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LOCATEDNESS AND OVERT SUBLOCALES

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Abstract. Locatedness is one of the fundamental notions in constructive mathematics. The existence of a positivity predicate on a locale, i.e. the locale being overt, or open, has proved to be fundamental in constructive locale theory. We show that the two notions are intimately connected.

Bishop defines a metric space to be compact if it is complete and totally bounded. A subset of a totally bounded set is again totally bounded iff it is located. So a closed subset of a Bishop compact set is Bishop compact iff it is located. We translate this result to formal topology. 'Bishop compact' is translated as compact and overt. We propose a definition of located predicate on subspaces in formal topology. We call a sublocale located if it can be presented by a formal topology with a located predicate. We prove that a closed sublocale of a compact regular locale has a located predicate iff it is overt. Moreover, a Bishop-closed subset of a complete metric space is Bishop compact — that is, totally bounded and complete — iff its localic completion is compact overt.

Finally, we show by elementary methods that the points of the Vietoris locale of a compact regular locale are precisely its compact overt sublocales.

We work constructively, predicatively and avoid the use of the axiom of countable choice. Consequently, all our results are valid in any predicative topos.

1. Introduction

As Freudenthal [Fre36] observed, an intuitionistic development of topology needs to be rebuilt starting with the foundations: one has to define the concept of space and one has to identify a useful concept of *sub*space. Brouwer identified the 'located compact' sets and their located subsets as important notions. Although Brouwer's work was set in a metric context, it seems natural to give a topological treatment. To do this Freudenthal followed Alexandroff's presentation of a compact metric space as the continuous image of Cantor space. Such an image is called a *fan* by Brouwer. Freudenthal's definition is point-free, it starts from formal opens and their incidence relation. A point is then defined as an infinite sequence of formal opens such that all finite initial sequences have a positive incidence.

The present state of the art in Bishop's constructive mathematics is not very different from that in Brouwer's intuitionistic mathematics seventy years ago. Bishop defined compactness for metric spaces: a metric space is Bishop-compact if it is complete and totally bounded. As in Brouwer's work, closed subspaces of Bishop-compact spaces are then compact iff they are located. Recently, Bridges and co-workers started a quest for a more topological development of Bishop's constructive mathematics and raised the question of finding a useful notion of compactness [BIM+07][BISV08]. We present our solution to this problem below. In short, we follow Martin-Löf and Sambin [Sam87] in adopting formal topology as a constructive theory of topology. We identify compact overt locales as an important notion of compactness, similar to the intuitionistic case. We propose a natural definition of locatedness and show that it is equivalent to previous definitions in familiar cases. A more precise list of our results can be found at the end of this introduction.

The present paper connects two notions: locatedness and overtness. Locatedness is a metric notion which comes from constructive mathematics in the sense of Brouwer and Bishop. Overtness is a topological notion which comes from constructive locale theory. Both notions are trivial in the presence of classical logic. We will start by very briefly introducing the two fields.

Constructive mathematics has its roots in Brouwer's intuitionistic mathematics. Brouwer's theory contains so-called law-like sequences. Kleene and Vesley [KV65] have formalized such sequences as recursive ones. Brouwer realized that the law-like sequences would form a rather sparse continuum and introduced his choice sequences to obtain a full continuum. A recursive continuum is indeed sparse. This leads to unexpected results, such as continuous functions on the unit interval which are unbounded and not Riemann integrable; see [BR87] for an excellent survey of these issues. Intuitively, one may view these functions as only defined on the *recursive* real numbers, but not on all real numbers. To avoid these issues Brouwer demanded his functions to be defined on *all* choice sequences. By a philosophical analysis of all possible ways to define such functions, Brouwer concluded that all such functions between choice sequences can be inductively defined; see [TvD88, sec.4.8]. In particular, Brouwer provides an analysis of all possible covers (=bars) of Baire space and Cantor space. All such bars are inductively given. Kreisel and Troelstra [KT70] use Brouwer's inductive encoding to give a syntactic translation from a theory with choice sequences to one without. This translation is known as the 'elimination of choice sequences'. Brouwer's analysis was also used by Martin-Löf to develop an inductive theory of constructive analysis [ML70] in the context of recursion theory. His proposal was continued by Sambin [Sam87], at which point insights from topos theory and domain theory were also included.

In topos theory one studies point-free spaces because it is often impossible to construct the points of certain topological spaces, since their construction requires the axiom of choice. Instead, one uses the theory of locales, complete distributive lattices satisfying the infinite distributive law, often referred to as point-free spaces. A connection with Brouwer's spreads was made by Fourman and Grayson [FG82]. Brouwer's key axioms, i.e. the 'bar-theorem' and the 'fan-theorem', are equivalent to Baire space and Cantor space having enough points. As in [ML70], this equivalence suggests that one can avoid the use of these axioms by working with formal spaces instead of point-set topology.

Those two works stimulated Sambin [Sam87] to develop what is called 'formal topology' in predicative constructive type theory. Predicative mathematics avoids quantification over powersets, one has power-classes instead. Formal topology can also be developed in predicative constructive set theory [AF05][Acz06], and as such it can be seen as a way of adapting locale theory to a predicative setting. Unlike type theory, constructive set theory, does not include the countable axiom of choice. One advantage of avoiding the axiom of choice is that theorems remain valid when reinterpreted in a sheaf model. Alternatively, one may see predicative formal topology as developing locale theory by working directly with coverages, or sites. This latter picture seems most natural when one thinks about a computational interpretation of constructive mathematics. This computational picture comes at a cost, for one has to check that the constructions do not depend on ones choice of coverage. On the other hand, Vickers [Vic07b] emphasizes the similarity between predicative reasoning and constructions preserved by inverse images of geometric morphisms between toposes. This similarity is due to the absence of the power set operation in both frameworks. This idea can be found already in the work of Joyal and Tierney [JT84]. Vickers [Vic06] relates inductively generated [CSSV03], or set-presented [Acz06], formal topology to the theory of continuous flat functors in topos theory.

Our final motivation for working predicatively is that the cover seems to be *needed* to reason about locatedness: As Example 2 shows, locatedness depends on the choice of the metric and hence on the choice of the base of the topology, the set of balls in this case. We will return to this issue after Definition 22.

Finally, let us come back to Bishop's work. Bishop and his followers developed an impressive body of analysis constructively. There are, however, a number of problems with his approach. For instance the metric spaces with continuous functions between them do not form a category. To prove that the composition of two continuous functions is again continuous, one needs the fan theorem. To avoid this problem, Bishop proposed a new definition of continuous function [Bri79], but later abandoned it [BB85]. This problem can be conveniently addressed in formal topology [Pal05, Pal07]. Moreover, Bishop's approach is mostly limited to the realm of separable metric spaces. In places where one is interested in more general spaces, for instance spectral spaces, formal topology seems to be more adequate, even when applied to the separable metric case, see e.g. [CS05][CS08].

We will avoid the use of the axiom of choice, even countable choice, and of the powerset axiom. Our results can therefore be interpreted both in predicative type theory and in topos theory. A predicative constructive set theory, as in [Acz06], suffices for our results.

1.1. **Located and overt.** Having introduced the general framework that we are working in, we now turn to locatedness and overtness.

1.1.1. *Locatedness.* Locatedness was introduced by Brouwer in [Bro19], as 'katalogisiertes Bereichkomplement', and has been used ever since in all flavors of constructive mathematics. Let A be subset of a metric space (X, ρ). A priori the distance

$$\rho_A(x) := \inf_{a \in A} \rho(a, x)$$

is an upper real in the sense of Definition 1. If $\rho_A(x)$ is actually a Dedekind real, for all x in X, then A is called *located*. In other words, A is located if for all x in X and $s < t$ either there exists y in A such that $d(x, y) < t$ or there is no y in Y such that $d(x, y) < s$.

A totally bounded subset is located and any located subset of a totally bounded space is totally bounded. This makes the notion crucial in Bishop's constructive mathematics. For instance, bounded located subsets of the a plane are exactly the ones that can be plotted accurately [O'C08].

1.1.2. *Overt.* In the point-free tradition of constructive mathematics one uses a positive way of stating that an open is non-empty, i.e. that it is inhabited. Surprisingly, the possibility of stating that an open is inhabited is a non-trivial property of a formal space. When this is possible we say that the formal topology carries a positivity predicate, see Definition 15. In locale theory one uses Definition 14 which is independent of the choice of the base and says that the locale is *overt*, or *open*. These definitions are equivalent. The notion of an open locale was developed by Johnstone [Joh84] after it had been introduced by Joyal and Tierney [JT84]. In a predicative context it was introduced as a positivity predicate by Sambin and Martin-Löf [Sam87]. Scott's consistency predicate in domain theory [Sco82] is another source of the positivity predicate in formal topology; see [Sam87] for a precise connection. The term open locale was coined because a locale is open iff its unique map to the terminal locale is open. However, this term leads to possible confusion with the notion of an open sublocale. We therefore prefer the term overt, introduced by Taylor, which seems to be becoming the standard terminology.

1.1.3. *Their connection.* We show that the notions of locatedness and overtness are intimately connected. We propose a definition of *locatedness* motivated by locatedness of subsets of metric spaces. A closed sublocale of a compact regular locale is located iff it is overt.

The similarity between Bishop-compact and compact overt was stated in the special case of the real numbers by Taylor in the context of his abstract Stone duality [Tay05]. Independently, we noticed this similarity in the special case of the spectrum of a Riesz space [CS05][Coq05], observing that in this case the spectrum is compact overt iff all elements are normable. Another motivation for the connection with locatedness may be found already in the work of Brouwer [Bro19, p.14] (which is conveniently presented in [Hey56, p.67]), Freudenthal [Fre36] and [ML70]. Brouwer proves that every bounded closed located subset of $\mathbb{R}^2$ coincides with a fan. A fan may be represented by a predicate on the finite binary sequences selecting the 'admissible' ones. If a finite sequence is admitted by the fan, then so is at least one of its successors — that is, the admissible sequences are positive.

We will develop the similarities above and extend them to more general — not necessarily compact — spaces. To be precise, we identify compact overt locales as an important notion of compactness, similar to the intuitionistic case. We propose a natural definition (Definition 22) of locatedness and show that it is equivalent to previous definitions in familiar cases: Bishop's metric definition (Proposition 7) and Martin-Löf's point-free definition (Proposition 12). This is a contribution to an ongoing project [CS05, Pal05, Pal07] of making the connections between Bishop's mathematics, intuitionistic mathematics and formal topology more precise.

In order to motivate our definition of locatedness, we generalize several properties of located metric spaces to more general spaces:

- A subset of a totally bounded metric space is Bishop-compact iff it is located (Theorems 2, 4). In the generalization 'Bishop-compact' is replaced by compact overt.
- A located Bishop-closed subset coincides with the complement of its complement (Propositions 10, 15, Corollary 1). Our generalization provides a condition under which closed and positively closed sublocales coincide. No general relation between closed and positively closed was known before [Vic07b].

- A theorem by Troelstra and van Dalen (Theorem 3).

These results are naturally found in four settings: totally bounded metric spaces (section 5), locally compact metric spaces (section 6), compact regular spaces (section 7) and regular spaces (section 8). In section 11 we suggest even more settings in which these results may be valid.

