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# K-theory of $\lambda$ -rings and tensorlike functors.

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## Abstract

We refine the invariant on  $K_2(A[C_{p^e}]/I_m, (T-1))$  constructed in a previous paper to one which is an isomorphism for all  $\lambda$ -rings  $A$ .

## 1 Introduction

This paper is part of a long term project aimed at the computation of  $K_1(A[G])$  for  $G$  a finite abelian group and  $A$  any  $\lambda$ -ring, for example a polynomial ring over the integers. To achieve this aim some  $K_2$  groups have to be calculated. In [4] an invariant was constructed on  $K_2(A \otimes B, A \otimes I_B)$  for  $B = \mathbf{Z}[G]/I_m$ . Here  $G$  is a cyclic group of order  $q = p^e$  and the  $I_m$  form a system of  $\lambda$ -ideals which is cofinal with the powers of the augmentation ideal  $I_B$ . This invariant map had values in an abelian group which is an explicit expression in  $A$  and  $\Omega_A$ , the module of Kähler differentials for  $A$ . It was proved that this invariant is an isomorphism as long as  $\Omega_A$  has no  $p$ -torsion. Unfortunately this excludes the case that  $A$  is itself a group ring of a  $p$ -cyclic group. It is the aim of the present paper to refine this invariant slightly so that it becomes an isomorphism for all  $A$ .

The computation of  $K_2(A[x]/x^n, (x))$  reveals that it can be expressed as a functor  $F$  of the homomorphism  $\delta: A \rightarrow \Omega_A$  of abelian groups, at least for rings  $A$  with the structure of  $\lambda$ -ring. The computation is done using a logarithm-like map  $L$ , see [1] and [3].

However for  $B$  as above even the best choice of logarithm is not an isomorphism, but has a certain small kernel and cokernel. In [4] it is tried to remedy this situation by combining with a second logarithmic invariant, which detects elements which are in the kernel of the original one. The result is a map to a fibered product of abelian groups, which is an isomorphism if  $\Omega_A$  has no  $p$ -torsion. However the resulting functor  $F$  has not the nice ‘tensorlike’ properties that the original one had.

The idea of this paper is to construct a new functor enforcing this nice property, by taking the fibered product not on the level of abelian groups, but on the level of the functors producing these groups. Finally we show that this new invariant is an isomorphism for all  $A$ .

## 2 Tensorlike functors

Write  $\mathcal{AG}$  for the category of abelian groups and homomorphisms. Let  $S$  be a ring, and write  $\mathcal{C}_S$  for the category of left  $S$ -modules. A right  $S$ -module  $K$  yields a functor  $F_K: \mathcal{C}_S \rightarrow \mathcal{AG}$  by the formula  $F_K(M) = K \otimes_S M$ .

We call a functor  $F$  from left  $R$ -modules to abelian groups

- Weakly additive if it commutes with finite direct sums. This we always assume.
- Additive if it is weakly additive and commutes with direct limits.
- Weakly tensorlike if it is weakly additive and commutes with right exact sequences.
- Tensorlike if it is additive and commutes with right exact sequences. This is the case if it commutes with arbitrary colimits, for example if it has a right adjoint.

**Proposition 1.** *Every weakly additive functor  $F$  has a natural tensorlike cover. If it is already weakly tensorlike it is an isomorphism for finitely presented modules. If it is already tensorlike then it is an isomorphism for all modules.*

*Proof.* Let  $F: \mathcal{C}_S \rightarrow \mathcal{AG}$  be weakly additive. For  $M$  a left  $S$ -module and  $a \in M$  there is an  $S$ -module homomorphism  $\rho_a$  given by  $\rho_a(s) = sa$ . This induces a map  $F(\rho_a): F(S) \rightarrow F(M)$ . By varying  $a$  we get a pairing  $F(S) \times M \rightarrow F(M)$ . From the weak additivity it follows that this map is additive also in the second entry. Therefore we get a map  $\mu: F(S) \otimes M \rightarrow F(M)$ .

