# Borel determinacy without the axiom of choice

# Borel determinacy without the axiom of choice

een wetenschappelijke proeve op het gebied van de
Wiskunde en Informatica

**Proefschrift**

ter verkrijging van de graad van doctor
aan de Katholieke Universiteit Nijmegen,
volgens besluit van het College van Decanen
in het openbaar te verdedigen
op maandag 28 juni 1993,
des namiddags te 3.30 uur precies

door

**Antonius Jacobus Cornelius Hurkens**

geboren op 29 april 1964
te Boxmeer

Universiteitsdrukkerij Nijmegen

Promotor: prof.dr. H.P. Barendregt
Co-promotor: dr. W.H.M. Veldman

CIP-GEGEVENS KONINKLIJKE BIBLIOTHEEK, DEN HAAG

Hurkens, Antonius Jacobus Cornelius

Borel determinacy without the axiom of choice / Antonius Jacobus Cornelius Hurkens. - [S.l. : s.n.] (Nijmegen : Universiteitsdrukkerij Nijmegen)
Proefschrift Nijmegen. - Met index, lit. opg. - Met samenvatting in het Nederlands.
ISBN 90-9006079-0
Trefw.: verzamelingsleer.

# Acknowledgements

First of all, I would like to thank Wim Veldman for teaching me set theory, for introducing me to the literature on infinite games, and for his guidance during my research as a student, assistant in training, and researcher in training.

The research leading to this thesis was carried out at the Catholic University of Nijmegen and financially supported by the Netherlands Organization for Scientific Research (NWO) via the Foundation Mathematical Centre (SMC).

Donald A. Martin's proof of Borel determinacy [1985] was the starting point of this research. On the recommendation of my thesis supervisor Henk Barendregt, Yiannis N. Moschovakis invited me to visit the Mathematics Department of the University of California at Los Angeles. During this stay in 1990, John R. Steel suggested a new proof of the main result of this thesis. I am grateful to Professors Moschovakis, Martin, and Steel for their stimulating co-operation.

I want to acknowledge financial support from NWO and SMC for this three months' stay. I also want to thank Carl and Linda Gordon very much for their kind hospitality and Albrecht Holderrieth and Susi Ratschka for their pleasant company in Los Angeles.

Tonny Hurkens, June 1993

# Contents

# Information for reading

The reader is supposed to have some familiarity with set theory, for example with ordinals. This is enough to understand most proofs, except in Section 2E, where constructible sets occur, and in Sections 7C and 7D, where forcing is mentioned. Sometimes when we need the fact that a certain statement is *unprovable* from the usual axioms of set theory, we will refer the reader to the literature.

It may be useful to know that some parts of this thesis may be studied more or less on their own. Occasionally, one will have to use the index and look up a definition.

The individual chapters are independent of the Introduction.

Sections 6A and 6B (a proof of quasi-Borel determinacy that uses AC) only depend on Chapter 1.

Chapter 3 (DC implies Borel pseudodeterminacy, using tactics) depends on Chapter 1, Sections 2A, 2B, and 2C.

Each of the Chapters 4 and 5 (coded Borel pseudodeterminacy, using preferential or generalized games, respectively) only depends on Chapter 1 and Sections 2A through 2D. After reading Chapter 5, one can read Sections 6A, 6B, and 6C (coded quasi-Borel pseudodeterminacy, using generalized games).

Sections 7A and 7B (coded quasi-Borel pseudodeterminacy, using strongly winning tactics) depend on Chapter 1, Sections 2A through 2D, Chapter 3, Sections 6A, 6B, and 6C up to (but not including) Lemma 6.11.

Section 7D (coded quasi-Borel pseudodeterminacy, using strongly winning pseudostrategies and forcing) depends on Chapters 1 and 2, Sections 6A, 6C up to (but not including) Lemma 6.11, Section 6D, and a small part of Section 7C, namely Lemma 7.17 and the first two paragraphs following Theorem 7.16.

# Introduction

## History of the problem of Borel determinacy

E. Zermelo [1913] demonstrated that in the game of chess, either both players have drawing strategies, or else one player has a winning strategy. This follows easily from the fact that chess is a *finite* game: The number of possible positions is finite. In 1935, S.M. Ulam introduced the following *infinite* game (see R.D. Mauldin [1981], page 113): Given a set $X$ of real numbers, two players produce a binary expansion of some $x \in [0, 1]$, by alternately writing down a 0 or a 1. At each moment the players can see the binary digits that have been written down so far. The first player tries to make sure that, 'in the end', $x \in X$, whereas the other player tries to ensure that $x \notin X$. This *binary game* is a modification of the Banach–Mazur game, the first infinite positional game of perfect information studied by mathematicians (see R. Telgárski [1987]). The problem proposed by Ulam is to decide for which $X$ one of the players has a method by which he will always win the binary game. If for example $X$ is countable, then it is easy to find a winning strategy for the second player. (He can even win without looking at the moves of his opponent.)

D. Gale and F.M. Stewart [1953] introduced the general notion of a *two-person infinite win-lose game of perfect information.* In such a game, two players, called I and II, make moves, one by one. At each moment, the players know the finite sequence of moves already played. It takes infinitely many moves before the game is over. Then one of the players wins, the other loses. The positions in such a game are the nodes of a tree: The root of the tree is the starting position and for each node of the tree, the immediate successor nodes correspond to the moves that can be made at that position. The infinite branches of the tree are the infinite sequences of moves that can be played in the game. Thus, an infinite game is given by specifying the tree, the set of nodes at which it is player I's turn to make the next move, and the *winning set* for player I: the set of infinite branches that result in a win for player I. If player I's winning set is, for example, open (in the usual topology), then the game itself is called open. If one of the players has a winning strategy, then the game is said to be *determined.* The following question remained unanswered for more than twenty years: Is each Borel game determined?

Gale and Stewart [1953], and independently J. Mycielski and A. Zięba [1955], proved that each *closed* game is determined. The idea of the proof is that if player II does not have a winning strategy, then player I can win simply by avoiding the positions from which player II has a winning strategy. Since the roles of I and II can be reversed, this also proves that each *open* game is determined.

Determinacy proofs for higher levels of the Borel hierarchy followed step by step, becoming more and more complex.