We refer to Johnstone [Joh82], Fourman and Grayson [FG82] for general background on point-free topology and to Sambin [Sam87] for formal topology. Background on formal topology developed without using type theoretic choice, may be found in [Acz06][AF05][Cur07]. Bishop and Bridges [BB85], and Troelstra and van Dalen [TvD88] are general references for constructive mathematics.

1.2. **Guide for the reader.** Researchers in Bishop style constructive mathematics will appreciate [BR87] followed by [Pal05, Pal07] as an introduction to the present work. On a first reading they may want to stop at section 7 and continue when they have an interest in constructive general topology.

2. Reals and metric spaces

In this and the following two sections we will collect some relevant background knowledge from the various traditions of constructive and classical mathematics. The expert reader can skip this section.

2.1. **The Dedekind, upper and lower reals.** It is natural to consider three kinds of real numbers: the upper real numbers, the lower real numbers and the Dedekind real numbers.

An *inhabited* set is one that is positively non-empty: we can construct a point in it.

Definition 1. *An* upper real number U *is an inhabited up-closed subset of the rationals which is* open*: if* $q \in U$*, then* $q' \in U$ *for some* $q' < q$*, and* proper*:* $U \neq \mathbb{Q}$*.*

Lower real numbers *are defined similarly.*

A Dedekind *real number is a disjoint pair* (L, U) *of a lower and an upper real number which are near: for all* $s < t$ *either* $s \in L$ *or* $t \in U$*. We denote the collection of Dedekind real numbers by* $\mathbb{R}$*.*

In the presence of the axiom of countable choice the Dedekind real numbers coincide with the Cauchy real numbers. In our present context we have no need for the Cauchy real numbers.

In the presence of classical logic, the lower, the upper and the Dedekind reals coincide.

Without the powerset axiom, the upper reals form a class, not a set. This is unproblematic because we have no need to quantify over them. Moreover, in general, it may be better to treat them as a formal space instead of a set of points, but we have no occasion to do this presently.

The map $(L, U) \mapsto L$ is an embedding of the Dedekind reals into the lower reals. The other projection is an embedding into the upper reals.

Definition 2. *Let* A *be an inhabited subset of the (Dedekind) real numbers. Let* x *be in* $\mathbb{R}$*. The distance* $\inf\{|x-a| | a \in A\}$ *is the upper real* $\{q | |x-a| < q and a \in A\}$*. If for each* x *this distance is actually a Dedekind real number, we say that* A *is* located.

In the presence of the principle of excluded middle, every non-empty subset is located since in this context upper reals and Dedekind reals coincide.

If (L, U) is a Dedekind real number, then L and U are located subsets of $\mathbb{R}$. Conversely, every located inhabited up-closed subset U of the reals has an infimum u and hence defines a Dedekind real $(\{x \in \mathbb{Q} | x < u\}, U)$. A similar statement holds for the lower reals.

Every, open or closed, interval is located, every finite union of intervals is located.

Example 1. *In a constructive context, one cannot prove that all subsets of the reals are located. We provide a counterexample in the style of Brouwer. Consider the set*

$$A := \{q \in \mathbb{Q} | q > 1 \,\mathrm{or} (q > 0 \,\mathrm{and}\, P)\}.$$

This set will only be located if we can decide whether the proposition P holds. This example also shows that constructively an upper real is not necessarily located and that the infimum of a subset of $\mathbb{Q}$ is not necessarily a Dedekind real number.

2.2. **Locatedness in pointwise metric spaces.** The notion of locatedness naturally extends to general metric spaces. A *metric space* is a set X equipped with a metric $\rho : X \times X \to \mathbb{R}^+$. (From section 4 onwards we will use a slightly more liberal definition of metric space.)

Definition 3. *A subset A of a metric space (X, ρ) is located if for each x in X the distance* $\inf \{\rho(x, a) | a \in A\}$*, which a priori is only an upper real, is a Dedekind real number.*

All finite unions of, open or closed, balls are located. On the other hand, Example 1 shows that not all subsets are located.

Definition 4. *A set is* finite *if it is in bijective correspondence with a set* $\{0, \ldots, n\}$, $n \geqslant 0$*. A set is* Kuratowski finite (K-finite) *if it is the image of a finite set.*

The set $\{a, b\}$ is the image of $\{0, 1\}$ and hence it is K-finite. It is finite only if we can decide whether $a = b$.

Definition 5. *A subset A of a metric space (X, ρ) is* totally bounded *if for all $\varepsilon > 0$ there are K-finitely many x_i such that for each x there exists i such that $\rho(x_i, x) < \varepsilon$.*

Since we can decide whether $\rho(x_i, x) > \frac{\varepsilon}{2}$ or $\rho(x_i, x) < \varepsilon$ we could have replaced K-finite by finite in the definition above; see [BB85].

All bounded intervals are totally bounded. On the other hand, the set $A \cap [0, 2]$ in Example 1 is not totally bounded.

Proposition 1. ([BB85] Prop. 4.4.5) *A subset Y of a totally bounded set is located iff it is totally bounded.*

Proof. Suppose that Y is located and let $\varepsilon > 0$ be given. Let $\{x_1, \ldots, x_n\}$ be an $\frac{\varepsilon}{3}$-approximation to X. For each i choose y_i in Y with $\rho(x_i, y_i) < \rho(x_i, Y) + \frac{\varepsilon}{3}$. Let y be an arbitrary point of Y. Then $\rho(y, x_j) < \frac{\varepsilon}{3}$. This gives

$$\rho(y, y_j) \leqslant \rho(y, x_j) + \rho(x_j, y_j) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

Thus the K-finite set $\{y_1, \ldots, y_n\}$ is an ε-approximation to Y. Since ε is arbitrary, it follows that Y is totally bounded.

Conversely, suppose that Y is totally bounded. Fix x in X. The function $y \mapsto \rho(x,y)$ is uniformly continuous and so $\inf\{\rho(x,y)|y \in Y\}$ is a Dedekind real number. $\square$

Example 2. *Locatedness is not a topological property, it is not even preserved by metric equivalence.*

Proof. The subset of non-zero elements is located in the natural numbers. Now consider the map $f_k : \mathbb{N} \to \mathbb{R}$ defined by $f_k(m) = m$ if $n \neq m$ and $f_k(k) = \frac{1}{2}$. Then $\rho_k(n,m) = |f_k(n) - f_k(m)|$ is a equivalent to the standard metric on $\mathbb{N}$. Suppose that α is an increasing binary sequence starting with 0. Define $\rho_\alpha(n,m) = \rho_k(n,m)$ if k is the least number less than $\max(n,m)$ such that $\alpha(k) = 1$ and $\rho_\alpha(n,m) = |n-m|$ otherwise. By computing the ρ_α-distance from 0 to the set of non-zero numbers it is possible to decide whether the sequence α contains a 1. We conclude that we cannot prove constructively that the set of non-zero elements is ρ_α-located. However, the metric ρ_α is equivalent to the standard metric on $\mathbb{N}$ — that is, there exist mutually inverse uniformly continuous functions. $\square$

Example 3. *Locatedness is not transitive — that is, if X is a located subspace of a located subspace Y of a space Z, then X need* not *be located in Z.*

Proof. We repeat Richman's example [Ric99]. Let P be a proposition, and let X be the subset of $\{-2,0,1,3\}$ consisting of $\{0,1,3\}$ together with -2 if P holds. That is, $X = \{0,1,3\} \cup \{-2 : P\}$. Let $B = X \cap \{-2,1,3\}$ and $A = X \cap \{-2,3\}$. To see that A is located in B, let b be a point in B. If $b = -2$ or $b = 3$, then b is in A and $d(A,b) = d(b,b) = 0$. If $b = 1$, then $d(A,b) = d(3,b) = 2$. Note that if $b = -2$, then P is true, while $d(A,1)$ can be computed independently of P. To see that B is located in X, let x be a point in X, If $x \in B$, then $d(B,x) = d(x,x) = 0$, while $d(B,0) = d(1,0) = 1$. However, if we could compute $d(A,0)$, then we could determine whether P was true or false. Indeed, if the distance from A to 0 is less than 3, then P is true, while if the distance from A to 0 is greater than 2, then P is false. $\square$

Lemma 1. *The image of a located subset A of a totally bounded set X under a uniformly continuous function f* is *a located subset of the image of that function.*

Proof. The set A is totally bounded by Proposition 1. As total boundedness is preserved by uniformly continuous mappings, $f(A)$ is totally bounded. By Proposition 1 again, $f(A)$ is located in $f(X)$. $\square$

3. Point-free topology

3.1. **Topology and locale theory.** To introduce topological spaces and locales we will assume that the powerset operation is present.

A topology on a set X is a collection $O(X)$ of subsets such that

(1) $\emptyset, X \in O(X)$;
(2) $U \cap V \in O(X)$, when $U, V \in O(X)$;
(3) $\bigcup_i U_i \in O(X)$, when $U_i \in O(X)$.

In other words a topology is a sup-lattice of open sets. A function between topological spaces is continuous if its inverse image is a morphism of sup-lattices — that is, a lattice morphism preserving $\bigvee$. Much of the general theory of topology can be captured in purely lattice theoretic terms. This leads to the study of the categories of frames and locales; see [Joh82].

Definition 6. *A* frame *is a complete distributive lattice satisfying the infinite distributive law: $x \wedge \bigvee y_i = \bigvee x \wedge y_i$. A* frame map *is a map of sup-lattices. The category of* locales *is the opposite category of the category of frames. The one point locale Ω is the collection of all subsets of the one-element set — the collection of all propositions. A* point *of a locale L is a map $\Omega \to L$ — that is, a frame-map $L \to \Omega$, in other words it is a filter.*

The category of locales fits well with one's intuitions about topological spaces. For instance, the product of topological spaces corresponds to the *co*-product in the category of frames, but to the product in the category of locales.

To any topological space one can assign its locale of open sets. Conversely, to any locale one can assign its collection of points. These constructions define an adjunction between the category of locales and the category of topological spaces. When a locale is isomorphic to its topological space of points, it is called *spatial*. Classically, one can show, for instance, that the compact regular locales are spatial and that this category coincides with the category of compact Hausdorff spaces. Constructively, it is rare that locales are spatial. Already to prove that the real numbers are spatial one has to use Brouwer's 'fan theorem', which is not acceptable for Bishop. To prove that spectral spaces are spatial often requires the classical axiom of choice. By staying entirely on the localic side one can usually avoid the use of this axiom; see e.g. [Mul03].

3.2. **Formal topology.** To give examples of topological spaces it is often convenient to present them by a base. In locale theory one uses a coverage for this purpose. We use Johnstone's definition of coverage [Joh02] which is a generalization of the one in [Joh82, Notes on sec.II.2].

Definition 7. *A* coverage *C on a pre-order $(S, \leqslant)$ assigns to each u in S a collection $C(u)$ of subsets, called covering families, such that when we write $u \lhd U$ for $U \in C(u)$, we have*

- ***(C)*:** *If $u \lhd U$ and $v \leqslant u$, then there exists $V \subset v \wedge U = \{x | \exists u \in U. x \leqslant v, x \leqslant u\}$ such that $v \lhd V$.*

A pair $(S, \lhd)$ is called a site.