For  $M = R$  this yields a right  $S$  module structure on  $F(S)$ . So we get a map  $\bar{\mu}: F(S) \otimes_S M \rightarrow F(M)$ . This map is an isomorphism for  $M = S$ , with inverse  $x \mapsto x \otimes 1$ . By weak additivity it is therefore an isomorphism for  $M = S^m$ . If  $F$  respects right exact sequences then  $\bar{\mu}$  is an isomorphism for all finitely presented modules  $M$ . If  $F$  is even right exact then  $\bar{\mu}$  is an isomorphism for all  $M$ .  $\square$

In our applications we have a category  $\mathcal{C}$  which is not given as a category of left  $S$ -modules, but as a category of diagrams of abelian groups, as discussed in the next section 3. However by the Freyd-Mitchell theorem one can realize this category as a category of left  $S$ -modules for some  $S$ . We make this correspondence explicit in proposition 2.

Each logarithmic map is a tensorlike functor of the diagram of abelian groups consisting of  $\delta: A \rightarrow \Omega_A$ . The combination of the logarithms is a map to a pullback which is no longer a tensorlike functor. It has however a tensorlike cover, which is constructed in section 4. We then lift our system of logarithms to this cover by considering the case where  $A$  is the universal  $\lambda$ -ring. This is done in section 5.

### 3 A category of arrows.

Let  $R$  be a ring and  $\tau: R \rightarrow R$  an endomorphism.

**Definition 1.** Given a left  $R$ -module  $N$  the  $R$ -module  $N^\tau$  is defined to be the one with the same additive structure as  $N$ , but with scalar multiplication given by  $r \cdot_\tau n = \tau(r) \cdot n$ .

If  $M$  is a left  $R$ -module then a module homomorphism  $D: M \rightarrow N^\tau$  is the same as an additive homomorphism  $D: M \rightarrow N$  such that  $D(r \cdot m) = \tau(r) \cdot D(m)$ , which we call a  $\tau$ -crossed homomorphism. Note that a module homomorphism  $\beta: N_1 \rightarrow N_2$  induces a module homomorphism  $\beta^\tau: N_1^\tau \rightarrow N_2^\tau$ .

**Definition 2.** Given  $\tau: R \rightarrow R$  the category  $\mathcal{A}(R, \tau)$  is the one with as objects the homomorphisms  $D: M \rightarrow N^\tau$  and as morphisms the commutative diagrams

$$\begin{array}{ccc} M_1 & \xrightarrow{\alpha} & M_2 \\ \downarrow D_1 & & \downarrow D_2 \\ N_1^\tau & \xrightarrow{\beta^\tau} & N_2^\tau \end{array}$$

where  $\alpha: M_1 \rightarrow M_2$  and  $\beta: N_1 \rightarrow N_2$  are  $R$ -module homomorphisms.

**Definition 3.** Given a ring  $R$  and an endomorphism  $\tau$  we construct a new ring  $S(R, \tau)$  by starting with the ring  $R\langle u, v \rangle$  of noncommutative polynomials and dividing out by the relations  $uv = 0$ ,  $v^2 = 0$ ,  $vu = v$ ,  $u^2 = u$ ,  $ur = ru$  and  $vr = \tau(r)v$ .

Given  $R$  modules  $M$  and  $N$  and a homomorphism  $D: M \rightarrow N^\tau$  we construct a left  $S(R, \tau)$ -module  $P$  by  $P = M \oplus N$ ,  $u(m \oplus n) = m \oplus 0$ ,  $v(m \oplus n) = 0 \oplus D(m)$ ,  $r(m \oplus n) = rm \oplus rn$ .

**Proposition 2.** *This produces an equivalence between the category  $\mathcal{A}(R, \tau)$  and the category of left  $S(R, \tau)$  modules.*

*Proof.* If  $\alpha: M_1 \rightarrow M_2$  and  $\beta: N_1 \rightarrow N_2$  are  $R$ -homomorphisms such that  $\beta \circ D = D \circ \alpha$  then we construct a  $S(R, \tau)$ -homomorphism from  $P_1 = M_1 \oplus N_1$  to  $P_2 = M_2 \oplus N_2$  by  $\gamma(m \oplus n) = \alpha(m) \oplus \beta(n)$ .