P. Wolfe [1955], and independently Mycielski, Świerczkowski, and Zięba [1956], proved determinacy of $G_\delta$ games. In such games, player I's winning set is a countable intersection of open sets $U_0, U_1, \ldots$. Wolfe's proof uses the determinacy of open games and can be sketched as follows. Suppose that player II has no winning strategy. While avoiding the positions from which player II has a winning strategy, player I first plays as if his winning set is $U_0$. As soon as a position is reached from which all infinite plays are in this open set, player I starts playing as if his winning set is $U_1$, and so on. In this way player I will win.

M. Davis [1964] proved determinacy of $G_{\delta\sigma}$ games (assuming that at each position, the number of possible moves is finite). Now player I's winning set is a countable union of $G_\delta$ sets $H_0, H_1, \ldots$. The basic step is to prove the following: Suppose player I has no winning strategy. Then player II can impose restrictions on his moves, such that the infinite sequence of moves will not be in $H_0$, and such that player I still does not have a winning strategy in this so-called II-imposed subgame of the original game. By repeating this basic step, we see that player II can avoid $H_1$ by imposing further restrictions on his moves, and so on.

J.B. Paris [1972] proved determinacy for the next level of the Borel hierarchy, by combining methods of Davis [1964] and Martin [1970]. The idea of the proof is as follows. In a $G_{\delta\sigma\delta}$ game, the winning set for I is a countable intersection of $G_{\delta\sigma}$ sets $A_0, A_1, \ldots$. Assuming that player II has no winning strategy, one proves that player II still has no winning strategy if player I is required to 'prove' that the sequence of moves will be in $A_0$, by making certain extra moves (elements of some large ordinal). Thus, instead of considering some *subgame* of the the original game, Paris constructs an auxiliary game on a *larger* tree. By repeating this construction for $A_1$, $A_2$, and so on, a winning strategy for player I in the original game is found.

Finally, D.A. Martin [1975] proved that all infinite Borel games of perfect information are determined. Later, Martin [1985] gave a simpler proof. The idea of both proofs is to reduce each Borel game to an open game (on a tree which may be very large). By the Gale–Stewart result, this auxiliary game is determined. The basic step, which is to be iterated transfinitely often, is as follows. For each closed set $A$ of infinite branches of some tree $T$ and each game on $T$, an 'equivalent' auxiliary game is constructed in which each player chooses a certain subtree of $T$ that imposes restrictions on his further moves. This is done in such a way that once the two extra moves have been played, it is clear whether the infinite sequence of original moves will be in $A$ or not.

## Determinacy and large cardinals

If one deletes the power set axiom from ZFC, Zermelo–Fraenkel set theory with the axiom of choice, then one cannot prove the existence of an uncountable set, but one can treat Borel sets of reals by means of codes (see Friedman [1981], page 210). The resulting theory is similar to second order arithmetic and Davis' proof of determinacy of binary $G_{\delta\sigma}$ games can be formalized in it. H. Friedman [1971] showed that determinacy of all binary $G_{\delta\sigma\delta\sigma}$ games is unprovable in second order arithmetic. Thus, in order to prove that the binary game is determined for certain sets $X$, one has to assume the existence of sets that are larger than $\omega$, the set of all natural numbers.

One can prove the determinacy of the binary game for Borel sets of finite Borel rank ($G$, $G_\delta$, $G_{\delta\sigma}$, etcetera), assuming the existence of $\omega$, the power set of $\omega$, the power set of the power set of $\omega$, and so on. Friedman [1971] showed that, in some precise sense, this assumption is necessary. He also proved that this assumption does not imply that each binary Borel game is determined. That is, one cannot prove Borel determinacy in ZFC without using the *axiom scheme of replacement*. This scheme states that if for each element $a$ of some set $A$ a set $x_a$ is given, then the collection $\{x_a : a \in A\}$ is a set. In fact, Martin's proof of Borel determinacy only uses this scheme for $A = \omega$. In other words, it is enough to be able to iterate the power set operation any countable transfinite number of times. According to Friedman [1981], page 213, Borel determinacy is "the first example of a mathematically interesting proposition from non-set-theoretic mathematics whose proof requires use of some set theoretic mathematics".

*Analytic* sets of reals can be defined as continuous images of Borel (or $G_\delta$) sets. The Borel sets are the analytic sets that have an analytic complement. Mycielski [1964] observed that under the axiom of constructibility (V=L), there is an analytic set $X$ for which the binary game is not determined. Consequently, the statement of analytic determinacy cannot be proved in ZFC (if ZFC is consistent). However, assuming the existence of a *measurable cardinal*, Martin [1970] proved analytic determinacy, by showing that every analytic binary game is equivalent to some open game on a much larger tree. In this auxiliary game, player II tries to 'prove' that he will win the original game, by choosing, as extra moves, elements of some measurable cardinal.

One can also consider *projective* sets: analytic sets, continuous images of complements of analytic sets, and so on. Many questions in descriptive set theory that are undecidable in ZFC, turn out to be settled by the assertion that each projective binary game is determined (PD). This assertion seems to be consistent with ZFC: Martin and Steel [1988, 1989] proved PD, assuming the existence of infinitely many so-called *Woodin cardinals*. This large cardinal hypothesis is essentially as weak as possible.

## Determinacy without the axiom of choice

Using the axiom of choice (AC), Gale and Stewart [1953], and independently Mycielski and Zięba [1955], proved that for some set $X$, the binary game is not determined. So the easy proof of determinacy of finite games cannot be extended to infinite games. On the other hand, the statement that each binary game is determined can also be considered as an interesting alternative to the axiom of choice: This so-called *axiom of determinacy* (AD) was introduced by Mycielski and Steinhaus [1962]. They conjectured its consistency with ZF, Zermelo–Fraenkel set theory, and thus the unprovability of AC in ZF (which was not known by then; see also H. Steinhaus [1965]). The statement that *each* infinite game is determined, is inconsistent since it implies both AD and AC.

AD has some very remarkable consequences. Mycielski and Świerczkowski [1964] proved that AD implies that each subset of the real interval $[0, 1]$ is Lebesgue measurable. R.M. Solovay showed that it also implies that $\omega_1$, the least uncountable ordinal, is measurable (see J.E. Fenstad [1971], page 53). Therefore AD implies that ZF is consistent. Note that AC does not imply that ZF is consistent (unless ZF is inconsistent), since AC holds in L, the smallest inner model of ZF. The problem of the consistency of AD with respect to ZF was finally settled by Martin and Steel [1988, 1989], using a result of W.H. Woodin [1988]. They showed that if there is a measurable cardinal larger than infinitely many Woodin cardinals, then AD holds in L($\mathbb{R}$), the smallest inner model of ZF that contains the set $\mathbb{R}$ of all real numbers. A.S. Kechris [1984] proved that if AD holds in L($\mathbb{R}$), then the principle of dependent choices (DC), a consequence of AC, also holds in this inner model. Thus, each of the incompatible axioms AD and AC has a weak form (PD and DC, respectively) that is compatible with the other if certain large cardinals exist.