Every frame carries a canonical a coverage by setting $V \in C(u)$ iff $u \leqslant \bigvee V$. Conversely, every coverage defines a frame as follows. A $\lhd$-ideal is a lower set I such that if $v \lhd U$ and $U \subset I$, then $v \in I$. The collection of all $\lhd$-ideals ordered by inclusion is the frame freely generated by $(S, \lhd)$; see [Joh82, II.2.11].

Example 4. *The frame of reals is defined by the following coverage on the open rational intervals ordered by inclusion.*

1. $(p, s) \lhd \{(p, r), (q, s)\}$ *if* $p \leqslant q < r \leqslant s$;
2. $(p, q) \lhd \{(p', q') | p < p' < q' < q\}$.

The points of this locale are precisely the Dedekind real numbers.

In topos theory, there is a long tradition of working directly with the generators and relations, presented by a coverage, instead of the locale that it generates; see [Vic07a] for an overview. One advantage is that the coverage is typically preserved by inverse images of geometric morphisms, whereas the generated locale is not. Another reason to work with coverages is that they fit better with the computational interpretation of constructive mathematics.

In a predicative setting a frame may not form a set, but a proper class. However, it is often the case that one can generate the frame L by a set-sized site $(S, \lhd)$. Set-generated locales coincide with inductively generated formal topologies [CSSV03, Acz06] and with flat sites [Vic06] from topos theory.

An inductively generated formal topology is precisely that: a formal topology generated inductively from a coverage. A formal topology comes with a *set* S of basic opens. Arbitrary opens are defined as, possibly class-sized, collections of those. In particular, $\mathcal{P}(S)$ in the following definition denotes the power-class of S.

Definition 8. *A* formal topology [Sam03] *consists of a pre-order* $(S, \leqslant)$ *of basic opens and* $\lhd \subset S \times \mathcal{P}(S)$*, the covering relation, which satisfies:*

- **Ref:** $a \in U$ *implies* $a \lhd U$*;*
- **Tra:** $a \lhd U, U \lhd V$ *imply* $a \lhd V$*, where* $U \lhd V$ *means* $u \lhd V$ *for all* $u \in U$*;*
- **Loc:** $a \lhd U$*,* $a \lhd V$ *imply* $a \lhd U \wedge V = \{x | \exists u \in U \exists v \in V. x \leqslant u, x \leqslant v\}$*;*
- **Ext:** $a \leqslant b$ *implies* $a \lhd \{b\}$*.*

These axioms are known as Reflexivity, Transitivity, Localization and Extensionality. **Ref** and **Ext** say that if a basic open belongs to a family, then the family covers it. **Tra** is the transitivity of the cover. **Loc** is the distributive rule for frames.

The *formal intersection* $U \wedge V$ is defined as $U_{\leqslant} \cap V_{\leqslant}$, where $Z_{\leqslant}$ is the set $\{x | \exists z \in Z. x \leqslant z\}$. Another common notation for $Z_{\leqslant}$ is $Z_{\downarrow}$. We write $a \lhd b$ for $a \lhd \{b\}$. We write $U \equiv V$ iff $U \lhd V$ and $V \lhd U$. Every coverage generates a formal topology; e.g. [CSSV03][Vic07b]. Conversely, every formal topology generates a coverage: If $u \lhd U$ and $v \leqslant u$, then $v \lhd \{v\}$, $v \lhd U$ and hence, by **Loc**, $v \lhd v \wedge U$.

For later reference we mention:

Lemma 2. ***Loc*** *is equivalent to* ***Loc'****: if* $a \lhd U$*, then* $a \lhd a \wedge U$*.*

Proof. We only prove that **Loc'** implies **Loc**, since the converse is trivial. Assume **Loc'** and assume that $a \lhd U$ and $a \lhd V$. Then $a \lhd a \wedge U$ and $a \lhd a \wedge V$. Consider any w in $a \wedge U$. Then $w \lhd a \lhd a \wedge V$. Thus $w \lhd a \wedge V \wedge w \lhd (a \wedge V) \wedge (a \wedge U)$ by transitivity of $\lhd$. So $a \lhd a \wedge U \lhd a \wedge U \wedge V \lhd U \wedge V$. □

It is straightforward to extend a site to a formal topology [Vic06].

Definition 9. *Let* $(S, \lhd)$ *be a formal topology. A* point *is an inhabited subset* $\alpha \subset S$ *that is filtering with respect to* $\leqslant$*, and such that for each* $a \in \alpha$ *if* $a \lhd U$*, then* $U \cap \alpha$ *is inhabited. The collection of points is denoted by* $\mathrm{Pt}(S)$*.*

Finally, we introduce the notion of a morphism of formal topologies.

Definition 10. *Let* S *and* S' *be formal topologies. A binary relation* $f \subset S \times S'$ *defines a map* $S \to \mathcal{P}(S')$ *by* $f(a) = \{b | afb\}$*. The relation* f *is a* continuous map *if*

1. $S' \lhd f(S)$;
2. $f(a) \wedge f(b) \lhd f(a \wedge b)$;
3. *If* $a \lhd U$*, then* $f(a) \lhd f(U)$*.*

Impredicatively, the category of locales and the category of formal topologies with continuous maps are equivalent; see [Acz06] for a proof of this fact in the context of IZF and for a development of class sized locales in a predicative context.

A formal topology defines a site on the pre-order $(S, \leqslant)$ considered as a category. A continuous map can be viewed as a morphism between sites [MM92, VII.10 Thm. 1].

3.3. Subspaces.

Definition 11. *Let L be a locale. A* nucleus *is an operator $j : L \to L$ such that $a \leqslant j(a)$ and $j \circ j(a) = j(a)$. A nucleus j defines a new locale $L_j = \{a \in L | a = j(a)\}$ and a monomorphism $L_j \rightarrowtail L$. A* sublocale *is a locale which is presented in this way.*

Alternatively, a sublocale may be seen as a subobject of L — that is, an equivalence class of locale embeddings into L. Finally, a sublocale may be seen as a new covering relation $a \lhd_j V$ iff $a \leqslant j(\bigvee V)$. Conversely, every covering relation $\lhd \subset \lhd'$ defines a nucleus $j(a) := \bigvee \{b | b \lhd' a\}$.

To see how this relates to point-set topology, consider a subspace Y of X, $u \in \mathcal{O}(X)$ and $V \subset \mathcal{O}(X)$. Then u is covered by the union of V in Y iff $u \cap Y \subset (\bigcup V) \cap Y$. In this way a number of subspace constructions (open, closed,...) can be captured by new covering relations.

3.3.1. *Subspaces in formal topology.* Let $(S, \lhd)$ be a formal topology.

Definition 12. *A* subspace *is a formal topology $(S, \lhd')$ such that $\lhd \subset \lhd'$ and $a \wedge' b \lhd' a \wedge b$.*

Predicative analogues of the other definitions are also possible. For instance, a nucleus will in general be a proper class-function $\mathcal{A} : \mathcal{P}(S) \to \mathcal{P}(S)$. We refer to [Cur07][AC] for proofs that nuclei, subcoverings and embeddings are also equivalent in a predicative setting.

We provide some examples of subspaces.

3.3.2. *Open subspaces.* Let $(S, \lhd)$ be a formal topology and let $U \subset S$. The *open* subspace defined by U is presented by the covering $u \lhd_U V$ if $u \wedge U \lhd V$.

Example 5. *The set $\{(0,1)\}$ represents the open unit interval as a subspace of the formal reals.*

3.3.3. *Closed subspaces.* Let $(S, \lhd)$ be a formal topology and let $U \subset S$. The *closed* subspace $S \setminus U$ is defined by the covering $u \lhd_{-U} V$ if $u \lhd V \cup U$. Intuitively this is the complement of the open U. In section 3.3.7 we will define 'positively closed' subspaces.

Example 6. *The set $\{(p, q) | q \leqslant 0 \vee p \geqslant 1\}$ represents the closed unit interval as a subspace of the real line.*

3.3.4. *Compact subspaces.* A (sub)locale is *compact* if every cover has a K-finite subcover. In formal topology one restricts this requirement to the generating set S.

The closed unit interval is compact [CN96].

There are important connections between compact locales and closed sublocales; see [Joh82].

Proposition 2. *A closed subspace of a compact locale is compact.*

Proposition 3. *A compact sublocale of a regular locale is closed.*

These results hold for locales as well as for formal topologies. The definition of a regular formal topology will be recalled in section 8.

3.3.5. *Open maps.* We will now consider a number of ideas related to overtness.

The inverse of a continuous map maps open sets to open sets. An *open map* maps open sets to open sets. If $f : X \to Y$ is an open map, then there exists a map $\exists_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ mapping each open set to its image. This map is left-adjoint to f^{-1} — that is, $\exists_f(U) \subset V$ iff $U \subset f^{-1}(V)$. Moreover, it satisfies

$$\exists_f(V \cap f^{-1}(U)) = \exists_f(V) \cap U.$$

Definition 13. *In locale theory, a map $f : X \to Y$ is* open *if the corresponding frame map $\mathcal{O}f : \mathcal{O}(Y) \to \mathcal{O}(X)$ has a left-adjoint $\exists_f : \mathcal{O}(X) \to \mathcal{O}(Y)$ and the so-called Frobenius law*

$$\exists_f(a \wedge \mathcal{O}f(b)) = \exists_f(a) \wedge b$$

holds. It suffices to require the Frobenius law to hold on a generating set, which gives us the definition of an open map between formal topologies: a continuous map is open *if it has a left-adjoint and the Frobenius law holds on basic opens.*

Example 7. *An open sublocale defines an open inclusion of locales. Let $i : X \to Y$ be an open sublocale represented by a nucleus $j(a) = a \wedge X$ on Y. The nucleus j is a locale map from X to Y, $X = Y_j = \{a \in Y | j(a) = a\}$. The inclusion is a left-adjoint j: $i(a) \leqslant b$ iff $a \leqslant j(b)$.*

3.3.6. *Overt subspaces and positivity.* In classical point-set topology, the unique map $! : X \to \Omega = \mathcal{O}(1)$ is always open. Constructively, this need not be the case. In constructive point-set topology, if X is inhabited, then the unique map $! : X \to \Omega$ is open. When the map $! : X \to \Omega$ has a left-adjoint the Frobenius law always holds. In this case the locale itself is said to be open. To avoid confusion with open sublocales, we call such locales *overt*.

Definition 14. *A locale X is* overt *if the unique map $! : X \to \Omega$ is open.*

In sheaf theory [Joh84] it is often useful to relate properties of locales to properties of maps of locales. Let $f : X \to Y$ be a continuous function between topological spaces. This function induces a geometric morphism $f : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$. The subobject classifier Ω_X is a locale in $\mathrm{Sh}(X)$ and so its direct image $f_*(\Omega_X)$ is a locale in $\mathrm{Sh}(Y)$. Many properties of maps $f : X \to Y$ are equivalent to properties of the locale $f_*(\Omega_X)$, provided that Y satisfies the T_D axiom that every point is the intersection of an open and a closed subset. The T_D axiom is strictly between T_0 and T_1. We provide two examples. Let Y be a T_D-space and let $f : X \to Y$ be given. Then:

- f is open iff $f_*(\Omega_X)$ is an open(=overt) locale;
- f is proper iff $f_*(\Omega_X)$ is compact regular.