On the other hand if  $P$  is a left  $S(R, \tau)$  module then we define  $M = uP$ ,  $N = (1 - u)P$ ,  $D(m) = v \cdot m$ . Then  $M$  and  $N$  are left  $R$  modules and  $D$  is a homomorphism from  $M$  to  $N^\tau$ . Moreover a  $S(R, \tau)$  homomorphism  $\gamma: P_1 \rightarrow P_2$  gives rise to  $R$  homomorphisms  $\alpha: uP_1 \rightarrow uP_2$  and  $\beta: (1 - u)P_1 \rightarrow (1 - u)P_2$  such that  $\beta \circ D = D \circ \alpha$ .  $\square$

We made the definitions with the following special case in mind. Let  $A$  be a  $\lambda$ -ring.

- The abelian group  $M = A$  given by addition, with a  $\mathbf{Z}[t]$  module structure given by  $t \cdot a = \psi^p(a)$ . Here  $\psi^p$  is the Adams operation.

- The abelian group  $N = \Omega_A$  of Kähler differentials, with a  $\mathbf{Z}[t]$  module structure given by  $t \cdot \alpha = \phi^p(\alpha)$ , Here  $\phi^p$  is the operation introduced in [1].
- The differential  $\delta: A \rightarrow \Omega_A$ . Since  $\delta\psi^p(a) = p\phi^p\delta(a)$  this is a crossed homomorphism  $D$  with respect to the endomorphism  $\sigma$  of  $\mathbf{Z}[t]$  given by  $\sigma(t) = pt$ .

The above construction defines a functor  $\Delta$  from  $\lambda$ -rings to  $\mathcal{A}(\mathbf{Z}[t], \sigma)$ . By the above theory we get a from this a functor from  $\lambda$ -rings to left  $S(\mathbf{Z}[t], \sigma)$ -modules.

We now recall the main result of [4]. Consider the following diagram:

$$\begin{array}{ccc}
K_2(A \otimes B, A \otimes I) & \xrightarrow{L_4 \circ p_{GM}} & K_{2,L}(A \otimes (B/I^2), A \otimes I) \\
\downarrow L_2 & & \downarrow L_3 \circ L_4^{-1} \\
K_{2,L}(A \otimes B, A \otimes I) & \xrightarrow{p_{GL}} & K_{2,L}(A \otimes (B/I^2), A \otimes I)
\end{array} \quad (1)$$

Here  $K_2$  is relative algebraic  $K$ -theory, and for any commutative ring  $W$  and ideal  $J$  the group  $K_{2,L}(W, J)$  is defined as  $\text{coker}(\delta^*: J \otimes J \rightarrow J \otimes_W \Omega_W)$ , where  $\delta^*(a \otimes b) = a \otimes \delta b + b \otimes \delta a$ . For the definitions of the homomorphisms  $L_2, L_3, L_4, p_{GL}$  and  $p_{GM}$  we refer to [4]. This diagram has the following properties:

- The map  $p_{GL}$  is surjective.
- If an element in the bottom left group and one in the upper right group have the same image then they come both from an element in the  $K_2$  group. In other words the  $K_2$  group maps surjectively to the fibre product of the  $K_{2,L}$  groups.
- In the case that  $\Omega_A$  has no  $p$ -torsion then this element is unique. In other words the diagram is then cartesian.

The  $K_{2,L}$  groups in this diagram are tensorlike functors of the diagram  $\Delta(A)$  and thus of the left  $S(\mathbf{Z}[t], \sigma)$  module associated to  $A$ . However their fibre product is not a tensorlike functor. The aim of this paper is to construct its tensorlike cover, lift our invariants to this new group, and show that the new invariant is an isomorphism for all  $A$ , without condition on  $\Omega_A$ . Thus the kernel of the old combination of invariants is ‘explained’ by the fact that the tensorlike cover maps not always injectively to the fibre product of  $K_{2,L}$  groups.

We now exhibit the tensorlike structure of the  $K_{2,L}$  groups in detail. A few remarks:

- For simplicity we assume that  $m \geq e$  and  $q > 2$ .
- $\epsilon(n)$  is defined as  $m + e - i$  where  $p^i \leq n < p^{i+1}$ .

