Lambda-conductors for group rings

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1 Introduction.

This paper is part of a project which aims to provide a method for computing the Nil groups of the group rings of finite abelian groups, by refining some of the techniques used in [1] and [2] in such a way that the allowed coefficient rings include polynomial rings. For the refinement of the p-adic logarithm discussed in [3] and [4] it is assumed that the rings involved have a structure of λ-ring; we refer to these papers for generalities about λ-rings. Thus it is useful to extend as much as possible of the other techniques to the context of λ-rings. In this paper we investigate how to describe a group ring of a finite abelian group as a pull back of a diagram of rings which are more accessible to calculations in algebraic K-theory.

Let be given a commutative ring S and subring R. For each ideal I of S which is contained in R one has a cartesian square

$$\begin{array}{ccc} R & \longrightarrow & S \\ \downarrow & & \downarrow \\ R/I & \longrightarrow & S/I \end{array}$$

thus describing R as a pull back of rings for which the K-theory is hopefully better understood. By taking for I the sum of all such ideals one finds a diagram where the rings on the bottom row are as small as possible.

We modify this construction by assuming that R has a structure of λ-ring and considering only ideals I stable under the λ-operations. We call the resulting ideal the λ-conductor of S into R. In particular we are interested in the case that R is the group ring $\mathbf{Z}[G]$ of a finite abelian group, and S is its normal closure in $R \otimes \mathbf{Q}$, which splits as a direct sum of rings $S_i = \mathbf{Z}[\chi_i]$ associated to equivalence classes of characters $\chi_i : G \to \mathbf{C}$.

In this situation R is a λ-ring such that $\psi^n(g) = g^n$ for every $n \in \mathbf{N}$ and $g \in G$. In general S is not stable under the λ-operations on $R \otimes \mathbf{Q}$, but it is stable under the associated Adams operations ψ^n since they are ring homomorphisms.

We will prove that in case G is a primary group its λ-conductor is precisely the intersection of the classical conductor and the augmentation ideal. We do this by exhibiting generators of the classical conductor and examining their behavior under the fundamental λ-operations.

2 The primary case

Throughout this section G is a group of order $n = p^e$, where p is prime. We consider representations $\rho\colon G \to \mathbf{C}^*$. We say that ρ is of level k if the image of ρ has p^k elements.

Two representations τ_1 and τ_2 are called equivalent if they have the same kernel. That means that there must be some $m \in \mathbf{Z}$ prime to p such that $\tau_2(x) = \tau_1(x)^m$ for all $x \in G$. Obviously equivalent representations have the same level.

Given a representation τ of level $k > 0$ one gets a representation $\psi\tau$ of level $k-1$ by the formula $(\psi\tau)(x) = \tau(x^p)$ for $x \in G$. If $\psi\tau_1$ and $\psi\tau_2$ are equivalent then we may replace τ_2 by an equivalent representation τ_2' so that $\psi\tau_2' = \psi\tau_1$. So we may choose a representation in each class in such a way that $\psi\tau$ and ρ coincide if they are equivalent.

Let ρ be a representation of level $k > 0$ and write $\omega = \exp(2\pi i/p)$. We define an element $b_\rho \in \mathbf{Z}[G]$ by the formula

$$b_\rho = \sum_{\rho x = 1} x - \sum_{\rho\xi = \omega} \xi.$$

If we choose $y_\rho \in G$ such that $\rho(y) = \omega$ then we get

$$b_\rho = \left(\sum_{\rho x = 1} x\right)(1 - y_\rho)$$

The only representation of level 0 is the trivial representation, which we denote by 1, and it gives rise to $b_1 = \sum_{x \in G} x$.

Proposition 1. *If ρ and τ are not equivalent then $b_\rho b_\tau = 0$. Furthermore $b_\rho^2 = p^{e-k}(1 - y_\rho)\, b_\rho$ for ρ of level $k > 0$.*

Proof. If $\ker(\rho) \neq \ker(\tau)$ we may assume that there is $g \in G$ with $\rho(g) = 1$ but $\tau(g) = \omega$. If g has order m then $\sum_{\rho(x)=1} x$ and thus b_ρ is a multiple of $\sum_{j=0}^{m-1} g^j$, whereas b_τ is a multiple of $1 - g$. The product of these two factors is 0.

