CONSTRUCTIVE POINTFREE TOPOLOGY ELIMINATES NON-CONSTRUCTIVE REPRESENTATION THEOREMS FROM RIESZ SPACE THEORY

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ABSTRACT. In Riesz space theory it is good practice to avoid representation theorems which depend on the axiom of choice. Here we present a general methodology to do this using pointfree topology. To illustrate the technique we show that almost f-algebras are commutative. The proof is obtained relatively straightforward from the proof by Buskes and van Rooij by using the pointfree Stone-Yosida representation theorem by Coquand and Spitters.

The Stone-Yosida representation theorem for Riesz spaces [LZ71, Zaa83] shows how to embed every Riesz space into the Riesz space of continuous functions on its spectrum.

Theorem 1. [Stone-Yosida] *Let R be an Archimedean Riesz space (vector lattice) with unit. Let Σ be its (compact Hausdorff) space of representations. Define the continuous function $\hat{r}(\sigma) := \sigma(r)$ on Σ. Then $r \mapsto \hat{r}$ is a Riesz embedding of R into $C(\Sigma, \mathbb{R})$.*

The theorem is very convenient, but sometimes better avoided, since it leads out of the theory of Riesz spaces. To quote Zaanen [Zaa97]:

> Direct proofs, although sometimes a little longer than proofs by means of representation [theorems], often reveal more about the situation under discussion.

Similar concerns where discussed by Buskes, de Pagter and van Rooij [BvR89, BdPvR91]. They proposed to avoid the use of the axiom of choice by restricting the size of the Riesz spaces [BvR89]. We provide a solution based on pointfree topology, which like [BvR89] avoids also the *countable* axiom of choice, but moreover avoids the axiom of excluded middle[1]. This allows the results to be applied in non-standard contexts. For instance, one can translate a theorem about one C*-algebra to a theorem about a continuous field of C*-algebras [BM97, BM00b, BM00a, BM06]. In turn, the results about commutative C*-algebras may be obtained directly using Riesz spaces [CS08]. This is used in applications of topos theory to quantum theory [HLS07].

Our strategy is as follows. First we replace the topological space of representations by a locale, a point-free space. This typically removes the need of the axiom of choice [Mul03]. Then we proceed to use pointfree topology, locale theory using a only basis for the topology [Sam87]. Since the space of representations of a Riesz space is compact Hausdorff, it can be described explicitly by the finite covering relation on the lattice of basic opens. This lattice can be defined directly using the Riesz space structure [Coq05, CS05].

[1] $P \vee \neg P$.

To illustrate the method we will reprove the results by Buskes and van Rooij [BvR00].

1. Preliminaries

Definition 1. *A* Riesz space *is a vector space with compatible lattice operations — i.e.* $f \wedge g + f \vee g = f + g$ *and if* $f \geqslant 0$ *and* $a \geqslant 0$*, then* $af \geqslant 0$*. A* (strong) unit *1 in a Riesz space is an element such that for all* f *there exists a natural number* n *such that* $|f| \leqslant n \cdot 1$*. A Riesz space is* Archimedean *if for all* n*,* $n|x| \leqslant y$ *implies that* $x = 0$*.*

1.1. Topologies, locales, lattices.

A topological space may be presented in its familiar set theoretic form, as such it is a complete distributive lattice of open sets with the operations of union and intersection. The category of *frames* has as objects lattices with a finitary meets and infinitary joins such that $\wedge$ distributes over $\bigvee$. Frame maps preserve this structure. A continuous function $f : X \to Y$ defines a frame map $f^{-1} : O(Y) \to O(X)$. Since this map goes in the reverse direction it is often convenient to consider the category of *locales*, the opposite of the category of frames. In fact, there is a categorical adjunction between the category of locales and the category of topological spaces. This restricts to an equivalence of categories for compact Hausdorff spaces and compact regular locales. In general, the axiom of choice is needed to move from locales to topological spaces. Hence, by staying on the localic side it is often possible to avoid the axiom of choice. However, one can go even further. Since compact locales are determined by the finitary coverings one may restrict one's attention to the finitary covering relation on a base of the topology. This base is a normal[2] distributive lattice.

For the spectrum of a Riesz space R a base for the topology can be described very explicitly. Recall that a base for the topology of the spectrum Σ is defined by the opens $\{\sigma | \hat{a}(\sigma) > 0\}$. Let P denote the set of positive elements of R. For a, b in P we define $a \preccurlyeq b$ to mean that there exists n such that $a \leq nb$. The following proposition is proved in [CS05] and involves only elementary considerations on Riesz spaces.