AD is the first serious alternative to AC: It leads to an extremely fruitful theory. "Compared to Cantor's universe, the world of determinacy (or at least its part comprising of sets of reals) is remarkably structured" (T.J. Jech [1981], page 346). Before its introduction, some other alternatives to AC had been studied by A. Church [1927], who drew an analogy to the study of hyperbolic geometry as an alternative to the usual Euclidean geometry (see G.H. Moore [1982], page 249). Another parallel is the study of $p$-adic numbers versus real numbers.

Working in ZF instead of ZFC, one can prove less and thus assume more without getting contradictions. Consider the following propositions concerning the set of all reals:

**(i)** $\mathbb{R}$ can be wellordered;

**(ii)** each uncountable subset of $\mathbb{R}$ has a non-empty closed subset without isolated points;

**(iii)** $\mathbb{R}$ is a countable union of countable sets;

**(iv)** there is an infinite subset of $\mathbb{R}$ that has no countable subset.

One easily verifies that, in ZF, any two of these propositions are contradictory. But none of these is known to lead to a contradiction on its own. Proposition (i) is consistent with ZF since it follows from AC. The second proposition follows from AD; in fact, ZF together with (ii) and DC is consistent if and only if ZF together with the existence of an inaccessible cardinal is consistent (see Solovay [1970]). Both (iii) and (iv) contradict the so-called countable axiom of choice (CAC), a consequence of DC, but are consistent with ZF (see Jech [1973], page 142, and Halpern and Lévy [1971], respectively).

Using alternatives like (ii), (iii), and (iv), one can show that certain theorems of ZFC are unprovable in ZF. For example, (iii) implies that for some Borel set $X$, the binary game is not determined, whereas both AC and AD imply that each binary Borel game is determined. In fact, (iii) even implies that some binary $G_{\delta\sigma}$ game is not determined. On the other hand, as observed by Mycielski [1964], page 213, we can prove in ZF that for each double sequence $U_{m,n}$ $(m, n \in \omega)$ of open sets of reals, the binary game with $X = \bigcup_{m\in\omega} \bigcap_{n\in\omega} U_{m,n}$ is determined. Thus one cannot prove in ZF that each $G_{\delta\sigma}$ set $X$ of real numbers can be represented this way (whereas this is a simple consequence of CAC).

## The subject of this thesis

Using AC, Martin [1975, 1985] proved that Borel games on arbitrarily large trees are determined. The use of AC is necessary, since even the determinacy of all *open* games (on arbitrarily large trees) implies AC. By a sophisticated set-theoretical argument, Martin showed that the full axiom of choice is not needed in order to prove that each Borel game on a *countable* tree is determined; CAC suffices in this special case.

But what about Borel games on uncountable trees if AC is not assumed? That is the subject of this thesis.

We do not restrict our attention to games on countable trees, since the essence of Martin's proof is to reduce Borel games to games of *lower* Borel complexity but on *larger* trees.

By avoiding the use of AC, we refrain from choosing between the alternatives AC and AD. We want to analyse the role of AC in the proof of Borel determinacy by studying reformulations of Martin's result that are provable without using the full axiom of choice. In the absence of AC, we must choose definitions carefully: Subtle distinctions may be important. For instance, the difference between *strategies* and so-called *pseudostrategies* is irrelevant if one assumes AC (and also if one only considers games on countable trees).

Let us first consider the situation for open games. Gale and Stewart gave a proof of open determinacy that does not use AC in case of *binary* games, and only uses CAC for games on *countable* trees. In fact, a slightly more complicated argument that uses ordinals shows that the determinacy of open games on countable trees is provable in ZF. A similar argument shows that open games on arbitrarily large trees are *pseudodetermined* (or *weakly determined*, see Moschovakis [1980], page 446): One of the players has a winning pseudostrategy.

In this thesis we prove, without using the full axiom of choice, that each Borel game is pseudodetermined. This cannot be done by simply replacing strategies by pseudostrategies in Martin's proof. In fact, we present several proofs, each based on Martin's idea of reducing Borel games to open games on much larger trees. We elaborate this idea in different ways, introducing 'tactics', 'preference relations', and 'generalized games'.

## Summary

We now give an overview of the contents of this thesis.

In Chapter 1, we give definitions of concepts like *game* and *strategy*. In order to simplify the basic step in Martin's proof of Borel determinacy, we allow trees to have terminal nodes. At a terminal node, no further move is possible and the player whose turn it is to make a move loses the game. Since Gale and Stewart only consider games on trees without terminal nodes, we adapt their proof of open determinacy and show that AC implies that each *basic open* game is determined. Then we describe our simplified basic step: the reduction of each open game to a basic open game on a larger tree. (Here the power set axiom is used: The larger tree has about the size of the power set of the original tree.) Just as in Martin's proof, the next step is the construction of the 'limit' $G$ of an infinite sequence of games $G_0, G_1, \ldots$, such that for each natural number $n$, the first $n$ moves in $G$ are the same as those in $G_m$ for all large $m$. (This is where the axiom scheme of replacement is used.) Finally, we put these steps together and prove, using AC, that each Borel game can be reduced to some basic open game and is therefore determined.

In Chapter 2, we see that we cannot prove Borel determinacy in ZF. We cannot even prove that each basic open game is determined: This is because a strategy must prescribe the moves of a player completely. But each basic open game is *pseudodetermined*. A pseudostrategy presents a player, when he has to move, with a set of acceptable moves from which the player has to choose one. DC implies that there is no game in which both players have a winning pseudostrategy, and AC implies that each pseudodetermined game is determined. In fact, the reverse implications hold as well.

Each binary game (or other game on a countable tree) is determined if and only if it is pseudodetermined. But in general, we cannot prove that a Borel game on a countable tree is determined, unless we have a code that expresses how player I's winning set is constructed from basic open sets by means of countable unions and countable intersections. CAC implies that each Borel set has such a code. In Section 2E, we show that coded Borel games on countable trees are determined. We use some inner model of ZF in which AC holds. In such a model, each Borel game is determined.