A continuous function $f : X \to Y$ is *proper* if the pre-image of every compact set in Y is compact in X. In fact, open maps may be seen as dual to proper maps; see [Ver94].

We now provide an alternative way of looking at overt locales. On every locale one can define $\mathrm{Pos}(a)$, as every cover of a is inhabited. Intuitively, this means that the open a is non-empty. However, it is not necessarily the case that there actually is a point in a. One proves that a locale is overt iff $a \leqslant \bigvee S$ implies that $a \leqslant \bigvee \{s \in S | \mathrm{Pos}(a)\}$. The impredicative definition above which quantifies over all coverings is treated axiomatically in formal topology. Importantly, the definition in formal topology restricts the positivity predicate to a base.

Definition 15. *Let $(S, \lhd)$ be a formal topology. Then* Pos $\subset S$ *is called a* positivity predicate *if it satisfies:*

Pos: $U \lhd U^+$, *where* $U^+ := \{u \in U | \operatorname{Pos}(u)\}$.
Mon: *If* $\operatorname{Pos}(u)$ *and* $u \lhd V$, *then* $\operatorname{Pos}(V)$ *— that is,* $\operatorname{Pos}(v)$ *for some* $v \in V$.

The interpretation of $\operatorname{Pos}(u)$ as 'u is inhabited' in a spatial formal topology gives a motivation for the previous axioms. A locale which, when considered with the standard covering relation, carries a positivity predicate is said to be *overt*.

Theorem 1. *A locale is overt iff there is a formal space presenting it that carries a positivity predicate. If this is the case all formal spaces presenting the locale carry a positivity predicate* [Neg02].

Overtness is a localic property, i.e. it is preserved by homeomorphisms, i.e. isomorphisms of locales.

The predicate Pos which is true on all rational intervals is a positivity predicate on the reals.

Example 8. *We now provide a formal analogue of Example 1. The closed sublocale defined by the open*

$$\{(p, 0)|p < 0\} \cup \{(2, q)|q > 2\} \cup \{(1, q)|q > 1 \,\text{and}\, P\}$$

will only be overt when we can decide whether the proposition P holds. This will follow from Theorem 2.

We recall some further properties of open maps and overt locales.

Proposition 4.

1. *An open subspace of an overt space is overt.*
2. *The direct image of an overt subspace under a continuous map is overt.*
3. *The inverse image of an overt subspace under an open map is overt.*

The reader will have no problems defining the positivity predicates explicitly.

Finally, for completeness, we mention the following facts about overtness. A locale A is *exponentiable* if the 'function space' B^A exists, as a locale, for all locales B. A locale is exponentiable iff it is locally compact [Hyl81]. Moreover, the exponential functor $-^A$ preserves separation properties (regularity, Hausdorffness, etc) iff A is overt [Joh84].

3.3.7. *Positively closed sublocales.* In locale theory one also considers a different notion of closedness, called weakly closed [Joh91][Ver92]. We will only consider weakly closed sublocales which are also overt. We will refer to them as *positively closed*. Positively closed sublocales correspond to points in the lower power locale [Vic07b].

Definition 16. *Let $(S, \lhd)$ be a formal topology. A subset F of S is* positively closed *when $F(u)$ and $u \lhd V$ imply that $F(v)$ for some v in V. For such F we define the subspace $\lhd_F$ to be the least subspace such that*

1. $u \lhd_F V$ *when* $u \lhd V$*;*
2. $u \lhd_F \{u | F(u)\}$.

This definition is a priori impredicative. However, when $\lhd$ is inductively defined, $\lhd_F$ can also be inductively defined by adding the clause

$$u \lhd_F U \quad \text{whenever}\, F(u) \to u \lhd_F U$$

to the clauses defining $\lhd$. A subspace defined in this way is called positively closed.

If $u \leqslant v$ and $F(u)$, then $F(v)$. If, moreover, S is overt, then $F(u)$ implies $\mathrm{Pos}(u)$. This can be seen as follows: $u \lhd \{u' | u' = u \,\mathrm{and}\, \mathrm{Pos}(u)\}$ and so $F(v)$ for some $v \in \{u' | u' = u \,\mathrm{and}\, \mathrm{Pos}(u)\}$ — that is, $\mathrm{Pos}(u)$.

It is *not* the case that $u \lhd_F U$ iff $F(u) \to u \lhd U$. The latter may not be transitive in general.

Lemma 3. *The set F is a positivity predicate on the subspace it defines.*

Proof. **Pos** is satisfied by the second generating case.

To prove **Mon** we prove that if $a \lhd_F U$ and $F(a)$, then $F(u)$ for some u in $a \wedge U$. The proof proceeds by induction on the proof of $a \lhd_F U$. The cases **Ref**, **Tra**, **Ext**, **Loc'** are straightforward. If $a \lhd U$, then $a \lhd a \wedge U$ and hence $F(u')$ for some u' in $a \wedge U$. Finally, if $F(a) \to a \lhd_F U$ and $F(a)$, then $a \lhd_F U$ and hence by the induction hypothesis, $F(u')$ for some u' in $a \wedge U$. ☐

A positively closed set coincides with Sambin's [Sam03] 'formal closed' in the case of set-generated formal topologies. Consider a binary positivity relation Pos on an inductively generated formal cover. The subcover induced on the formal closed captured by F is that what we defined after substituting F with $\mathrm{Pos}(F, -)$; see also [Sam03]. We use 'positively closed' because of both the positive formulation and the existence of a positivity predicate on a positively closed sublocale. 'Positively closed' is the point-free analogue of the following definition of closed set: 'a set A is closed if it coincides with its closure — that is, the set of all points all of whose neighborhoods meet A.' This definition is the usual one in Bishop's constructive mathematics [BB85].

Example 9. *The set $F := \{(p,q) | p < q, p < 1, q > 0\}$ is positively closed. The corresponding subspace is homeomorphic to the closed unit interval.*

In the next section we construct a locale from a metric space. This will allow us to provide a class of examples of positively closed sublocales; see Definition 19.

4. Locales from metric spaces

In this section we construct a formal space from a metric space. In section 2.2 we defined a metric space as a set X with a metric $\rho : X \times X \to \mathbb{R}^+$. Many concepts in the theory of metric spaces, such as ε-δ-continuity, do not depend on the ability to compute the distance between two points precisely, but only require certain distances to be small. This part of the theory can also be naturally defined using only a ternary relation $\rho(x,y) < \varepsilon$ without the requirement that ρ is actually a function. The following construction of a formal topology from a metric space in Definition 18, which follows Vickers [Vic05] and Palmgren [Pal07], is a case in point.

Concretely, a metric space is defined as follows. We let $\mathbb{Q}^{>0}$ denote the set of strictly positive rational numbers.

Definition 17. *A* metric space *consists of a set X together with a ternary relation denoted by $d(x,y) < r$, where $x, y \in X$ and $r \in \mathbb{Q}^{>0}$, such that*

(1) *For all x, y in X, there exists r in $\mathbb{Q}^{>0}$ such that $d(x,y) < r$;*
(2) *If $d(x,y) < r$, then there exists $s < r$ such that $d(x,y) < s$;*
(3) *$d(x,y) < r$ for all $r > 0$ iff $x = y$;*
(4) *$d(x,y) < r$ iff $d(y,x) < r$;*

(5) *If* $d(x,y) < r$ *and* $d(y,z) < s$, *then* $d(x,z) < r+s$.

We derive that if $d(x,y) < r$ and $r < s$, then $d(x,y) < s$.

For x, y in X, the set $\{r|d(x,y) < r\}$ is an upper real. If for all x, y, the set $\{r|d(x,y) < r\}$ is located, a Dedekind real, then $d(x,y) := \inf\{r|d(x,y) < r\}$ defines a function from $X \times X \to \mathbb{R}$. Therefore whenever the need to distinguish them arises, we will refer to the new definition as upper metric spaces and to the old definition as Dedekind metric spaces.

Constructively, Definition 17 is a genuine generalization of ordinary metric spaces. For instance, Richman [Ric98] provides the example of the distance between two sets. Definition 17 is also the most natural in geometric logic. As an added benefit, this definition allows us to *define* the Dedekind real numbers as the completion of the rational numbers, avoiding the otherwise circular use of the real numbers in the definition of a metric.

The notion of locatedness generalizes to upper metric spaces. However, it seems difficult to find interesting examples of located subsets in such spaces, as even points need not be located. Nevertheless, we will develop the following theory in full generality in Section 5, but first we construct a locale from a metric space.

Definition 18. *To any metric space X, we define a locale* $\mathrm{loc}(X)$ *called the* localic completion *of X. A formal open is a pair $(x,r) \in X \times \mathbb{Q}^{>0}$, written $B_r(x)$. We define the relation $B_r(x) < B_s(y)$ iff $d(x,y) < s - r$ as illustrated in Figure 18.*

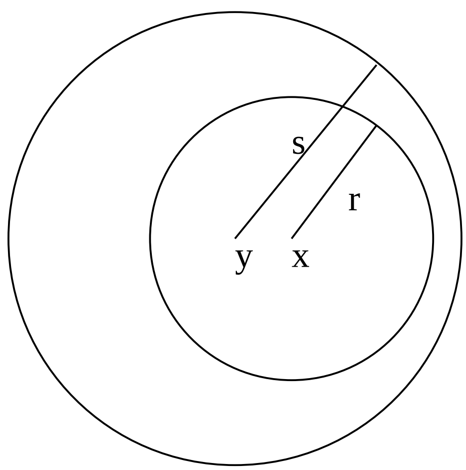

FIGURE 1.

The order $\leqslant$ is defined by $B_r(x) \leqslant B_s(y)$ iff $d(x,y) < t$ for all $t > s - r$. The covering relation $\lhd$ is inductively generated by the axioms

M1: $u \lhd \{v|v < u\}$;

M2: $\mathrm{loc}(X) \lhd \{B_r(x)|x \in X\}$ *for any r.*

M1: *Every ball is covered by all the balls strictly inside it (since the ball is open).*
M2: *For each $r > 0$, the space is covered by all balls of size r.*

Proposition 5. *The localic completion of a metric space is always overt.*

Proof. The positivity predicate is defined by $\mathrm{Pos}(B_r(x))$ being true for all balls. $\square$

The localic reals in Example 4 may be seen as $\mathrm{loc}(\mathbb{Q})$.

We prove two easy lemmas.

Lemma 4. *$B_r(x) \leqslant B_s(y)$ iff for all $B_t(z)$, $B_t(z) < B_r(x)$ implies $B_t(z) < B_s(y)$.*

Proof. **$\Rightarrow$:** By the triangle inequality.

$\Leftarrow$: Let $\varepsilon > 0$ and choose $B_t(z) := B_{r-\varepsilon}(x)$.

$\square$

Vickers [Vic05] identifies the points of $\mathrm{loc}(X)$ with the Cauchy filters of X. An inhabited subset F of $X \times \mathbb{Q}^{>0}$ is a *filter* if

(1) it is *upper* — if $B_\delta(x) \in F$ and $B_\delta(x) < B_\varepsilon(y)$, then $B_\varepsilon(y) \in F$;
(2) any two members of F have a common refinement — for all $b, b' \in F$ there exists $b'' \in F$ such that $b'' < b$ and $b'' < b'$.