Proposition 1. $L(R) := (P, \vee, \wedge, 1, 0, \preccurlyeq)$ *is a distributive lattice. In fact, if we define* $D : R \to L(R)$ *by* $D(a) := a^+$*, then* $L(R)$ *is the free lattice generated by* $\{D(a) | a \in R\}$ *subject to the following relations:*

1. $D(a) = 0$*, if* $a \leq 0$*;*
2. $D(1) = 1$*;*
3. $D(a) \wedge D(-a) = 0$*;*
4. $D(a + b) \leqslant D(a) \vee D(b)$*;*
5. $D(a \vee b) = D(a) \vee D(b)$*.*

We have $D(a) \leqslant D(b)$ if and only if $a^+ \preccurlyeq b^+$ and $D(a) = 0$ if and only if $a \leqslant 0$. We write $a \in (p, q) := (a - p) \wedge (q - a)$ and observe that this is an element of R.

For a in R we define its norm $\|a\| = \inf\{q | a \leqslant q1\}$.

The corresponding locale[3] (complete distributive lattice) Σ is the one defined by the same generators and relations together with the relation $D(a) = \bigvee_{s>0} D(a - s)$. The generators and relations above may also be read as a propositional geometric

[2] A lattice is *normal* if for all b_1, b_2 such that $b_1 \vee b_2 = \top$ there are c_1, c_2 such that $c_1 \wedge c_2 = \bot$ and $c_1 \vee b_1 = \top$ and $c_2 \vee b_2 = \top$. The opens of a normal topological space form a normal lattice.

[3] From this point onwards Σ is the spectrum considered as a locale. If we want to treat it as a topological space we write $\mathrm{pt}\,\Sigma$.

theory [Vic07] by reading $\leqslant$ as $\Rightarrow$. A model m of this theory defines a representation σ_m of the Riesz space by

$$\sigma_m(a) := (\{q | m \models D(q \cdot 1 - a)\}, \{q | m \models D(a - q \cdot 1)\}),$$

where the right hand side is a Dedekind cut in the rationals and hence a real number. Such a σ_m is a point of the locale Σ. This motivates the interpretation of $D(a)$ as $\{\sigma | \hat{a}(\sigma) > 0\}$: the models which make the proposition $D(a)$ true coincide with the points σ such that $\hat{a}(\sigma) > 0$. Proving that there are enough such models/points requires the axiom of choice. We avoid this axiom by working with the propositions/opens instead.

Theorem 2. [Localic Stone-Yosida] *The map* $\hat{\cdot} : R \to \mathrm{Loc}(\Sigma, \mathbb{R})$ *defined by the frame map* $\hat{a}(p, q) := a \in (p, q)$ *is a norm-preserving Riesz morphism. Its image is dense with respect to the uniform topology on* $\mathrm{Loc}(\Sigma, \mathbb{R})$.

Proof. The map $\hat{\cdot} : R \to \mathrm{Loc}(\Sigma, \mathbb{R})$ is norm-preserving; see [CS05]. It remains to prove the density. For this consider a natural number N and a continuous f on Σ such that $0 \leqslant f \leqslant 1$. We need to find an element a of R such that $\hat{a}$ is close to f. The set $\bigcup_{k=0}^{N} f \in ((k-1)/N, (k+1)/N)$ covers Σ. By Proposition 3.1 of [Coq05] there exists a partition of unity p_i in the Riesz space such that $\sum p_i = 1$ and $D(p_i)$ is contained in some open $f \in ((k_i - 1)/N, (k_i + 1)/N)$ in Σ. Concretely, $p_i \preccurlyeq (f - (k_i - 1)/N) \wedge ((k_i + 1)/N - f)$. Consequently,

$$|f - \sum k_i \widehat{p_i}| = |f \sum \widehat{p_i} - \sum k_i \widehat{p_i}| = |\sum (f - k_i) \widehat{p_i}| \leqslant \frac{1}{N}.$$

$\square$

The map $\hat{\cdot}$ is a Riesz embedding if R is Archimedean.

Corollary 1. *There is a norm-preserving Riesz morphism of R into an f-algebra such that the image is dense.*

The axiom of choice implies that compact regular locales have enough points and hence we obtain the more familiar formulation of the theorem by working with the topological space $\mathrm{pt}\,\Sigma$ of the points of the spectrum. However, in practice, only the localic version is needed.

Corollary 2. [Stone-Yosida] *The map* $\hat{\cdot} : R \to C(\mathrm{pt}\,\Sigma, \mathbb{R})$ *defined by the frame map* $\hat{a}(p, q) := a \in (p, q)$ *is a norm-preserving Riesz morphism. Its image is dense with respect to the uniform topology on* $C(\mathrm{pt}\,\Sigma, \mathbb{R})$.