In the next three chapters, we give some proofs of Borel pseudodeterminacy that only use some weak form of AC. These three proofs are independent of each other.

In Chapter 3, we prove Borel pseudodeterminacy, using DC. By taking a second look at the auxiliary basic open game in Chapter 1, we show that in each Borel game, either player II has a winning pseudostrategy or player I has a winning *tactic*. A tactic is a kind of pseudostrategy with auxiliary moves. Reversing the roles of the players, we see that each Borel game is pseudodetermined, since, by DC, no game exists in which both players have a winning tactic.

In Chapter 4, we consider *preferential games*: games equipped with some relation $R$ between positions, expressing that certain positions are 'easier' for player I (and more 'difficult' for II) than other positions. Using such a preference relation $R$, we define $R$-pseudostrategies: pseudostrategies that satisfy some 'reasonable' condition. In contrast to the condition of being a strategy, this condition is so weak that we can prove in ZF that if such a preferential game is basic open, then one of the players has a winning $R$-pseudostrategy. At the same time, it is so strong that we can reduce each open preferential game to a basic open one. This reduction is much more complicated than the corresponding one in the first chapter. We then *define* for each Borel preferential game an auxiliary basic open preferential game, using a Borel code. In this way, we see that CAC implies that each Borel game is pseudodetermined.

We prove this theorem again in Chapter 5, but now by generalizing the concept of a game. In a game, there are two types of positions, depending on whose turn it is to make a move, and these types play an important role in the definition of 'pseudostrategy'. In a *generalized game*, there may be other types of positions as well, and we use these to expand the definitions of 'pseudostrategy for I' and 'pseudostrategy for II'. Using the duality of these definitions, we prove that each basic open generalized game is pseudodetermined (and we do not mind that *both* players may have a winning pseudostrategy). Using generalized games, we can further simplify the basic step in Martin's proof: In the auxiliary generalized game, there is only one extra move and the roles of both players are essentially the same. Iterating this step as before, we see that each coded Borel (generalized) game is pseudodetermined.

Martin [1990] extended his proof of Borel determinacy to *quasi-Borel* games. He showed that quasi-Borel sets are the same as the so-called $\Delta^1_1$ sets, and if the underlying tree is countable, then they coincide with the Borel sets. In Chapter 6, we examine the role of the axiom of choice in all this. We prove (in ZF) that each coded quasi-Borel game is pseudodetermined. We also show that DC is not strong enough to prove that each quasi-Borel set has a code. DC implies that coded quasi-Borel sets are the same as $\Delta^1_1$ sets. We can even avoid DC by considering *absolutely* $\Delta^1_1$ sets. This last characterization of coded quasi-Borel sets is, in a certain sense, absolute for transitive class models of ZF containing all ordinals.

In the last chapter, *strongly winning* pseudostrategies are introduced, but only in coded quasi-Borel games. We will see that a pseudostrategy is strongly winning if and only if it is winning in every generic extension of the universe. Each strongly winning pseudostrategy is winning; the converse holds if the game is played on a countable tree or if DC holds. The following version of Borel determinacy is the strongest one that we were able to prove in ZF: In each coded quasi-Borel game, exactly one of the players has a strongly winning pseudostrategy. We give two proofs. The first one is like the proof in Chapter 3, but now we can exclude the possibility that both players have a *strongly* winning tactic, without using DC. The other proof is an elaboration of an idea of J.R. Steel and uses forcing. For each coded quasi-Borel game on some tree, we find a generic extension $\mathrm{V}[G]$ of the universe in which that tree is at most countable. So the corresponding game in $\mathrm{V}[G]$ is coded Borel and therefore determined. We show that the player who has a winning strategy in $\mathrm{V}[G]$, also has a strongly winning pseudostrategy in the 'real' universe V.

# 1 A proof of Borel determinacy using AC

In this chapter we prove, using the axiom of choice, that every Borel game is determined. This result can de stated as follows: For every set $M$ and every Borel set $X$ of infinite sequences of elements of $M$, one of the players has a winning strategy in the following game. Player I starts by choosing $a_0 \in M$; then player II chooses $a_1 \in M$; player I chooses $a_2 \in M$ and so on. In the end, player I wins if and only if the infinite sequence $\langle a_0, a_1, a_2, \ldots \rangle$ of moves belongs to $X$.

In order to give a precise formulation of this theorem, we start by giving set theoretical definitions concerning games and Borel sets.

We will work in ZF, Zermelo–Fraenkel set theory without the axiom of choice, so we must take care: Definitions that are equivalent if one assumes the axiom of choice, need not be equivalent in ZF.

The **axiom of choice** (AC) is the statement that for every collection $\mathcal{C}$ of non-empty sets, there exists a function $f$ on $\mathcal{C}$ such that for all $A \in \mathcal{C}$, $f(A) \in A$. Such a function $f$ is called a **choice function**.

## 1A Infinite games

We consider infinite two-person win-lose games of perfect information. These games can be described as follows: Two players alternately choose a member of a given non-empty set, knowing the finite sequence of moves already played. After infinitely many moves, exactly one of the players wins.

It turns out to be more convenient to consider games in which:

- the set of possible moves may depend on the *position*, i.e. the finite sequence of moves already played;
- there may be positions where no move is possible; the player whose turn it is to play in such a *terminal* position, loses the game;
- the players need not play alternately; it depends on the position who has to make the next move.

These games are slightly more general than the games described earlier.

The positions in a game are the nodes of a *tree*. The infinite sequences of moves are the *infinite branches* of that tree.

**1.1** DEFINITION A **finite sequence** is a function $\sigma$ whose domain is a natural number $n = \{0, 1, \ldots, n-1\}$, which is called the **length** of $\sigma$.

A **tree** is a non-empty set $T$ of finite sequences such that for all finite sequences $\sigma$ and $\tau$, if $\sigma \subseteq \tau$ and $\tau \in T$ then $\sigma \in T$.

An **infinite sequence** is a function whose domain is $\omega = \{0, 1, \ldots\}$, the set of all natural numbers.

Let $T$ be a tree. An **infinite branch** of $T$ is an infinite sequence $x$ such that for every finite sequence $\sigma$, if $\sigma \subseteq x$ then $\sigma \in T$.
The set of all infinite branches of $T$ is denoted by $[T]$.