The second part follows from $(\sum_{\rho(\xi)=1} \xi)(\sum_{\rho(x)=1} x) = p^{n-k} \sum_{\rho(x)=1} x$ which is true because each ξ gives the same contribution and there are p^{e-k} of them. $\square$

Every representation ρ of level k gives rise to a homomorphism j_ρ from $\mathbf{Z}[G]$ to $S_\rho = \mathbf{Z}[\omega_k]$, where $\omega_k = \exp(2\pi i/p^k)$.

Proposition 2. *If ρ and τ are not equivalent then $j_\rho(b_\tau) = 0$. Furthermore $j_1(b_1) = p^e$, and $j_\rho(b_\rho) = p^{e-k}(1-\omega)$ if ρ is of level $k > 0$,.*

Proof. The second part is obvious since every x in the definition of b_ρ maps to 1, and y_ρ maps to ω. The first part follows since $j_\rho(b_\rho) j_\rho(b_\tau) = 0$ by Proposition 1 and S_ρ is a domain. $\square$

It is well known that the maps j_ρ (one from each equivalence class) combine to an embedding from R into its integral closure $S = \oplus_\rho S_\rho$.

Proposition 3. *The b_ρ generate the conductor ideal I of S into R.*

Proof. By the theorem of Jacobinski (Theorem 27.8 in [5]) the conductor is $\oplus_\rho n\mathcal{D}_\rho^{-1} \subset \oplus_\rho S_\rho = S$, where $\mathcal{D}_\rho^{-1}$ is the lattice in $\mathbf{Q}[\omega_k]$ dual to $\mathbf{Z}[\omega_k]$ under the trace form. It is a simple exercise that this fractional ideal is in fact generated by $p^{-k}(1-\omega)$, which means that $n\mathcal{D}_\rho^{-1}$ is just $j(b_\rho)S = j(B_\rho)S_\rho = j(b_\rho R)$. □

We remind the reader that in particular nS is contained in the conductor.

The λ-conductor I_λ from S into R is defined as the largest ideal of S contained in R which is stable under the fundamental λ-operations θ^ℓ. It is of course a subset of the largest ideal of S contained in R, which is the ordinary conductor I described above. Thus we have to investigate the behaviour of the operations θ^ℓ on the generators b_ρ.

Lemma 1. *If ρ is of level $k > 0$ and there is no τ with $\psi\tau = \rho$ then $\psi^p b_\rho = 0$.*

Proof. Write G as a direct product of cyclic groups, with generators g_i. If the order of $\rho(g_i)$ is strictly smaller than the order of g_i for all i then one can find a suitable τ by taking for each $\tau(g_i)$ a p-th root of $\rho(g_i)$. If however the orders are the same for some i then there is certainly some $h \in G$ such that $h^p = 1$ and $\rho(h) = \omega$. By definition of b_ρ we have

$$\psi^p b_\rho = \sum_{\rho\xi=1} \xi^p - \sum_{\rho\eta=\omega} \eta^p$$

Here the term in the second sum associated to $\eta = h\xi$ cancels the term in the first sum associated to ξ. □

Proposition 4. *If ρ is of level $k > 0$ then*

$$\psi^p(b_\rho) = \sum_{\psi\tau=\rho} p b_\tau$$

Proof. By definition we have

$$\sum_{\psi\tau=\rho} b_\tau = \sum_{\psi\tau=\rho} \sum_{\tau x=1} x - \sum_{\psi\tau'=\rho} \sum_{\tau' x=\omega} x$$

We claim that all terms with $x \notin G^p$ cancel. To prove this assume that the class of x in G/G^p is nontrivial. Then there exists a homomorphism $\sigma\colon G/G^p \to \mathbf{C}^*$ such that $\sigma(x) = \omega$. Now the term associated to τ in the first sum equals the term associated to $\tau' = \tau \cdot \sigma$ in the second sum.

So we only have to consider terms of the form $x = \xi^p$ with $\xi \in G$. The condition $\tau(x) = 1$ is then independent of τ (since it is equivalent to $\rho(\xi) = 1$) and the sum over all τ with $\psi\tau = \rho$ reduces to a multiplication with the number of equivalence classes of such τ. By the Lemma we may assume that this number is nonzero. Now τ_1 and τ_2 with $\psi\tau_1 = \rho = \psi\tau_2$ are equivalent iff $\tau_2 = \tau_1^{1+mp^k}$ for some m with $0 \leq m < p$. So this number is $1/p$ times the number of homomorphisms $\sigma\colon G/G^p \to \mathbf{C}^*$, hence equals p^{r-1}, where r denotes the rank of G.