2. The results

Definition 2. *An* almost f-algebra *is a Riesz space with multiplication such that* $a \cdot b \geqslant 0$ *if* $a, b \geqslant 0$, *and* $a \wedge b = 0$ *implies* $a \cdot b = 0$.

If E is a Riesz space, a bilinear map A of $E \times E$ into a vector space F is called *orthosymmetric* if

$$f \wedge g = 0 \Rightarrow A(f, g) = 0$$

for all $f, g \in E$.

A *partition of unity* is a list u_i such that $\sum u_i = 1$ and $0 \leqslant u_i \leqslant 1$. If u, v are partitions of unity in an almost f-algebra, then so is $u \cdot v$: $\sum_i \sum_j u_i v_j = 1 \cdot \sum_j v_j$.

Theorem 3. *Let E be Riesz spaces with unit and let F be Archimedean and let A be a orthosymmetric positive bilinear map $E \times E \to F$. Let $\bar{E}$ be an f-algebra in which E is dense and let F' be the uniform completion of F. Then A extends uniquely to a orthosymmetric positive bilinear map from $\bar{E} \times \bar{E}$ to F' and $A(f,g) = A(1, fg)$ for all f, g in E.*

Proof. Let f, g be in E.

$$A(f,g) = A(f^+, g^+) + A(f^-, g^-) - A(f^-, g^+) - A(f^+, g^-).$$

So, it suffices to consider the case $0 \leqslant f, g \leqslant 1$. Let k be a natural number. Define $u_n := k(f \vee \frac{n}{k} \wedge \frac{n+1}{k})$, whenever $0 \leq n < k$. Define $v_0 := 1 - u_0$ and $v_n := u_n - u_{n+1}$ and $v_k := u_k$. The set $\{v_0, \ldots, v_k\}$ is a partition of unity — that is, $\sum v_i = 1$ and $0 \leqslant v_i \leqslant 1$. Moreover, $v_n \perp v_m$, whenever $|n - m| > 1$ and such that $|f v_n - \frac{n}{k} v_n| \leq \frac{1}{k}$. By repeating a similar construction for g we find a partition of unity v'. Then $w_{ij} := v_i v'_j$ is again a partition of unity. For convenience, we reindex w by one natural number to obtain a sequence w_n. We define α_n, β_n such that $|f - \sum_n \alpha_n w_n| < \frac{1}{k}$ and $|g - \sum_n \beta_n w_n| < \frac{1}{k}$.

Let $\varepsilon = \frac{1}{k}$. Set $f' := \sum \alpha_n w_n$, $g' := \sum \beta_n w_n$ and $h' := \sum \alpha_n \beta_n w_n$. Then

$$\begin{aligned} |A(f,g) - A(f',g')| &= |A(f - f', g) + A(f', g - g')| \\ &\leqslant \varepsilon A(1,1) + \varepsilon A(1,1) \end{aligned}$$

since $|f - f'|, |g - g'| \leqslant \varepsilon$ and A is positive. Thus, it suffices to show that

$$|A(f', g') - A(1, h')| \leqslant 2\varepsilon A(1,1).$$

Observe that for all n, m in $\{1, ..., N\}$,

if $|n - m| > 1$, then $w_n \perp w_m$, so $A(w_n, w_m) = 0$;

if $|n - m| \leqslant 1$, then $|\alpha_n - \alpha_m| \leqslant 2\varepsilon$.

It follows that

$$\begin{aligned} |A(f', g') - A(1, h')| &= |\sum \alpha_n \beta_m A(w_n, w_m) - \sum_{n,m} \alpha_m \beta_m A(w_n, w_m)| \\ &\leqslant \sum |\alpha_n - \alpha_m| |\beta_m| A(w_n, w_m) \\ &\leqslant 2\varepsilon \sum_{n,m} A(w_n, w_m) = 2\varepsilon A(1,1) \end{aligned}$$

The last inequality follows from the observation above and the inequality $|\beta_m| \leqslant 1$.

Changing the roles of the αs and βs we have that $|A(g', f') - A(1, h')| \leqslant 2\varepsilon A(1,1)$. Hence $|A(f', g') - A(g', f')| \leqslant 4\varepsilon A(1,1)$ and $|A(f,g) - A(g,f)| \leqslant 8\varepsilon A(1,1)$. Finally, $|h' - f'g'| \leqslant |\sum \alpha_m \beta_m w_n w_m - \sum \alpha_n \beta_m w_n w_m| \leqslant \sum |\alpha_n - \alpha_m| |\beta_m| w_n w_m \leqslant 2\varepsilon$. Hence $|h' - fg|$ is small. Since F is Archimedean, $A(f,g) = A(1, fg)$. □

The completion of E and F in the previous proof are used to define the multiplication on E and to be able to extend A to this completion of E. This, however, can be avoided as follows. The joint partition of unity can be obtained as in Theorem 2. The proof above then shows that for each ε, $|A(f,g) - A(g,f)| \leqslant 8\varepsilon A(1,1)$. This implies the following result.