In other words, a tree is a set $T$ of finite sequences such that $\langle\rangle \in T$ and for all $\langle a_0, a_1, \ldots, a_n\rangle \in T$, $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in T$. An infinite branch of $T$ is an infinite sequence $\langle a_0, a_1, \ldots\rangle$ such that for all $n \in \omega$, $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in T$. Here $\langle\rangle = \emptyset$, the unique sequence of length 0. The expression $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ denotes the unique sequence $\sigma$ of length $n$ such that for all $i < n$, $\sigma(i) = a_i$. The expression $\langle a_0, a_1, \ldots\rangle$ denotes the unique infinite sequence $x$ such that for all $n \in \omega$, $x(n) = a_n$.

Note that for each infinite sequence $x$ and $n \in \omega$, the restriction $x|n$ of the function $x$ to the set $n$ is the unique finite sequence $\sigma$ of length $n$ such that $\sigma \subseteq x$.

The set of all functions from a set $X$ to a set $Y$ is denoted by ${}^{X}Y$.

**1.2** Example For every set $M$, the set ${}^{<\omega}M := \bigcup_{n\in\omega} {}^{n}M$ of all finite sequences of elements of $M$ is a tree and $[{}^{<\omega}M] = {}^{\omega}M$, the set of all infinite sequences of elements of $M$.

We give a formalization of the notion of a *game*.

**1.3** Definition A **game** $G$ is a triple $(T, P, X)$ where $T$ is a tree, $P \subseteq T$, and $X \subseteq [T]$.

We say that $G$ is a **game on** $T$.

We say that **it is player I's turn at position** $\sigma$ (or **player I has to make a move at position** $\sigma$) if $\sigma \in P$ and **it is player II's turn at position** $\sigma$ if $\sigma \in T \setminus P$.

We say that **player I wins play** $x$ if $x \in X$ and **player II wins play** $x$ if $x \in [T] \setminus X$. $X$ is called the **winning set for player I** and $[T] \setminus X$ is the **winning set for player II**.

Informally, such a game $G$ is played by two players, I and II, as follows. The game starts at the position $\langle\rangle \in T$. After a finite number of moves, at a position $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in T$, the player whose turn it is, makes a move $a_n$ such that $\langle a_0, a_1, \ldots, a_n\rangle \in T$; if he cannot make such a move, then he loses and the other player wins. If infinitely many moves are played, then one of the players wins the play $\langle a_0, a_1, \ldots\rangle$ and the other loses.

Often one only considers games $(T, P, X)$ for which $P = \{\sigma \in T : \text{length}(\sigma) \text{ is even}\}$. In such a 'standard game', the players play turn by turn, player I starting.

**1.4** Definition Let $T$ be a tree. Let $\sigma \in T$ and put $n = \text{length}(\sigma)$, so $\sigma = \langle \sigma(0), \sigma(1), \dots, \sigma(n-1) \rangle$.

A **move in** $T$ **at** $\sigma$ is a set $a$ such that $\sigma^\frown\langle a \rangle \in T$, where $\sigma^\frown\langle a \rangle = \sigma \cup \{(n,a)\} = \langle \sigma(0), \sigma(1), \dots, \sigma(n-1), a \rangle$, the concatenation of the finite sequences $\sigma$ and $\langle a \rangle$.

If there is no move in $T$ at $\sigma$ then $\sigma$ is called a **terminal node of** $T$.

The **subtree of** $T$ **via** $\sigma$ is the set $\{\tau \in T : \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$. We denote this tree by $T^{\text{via}\,\sigma}$.

Note that for each $\tau \in T^{\text{via}\,\sigma}$, if $\text{length}(\tau) < n$ then there is exactly one move in $T^{\text{via}\,\sigma}$ at $\tau$, and if $\text{length}(\tau) \geq n$ then the moves in $T^{\text{via}\,\sigma}$ at $\tau$ are the same as the moves in $T$ at $\tau$.

**1.5** Example Let $M$ be a set and define $T$ as the tree ${}^{<\omega}M$. Let $\sigma \in T$. Then the set of all moves in $T$ at $\sigma$ is $M$.

A *strategy* for some player in a game is a way of playing which completely prescribes the moves of that player. It is called a *winning* strategy if the player wins every play when he follows that strategy. If one of the players has a winning strategy, then the game is called *determined*.

A strategy for a player is often identified with a function that assigns a move to each position where the player has to make a move, but we will identify a strategy with the set of all positions that may occur when the player follows that strategy.

**1.6** Definition Let $G = (T, P, X)$ be a game.

A **strategy for player I** in $G$ is a tree $S \subseteq T$ such that for every $\sigma \in S \cap P$ there is exactly one move in $S$ at $\sigma$ and such that for every $\sigma \in S \setminus P$, every move in $T$ at $\sigma$ is a move in $S$ at $\sigma$.

A **winning strategy for player I** in $G$ is a strategy $S$ for player I in $G$ such that $[S] \subseteq X$.

A (**winning**) **strategy for player II** in $G$ is a (winning) strategy for player I in the game $(T, T \setminus P, [T] \setminus X)$.

The game $G$ is **determined** if there is a winning strategy for player I or a winning strategy for player II in $G$.

Note that if $S$ is a (winning) strategy for some player in $G$ and $\sigma \in S$, then $S^{\text{via}\,\sigma}$ is a (winning) strategy for that player in the game $(T^{\text{via}\,\sigma}, P \cap T^{\text{via}\,\sigma}, X \cap [T^{\text{via}\,\sigma}])$. In this game, the players make some obligatory moves until position $\sigma$ is reached. Then they continue as if they are playing game $G$.

Suppose that $S_{\text{I}}$ and $S_{\text{II}}$ are strategies for player I and player II in some game. Let $D$ be the tree $S_{\text{I}} \cap S_{\text{II}}$. Then for every $\sigma \in D$ there is exactly one move in $D$ at $\sigma$. So $D$ has exactly one infinite branch $s$ and $s \in [S_{\text{I}}] \cap [S_{\text{II}}]$. This implies that there is no game in which *both* players have a winning strategy.

**1.7** EXAMPLE Consider the following game $G$: Player I starts by choosing a non-empty set $A$ of infinite sequences of 0's and 1's. Then player II chooses elements $a_0, a_1, a_2, \dots$ of $\{0,1\}$, one by one. In the end, player II wins the game if and only if $\langle a_0, a_1, \dots \rangle \in A$.