A *Cauchy filter* is a filter that contains arbitrary small balls. Thus the points of $\mathrm{loc}(X)$ coincide with the points of the completion of X.

Proposition 6. *The points of* $\mathrm{loc}(X)$ *are the Cauchy filters of X.*

Using countable dependent choice one can identify Cauchy filters with Cauchy sequences, but we will not do this.

As promised, we are now able to provide examples of closed and positively closed sublocales.

Lemma 5. *Let Y be a subset of a metric space X. Consider the set* Pos_Y *of all $B_r(x)$ such that there exists y in Y such that $d(x, y) < r$. Then* Pos_Y *is positively closed.*

Proof. We need to check that if $\mathrm{Pos}_Y(u)$ and $u \lhd V$, then $\mathrm{Pos}_Y(v)$ for some v in V. As in Lemma 3 we proceed by induction on the proof that $u \lhd V$. The cases **M1** and **M2** are satisfied, by induction we check the cases **Ref**, **Tra**, **Loc**, **Ext**. $\square$

Definition 19. *Let Y be a subset of a metric space X. We write $\overline{Y}$ for the positively closed sublocale defined by* Pos_Y *and $\overline{\lhd}_Y$ for the covering so defined.*

Example 9 provides a positively closed description of the closed unit interval, it is $\overline{[0, 1]}$.

We have $u \overline{\lhd}_Y \{u | F(u)\} \cup \neg\,\mathrm{Pos}$, since $(\neg\,\mathrm{Pos}) \overline{\lhd_Y} \emptyset$. We use the notation $\overline{Y}$ since $\mathrm{Pt}(\overline{Y})$ is the closure of Y in the completion of X: Every point of the closure defines a point of $\overline{Y}$. Conversely, let $\alpha \in \mathrm{Pt}(\overline{Y})$. Then α is a point of the completion of X and for each $\varepsilon > 0$, there exists y in Y such that $\alpha \in B_\varepsilon(y)$ — that is, α is in the closure of Y.

A set Y and its closure define the same set Pos_Y.

Definition 20. *Let Y be a subset of X. We write Y^{cc} for the closed sublocale the complement of which consists of all the balls that do not meet Y.*

Example 10. *The localic unit interval of Example 6 may be seen as $[0, 1]^{\mathrm{cc}}$.*

The notation $-^{\mathrm{cc}}$ refers to the double complement: Let P be a proposition such that $\neg\neg P$ holds. Let $X = [0, 2]$ and define $Y = \{x \in [0, 1] | P\}$. Then $\neg Y = (1, 2]$ and $Y^{\mathrm{cc}} = [0, 1]$. We will return to this locale in Example 13.

Write U for the collection of balls that do not meet Y. The set of points of Y^{cc} consists of those x in the completion of X such that $x \notin U$.

Unlike the localic completion $\mathrm{loc}(Y)$, Y^{cc} need not be overt.

4.1. **Locales from locally compact metric spaces.** Definition 5 is straightforwardly generalized from Dedekind metric spaces to general metric spaces.

Definition 21. *A metric space is said to be* totally bounded *if for each $\varepsilon > 0$ the space can be covered by a finitely enumerable set of balls with radius at most ε. Equivalently, for all $\varepsilon > 0$ there exist $x_1, ..., x_k$ such that for all x there exists i such that $d(x, x_i) < \varepsilon$. A metric space is said to be* locally totally bounded *if for each ball and each $\varepsilon > 0$ the ball can be covered by a finitely enumerable set of balls with radius at most ε.*

For Bishop a metric space is locally compact if it is complete and locally totally bounded. The reals are Bishop locally compact. However, $(0, 1)$ is not Bishop locally compact with the usual metric, but there is an new metric under which it is; see [BB85, p.112][Man93].

Palmgren showed in the context of Martin-Löf type theory that there is a full and faithful embedding of Bishop's locally compact metric spaces into the locally compact locales. His metric is assumed to take values in the positive Dedekind real numbers. Choice is not required to define the embedding, but choice is used to prove that $\mathrm{Pt}(\mathrm{loc}(X))$ is isomorphic to the completion of X. Here Pt assigns to each locale $\mathrm{loc}(X)$ its space of points. Palmgren uses the standard definition of 'completeness' for metric spaces using Cauchy sequences. However, in the absence of countable choice one can use Cauchy filters like Vickers'. It may be possible to extend Palmgren's result to a choicefree context in this way, but we will not pursue it here.

Working on the formal side is advantageous in the following way. To prove that for instance $\mathrm{loc}([0, 1])$ is spatial, i.e. that it has 'enough points', one needs to show that the Heine-Borel theorem holds [FG82]. This is not possible in Bishop's constructive mathematics. By staying on the formal side we can avoid such issues. See [CN96] for a constructive proof of the Heine-Borel theorem for the formal unit interval and [FH79] for the analogue result for locales.

An open subset of X is the union of the open balls contained in it, and hence defines an open sublocale of $\mathrm{loc}(X)$. Going from locales to spaces strange phenomena may occur:

Example 11. *Kleene's singular tree is a recursive, and hence, decidable subset of $2^{<\omega}$ and defines a closed (and open) sublocale of Cantor space; see e.g.* [TvD88]. *In a recursive context this sublocale does not have any points (infinite paths), but as a locale it is nontrivial. In the presence of countable choice, these phenomena do not occur for closed* overt *sublocales of Cantor space.*

5. Located and overt

5.1. **Locatedness for the localic completion.** Recall from Definition 3 that a subset A of a metric space (X, ρ) is located if for each x in X the distance $\inf \{\rho(x, a) | a \in A\}$ exists (as a Dedekind real number). In other words, iff for all x, z in X such that $d(x, z) < s - r$, either there exists y in Y such that $d(z, y) < s$ or there is no y in Y such that $d(x, y) < r$. To express this formally, we will need to be able to express whether $B_s(z)$ meets Y. Moreover, locatedness is a property of the closure of a set: a set is located iff its closure is. It thus seems natural to consider locatedness of positively closed sublocales in the localic completion of a metric space. This and Martin-Löf's definition of locatedness, recalled in Section 6.1, motivate the following definitions.

Definition 22. *Let X be a metric space. A positively closed predicate* Pos *on* $S = \{B_r(x) | x \in X, r \in \mathbb{Q}^+\}$ *is called* located *when* $v < u$ *implies that* $\neg \operatorname{Pos} v$ *or* $\operatorname{Pos} u$. *Let T be a closed sublocale of* $\operatorname{loc}(X)$. *Then T is called* located *if there is a located predicate* Pos *such that T coincides with the closed sublocale defined by the open* $\neg \operatorname{Pos} \subset S$.

A located predicate Pos defines a positively closed sublocale also denoted by Pos.

The aim of this paper is to make a connection between located and overt. It may seem that we have just included a positivity predicate in the definition of locatedness. However, as we will show in Proposition 12, there is an alternative definition of located that does not start from a positively closed set.

Example 12. *The unit interval defined as a positively closed subspace in Example 9 is located and hence so is its description as a closed subspace in Example 6. The subspace in Example 8 is not located:* $(1\frac{1}{2}, 3) < (1, 3)$, *but we are unable to decide that either the former is negative or that the latter is positive.*

Locatedness is not a topological, or localic, notion, since as Example 2 shows, it depends both on the choice of the base S of the topology, the set of balls in this case, and on the relation $<$. For compact regular locales, the choice of base turns out to be irrelevant, as we will see in Corollary 2.

We connect the pointwise and point-free definitions of locatedness.

Proposition 7. *Let X be a metric space and Y a subset of X. Then $\overline{Y}$ is a located sublocale of* $\operatorname{loc}(X)$ *iff Y is located as a subset of X.*

Proof.

- **$\Leftarrow$:** Define the located predicate $\operatorname{Pos}(B_s(x))$ as $d(x, y) < s$ for some y in Y.
- **$\Rightarrow$:** The sublocale $\overline{Y}$ is located if and only if for each x in X, and $r < s$ either $\operatorname{Pos}(B_r(x))$ or $\neg \operatorname{Pos}(B_s(x))$. Since $\operatorname{Pos}(B_r(x))$ iff there exists y in Y such that $d(x, y) < s$ we conclude that Y is located.

□

We now prove similar theorems connecting locatedness of *closed* sublocales with the pointwise definition. The notation Y^{cc} was introduced in Definition 20.

Proposition 8. *Let X be a metric space and Y a subset of X. Then the closed sublocale Y^{cc} of* $\operatorname{loc}(X)$ *is located iff the set* $\neg\neg Y := \{x | \neg\neg x \in Y\}$ *is located.*

Proof. Define $\operatorname{Pos}(B_s(x))$ as: there exists y in Y such that $d(x, y) < s$. □

Example 13. *The use of the double negation in the previous proposition is necessary, as the following Brouwerian counterexample shows. Let P be a proposition such that $\neg\neg P$ holds. Let $X = [0, 2]$. Define $Y = \{x \in [0, 1] | P\}$. Then $\neg Y = (1, 2]$ and $Y^{\mathrm{cc}} = [0, 1]$. We introduced this sublocale already in Example 10. The sublocale Y^{cc} has a located predicate, but if Y is located, then we can decide whether P holds.*

If Y is located, then so is the set $\neg\neg Y$, i.e. the distance of a point to the set $\neg\neg Y$ is the same as the distance to Y.

Definition 23. *A subset of a metric space is* Bishop-closed *if it contains all its limit points, i.e. if it coincides with its closure.*

A Bishop-closed located subset of a metric space coincides with the complement of its complement: a Bishop-closed located set coincides with the set of all points which have zero distance to it. After some preparations, we will prove a formal analogue of this fact in Corollary 1.

The following proposition shows that a located closed sublocale is overt. Consequently, locatedness of a sublocale T is equivalent to

$$\forall ab \in S[a < b \to (a =_T 0 \vee \mathrm{Pos}_T(b))].$$

This is reminiscent of Johnstone's Townsend-Thoresen Lemma [Joh84].

Proposition 9. *Let* Pos *be a located predicate and write* $U^+ := \{u \in U | \mathrm{Pos}(u)\}$*. Then* $U \lhd U^+ \cup \neg \mathrm{Pos}$ *and thus the closed sublocale defined by* $\neg \mathrm{Pos}$ *is overt with* Pos *as its positivity predicate.*

Proof. Choose u in U and let $v < u$. Then either $v \in \neg \mathrm{Pos}$, or $u \in \mathrm{Pos}$.

In the former case $v \lhd \neg \mathrm{Pos} \lhd U^+ \cup \neg \mathrm{Pos}$.

In the latter case $u \in U^+$, so $u \lhd U^+ \cup \neg \mathrm{Pos}$, and thus $v \lhd U^+ \cup \neg \mathrm{Pos}$.

In both cases, $v \lhd U^+ \cup \neg \mathrm{Pos}$. Since u is covered by the set of such v, we have $u \lhd U^+ \cup \neg \mathrm{Pos}$. □

Proposition 10. *Let X be a metric space and let* Pos *be a located predicate in its localic completion. Then the positively closed locale defined by* Pos *coincides with the closed sublocale defined by* $\neg \mathrm{Pos}$.