On the other hand we have

$$\psi^p b_\rho = \sum_{\rho\xi=1} \xi^p - \sum_{\rho\eta=\omega} \eta^p$$

Here the first sum is a certain factor times the sum over all $x \in G$ for which there exists $\xi \in G$ with $x = \xi^p$ and which satisfy $\tau(x) = 1$ for any (and thus all) τ with $\psi\tau = \rho$. The factor is the number of ξ which satisfy these conditions, which equals p^r. □

We write h for the polynomial of degree $p-2$ given by

$$h(t) = \frac{1}{1-t}\left(p - \frac{1-t^p}{1-t}\right)$$

Proposition 5.

$$\psi^p(b_1) = b_1 + \sum_{\tau\neq 1, \psi\tau=1} h(y_\tau) b_\tau$$

Proof. We have

$$b_\tau = \left(\sum_{\tau x=1} x\right)(1 - y_\tau)$$

and thus

$$h(y_\tau)b_\tau = \left(\sum_{\tau x=1} x\right)\left(p - \sum_{j=0}^{p-1} y_\tau^j\right) = p\left(\sum_{\tau x=1} x\right) - \left(\sum_{x\in G} x\right)$$

We must take the sum of $\sum_{\tau x=1} x$ over all equivalence classes of $\tau \neq 1$ with $\psi\tau = 1$. Interchange the sum over x and the sum over τ. There are two cases:

- If $x \in G^p$ then $\tau x = 1$ for all τ, and we must simply count the number of equivalence class of τ. There are $p^r - 1$ of them, with $p-1$ in each class.
- If $x \notin G^p$ then the number of τ such that $\tau x = 1$ is $p^{r-1} - 1$, with again $p-1$ in each class.

So we get

$$\begin{aligned}\sum_{\tau\neq 1, \psi\tau=1} h(y_\tau)b_\tau =& p\left(\frac{p^r-1}{p-1}\sum_{x\in G^p} x + \frac{p^{r-1}-1}{p-1}\sum_{x\notin G^p} x\right)\\ &- \left(\frac{p^r-1}{p-1}\sum_{x\in G^p} x + \frac{p^r-1}{p-1}\sum_{x\notin G^p} x\right)\\ =& p^r\sum_{x\in G^p} x - \sum_{x\in G} x = \sum_{\xi\in G}\xi^p - \sum_{x\in G} x = \psi^p b_1 - b_1\end{aligned}$$

$\square$

Now we consider the effect of Adams operations ψ^q for primes $q \neq p$. For any prime q we write f_q and g_q for the polynomials given by

$$f_q(t) = \frac{1-t^q}{1-t}, \qquad g_q(t) = \frac{(1-t)^{q-1} - f_q(t)}{q}$$

Proposition 6. *If ρ is of level $k > 0$ then*

$$\psi^q(b_\rho) = f_q(y_\rho) b_\rho$$

and $\psi^q(b_1) = b_1$.

Proof. We have $\psi^q(b_1) = \psi^q\left(\sum_{x \in G} x\right) = \sum_{x \in G} x^q = \sum_{\xi \in G} \xi = b_1$ and

$$\begin{aligned}
\psi^q(b_\rho) &= \psi^q\left(\left(\sum_{\rho x = 1} x\right)(1-y_\rho)\right) = \left(\sum_{\rho x=1} x^q\right)(1-y_\rho^q) \\
&= \left(\sum_{\rho\xi=1} \xi\right)(1-y_\rho) f_q(y_\rho) = b_\rho f_q(y_\rho)
\end{aligned}$$

$\square$

Corollary 1. *For the idempotents $e_\rho \in S_\rho$ one has*

$$\begin{aligned}
\psi^p(e_\rho) &= \sum_{\psi\tau=\rho} e_\rho \qquad \text{if } \rho \neq 1, \\
\psi^p(e_1) &= e_1 + \sum_{\tau\neq 1, \psi\tau = 1} e_\tau \\
\psi^q(e_\rho) &= e_\rho \qquad \text{if } q \neq p
\end{aligned}$$