In other words, if $\mathcal{C} = \{A \subseteq {}^{\omega}2 : A \neq \emptyset\}$ then $G = (T, P, X)$ where

- $T = \{\langle\rangle\} \cup \{\langle A \rangle^\frown \sigma : A \in \mathcal{C} \text{ and } \sigma \in {}^{<\omega}2\}$;
- $P = \{\langle\rangle\}$;
- $X = \{\langle A \rangle^\frown x : A \in \mathcal{C} \text{ and } x \in [T] \setminus A\}$.

It is clear that player I does not have a winning strategy in this game: If $S$ is a strategy for player I and $A$ is the unique move in $S$ at position $\langle\rangle$, then there is some $x \in A$; now player I loses the play $\langle A \rangle^\frown x \in [S]$.

It is also clear that every strategy $S$ for player II in this game corresponds to a function $f : \mathcal{C} \longrightarrow {}^{\omega}2$ and that $S$ is winning if and only if $f$ is a choice function.

It follows from the axiom of choice that there is such a choice function. So, AC implies that the game $G$ is determined.

Not every game is determined. This is provable in ZF as follows:

CASE 1: The set ${}^{\omega}2$ can be wellordered. Using this, one can construct an undetermined game on the countable tree ${}^{<\omega}2$ (see Gale and Stewart [1953]). The idea is to build winning sets in such a way that for every strategy for a player at least one of its infinite branches is put into the winning set of the other player.

CASE 2: The set ${}^{\omega}2$ cannot be wellordered. Then there is no choice function on the set of all non-empty subsets of ${}^{\omega}2$, so the game $G$ in Example 1.7 is undetermined.

So the question arises which games are (provably) determined.

## 1B Borel sets

For every tree $T$ we give $[T]$ the usual topology by letting basic open sets be the sets of the form $\{x \in [T] : x|n \in \Delta\}$ for some $n \in \omega$ and some set $\Delta$ of finite sequences of length $n$. Another basis for this (metrizable) topology consists of the sets $[T^{\text{via}\,\sigma}]$ for $\sigma \in T$. If $T$ is of the form ${}^{<\omega}M$, then this topology on ${}^{\omega}M$ is the product topology, taking $M$ discrete.

**1.8** DEFINITION Let $T$ be a tree.

We call $\emptyset$ and $[T]$ the **trivial** subsets of $[T]$.

A **basic open** subset of $[T]$ is a set $X \subseteq [T]$ such that for some $n \in \omega$, for all $x \in X$, for all $y \in [T]$, if $x|n = y|n$ then $y \in X$.

An **open** subset of $[T]$ is a countable union of basic open subsets of $[T]$.

A **closed** subset of $[T]$ is a countable intersection of basic open subsets of $[T]$.

A **Borel** subset of $[T]$ is a set that belongs to every collection of subsets of $[T]$ that contains every basic open subset of $[T]$ and is closed under countable union and countable intersection.

One easily verifies that for every tree $T$ and every $X \subseteq [T]$:

- if $X$ is trivial, then $X$ is basic open;
- if $X$ is basic open, then $X$ is open and closed;
- if $X$ is open or closed, then $X$ is Borel;
- if $X = \bigcup_{n\in\omega} X_n$ or $X = \bigcap_{n\in\omega} X_n$ for some infinite sequence $\langle X_0, X_1, \ldots\rangle$ of Borel sets, then $X$ is Borel;
- $X$ is trivial, basic open, open or Borel if and only if $[T] \setminus X$ is trivial, basic open, closed or Borel, respectively.

**1.9** Remark The open subsets of $[T]$ are precisely the sets of the form $\{x \in [T] : \text{for some } \tau \in \Delta,\ \tau \subseteq x\}$ for some $\Delta \subseteq T$.

The closed subsets of $[T]$ are precisely the sets of the form $[S]$ for some tree $S \subseteq T$.

We can stratify the collection of all Borel subsets of $[T]$ into a hierarchy as follows:
Let $\mathcal{B}_0$ be the set of all basic open subsets of $[T]$ and define by transfinite induction for every ordinal $\alpha > 0$, $\mathcal{B}_\alpha = \{X \subseteq [T] : X$ is is a countable union or intersection of elements of $\bigcup_{\beta<\alpha} \mathcal{B}_\beta\}$.
Then for every $X$: $X$ is a Borel subset of $[T]$ if and only if $X \in \mathcal{B}_\alpha$ for some ordinal $\alpha$.

The **Borel rank** of a Borel subset $X$ of $[T]$ is the least ordinal $\alpha$ such that $X \in \mathcal{B}_\alpha$. So the basic open sets have Borel rank 0.

The (countable) axiom of choice implies that every Borel set has a *finite or countable* Borel rank.

## 1C Determinacy of basic open games

**1.10** Definition A game $(T, P, X)$ is called **trivial**, **basic open**, **open**, **closed** or **Borel** if $X$ is a trivial, basic open, open, closed or Borel subset of $[T]$, respectively.

D. Gale and F.M. Stewart [1953] proved, using the axiom of choice, that every open game on a tree without terminal nodes is determined. The idea of the proof is the following:

In every open game on a tree without terminal nodes, player I only wins a play if some position is reached such that from that position on, he wins the game no matter how he plays. Now suppose that player I has no winning strategy. Then player II can play in such a way that no position is reached from which player I has a winning strategy. Playing in this way, player II wins every play.

Using the same idea, we prove the following theorem:

**1.11** THEOREM AC implies that every trivial game is determined.

PROOF In a trivial game, the winning set for one of the players is empty. By symmetry, we only have to consider games for which player I's winning set is empty. In such a game, player I only wins if a terminal position is reached at which it is player II's turn. If infinitely many moves are made, then player II wins the play.

So suppose AC holds and let $G$ be a game of the form $(T, P, \emptyset)$. For every $\tau \in T$, let $G_\tau$ be the game $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, \emptyset)$. Put $B = \{\tau \in T : \text{player I has a winning strategy in the game } G_\tau\}$.

CASE 1: $\langle\rangle \in B$.

Then player I has a winning strategy in $G$ since $G_{\langle\rangle} = G$.

CASE 2: $\langle\rangle \notin B$.

Let $\tau \in T$ and suppose that $\tau \notin B$, so player I has no winning strategy in the game $G_\tau$.

If $\tau \in P$, then for every move $a$ in $T$ at $\tau$, $\tau^\frown\langle a\rangle \notin B$, since otherwise for some move $a$ in $T$ at $\tau$, player I has a winning strategy $S$ in $G_{\tau^\frown\langle a\rangle}$, but then $S$ is also a winning strategy for player I in $G_\tau$.