- $z \in B$  is defined as  $T - 1$ , where  $T$  is the generator of the cyclic group  $G$ .

$K_{2,L}(A \otimes B, A \otimes I)$  is a direct sum  $CF_a \oplus CF_b \oplus CF_2 \oplus \dots \oplus CF_{q-1}$ , although  $CF_a \oplus CF_b$  is called  $CF_{\text{low}}$  in proposition 4 of [4]. Here the summands are defined as follows:

$$CF_a = \frac{\Omega_A}{p^{m+e}\Omega_A + p^e\delta A}, \quad CF_b = \frac{A}{p^e A}, \quad CF_n = \frac{\frac{\Omega_A}{p^{e(n)}\Omega_A} \oplus \frac{A}{p^e A}}{\{\delta a \oplus na\}}$$

The correspondence is as follows:

- The class of  $\alpha \in \Omega_A$  in  $CF_a$  maps to the class of  $z \otimes \alpha$  in  $K_{2,L}(A \otimes B, A \otimes I)$ .
- The class of  $a \in A$  in  $CF_b$  maps to  $z^{q-1} \otimes a\delta z$ .
- The class of  $\alpha \oplus a$  in  $CF_n$  maps to  $z^n \otimes \alpha + z^{n-1} \otimes a\delta z$ .

Similarly  $K_{2,L}(A \otimes (B/I^2), A \otimes I)$  is a direct sum  $DF_1 \oplus DF_2$ . Here the summands are defined as follows:

$$DF_1 = \frac{\Omega_A}{p^e\Omega_A}, \quad DF_2 = \frac{A}{p^e A + 2A}$$

In particular  $DF_2$  vanishes if  $p > 2$  and is  $\frac{A}{2A}$  if  $p = 2$ . The correspondence is as follows:

- The class of  $\alpha \in \Omega_A$  in  $DF_1$  maps to  $z \otimes \alpha$  in  $K_{2,L}(A \otimes (B/I^2), A \otimes I)$ .
- The class of  $a \in A$  in  $DF_2$  maps to the class of  $z \otimes a\delta z$ .

The map  $p_{GL}$  maps  $CF_a$  in the obvious way to  $DF_1$  and maps  $CF_2$  in the obvious way to  $DF_2$ ; it vanishes on  $CF_b$  and on the  $CF_n$  with  $n > 2$ . The map  $L_3(L_4)^{-1}$  maps  $DF_1$  to itself by  $1 - \phi^p$  and maps  $DF_2$  to itself by  $1 - \psi^p$ .

From the above enumeration we see the only nontrivial parts in the determination of the tensorlike cover are the parts involving  $CF_a$  and  $CF_2$ , and the latter only if  $p = 2$ .

## 4 Determination of the covers.

Motivated by the above analysis we define the following functors on the category  $\mathcal{A}(\mathbf{Z}[t], \sigma)$ .

$$CF_a(D: M \rightarrow N) = \frac{N}{p^{m+e}N + p^e D(M)}, \quad DF_1(D: M \rightarrow N) = \frac{N}{p^e N}$$

There is an obvious transformation  $\pi: CF_a \rightarrow DF_1$ . We write  $\chi: DF_1 \rightarrow DF_1$  for the transformation given by multiplication by  $1 - t$ .

We also define the following functor with the aim of showing that it is the fibre product of  $\pi$  and  $\chi$ :

$$EF_1(D: M \rightarrow N) = \frac{N \oplus \frac{N}{p^m N + D(M)}}{\{p^e \gamma, -(1-t)\gamma\}} = \frac{\frac{N}{p^{m+e}N} \oplus \frac{N}{p^m N + D(M)}}{\{p^e \gamma, -(1-t)\gamma\}}$$

**Proposition 3.** Let  $\pi^*: EF_1 \rightarrow DF_1$  be the projection on the first summand, and let  $\chi^*: EF_1 \rightarrow CF_a$  be given by  $\chi^*(\alpha, \beta) = (1-t)\alpha + p^e\beta$ . Then the diagram of tensorlike functors

$$\begin{array}{ccc} EF_1 & \xrightarrow{\pi^*} & DF_1 \\ \downarrow \chi^* & & \downarrow \chi \\ CF_a & \xrightarrow{\pi} & DF_1 \end{array}$$

is cartesian in the sense that the corresponding diagram of right  $S(\mathbf{Z}[t], \sigma)$ -modules is cartesian.