Proof. We need to show that $u \lhd_{\mathrm{Pos}} V$ iff $u \lhd V \cup \neg \mathrm{Pos}$. We write $u^+ := \{u | \mathrm{Pos}(u)\}$.

Suppose that $u \lhd V \cup \neg \mathrm{Pos}$. Then $u \lhd_{\mathrm{Pos}} V \cup \neg \mathrm{Pos}$ and thus $u \lhd_{\mathrm{Pos}} (V \cup \neg \mathrm{Pos})^+ \subset V$.

For the converse it is sufficient to show that the two base cases are satisfied since $\lhd_{\mathrm{Pos}}$ is the *least* covering relation satisfying those cases. If $u \lhd V$, then $u \lhd V \cup \neg \mathrm{Pos}$. To prove the second case we assume that $V = u^+$. By Proposition 9 $u \lhd u^+ \cup \neg \mathrm{Pos}$ and the proof is complete. □

As promised, we are now ready to prove a formal analogue of the statement that a Bishop-closed located subset of a metric space coincides with the complement of its complement

Corollary 1. *Let X be a metric space and Y a subset of X. If Y is located, then* $Y^{\mathrm{cc}} = \overline{Y}$.

If X is totally bounded, then the converse implication holds too, as is stated in Theorem 2.

The following theorem gives a connection between locatedness, which is not a topological property, see Example 2, and overtness which is localic. It follows that in this case, a posteriori, locatedness does not depend on the choice of the base or the ambient topological space. Theorem 4 generalizes this to compact regular locales.

Theorem 2. *Let X be a totally bounded metric space and Y a subset. Then the following are equivalent:*

1. Y^{cc} *is overt;*
2. Y^{cc} *is located;*
3. *the set* $\neg\neg Y$ *is located as a subset of X.*

The following statements are equivalent:

a. $\overline{Y} = Y^{cc}$*;*
b. $\overline{Y}$ *is compact;*
c. Y *is located.*

Finally, if Y is located, then $\neg\neg Y$ is located and hence the second group of statements implies the first group.

Proof. We first prove the first group of statements to be equivalent.

The implication $1 \Leftarrow 2$ follows from Proposition 9.

For the implication $1 \Rightarrow 2$, suppose that Y^{cc} is overt. Let $u < v$ be given. Then $\mathrm{loc}(X) \lhd \{v\} \cup \{w | w \wedge u = 0\}$. This fact has an elementary proof, but is also a consequence of the coincidence of the way below and well inside relations on compact regular locales. Since $\mathrm{loc}(X)$ is compact, $v \vee \bigvee w_i = 1$, for some K-finite set $\{w_i\}$. The compact sublocale Y^{cc} is covered by a K-finite *positive* subset of $\{w_i, v\}$. If this set contains v, then $\mathrm{Pos}_{Y^{cc}}(v)$. If it does not contain v, then $Y \lhd \bigvee w_i$, i.e. $\neg \mathrm{Pos}_{Y^{cc}}(u)$. We see that Y^{cc} is located.

The equivalence $2 \Leftrightarrow 3$ is Proposition 8.

To prove the implication $a \Rightarrow b$ we observe that $\overline{Y}$ coincides with the closed sublocale Y^{cc} of the compact locale $\mathrm{loc}(X)$, and hence is compact.

To prove the implication $b \Rightarrow c$ suppose that $\overline{Y}$ is compact. Let $u < v$ be given. Then $\mathrm{loc}(X) \lhd \{v\} \cup \{w | w \wedge u = 0\}$. Since $\mathrm{loc}(X)$ is compact, $v \vee \bigvee w_i = 1$, for some K-finite set $\{w_i\}$. Since $\overline{Y}$ is compact and it is always overt, it is covered by a K-finite *positive* subset of $\{w_i, v\}$. If this set contains v, then $\mathrm{Pos}_{Y^{cc}}(v)$. If it does not contain v, then $1 \lhd_{Y^{cc}} \bigvee w_i$. Consequently, $\neg \mathrm{Pos}_{Y^{cc}}(u)$, since for each i, $u \wedge w_i = 0$. Consequently, $\overline{Y}$, and hence Y, is located.

The implication $c \Rightarrow a$ is Corollary 1. □

The use of the double negation was explained in Example 13.

As before, there is a similar statement for a positively closed sublocale Z of $\mathrm{loc}(X)$.

5.1.1. *Discussion.* It follows from Theorem **??** that unlike a closed sublocale of a compact locale, a *positively* closed sublocale need not be compact, though it *is* weakly compact [Ver92]. Conversely, a positively closed sublocale is always overt, but a closed sublocale need not be. A similar phenomenon occurs in Bishop's analysis, a closed subset of a Bishop-compact subset need not be Bishop-compact since a Bishop-compact subset is always located.

We conclude that Bishop's totally bounded and complete subspaces correspond to compact overt sublocales. However, compact overt sublocales behave slightly better under continuous functions. For instance, the image of an overt locale is overt and similarly the image of a compact locale is compact. However, the image of a complete totally bounded metric space may *not* be complete constructively. For an example consider any uniformly continuous real function f on $[0, 1]$ which does not *attain* its supremum [BB85]. The supremum is in the completion of the image, but *not* in the image itself. When considering this example using Palmgren's full and faithful embedding of Bishop's locally compact metric spaces into the locally compact locales, we see that the continuous map corresponding to f maps the localic completion of $[0, 1]$ to a compact sublocale of the localic reals, the closure of the pointwise image in this case.

We note that we only consider localic completions of metric spaces, so we do not treat non-complete spaces, like the open unit interval $(0,1)$, directly, but only as sublocales of localic completions. A similar phenomenon occurs in Bishop's framework [BB85]. For him, $(0,1)$ is not (Bishop) locally compact. In a localic framework $(0,1)$ is represented by an open sublocale of the compact locale $[0,1]$ and thus locally compact in the sense that $a \lhd \{b | b \ll a\}$, or more precisely, that its frame of opens is a continuous lattice [Joh82]. Palmgren [Pal09] studies such open subspaces in formal topology.

5.2. **An application.** We close this section with an application. Troelstra and van Dalen [TvD88] prove the following result:

Let X be a complete metric space. Let $Y \subset X$ be located. Then for all Z:

$$\overline{Y} \subset Z^\circ \text{ iff } \neg Y \cup Z = X.$$

Without loss of generality, we may assume that $Y = \overline{Y}$ and $Z = Z^\circ$.

In our setting this becomes:

Theorem 3. *Let X be a metric space. Let* Pos *be a located predicate on* $\mathrm{loc}(X)$ *and let Z be an open. Then the positively closed sublocale generated by Pos is a sublocale of Z iff* $\neg \mathrm{Pos} \cup Z = \mathrm{loc}(X)$.

Proof. Let $\lhd_{\mathrm{Pos}}$ denote the positively closed locale generated by Pos.

$\Rightarrow$Suppose that $u \wedge Z \lhd V$, then, by the assumption: $u \lhd_{\mathrm{Pos}} V$. By locatedness $u \lhd V \cup \neg \mathrm{Pos}$. In particular, $\mathrm{loc}(X) \wedge Z \lhd Z$ and thus $\mathrm{loc}(X) \lhd Z \cup \neg \mathrm{Pos}$.

$\Leftarrow$Suppose that $\mathrm{loc}(X) \lhd Z \cup \neg \mathrm{Pos}$. Then $\mathrm{loc}(X) \lhd_{\mathrm{Pos}} Z$ by locatedness. So, if $u \wedge Z \lhd V$, then $u \wedge Z \lhd_{\mathrm{Pos}} V$ ($\lhd_{\mathrm{Pos}}$ is a sublocale). Thus $u \lhd_{\mathrm{Pos}} V$ (since $V =_{\mathrm{Pos}} Z$). □

This proof is simpler then the one by Troelstra and van Dalen and works in a more general context. For instance, it directly generalizes to compact regular and regular locales by the methods in the following sections.

6. Locatedness for locally compact metric spaces

The goal of this section is to prove Proposition 12, which allows us to connect our definition of locatedness with Martin-Löf's corresponding intuitionistic definition which we introduce in subsection 6.1. We need some preparations first.

We begin by discussing locally compact locales. A topological space is locally compact if every point of X has a compact neighborhood. In locale theory it is convenient to consider a locally compact space as a continuous lattice [Joh82]. A *continuous lattice* is a $\vee$-semi-lattice such that $a = \bigvee \{a' | a' \ll a\}$, where $a' \ll b$, 'a is *way below* b', if a' is a member of every ideal I with $\bigvee I \geqslant a$.

Let X be a topological space. Then

(1) If U and V are open and there exists a compact K such that $U \subset K \subset V$, then $U \ll V$ in the lattice $\mathcal{O}(X)$;
(2) If X is locally compact, then $\mathcal{O}(X)$ is a continuous lattice.

In the light of this result, a locale is defined to be *locally compact* if it is a continuous lattice. The definition of the way-below-relation is impredicative. However, for set-presented formal topologies a predicative definition of the way-below-relation, and subsequently of locally compact formal topologies, is possible [Neg02, Cur07]: one defines $a \ll b$ (a is *way below* b) iff $b \lhd U$ implies that a is covered by a K-finite subset of U. We will focus on the special case of the localic completion of locally compact metric spaces where a more concrete definition is possible. In the context of Martin-Löf type theory Palmgren [Pal07, Theorem 4.18] shows that when X is a locally totally bounded metric space, then for all a, b in $\mathrm{loc}(X)$: if $a < b$, then $a \ll b$. His proof does not depend on type theoretic choice and hence also works in our context. A converse of Palmgren's result also holds.

Lemma 6. *Let X be a metric space. Let a, b be opens in its localic completion. Assume that $a \ll b$. Then there exists a c such that $a \lhd c < b$.*

Proof. The open a can be covered by a K-finite subset $\{c_0, \dots, c_n\}$ of $\{c | c < b\}$. Write c_i as $B_{r_i}(x_i)$ and b as $B_s(y)$ and define $m = \inf\{s - r_i | 0 \leqslant i \leqslant n\}$. Then $c_i < B_{s-m}(y) < b$. Consequently, $a \lhd \{c_0, \dots c_n\} \lhd B_{s-m}(y) < b$. □

Example 14. *The following example shows that we cannot expect $a < b$ in general. Consider the formal unit interval $[0, 1]$. Then $[0, 1] = B_3(0) \ll B_2(0)$, but $B_3(0) > B_2(0)$.*

Similarly, we have $B_3(0) \lhd B_2(0)$, but it is not the case that $B_3(0) \leqslant B_2(0)$. This shows that $a \lhd b$ does not imply $a \leqslant b$.

Thus let X be a locally totally bounded metric space. Then $a < b$ and $b \lhd U$ imply that a is covered by a K-finite subset of U. This allows us to express in a simple predicative way that $\mathrm{loc}(X)$ is locally compact.

The following is proved in exactly the same way as Proposition 9.