Corollary 3. *Let E and F be Riesz spaces of which F is Archimedean. Let A be an orthosymmetric positive bilinear map $E \times E \to F$. Then*

$$A(f,g) = A(g,f) \qquad (f, g \in E).$$

Proof. Take $f, g \in E$. Let E_0 be the Riesz subspace of E generated by $\{f, g\}$. Then $|f| + |g|$ is a unit in E_0. Without restriction, suppose that E_0 is E. The result is now follows from the previous theorem. □

We have proved the following result in an entirely explicit way by a straightforward analysis of the proof by Buskes and van Rooij.

Corollary 4. *Every Archimedean almost f-algebra is commutative.*

3. Internal real numbers

Buskes and van Rooij use the Stone-Yosida representation theorem combined with Dini's theorem to show that a certain sequence of elements in a Riesz space converges. This can be replaced by applying the following result which does not require the sequence to be decreasing.

Theorem 4. *Let e_n be a sequence of expressions in the language of Riesz spaces such that e_n converges* constructively *when interpreted in the Riesz space of real numbers. Then e_n converges uniformly when interpreted in any Riesz space with strong unit.*

Proof. The Riesz space can be (densely) embedded into a space $C(\Sigma)$ and hence its elements may be interpreted as global sections of the real number object in the topos $\mathrm{Sh}(\Sigma)$ of sheaves over Σ [Joh02, MLM94]. Now, if a_n converges to 0 in the internal language of $\mathrm{Sh}(\Sigma)$. Then for each q there exists n such that $a_n \leqslant q$ internally. This is interpreted as: for each q there exists a (finite) cover U_i of Σ and n_i such that $a_{n_i} \leqslant q$ on U_i. Taking $n = \min n_i$ we see that $a_n \leqslant q$ on Σ. □

Sheaf theory may seem to be a very complex tool to use for such a simple lemma, however, when applied in concrete cases we obtain natural results. For instance, Buskes and van Rooij apply Dini's theorem and the Stone-Yosida representation theorem to prove that the sequence $[(f \wedge g)h - nf(g \wedge h)]^+$ converges uniformly. We first prove this for the Riesz space of real numbers. Fix m in $\mathbb{N}$. We may assume that $f, g, h \leqslant 1$. Moreover, either[4] $f \geqslant \frac{1}{m}$ or $f \leqslant \frac{2}{m}$. We may assume that $f \geqslant \frac{1}{m}$ and similarly that $g, h \geqslant \frac{1}{m}$. Choosing $n = m^2$ shows that $(f \wedge g)h \leqslant 1$ and $nf(g \wedge h) \geqslant n \cdot \frac{1}{m} \cdot \frac{1}{m} \geqslant 1$. Hence if $n \geqslant m^2$, then $[(f \wedge g)h - nf(g \wedge h)]^+ \leqslant \frac{2}{m}$ in all cases. The interpretation of this statement in the sheaf model $\mathrm{Sh}(\Sigma)$ defined from the spectrum Σ of a Riesz space is: there is a finite cover U_i of Σ such that $[(f \wedge g)h - nf(g \wedge h)]^+ \leqslant \frac{2}{m}$ is true on each U_i. A finite cover gives rise to a partition u_i of unity such that $D(u_i) \subset U_i$. So, that $u_i[(f \wedge g)h - nf(g \wedge h)]^+ \leqslant \frac{2}{m}$ and hence

$$[(f \wedge g)h - nf(g \wedge h)]^+ = \sum u_i[(f \wedge g)h - nf(g \wedge h)]^+ \leqslant \frac{2}{m}.$$

Takeuti's use of Boolean valued models [Tak78] to obtain non-standard results from familiar theorems has a similar flavor as Theorem 4. Boolean valued models are a special class of sheaf models.

[4] The case distinction $f \geqslant \frac{1}{m}$ or $f \leqslant \frac{1}{m}$ is not constructive/continuous.

4. Conclusion

We have illustrated how the use of locale theory, presented by a normal distributive lattice of basic elements, naturally translates proofs which depend on the axiom of choice to simpler lattice theoretic proofs which avoid the axiom of choice, even in its countable form, and the principle of excluded middle. Buskes and van Rooij had previously proposed different methods to avoid the axiom of choice. An advantage of our approach is that it is valid in any topos. It also provides a logical tool to remove the use of representation theorems from Riesz space theory, the importance of avoiding representation theorems was stressed by Zaanen.

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