*Proof.* The functors  $CF_a$  and  $DF_1$  and  $EF_1$  can be viewed as functors on the category of left  $S(\mathbf{Z}[t], \sigma)$ -modules. By proposition 1 we only must show that the diagram of abelian groups which one gets by applying those functors to  $S = S(\mathbf{Z}[t], \sigma)$  itself is cartesian. The diagram corresponding to  $S$  is the direct sum of two diagrams: one has  $M = 0$  and  $N = \mathbf{Z}[t]$  and the other has  $M = \mathbf{Z}[t]$  and  $N = \mathbf{Z}[t]$ . We only check the second case: the first is similar but easier. Thus we have to consider the following diagram:

$$\begin{array}{ccc} \frac{\mathbf{Z}[t] \oplus \frac{\mathbf{Z}[t]}{p^m \mathbf{Z}[t] + \{g(pt)\}}}{\{p^e f \oplus -(1-t)f\}} & \xrightarrow{\pi^*} & \frac{\mathbf{Z}[t]}{p^e \mathbf{Z}[t]} \\ \downarrow \chi^* & & \downarrow \chi \\ \frac{\mathbf{Z}[t]}{p^{m+e} \mathbf{Z}[t] + p^e \{h(pt)\}} & \xrightarrow{\pi} & \frac{\mathbf{Z}[t]}{p^e \mathbf{Z}[t]} \end{array}$$

To check that this diagram is cartesian is elementary. □

Again motivated by the analysis in the last section we define the following functors in the case  $p = 2$ :

$$CF_2(D: M \rightarrow N) = \frac{\frac{N}{2^{m+e-1}} \oplus \frac{M}{2^e}}{\{Da \oplus 2a\}}, \quad DF_2(D: M \rightarrow N) = \frac{M}{2M}$$

There is an obvious transformation  $\pi: CF_2 \rightarrow DF_2$ . We write  $\chi: DF_2 \rightarrow DF_2$  for the transformation given by multiplication by  $1-t$ .

We also define the following functor with the aim of showing that it is the fibre product of  $\pi$  and  $\chi$ :

$$\begin{aligned} EF_2(D: M \rightarrow N) &= \frac{\frac{N}{2^{m+e-1}} \oplus M \oplus \frac{M}{2^{e-1}}}{\{D(c) \oplus 2h \oplus (c - (1-t)h)\}} \\ &= \frac{\frac{N}{2^{m+e-1}} \oplus \frac{M}{2^e} \oplus \frac{M}{2^{e-1}}}{\{D(c) \oplus 2h \oplus (c - (1-t)h)\}} \end{aligned}$$

**Proposition 4.** Let  $\pi^*: EF_2 \rightarrow DF_2$  be the projection on the second summand, and let  $\chi^*: EF_2 \rightarrow CF_2$  be given by  $\chi^*(\alpha \oplus a \oplus b) = \alpha \oplus ((1-t)a + 2b)$ . Then the diagram of tensorlike functors

$$\begin{array}{ccc} EF_2 & \xrightarrow{\pi^*} & DF_2 \\ \downarrow \chi^* & & \downarrow \chi \\ CF_2 & \xrightarrow{\pi} & DF_2 \end{array}$$

is cartesian in the sense that the corresponding digram of right  $S(\mathbf{Z}[t], \sigma)$ -modules is cartesian.