Since R has no $\mathbf{Z}$-torsion the Adams operations ψ^ℓ determine the operations θ^ℓ and we find

Proposition 7. *If ρ has level $k > 0$ then*

$$\begin{aligned}
\theta^p(b_\rho) &= p^{(e-k)(p-1)-1}(1-y_\rho)^{p-1} b_\rho - \sum_{\psi\tau=\rho} b_\tau \qquad \text{if } k < e \\
\theta^p(b_\rho) &= g_p(y_\rho) b_\rho \qquad \text{if } k = e \\
\theta^q(b_\rho) &= \left(\frac{p^{(e-k)(q-1)}-1}{q}(1-y_\rho)^{q-1} + g_q(y_\rho)\right) b_\rho \qquad \text{for } q \neq p
\end{aligned}$$

Moreover

$$\begin{aligned}
\theta^p(b_1) &= p^{e(p-1)-1} b_1 - p^{-1} b_1 - p^{-1} \sum_{\tau\neq 1, \psi\tau=1} h(y_\tau) b_\tau \\
\theta^q(b_1) &= \frac{p^{e(q-1)}-1}{q} b_1 \qquad \text{for } q \neq p
\end{aligned}$$

Proof. This is just a matter of combining the last three Propositions with the formula $\ell\theta^\ell(a) = a^\ell - \psi^\ell a$. Note that $k = e$ can only happen if G is cyclic, in which case $y_\rho^p = 1$, which implies that $b_\rho^p = (1 - y_\rho)^p = p(1 - y_\rho)g_p(y_\rho)$. □

Theorem 1. *The b_ρ with $\rho \neq 1$ generate the λ-conductor ideal I_λ. In other words I_λ is the intersection of the augmentation ideal and the ordinary conductor ideal I.*

Proof. Write J for the R-ideal generated by the b_ρ with $\rho \neq 1$. From Proposition 7 one reads of that $\theta^\ell(b_\rho) \in J$ for $\rho \neq 1$ and for every prim ℓ. From the identity

$$\theta^\ell(ab) = \theta^\ell(a)b^\ell + \psi^\ell(a)\theta^\ell(b)$$

it then follows that $\theta^\ell(Rb_\rho) \subset J$ for $\rho \neq 1$ and all ℓ. Finally from

$$\theta^\ell(u+v) = \theta^\ell(u) + \theta^\ell(v) + \sum_{i=1}^{\ell-1} \frac{1}{\ell}\binom{\ell}{i} u^i v^{\ell-i}$$

it follows that $\theta^\ell(J) \subset J$ for every ℓ. Since $J \subset I$ by Proposition 3 this shows that $J \subset I_\lambda$.

Suppose that $x \in I_\lambda$ and $x \notin J$. Then $x \in I$, so by Proposition 3 there are $x_\rho \in R$ such that $x = \sum_\rho x_\rho b_\rho$. Since $\sum_{\rho\neq 1} x_\rho b_\rho \in J \subset I_\lambda$ by the first half of the proof, it follows that $x_1 b_1 \in I_\lambda$. Since $gb_1 = b_1$ for every $g \in G$ we may assume that $x_1 \in \mathbf{Z}$. Moreover $x_1 \neq 0$ which means that its p-valuation $v_p(x_1)$ is a natural number. We may assume that x is chosen in such a way that $v_p(x_1)$ is minimal. Now I_λ must also contain

$$\theta^p(x_1 b_1) = p^{-1}\left(x_1^p p^{e(p-1)} b_1 - x_1(b_1 + \sum_{\tau\neq 1, \psi\tau = 1} b_\tau)\right)$$

However the valuation of the coefficient of b_1 is $v_p(x_1^p p^{e(p-1)} - x_1) - 1 = v_p(x_1) - 1$, in contradiction with the way x was chosen. Thus $I_\lambda \subset J$. □

3 Direct products of relatively prime order

Let G_1 be a group of order $n_1 = p^e$, and let G_2 a group of order $n_2 = q^f$, where p and q are different primes. We write $R_1 = \mathbf{Z}[G_1]$ and $R_2 = \mathbf{Z}[G_2]$, and denote their normal closures by S_1 and S_2 respectively. Finally we write I_1 for the conductor of S_1 into R_1 and I_2 for the conductor of S_2 into R_2. Since the S_i are free abelian groups, the same is true for the other additive groups involved, and we can view $I_1 \otimes I_2$ as a subgroup of $R_1 \otimes I_2$ and of $R_1 \otimes R_2$.