If $\tau \notin P$, then for some move $a$ in $T$ at $\tau$, $\tau^\frown\langle a\rangle \notin B$, since otherwise for every move $a$ in $T$ at $\tau$, player I has a winning strategy in $G_{\tau^\frown\langle a\rangle}$. Then, by AC, there is a function that assigns to every move $a$ in $T$ at $\tau$ a winning strategy $S_a$ for player I in $G_{\tau^\frown\langle a\rangle}$. But then the tree $\{\sigma \in T : \sigma \subseteq \tau$ or for some move $a$ in $T$ at $\tau$, $\sigma \in S_a\}$ is a winning strategy for player I in $G_\tau$.

By AC, there is a function that assigns to every $\tau \in (T \setminus P) \setminus B$ a move $a_\tau$ in $T$ at $\tau$ such that $\tau^\frown\langle a_\tau\rangle \notin B$. Let $S$ be the tree $\{\sigma \in T : \text{for all } n < \text{length}(\sigma),\ \sigma|n \notin B \text{ and if } \sigma|n \notin P \text{ then } \sigma(n) = a_{\sigma|n}\}$. Then $S \subseteq T \setminus B$ and $S$ is a strategy for player II in $G$. Since player II wins every play in $G$, strategy $S$ is winning.

So in both cases $G$ is determined.

**1.12** THEOREM AC implies that every basic open game is determined.

PROOF Suppose AC holds and let $G = (T, P, X)$ be a basic open game. For every $\tau \in T$, let $G_\tau$ be the game $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, X \cap [T^{\text{via}\,\tau}])$.

CLAIM Let $\tau \in T$ and suppose that for every move $a$ in $T$ at $\tau$, the game $G_{\tau \frown \langle a \rangle}$ is determined. Then $G_\tau$ is determined.

PROOF OF CLAIM Suppose that $\tau \in P$. (The other case is similar: One only has to interchange the roles of I and II.)

If for some move $a$ in $T$ at $\tau$, player I has a winning strategy $S$ in $G_{\tau \frown \langle a \rangle}$, then $S$ is a winning strategy for player I in $G_\tau$ too.

If not, then for every move $a$ in $T$ at $\tau$, player II has a winning strategy in $G_{\tau \frown \langle a \rangle}$. By AC, there is a function that assigns to every move $a$ in $T$ at $\tau$ a winning strategy $S_a$ for player II in $G_{\tau \frown \langle a \rangle}$. But then the tree $\{\sigma \in T : \sigma \subseteq \tau$ or for some move $a$ in $T$ at $\tau$, $\sigma \in S_a\}$ is a winning strategy for player II in $G_\tau$.

Since $X$ is a basic open subset of $[T]$, there is an $n \in \omega$ such that for all $x \in X$, for all $y \in [T]$, if $x|n = y|n$ then $y \in X$. Thus for every $\tau \in T$ of length $n$, either $X \cap [T^{\text{via}\,\tau}] = \emptyset$ or $[T^{\text{via}\,\tau}] \subseteq X$. In both cases the game $G_\tau$ is trivial and thus, by Theorem 1.11 and AC, determined.

Since $G = G_{\langle\rangle}$ we may conclude, using the claim repeatedly, that $G$ is determined.

## 1D Reducing open games to basic open games

**1.13** THEOREM AC implies that every open game is determined.

PROOF Suppose that AC holds and let $G = (T, P, X)$ be an open game. Choose $\Delta \subseteq T$ such that $X = \{x \in [T] :$ for some $\tau \in \Delta$, $\tau \subseteq x\}$. For every $\tau \in \Delta$, let $G_\tau$ be the trivial game $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, [T^{\text{via}\,\tau}])$. By Theorem 1.11 and AC, $G_\tau$ is determined.

Let $D$ be the tree $\{\sigma \in T :$ for all $n < \text{length}(\sigma)$, $\sigma|n \notin \Delta\}$. So $[D] = [T] \setminus X$. Define $Q = (P \cap D \setminus \Delta) \cup \{\tau \in D \cap \Delta :$ player II has a winning strategy in $G_\tau\}$. Let $H$ be the trivial game $(D, Q, \emptyset)$. This game is played like $G$ until, if ever, a position $\tau \in \Delta$ is reached. Then the game ends and the player who has a winning strategy in $G_\tau$, wins. If infinitely many moves are made, then player II wins the play.

By Theorem 1.11 and AC, $H$ is determined. Thus some player $W$ has a winning strategy $S$ in $H$. For every $\tau \in S \cap \Delta$, $\tau$ is a terminal node of $S$, so player $W$ has a winning strategy in $G_\tau$.

By AC, there is a function that assigns to every $\tau \in S \cap \Delta$ a winning strategy $S_\tau$ for $W$ in $G_\tau$. Now one easily verifies that $S \cup \bigcup_{\tau \in S \cap \Delta} S_\tau$ is a winning strategy for player $W$ in the open game $G$. This strategy may be described as follows: $W$ follows strategy $S$ until, if ever, a position $\tau \in \Delta$ is reached. Then player $W$ switches to strategy $S_\tau$.

Thus $G$ is determined.

Since every basic open game is open, this gives another proof of Theorem 1.12.

D.A. Martin [1975,1985] proved, using the axiom of choice, that every Borel game on a tree without terminal nodes is determined. As a basic step in his proof he constructs, for every open game, a basic open game such that every (winning) strategy in that game can be translated into a (winning) strategy in the original open game. The proof itself is a transfinite iteration of this basic step.

Y.N. Moschovakis [1980], page 358, simplified Martin's original proof by using trees with terminal nodes, removing the necessity for some auxiliary games used in the basic step. It was not known whether this idea could be mixed with Martin's new proof (see Martin [1985], page 307). We now present such a simplification of the basic step. We will use this in the proof of Lemma 1.20.

Let $G_0 = (T_0, P_0, X_0)$ be an open game and let $k \in \omega$. We will construct a basic open game $G_1$ such that the first $k$ moves in these two games are the same, and such that each (winning) strategy for some player in $G_1$ corresponds to some (winning) strategy for the same player in $G_0$. We assume the axiom of choice. So, by Theorem 1.12, the basic open game $G_1$ is determined. Therefore this construction will give another proof of Theorem 1.13.

Let $G_1$ be the game that is played as follows:

The first $k$ moves are the same as in the game $G_0$.