*Proof.* As in the proof of the last proposition we only have to check the case  $M = \mathbf{Z}[t]$  and  $N = \mathbf{Z}[t]$ . Thus we have to consider the following diagram:

$$\begin{array}{ccc} \frac{\mathbf{Z}[t]}{2^{m+e-1}\mathbf{Z}[t]} \oplus \mathbf{Z}[t] \oplus \frac{\mathbf{Z}[t]}{2^{c-1}\mathbf{Z}[t]} & \xrightarrow{\pi^*} & \frac{\mathbf{Z}[t]}{2\mathbf{Z}[t]} \\ \{c(pt) \oplus 2h(t) \oplus (c(t) - (1-t)h(t))\} & & \downarrow \chi \\ \downarrow \chi^* & & \downarrow \chi \\ \frac{\mathbf{Z}[t]}{2^{m+e-1}\mathbf{Z}[t]} \oplus \frac{\mathbf{Z}[t]}{2^e\mathbf{Z}[t]} & \xrightarrow{\pi} & \frac{\mathbf{Z}[t]}{2\mathbf{Z}[t]} \\ \{g(2t) \oplus 2g(t)\} & & \end{array}$$

To check that this diagram is cartesian is elementary.  $\square$

**Corollary 1.** The kernel of the surjective map from  $EF_1(D: M \rightarrow N)$  to the fibre product of  $CF_a(D: M \rightarrow N)$  and  $DF_1(D: M \rightarrow N)$  is a quotient of the  $p^e$ -torsion in  $N$ .

The kernel of the surjective map from  $EF_2(D: M \rightarrow N)$  to the fibre product of  $CF_2(D: M \rightarrow N)$  and  $DF_2(D: M \rightarrow N)$  is a quotient of the 2-torsion in  $M$ .

*Proof.* Straightforward.  $\square$

Write  $FP(A)$  for the fibre product of the bottom arrow and right arrow in diagram (1). Thus the old invariant can be seen as a map  $K_2(A \otimes B, A \otimes I) \rightarrow FP(A)$ . Write  $TC(A)$  for its tensorlike cover viewing  $FP(A)$  as a functor of the diagram  $\Delta(A)$  associated to  $A$ . Thus the new invariant to be constructed can be seen as a map  $K_2(A \otimes B, A \otimes I) \rightarrow TC(A)$ .

Let us call a ring  $A$  torsionfree if there is no  $p$ -torsion in  $A$  nor in  $\Omega_A$ . We have seen that the canonical map  $TC(A) \rightarrow FP(A)$  is an isomorphism if  $A$  is torsionfree.

**Corollary 2.** If  $p > 2$  then  $TC(A)$  is isomorphic to

$$EF_1(A) \oplus CF_b(A) \oplus CF_2(A) \oplus CF_3(A) \oplus \dots \oplus CF_{q-1}(A).$$

If  $p = 2$  then  $TC(A)$  is isomorphic to

$$EF_1(A) \oplus CF_b(A) \oplus EF_2(A) \oplus CF_3(A) \oplus \dots \oplus CF_{q-1}(A).$$



## 5 The invariant map and its inverse.

Recall that there exists a  $\lambda$ -ring  $U$  and an element  $u \in U$  with the following property: for any  $\lambda$ -ring  $A$  and any  $x \in A$  there is a unique  $\lambda$ -ring homomorphism  $j_x: U \rightarrow A$  such that  $j_x(u) = x$ . Similarly  $U_d = U^{\otimes d}$  has the following property for any  $\lambda$ -ring  $A$  and any  $x_1, \dots, x_d \in A$  there is a unique  $\lambda$ -ring homomorphism  $j_{x_1, \dots, x_d}: U_d \rightarrow A$  such that  $j_{x_1, \dots, x_d}(u_i) = x_i$  for  $i = 1, \dots, d$ . Here  $u_i = 1 \otimes \dots \otimes u \otimes \dots \otimes 1$ . The ring  $U$  is a polynomial ring over  $\mathbf{Z}$  in infinitely many variables; in particular the  $U_d$  are torsionfree.

The key to the construction of the new invariant map is the following principle:

**Proposition 5.** *Let  $F$  and  $G$  be functors from the category of  $\lambda$ -rings to the category of abelian groups. Let  $T: F \rightarrow G$  be a transformation defined on the full subcategory consisting of the rings  $U_d$ . Suppose that  $F(A)$  is given by generators and relations associated to sequences of elements in  $A$ . Then  $T$  can be extended to all  $\lambda$ -rings.*

*Proof.* To simplify the presentation of the proof we consider the special case that  $F(A)$  has a generator  $\langle a, b \rangle$  for each  $a, b \in A$ , and that there is only one type of relation, say  $\langle a, bc \rangle = \langle ab, c \rangle + \langle ac, b \rangle$  for each  $a, b, c \in A$ . The general case would involve lots of extra indices, which would only obfuscate the idea of the proof.

