}$
100000	$-v_0u_0^5+5u_0^4$	$5u_0^3-v_0u_0^4$	$-v_0u_0^3+5u_0^2$	$-u_0^2v_0+5u_0$	$-u_0v_0+5$
200000	0	0	0	0	0
300000	0	0	0	0	0
400000	0	0	0	0	0
500000	0	0	0	0	0
600000	0	0	0	0	0
010000	0	0	0	0	0
020000	0	0	0	0	0
030000	0	0	0	0	0
110000	0	0	0	0	0
210000	0	0	0	0	0
310000	0	0	0	0	0
410000	0	0	0	0	0
120000	0	0	0	0	0
220000	0	0	0	0	0
001000	0	0	0	0	0
002000	0	0	0	0	0
101000	0	0	0	0	0
201000	0	0	0	0	0
301000	0	0	0	0	0
011000	0	0	0	0	0
111000	0	0	0	0	0
000100	0	0	0	0	0
100100	0	0	0	0	0
200100	0	0	0	0	0
010100	0	0	0	0	0
000010	0	0	0	0	0
100010	0	0	0	0	0
000001	0	0	0	0	0

$\mathbf{i}_6$	$H_{6\mathbf{i}_6 44}$	$H_{5\mathbf{i}_6 43}$	$H_{5\mathbf{i}_6 42}$	$H_{5\mathbf{i}_6 41}$	$H_{5\mathbf{i}_6 33}$
100000	0	0	0	0	0
200000	$-4v_0u_0^3+\frac{1}{2}u_0^4v_0^2+6u_0^2$	$-4v_0u_0^2+5u_0+\frac{1}{2}u_0^3v_0^2$	$3-4v_0u_0+\frac{1}{2}u_0^2v_0^2$	$-4v_0+\frac{1}{2}u_0v_0^2$	0
300000	0	0	0	0	$\frac{3}{2}u_0^2v_0^2-3v_0u_0-\frac{1}{6}u_0^3v_0^3+1$
400000	0	0	0	0	0
500000	0	0	0	0	0
600000	0	0	0	0	0
010000	$-\frac{1}{2}u_0^4v_0+4u_0^3$	$-\frac{1}{2}u_0^3v_0+5u_0^2$	$-\frac{1}{2}u_0^2v_0+7u_0$	$-\frac{1}{2}u_0v_0+10$	0
020000	0	0	0	0	0
030000	0	0	0	0	0
110000	0	0	0	0	$\frac{1}{2}u_0^3v_0^2-\frac{9}{2}u_0^2v_0+6u_0$
210000	0	0	0	0	0
310000	0	0	0	0	0
410000	0	0	0	0	0
120000	0	0	0	0	0
220000	0	0	0	0	0
001000	0	0	0	0	$-\frac{1}{3}u_0^3v_0+3u_0^2$
002000	0	0	0	0	0
101000	0	0	0	0	0
201000	0	0	0	0	0
301000	0	0	0	0	0
011000	0	0	0	0	0
111000	0	0	0	0	0
000100	0	0	0	0	0
100100	0	0	0	0	0
200100	0	0	0	0	0
010100	0	0	0	0	0
000010	0	0	0	0	0
100010	0	0	0	0	0
000001	0	0	0	0	0