Lemma 2.

$$I_1 \otimes I_2 = (R_1 \otimes I_2) \cap (I_1 \otimes R_2)$$

Proof. There are $m_1, m_2 \in \mathbf{Z}$ such that $m_1n_1 + m_2n_2 = 1$. If x is an element of the left hand side then $x \in R_1 \otimes I_2$, so $n_1x \in n_1R_1 \otimes I_2 \subset n_1S_1 \otimes I_2 \subset I_1 \otimes I_2$ and therefore $m_1n_1x \in I_1 \otimes I_2$. Similarly $m_2n_2x \in I_1 \otimes I_2$ and thus $x = m_1n_1x + m_2n_2x \in I_1 \otimes I_2$. The other implication is obvious □

Proposition 8. *The conductor I of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is $I_1 \otimes I_2$.*

Proof. Suppose that $x \in I$, so that $x(S_1 \otimes S_2) \subset R_1 \otimes R_2$. We write $x \in R_1 \otimes R_2$ as $\sum x_g \otimes g$, where g runs trough G_2. For any $a \in S_1$ we have $\sum (x_g a) \otimes g = (\sum x_g \otimes g)(a \otimes 1) = x(a \otimes 1) \in R_1 \otimes R_2$. Therefore $x_g a \in I_1$ for any $a \in S_1$, which means that $a_g \in I_1$ for all $g \in G_1$. Thus $x \in I_1 \otimes R_2$. Similarly $x \in R_1 \otimes I_2$. Thus $x \in I_1 \otimes I_2$ by the Lemma. The other inclusion is obvious. □

We show now that for the λ-conductor a similar theorem holds:

Theorem 2. *The λ-conductor I_λ of $S_1 \otimes S_2$ into $R_1 \otimes R_2$ is the tensor product of the λ-conductors $I_{1\lambda}$ of S_1 into R_1 and $I_{2\lambda}$ of S_2 into R_2.*

Proof. The λ-conductor I_λ is a subset of the classical conductor I , which is $I_1 \otimes I_2$. However I_1 is the direct sum $\mathbf{Z}b_1 \oplus I_{1\lambda}$ by theorem 1. and similarly for I_2. Thus any $x \in I_\lambda$ can uniquely be written as

$$x = x_0(b_1 \otimes b_1) \oplus (x_1 \otimes b_1) \oplus (b_1 \otimes x_2) \oplus y$$

with $x_0 \in \mathbf{Z}$, $x_1 \in I_{1\lambda}$, $x_2 \in I_{2\lambda}$, $y \in I_{1\lambda} \otimes I_{2\lambda}$. Since I_λ is an ideal of $S_1 \otimes S_2$, each of these four summands must be in I_λ.

Therefore we consider the intersection of I_λ with $b_1 \otimes I_{2\lambda}$. Suppose that a is an element of this intersection, say $a = b_1 \otimes x$ with $x \in I_{2\lambda}$. Then $\theta^p(a) \in I_\lambda$ too. We have

$$\theta^p(a) = p^{-1}(a^p - \psi^p a) = p^{-1}(p^{e(p-1)} b_1 \otimes x^p - (b_1 + \sum_{\tau \neq 1, \psi\tau = 1} b_\tau) \otimes \psi^p x)$$

and thus $p^{e(p-1)-1} b_1 \otimes x^p - p^{-1} b_1 \otimes \psi^p x$ should be in I_λ. Now the first term is a multiple of a and thus in I_λ. So the other term $p^{-1} b_1 \otimes \psi^p x$ is in the aforementioned intersection. Since ψ^p is an automorphism (of finite order) of R_2 this shows that the intersection is p-divisible. Since the intersection is a finitely generated abelian group this can only happen if it vanishes.

The same argument applies to the first and second summand of x. Thus $x = y \in I_{1\lambda} \otimes I_{2\lambda}$ and we have shown that $I_\lambda \subset I_{1\lambda} \otimes I_{2\lambda}$. The other inclusion is obvious. □

References

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