Proposition 11. *Let Pos be a located predicate and write $U^+ := \{u \in U | \operatorname{Pos}(u)\}$. Then $U \lhd U^+ \cup \neg \operatorname{Pos}$ and thus the closed sublocale defined by $\neg \operatorname{Pos}$ is overt.*

Proposition 12 shows that an alternative definition of located predicate, motivated by Martin-Löf's definition 6.1 is, in fact, equivalent to our definition. We need an introductory lemma.

Lemma 7. *Let X be a locally totally bounded metric space. Suppose that $v < u \lhd U$. Then there are $u_0, \dots, u_n$ in U and $v_0, \dots, v_n$ in $U_< := \{u' | \exists u \in U. u' < u\}$ such that for all i, $v_i < u_i$ and $v \lhd \{v_i\}$.*

Proof. There exists w such that $v < w < u$. Moreover, $U \lhd U_< := \{u' | \exists u \in U. u' < u\}$, so there exists a K-finite $U_0 \subset U_<$ such that $w \lhd U_0$. □

Proposition 12. *Let X be a locally totally bounded metric space. A subset Pos of $S = \{B_r(x) | x \in X, r \in \mathbb{Q}\}$ such that*

(1) *If $v < u$, then $v \notin \operatorname{Pos}$ or $u \in \operatorname{Pos}$;*
(2) *Pos is upwards closed:*
  (a) *If $u \in \operatorname{Pos}$ and $u \leqslant u'$, then $u' \in \operatorname{Pos}$;*
  (b) *If $\operatorname{Pos}(u)$, then $\operatorname{Pos}(v)$ for some $v < u$.*

is a located predicate.

Proof. Suppose that $u \lhd U$ and $\operatorname{Pos}(u)$. We need to prove $\operatorname{Pos}(U)$.

There exists $v < u$ such that $\operatorname{Pos}(v)$. By Lemma 7 there are $u_0, \dots, u_n$ in U and $v_0, \dots, v_n$ such that for all i, $v_i < u_i$ and $v \lhd \bigvee v_i$. Since $\operatorname{Pos}(v)$, it is impossible that all v_i are negative. Therefore some u_i is positive — that is, $\operatorname{Pos}(U)$. □

6.1. **Intuitionistic locatedness.** As mentioned before, the definition of locatedness was motivated by Martin-Löf's definition of locatedness for Euclidean spaces [ML70], which in turn was inspired by Brouwer [Bro19]. Martin-Löf defines the complement of an open set in Cantor space to be located if it can be decided for every neighborhood whether or not it belongs to the open set. In a Euclidean space, a closed set, the complement of the open set G, is located if we can find a (recursively enumerable) set of neighborhoods F such that for every pair of neighborhoods I and J, I being finer than J, either I belongs to G or J belongs to F. Without loss of generality F can be taken to satisfy the following two conditions dual to those defining an open set.

(1) If I is finer than J and I belongs to F, then so does J.
(2) If I belongs to F, then we can find J in F that is finer than I.

These properties are the ones we have generalized in Proposition 12.

Finally, Martin-Löf defines the complement of an open set G in Baire space, $\mathbb{N}^{\mathbb{N}}$, to be located if we can find a (recursively enumerable) set of neighborhoods F disjoint from G such that every neighborhood I belongs to either F or G, and if I belongs to F then we can find a natural number n such that I, n likewise belongs to F.

A located closed set in Cantor space defines a spread-law in Heyting's terminology. Spreads form a last motivation for considering located and overt sublocales. Spreads are at the heart of Brouwer's intuitionistic mathematics. Baire space is called the universal spread, and other spaces are constructed from it as continuous images or as nice subspaces, called *spreads*. In particular, every complete separable metric space can be presented as the image of Baire space. A spread-law, as defined by Brouwer [Hey56, p.34], is precisely a decidable positivity predicate on Baire space considered as a formal topology. A spread-law, being decidable, defines both a closed sublocale of Baire space and a positively closed sublocale of Baire space. Brouwer's choice sequences may be seen as points in a spread considered as a topological space. Our present emphasis on formal topology, as opposed to point-set topology, may then be seen as the interpretation of choice sequences as a 'figure of speech'; see [TvD88, p.644].

7. Locatedness for compact regular locales

We will now extend the point-free definition of locatedness to more general spaces. To define locatedness as above, we need a notion of refinement. There are several natural candidates for this. We have considered the relation $<$ for metric spaces before. We will now consider the well-inside relation for regular locales. In this section we treat the compact case. In section 8 we treat the general case.

Definition 24. *A distributive lattice L is* normal *if for all b_1, b_2 such that $b_1 \vee b_2 = 1$ there are c_1, c_2 such that $c_1 \wedge c_2 = 0$ and $c_1 \vee b_1 = 1$ and $c_2 \vee b_2 = 1$. We define $u \prec v$ as: there exists w such that $u \wedge w = 0$ and $v \vee w = 1$ in which case we say: u is* well-inside *v.*

Proposition 13. *For a normal lattice L we define $x \lhd U$ as for all $y \prec x$ there exists $u_1, ..., u_k$ in U such that $y \leqslant u_1 \vee ... \vee u_k$. Then $(L, \lhd)$ is a compact regular locale* [CC00].

We recall that in locale theory the well inside relation and the way below relation coincide for compact regular locales.

Compact regular locales can be conveniently presented by normal distributive lattices [CC00] corresponding to the finitary covering relation (a coherent locale). Giving a normal distributive lattice we define the covering relation $u \lhd V$, which presents the compact regular locale, as: for each $v \prec u$ there exists a finite $V_0 \subset V$ such that $v \leqslant \bigvee V_0$ in the distributive lattice.

A prime example is the formal topology of the closed unit interval $[0,1]$ as described in Example 6 and in [CN96]. The distributive lattice is the one generated by the rational intervals. We have $(p,q) \prec (r,s)$ iff $r < p < q < s$. The formal topology of $[0,1]$ is then constructed as in the previous paragraph.

Definition 25. *A lattice is called* strongly normal *when for all a, b there exist x, y such that $a \leqslant b \vee x$ and $b \leqslant a \vee y$ and $x \wedge y = 0$.*

Lemma 8. *Every strongly normal lattice is normal.*

Proof. Let $b_1 \vee b_2 = 1$. Choose x, y such that $b_1 \leqslant b_2 \vee x$ and $b_2 \leqslant b_1 \vee y$ and $x \wedge y = 0$. Then $1 \leqslant b_1 \vee b_2 \leqslant (b_2 \vee x) \vee b_2 = b_2 \vee x$. Similarly, $1 = b_1 \vee y$. □

Many examples of normal lattices are actually strongly normal.

Definition 26. *Let $(S, \lhd)$ be a formal topology. The* complement *of $a \in S$ is $a^* := \{b \in S | a \wedge b \lhd \emptyset\}$. We write $a \prec b$ for $1 \lhd a^* \cup \{b\}$ and say that a is* well inside *b. A formal topology is* regular *when for all $a \in S$, $a = \bigvee \{b \in S | b \prec a\}$.*

For compact regular locales it suffices to consider the well-inside relation on a basis of the locale as we did above.

Definition 27. *Let X be a compact regular locale, presented by a normal distributive lattice S. A subset* Pos *of S is called* located *when*

(1) *If $v \prec u$, then $v \notin \mathrm{Pos}$ or $u \in \mathrm{Pos}$;*
(2) Pos *is upwards closed:*
   If $u \lhd U$ and $u \in \mathrm{Pos}$, then $u' \in \mathrm{Pos}$, for some $u' \in U$.

The definitions of located closed and located positively closed sublocales from a located predicate are as before. The following two propositions directly generalize from the metric case.

Proposition 14. *Let X be a compact regular locale, presented by a normal distributive lattice S. A subset* Pos *of S such that*

(1) *If $v \prec u$, then $v \notin \mathrm{Pos}$ or $u \in \mathrm{Pos}$;*
(2) Pos *is upwards closed:*
   (a) *If $u \in \mathrm{Pos}$ and $u \leqslant u'$, then $u' \in \mathrm{Pos}$;*
   (b) *If $\mathrm{Pos}(u)$, then $\mathrm{Pos}(v)$ for some $v \prec u$.*

is a located predicate.

Proposition 15. *Let* Pos *be a located predicate. Then the positively closed locale defined by* Pos *coincides with the closed sublocale defined by* $\neg\,\mathrm{Pos}$.

Theorem 4. *A closed sublocale of a compact regular locale is overt iff it is located.*

Proof. A closed located sublocale is positively closed and thus overt.

An overt closed sublocale Y is located: Let $u \prec v$ be given. Then $u \wedge w = 0$ and $v \vee w = 1$, for some w. The compact Y is covered by a K-finite *positive* subset of $\{w, v\}$. If this set contains v, then $\mathrm{Pos}_Y(v)$. If it does not contain v, then $Y \lhd w$, i.e. $\neg\,\mathrm{Pos}_Y(u)$. □

Theorem 5. *A positively closed sublocale of a compact regular locale is compact iff it is located.*

7.1. **Totally bounded metric spaces as compact regular locales.** In this subsection we show that the localic completion of a totally bounded metric space is compact regular. Consequently, there are, a priori, two definitions of locatedness on such a locale. Fortunately, they coincide (Corollary 2).

We will first show that the localic completion of such a metric space is compact regular and can be represented by a normal distributive lattice.

Lemma 9. *Let X be a Dedekind metric space. Then* $\mathrm{loc}(X)$ *is regular.*

Proof. It is suffices to show that $B_r(x) < B_s(y)$ implies that $B_r(x) \prec B_s(y)$. This holds because the space can be covered by balls that are smaller than $\frac{s-r}{2}$. Each such ball is either contained in $B_s(y)$ or in the complement of $B_r(x)$. □

Vickers [Vic03] proves that $\mathrm{loc}(X)$ is compact iff X is totally bounded. Palmgren's proof [Pal07] of this fact can be adapted to our context.

For the rest of this section X will be a totally bounded metric space. We will now prove that $\mathrm{loc}(X)$ can be presented by a normal distributive lattice. We first need a lemma.

Lemma 10. *For all balls a, b:*
$a \lhd b$ iff for all balls c such that $c < a$, there exists d such that $c \lhd d < b$.

Proof. Suppose that for all balls c such that $c < a$, there exists d such that $c \lhd d < b$. Then $a \lhd \{c|c < a\} \lhd \{d|d < b\} \lhd b$.

For the converse we recall that $a < b$ implies $a \ll b$. Suppose that $a \lhd b$ and $c < a$. Then $c \ll a$ and thus $c \ll b$. By Lemma 6, we conclude that there exists d such that $c \lhd d < b$. □

Lemma 11. *The locale* $\mathrm{loc}(X)$ *can be presented by a (small) normal distributive lattice.*

Proof. We write $c < b_1 \wedge \ldots \wedge b_n$ when $c < b \lhd b_1 \wedge \ldots \wedge b_n$ for some b. We see that

$$b_1 \wedge \ldots \wedge b_n = \bigvee \{c|c < b_1 \wedge \ldots \wedge b_n\}.$$

We now consider the distributive lattice of finite unions of finite intersections of balls. We prove that this lattice is normal. Let d_i, e_j be finite lists of finite intersections of balls. Write $D = \bigvee d_i$ and $E = \bigvee e_j$. Suppose that $D \vee E = 1$. Since each e_j is covered by $\{c|c < e_j\}$, we have

$$D \vee \bigvee_j (\bigvee \{c|c < e_j\}) = 1.$$

By compactness there is a K-finite set $\{c_k\}$ such that

$$D \vee \bigvee c_k = 1.$$

We write $C = \bigvee c_k$. Then, by regularity, $(D \wedge C^*) \vee E = 1$. Thus by compactness there is a K-finite subset of $\{d_i \wedge c' | c' \wedge C = 0\}$ with supremum F such that $F \vee E = 1$. Since $F \wedge C = 0$ and $D \vee C = 1$ we have shown that the lattice is normal. □

The locale $\mathrm{loc}(X)$ itself is a normal distributive lattice, but it forms a proper class in a predicative constructive set theory.