In order to construct  $T_A: F(A) \rightarrow G(A)$  we define it on generators by  $T_A \langle a, b \rangle = G(j_{a,b})T_{U_2} \langle u_1, u_2 \rangle$ . We must then check that it maps the relations to zero.

The description in terms of generators and relations also applies to  $F(U_3)$ ; in particular  $\langle u_1, u_2 u_3 \rangle = \langle u_1 u_2, u_3 \rangle + \langle u_1 u_3, u_2 \rangle$  in  $F(U_3)$ . Now we have

$$\begin{aligned} T_A \langle a, bc \rangle &= G(j_{a,bc})T_{U_2} \langle u_1, u_2 \rangle = G(j_{a,b,c} \circ j_{u_1, u_2 u_3})T_{U_2} \langle u_1, u_2 \rangle \\ &= G(j_{a,b,c})G(j_{u_1, u_2 u_3})T_{U_2} \langle u_1, u_2 \rangle \\ &= G(j_{a,b,c})T_{U_3} F(j_{u_1, u_2 u_3}) \langle u_1, u_2 \rangle = G(j_{a,b,c})T_{U_3} \langle u_1, u_2 u_3 \rangle \end{aligned}$$

and similarly

$$\begin{aligned} T_A \langle ab, c \rangle &= G(j_{a,b,c})T_{U_3} \langle u_1 u_2, u_3 \rangle \\ T_A \langle ac, b \rangle &= G(j_{a,b,c})T_{U_3} \langle u_1 u_3, u_2 \rangle \end{aligned}$$

Thus the desired result follows by applying  $G(j_{a,b,c}) \circ T_{U_3}$  to the relation in  $F(U_3)$ .

We check that the extended  $T$  is still a transformation. Let  $f: A_1 \rightarrow A_2$  be a  $\lambda$ -ring homomorphism. For  $a, b \in A_1$  we have

$$\begin{aligned} G(f)T_{A_1} \langle a, b \rangle &= G(f)G(j_{a,b})T_{U_2} \langle u_1, u_2 \rangle = G(j_{f(a), f(b)}) \langle u_1, u_2 \rangle \\ &= T_{A_2} \langle f(a), f(b) \rangle = T_{A_2} F(f) \langle a, b \rangle \end{aligned}$$

since  $f \circ j_{a,b} = j_{f(a), f(b)}$ . Since  $G(f)T_{A_1}$  and  $T_{A_2} F(f)$  agree on generators they agree on  $F_1(A)$ .  $\square$

**Proposition 6.** *Let  $F$  and  $G$  be functors from the category of  $\lambda$ -rings to the category of abelian groups. Suppose that  $F$  and  $G$  are both given by generators and relations. Let  $T: F \rightarrow G$  and  $S: G \rightarrow F$  be transformations such that  $TS = 1$  and  $ST = 1$  on the rings  $U_d$ . Then  $TS = 1$  and  $ST = 1$  on all  $\lambda$ -rings.*

*Proof.* Again to simplify the proof we assume that  $F(A)$  has a generator  $\langle a, b \rangle$  for each  $ab \in A$ , and that  $G(A)$  has a generator  $[a, b]$  for each  $a, b \in A$ . We make no assumptions on the form of the relations. We will prove that  $ST = 1$  on  $F(A)$ .