	$H_{6\mathbf{i}_6 32}$	$H_{6\mathbf{i}_6 31}$	$H_{6\mathbf{i}_6 22}$	$H_{6\mathbf{i}_6 21}$	$H_{6\mathbf{i}_6 11}$	$H_{6\mathbf{i}_6 00}$
100000	0	0	0	0	0	0
200000	0	0	0	$-\frac{1}{4}v_2$	0	0
300000	$\frac{3}{2}v_0^2u_0 - \frac{1}{6}v_0^3u_0^2 - 2v_0$	$-\frac{1}{6}v_0^3u_0 + \frac{3}{2}v_0^2$	0	0	0	0
400000	0	0	$\frac{1}{2}v_0^2 - \frac{1}{3}v_0^3u_0 + \frac{1}{24}v_0^4u_0^2$	$\frac{1}{24}v_0^4u_0 - \frac{1}{3}v_0^3$	0	0
500000	0	0	0	0	$\frac{1}{24}v_0^4 - \frac{1}{120}v_0^5u_0$	0
600000	0	0	0	0	0	$\frac{1}{720}v_0^6$
010000	0	0	0	$\frac{1}{4}v_2u_0$	0	0
020000	0	0	$\frac{1}{8}u_0^2v_0^2 - u_0v_0 + 1$	$\frac{1}{8}v_0^2u_0 - \frac{3}{2}v_0$	0	0
030000	0	0	0	0	0	$-\frac{1}{48}v_0^3$
110000	$\frac{1}{2}v_0^2u_0^2 - \frac{11}{2}u_0v_0 + 5$	$-\frac{15}{2}v_0 + \frac{1}{2}u_0v_0^2$	0	0	0	0
210000	0	0	$2v_0^2u_0 - \frac{5}{2}v_0 - \frac{1}{4}v_0^3u_0^2$	$\frac{5}{2}v_0^2 - \frac{1}{4}v_0^3u_0$	0	0
310000	0	0	0	0	$-\frac{5}{12}v_0^3 + \frac{1}{12}v_0^4u_0$	0
410000	0	0	0	0	0	$-\frac{1}{48}v_0^5$
120000	0	0	0	0	$-\frac{1}{8}v_0^3u_0 + \frac{5}{8}v_0^2$	0
220000	0	0	0	0	0	$\frac{1}{16}v_0^4$
001000	$-\frac{1}{3}u_0^2v_0 + 5u_0$	$-\frac{1}{3}u_0v_0 + 10$	0	0	0	0
002000	0	0	0	0	0	$\frac{1}{18}v_0^2$
101000	0	0	$\frac{1}{3}v_0^2u_0^2 - \frac{8}{3}u_0v_0 + 2$	$\frac{1}{3}v_0^2u_0 - \frac{14}{3}v_0$	0	0
201000	0	0	0	0	$-\frac{1}{6}u_0v_0^3 + \frac{5}{6}v_0^2$	0
301000	0	0	0	0	0	$\frac{1}{18}v_0^4$
011000	0	0	0	0	$\frac{1}{6}v_0^2u_0 - \frac{5}{6}v_0$	0
111000	0	0	0	0	0	$-\frac{1}{6}v_0^3$
000100	0	0	$-\frac{1}{4}u_0^2v_0 + 2u_0$	$-\frac{1}{4}u_0v_0 + 5$	0	0
100100	0	0	0	0	$\frac{1}{4}v_0^2u_0 - \frac{5}{4}v_0$	0
200100	0	0	0	0	0	$-\frac{1}{8}v_0^3$
010100	0	0	0	0	0	$\frac{1}{8}v_0^2$
000010	0	0	0	0	$-\frac{1}{5}v_0 + 1$	0
100010	0	0	0	0	0	$\frac{1}{5}v_0^2$
000001	0	0	0	0	0	$-\frac{1}{6}v_0$

8.2. The matrix of the linear system w.r.t. $\bar{\alpha}$, for difference equations $G\big(P(x-1), P(x-2), P(x-3)\big) = 0$

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
-2 & -3 & -4 & -4 & -5 & -6 \\
-1 & -5/2 & -5 & -4 & -13/2 & -9 \\
2 & 9/2 & 8 & 8 & 25/2 & 18 \\
-2/3 & -3 & -28/3 & -16/3 & -35/3 & -18 \\
2 & 15/2 & 20 & 16 & 65/2 & 54 \\
-4/3 & -9/2 & -32/3 & -32/3 & -125/6 & -36 \\
-1/2 & -17/4 & -41/2 & -8 & -97/4 & -81/2 \\
11/6 & 97/8 & 299/6 & 88/3 & 1907/24 & 297/2 \\
-2 & -45/4 & -40 & -32 & -325/4 & -162 \\
2/3 & 27/8 & 32/3 & 32/3 & 625/24 & 54 \\
-2/5 & -33/5 & -244/5 & -64/5 & -55 & -486/5 \\
5/3 & 81/4 & 386/3 & 160/3 & 2365/12 & 405 \\
-7/3 & -183/8 & -374/3 & -224/3 & -6035/24 & -567 \\
4/3 & 45/4 & 160/3 & 128/3 & 1625/12 & 324 \\
-4/15 & -81/40 & -128/15 & -128/15 & -625/24 & -324/5
\end{pmatrix} \bar{\alpha} = \bar{0}$$

The matrix of this system is computed and the system is solved by the generic Maple script

$$\texttt{corollaries} := \texttt{proc}(\texttt{s} :: \texttt{posint}, \texttt{v}_0\texttt{c} :: \texttt{posint})$$

For this corollary run it as `corollaries(3, 2)`.