We have two ways to present $\mathrm{loc}(X)$ as a formal topology: as a metric space and as a compact regular space. Fortunately, the two corresponding notions of locatedness coincide. This is a direct consequence of Theorem 2.

Corollary 2. *Let X be a totally bounded metric space with Dedekind metric. A closed sublocale of $\mathrm{loc}(X)$ is metrically located iff it is overt iff it is located as a sublocale of the compact regular space $\mathrm{loc}(X)$.*

8. Regular locales

In this section we extend the ideas above to regular locales. Regular locales generally do not have a canonical presentation as a (small) distributive lattice, but it is still possible to proceed along the lines of the previous section. Regularity was defined on page 23. In point-set topology regularity says that a point and a closed set can be separated by opens. Unlike the Hausdorff property, regularity is conveniently expressed in terms of formal opens. Every compact Hausdorff spaces is compact regular, but the converse requires the axiom of choice.

Definition 28. *Let X be a regular locale. A subset* Pos *of S is called* located *when*

(1) *If $v \prec u$, then $v \notin \mathrm{Pos}$ or $u \in \mathrm{Pos}$;*
(2) Pos *is upwards closed:*
    If $u \lhd U$ and $u \in \mathrm{Pos}$, then $u' \in \mathrm{Pos}$, for some $u' \in U$.

A direct translation of the results in Section 7 gives the following results.

Proposition 16. *Let* Pos *be a located predicate. Then the positively closed locale defined by* Pos *coincides with the closed sublocale defined by* $\neg\,\mathrm{Pos}$.

Proposition 17. *Every compact overt sublocale of a regular locale is located.*

Theorem 6. *A positively closed sublocale of a compact regular locale is compact iff it is located.*

It is not known to us whether a definition of locatedness as in Proposition 12 is equivalent to the present definition in the context of regular locales.

9. A non-decidable positivity predicate on Baire space

Finally, let us live up to our promise in section 6 to give a counterexample which shows that overtness does not imply locatedness. For this we consider Baire space, $\mathbb{N}^{\mathbb{N}}$, with the product topology. This topology may also be derived from the metric d such that

$$d(\alpha,\beta) < 2^{-n} \qquad \text{iff} \qquad \forall k \leqslant n.\alpha(k) = \beta(k)$$

We first need a lemma.

Lemma 12. *A subset Y of Baire space is metrically located if and only if the positivity predicate $\mathrm{Pos}_{Y^{\mathrm{cc}}}$ defined from the positively closed sublocale Y^{cc} is decidable.*

Proof. Suppose that Y is metrically located. Let a be a finite sequence. Write $a*0$ for the sequence which starts with a and then continues with 0s. To decide whether $\mathrm{Pos}(a)$ holds we decide whether the distance $a*0$ to Y is less than $2^{-|a|}$ or bigger than $2^{-(|a|+1)}$. This shows that Pos is decidable.

The converse implication is immediate. □

Proposition 18. *A positively closed sublocale of Baire space need not be decidable.*

Proof. Let $\alpha \in 2^\omega$ and define Y_α to be the complement of the open $\{0n|\alpha(n) = 0\}$. For each n we *can* decide whether $\neg\,\mathrm{Pos}(0n)$ or $\mathrm{Pos}(0)$, which shows that it is overt. However, to decide whether $0 \in Y_\alpha$ we need to know whether there exists n such that $\alpha(n) = 1$. This is not possible for general α. It follows that Pos is not decidable, so Y_α need not be located in general. □

The previous example is not metrically located, but it is located as a regular locale presented by a canonical base of finite lists of numbers.

A simpler example of a non-decidable positivity predicate can be constructed in the real numbers. Consider for $x \in (0, 1)$ the positively closed sublocale of $\mathrm{loc}([0, 1])$ generated by $[0, x]$. If its positivity predicate would be decidable, we would be able to decide whether $x < q$ or $x \geqslant q$ for all rational numbers q. Other examples can be found in [CS05].

The coherent locales form a class of positive examples. Every positivity predicate on a coherent locale is decidable. In particular, this holds for Cantor space. A locale is *coherent* if it is isomorphic to the locale of ideals of a distributive lattice.

10. Vietoris

Let X be a compact regular locale presented by a normal distributive lattice L. We show that the points of its Vietoris locale are precisely its compact overt sublocales. The Vietoris construction [Joh82] generalizes the construction of the compact subsets of a compact metric space with the topology given by the Hausdorff metric to general compact regular locales.

Define the distributive lattice $V(L)$ with generators $\Diamond u, \Box u$ for $u \in L$ and relations:

(1) $\Diamond u \vee \Diamond v = \Diamond(u \vee v)$
(2) $\Box u \wedge \Box v = \Box(u \wedge v)$
(3) $\Box u \wedge \Diamond v \leqslant \Diamond(u \wedge v)$
(4) $\Box(u \vee v) \leqslant \Box u \vee \Diamond v$
(5) $\Diamond 0 = 0$
(6) $\Box 1 = 1$.

The lattice $V(L)$ is normal [CC00] and defines the Vietoris locale, which is compact regular. The Vietoris locale, also denoted $V(L)$, has the same generators and relations as the lattice $V(L)$, but supplemented by the relations:

(1) $\Diamond u = \bigvee \{\Diamond v | v \prec u\}$;
(2) $\Box u = \bigvee \{\Box v | v \prec u\}$.

We will now show that the models of this theory, i.e. points of the corresponding locale, are compact overt locales. Let K be such a compact overt sublocale. Then we will have $K \in \Diamond u$ iff $\mathrm{Pos}_K(u)$ and $K \in \Box u$ iff $K \subset u$. This may help the reader to obtain some intuition for the relations above.

In order to prove that the points are the compact overt sublocales we prove that the theory $V(L)$ is equivalent to the theory Loc of located sublocales. In Loc we only have one predicate, called Pos, and a single implication:

If $u \prec v$, then $\mathrm{Pos}\, v$ or $\neg\,\mathrm{Pos}\, u$.

Proposition 19. *The theory Loc and the geometric theory $V(L)$ are bi-interpretable.*

Proof. We interpret Loc in the locale $V(L)$. Define the positively closed predicate $\mathrm{Pos}\, u$ iff $\Diamond u$. Suppose that $v \prec u$, that is there exists w such that $w \wedge v = 0$ and $w \vee u = 1$. Hence $\Box w \wedge \Diamond v \leqslant \Diamond(u \wedge v) = \Diamond 0 = 0$ and $1 = \Box 1 = \Box(w \vee u) \leqslant \Box w \vee \Diamond u$. That is, $\Diamond v \prec \Diamond u$, and so $\neg \Diamond v \vee \Diamond u = 1$ in the locale $V(L)$.

Conversely, let Pos be a located predicate. We define $\Diamond u$ iff $\mathrm{Pos}(u)$ and we define $\Box u$ iff $u \vee \neg \mathrm{Pos} = X$. We only check the two non-trivial rules:

To prove $\Box(u \vee v) \leqslant \Box u \vee \Diamond v$, we assume that $\Box(u \vee v)$. Then $u \vee v \vee \neg \mathrm{Pos} = X$. So, $u \vee v' \vee \neg \mathrm{Pos} = X$, for some $v' \prec v$. Hence, either $v' \in \neg \mathrm{Pos}$ or $v \in \mathrm{Pos}$ — that is, $\Box u$ or $\Diamond v$.

To prove $\Box u \wedge \Diamond v \leqslant \Diamond(u \wedge v)$ we assume that $\Box u \wedge \Diamond v$. Then $u \vee \neg \mathrm{Pos} = X$ and $\mathrm{Pos}(v)$. Since $v \lhd (u \vee \neg \mathrm{Pos}) \wedge v$ we see that $\Diamond(u \wedge v)$. $\Box$

Theorem 7. *Let X be a compact regular locale. The points of its Vietoris locale are precisely its compact overt sublocales.*

This result is not new, Vickers [Vic97] proves, impredicatively, that for a stably locally compact locale, the points of its Vietoris locale are the weakly semi-fitted sublocales with compact overt domain. A sublocale is weakly semi-fitted if it is the meet of a weakly closed sublocale with a fitted sublocale. In a compact regular locale, every compact sublocale is closed and thus weakly closed. It is also fitted, i.e. the meet of the open sublocales containing it.

Our proof is elementary. We have the interesting situation that the theories for the Vietoris locale and the theory for located sets are intuitionistically bi-interpretable. However, the former is geometric, but the latter theory is not.

In Taylor's Abstract Stone Duality [Tay05] the Vietoris construction and in particular the modal operators $\Box, \Diamond$ are taken as a starting point for the development of constructive analysis. It is interesting to note that his system allows us to interpret 'overt' as 'computable'. An analogue of this can be found by a recursive or type theoretical interpretation of the constructive mathematics underlying our work. In our case, the computation is present in the existential quantification in the definition of the positivity predicate.

11. Conclusion and Further work

We have generalized the notion of locatedness from metric spaces to general compact regular formal spaces and shown that a closed sublocale is located iff it is overt, thus proving that a closed subset is Bishop-compact iff its localic completion is compact overt (Theorems 2, 4)

The three types of locales we have studied above (locales defined from metric spaces, compact regular locales and regular locales) seem to allow a somewhat more uniform treatment by abstracting some of the properties of the relation $<$. Banaschewski's axioms for such strong inclusion relations and their relation to compactifications [Ban90][Cur08] may be of help here. They also include Curi's $<$-relation for uniform spaces [Cur06]. We believe that this may be a way of extending our results on compact locales and locally compact metric spaces to more general locales such as locally compact ones.

We draw the following preliminary conclusions from our investigation of the connections between located and overt sublocales. In the compact case, located and overt closed sublocales coincide; see Theorem 2. In the locally compact case, locatedness is not a metric property; see Example 2. On the other hand, there are a number of applications of locatedness outside the realm of compactness of which it is not so clear how to capture them by overt locales. A key example is the use of locatedness in Banach spaces. For instance, there exists a projection on a closed subspace of a Hilbert space iff the subspace is located [BB85, Thm. 7.8.7]. This notion of locatedness is used, for instance, in the theory of unbounded operators on Hilbert spaces [Spi02]. However, it may be possible to draw the connection between locatedness and compactness following Richman's observation that locatedness of certain subspaces of Hilbert spaces is equivalent to their weak total boundedness; see [Ric01, Ish01, IV03].

In the present paper, we have treated metric spaces mainly through their localic completion. We intend to return to the spatial side in another paper.

It would be interesting to extend our definition of locatedness on a formal topology, i.e. a site on a poset, to general sites.

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