The description in terms of generators and relations also applies to  $G(U_2)$ ; therefore  $T_{U_2}\langle u_1, u_2 \rangle$  can be expressed in these generators, say

$$T_{U_2}\langle u_1, u_2 \rangle = \sum_{i=1}^r c_i [v_i, w_i], \quad \text{with } c_i \in \mathbf{Z}$$

This assumption implies that

$$T_A\langle a, b \rangle = G(j_{a,b}) \sum_{i=1}^r c_i [v_i, w_i] = \sum_{i=1}^r c_i [x_i, y_i]$$

where  $x_i = j_{a,b}(v_i)$  and  $y_i = j_{a,b}(w_i)$ . Therefore

$$\begin{aligned} S_A T_A \langle a, b \rangle &= S_A \sum_{i=1}^r c_i [x_i, y_i] = \sum_{i=1}^r c_i S_A G(j_{a,b}) [v_i, w_i] \\ &= \sum_{i=1}^r c_i F(j_{a,b}) S_{U_2} [v_i, w_i] = F(j_{a,b}) S_{U_2} \sum_{i=1}^r c_i [v_i, w_i] \\ &= F(j_{a,b}) S_{U_2} T_{U_2} \langle u_1, u_2 \rangle = F(j_{a,b}) \langle u_1, u_2 \rangle = \langle a, b \rangle \end{aligned}$$

Since  $S_A T_A$  is the identity on generators, it is the identity on  $F(A)$ .  $\square$

The aim is now to use proposition 5 to construct our new invariant map from  $K_2(A \otimes B, A \otimes I)$  to  $TC(A)$ , and proposition 6 to construct its inverse. The point is of course that we already know what it should do on the rings  $U_d$  and indeed on torsionfree  $\lambda$ -rings since there its should agree with the old invariant map under the isomorphism  $TC(A) \rightarrow FP(A)$ . The only thing left to do is to point out why  $K_2(A \otimes B, A \otimes I)$  and  $TC(A)$  are described by generators and relations.

For  $K_2$  this well known: in [5] its is proved that for a commutative ring  $W$  and a nilpotent ideal  $J$  such that the projection  $W \rightarrow W/J$  splits the group  $K_2(W, J)$  has a presentation with as generators  $\langle a, b \rangle$  with  $a \in W$  and  $b \in I$  or  $a \in I$  and  $b \in J$ . The relations are

- $\langle a, b \rangle + \langle b, a \rangle$  for  $b \in J$  or  $a \in J$ .
- $\langle a, b \rangle + \langle a, c \rangle - \langle a, b + c - abc \rangle$  for  $a \in J$  or and  $b, c \in J$ .
- $\langle a, bc \rangle - \langle ab, c \rangle - \langle ac, b \rangle$  for  $a \in J$  or  $b \in J$  or  $c \in J$ .

So for  $W = A \otimes B$  and  $J = A \otimes I$  we have generators

$$\langle a_0 + a_1z + a_2z^2 + \cdots + a_{q-1}z^{q-1}, b_0 + b_1z + b_2z^2 + \cdots + b_{q-1}z^{q-1} \rangle$$

with  $a_i, b_i \in A$  and  $a_0 = 0$  or  $b_0 = 0$ . This can be seen as a generator depending on  $2q-1$  elements of  $A$ . The relations should of course be rewritten accordingly. Also one should not forget the relation that two such generators agree if their  $a_i$  or their  $b_i$  are equal module the order of  $z^i$ , which is  $\epsilon(i)$  for  $i > 0$ .

The group  $TC(A)$  is a direct sum of summands which each have a description in terms of generators and relations. For example  $\Omega$  has a presentation with generators  $[a, b]$  for  $a, b \in A$ ; the relations are

- $[a, b] + [a, c] - [a, b + c]$ .
- $[a, c] + [b, c] - [a + b, c]$ .
- $[a, bc] - [ab, c] - [ac, b]$ .

The generator  $[a, b]$  corresponds with the element  $a\delta b$  of  $\Omega_A$ . The image of  $a\delta b$  under  $\phi^p$  is

$$\psi^p(a)\phi^p(\delta b) = \psi^p(a)(b^{p-1}\delta b - \delta\theta^p(b))$$

Thus in terms of the above presentation  $\phi^p$  maps  $[a, b]$  to

$$[\psi^p(a)b^{p-1}, b] - [\psi^p(a), \theta^p(b)]$$

Here  $\theta^p$  is the  $\lambda$ -operation introduced in [1]. Using this remark it is now easy to write down presentations of the groups  $CF_n$ ,  $CF_b$ ,  $EF_1$  and  $EF_2$ .

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