At a position $\sigma$ of length $k$, player I chooses a subset $A$ of $\Delta_\sigma$, where $\Delta_\sigma = \{\tau \in T_0 : \sigma \subseteq \tau \text{ and for all } x \in [T_0], \text{ if } \tau \subseteq x \text{ then } x \in X_0\}$. This extra move can be interpreted as follows: Player I proposes to play on, with the restriction that player II will give up as soon as a position $\tau \in A$ is reached and that player I will give up as soon as another position in $\Delta_\sigma$ is reached.

Then player II chooses a member of $A \cup \{1\}$. Note that $1 \notin A$. The move 1 is interpreted as acceptance of the proposal; a move $\tau \in A$ means that player II does not want to give up at position $\tau$.

If player II has chosen 1, then the players play on as if they are playing the game $G_0$ at position $\sigma$, until (if ever) a position $\sigma^\frown\langle A, 1\rangle^\frown\rho$ is reached such that the corresponding position $\sigma^\frown\rho$ in game $G_0$ is an element of $\Delta_\sigma$. Then the game ends. If $\sigma^\frown\rho \in A$ then player I wins and if $\sigma^\frown\rho \in \Delta_\sigma \setminus A$ then player II wins.

If player II has chosen a member $\tau$ of $A$, say of length $n$, then the next $n - k$ moves are $\tau(k), \tau(k+1), \ldots, \tau(n-1)$. After these obligatory moves, the players play on as if they are playing the game $G_0$ at position $\tau$.

If infinitely many moves are made, the play is won by the same player that wins the corresponding play in $G_0$, i.e. the play without the two extra moves.

In other words, $G_1$ is the game $(T_1, P_1, X_1)$, where:

- $T_1$ is the tree $\{\sigma \in T_0 : \text{length}(\sigma) \leq k\} \cup \{\sigma^\frown\langle A\rangle : \sigma \in T_0 \text{ and } \text{length}(\sigma) = k \text{ and } A \subseteq \Delta_\sigma\} \cup \{\sigma^\frown\langle A, 1\rangle^\frown\rho : \sigma \in T_0 \text{ and } \text{length}(\sigma) = k \text{ and } A \subseteq \Delta_\sigma \text{ and } \sigma^\frown\rho \in T_0 \text{ and for all } n < \text{length}(\rho),\ \sigma^\frown(\rho|n) \notin \Delta_\sigma\} \cup \{\sigma^\frown\langle A, \tau\rangle^\frown\rho : \sigma \in T_0 \text{ and } \text{length}(\sigma) = k \text{ and } A \subseteq \Delta_\sigma \text{ and } \tau \in A \text{ and } \sigma^\frown\rho \in {T_0}^{\text{via}\,\tau}\}$;
- $P_1$ is the set $\{\sigma \in T_1 : (\text{length}(\sigma) < k \text{ and } \sigma \in P_0) \text{ or } \text{length}(\sigma) = k\} \cup \{\sigma^\frown\langle A, 1\rangle^\frown\rho \in T_1 : \text{length}(\sigma) = k \text{ and } \sigma^\frown\rho \in (P_0 \setminus \Delta_\sigma) \cup (\Delta_\sigma \setminus A)\} \cup \{\sigma^\frown\langle A, \tau\rangle^\frown\rho \in T_1 : \text{length}(\sigma) = k \text{ and } \tau \in A \text{ and } \sigma^\frown\rho \in P_0\}$;
- $X_1$ is the inverse image $p^{-1}X_0$ of $X_0$ under the function $p$ that assigns to each infinite branch $x_1$ of $T_1$ the infinite branch $x_0$ of $T_0$ that is defined by $x_0(n) = \begin{cases} x_1(n) & \text{if } n < k, \\ x_1(n+2) & \text{if } n \geq k. \end{cases}$

Let $x \in [T_1]$. Put $\sigma = x|k$ and $A = x(k)$.

If $x(k+1) = 1$ then for some infinite sequence $r$, $x = \sigma^\frown\langle A, 1\rangle^\frown r$ and there is no finite sequence $\rho \subseteq r$ such that $\sigma^\frown\rho \in \Delta_\sigma$. Thus $p(x) = \sigma^\frown r \in [T_0] \setminus X_0$.

If $x(k+1)$ is a member $\tau$ of $A$, then for some infinite sequence $r$, $x = \sigma^\frown\langle A, \tau\rangle^\frown r$ and $\tau \subseteq \sigma^\frown r$. Thus, since $\tau \in \Delta_\sigma$, $p(x) = \sigma^\frown r \in X_0$.

So $X_1 = \{x \in [T_1] : x(k+1) \neq 1\}$ and thus the game $G_1$ is basic open.

For every strategy $S$ for player I in $G_1$, we describe a strategy $\phi_{\text{I}}(S)$ for player I in $G_0$ as follows:

Player I follows strategy $S$ until a position $\sigma$ of length $k$ is reached. Let $A$ be the unique move in $S$ at $\sigma$. Of course player I does not actually play this extra move $A$, but he proceeds by following strategy $S$ as if player II has played the extra move 1, until, if ever, a position $\tau \in \Delta_\sigma$ is reached. In $G_1$, this corresponds to a terminal position $\sigma^\frown\langle A, 1\rangle^\frown\rho \in S$, where $\tau = \sigma^\frown\rho$. Since $S$ is a strategy for player I, $\tau \in A$. Now player I proceeds by following strategy $S$ as if player II had played the extra move $\tau$ at position $\sigma^\frown\langle A\rangle$ and as if the moves $\rho(0), \rho(1), \ldots$ were obligatory.

In other words, $\phi_{\text{I}}(S) = \{\sigma \in S : \text{length}(\sigma) \leq k\} \cup \{\sigma^\frown\rho : \text{length}(\sigma) = k \text{ and for some } A,\ \sigma^\frown\langle A, 1\rangle^\frown\rho \in S\} \cup \{\sigma^\frown\rho : \text{length}(\sigma) = k \text{ and for some } A, \rho' \text{ and } \tau,\ \sigma^\frown\langle A, 1\rangle^\frown\rho' \in S \text{ and } \tau = \sigma^\frown\rho' \text{ and } \sigma^\frown\langle A, \tau\rangle^\frown\rho \in S\}$.

Let $x_0$ be an infinite branch of $\phi_{\text{I}}(S)$. Let $\sigma = x_0|k$ and let $A$ be the unique move in $S$ at $\sigma$.

If $x_0 \in X_0$, then there is minimal $\tau \in \Delta_\sigma$ such that $\tau \subseteq x_0$, so for every natural number $n \geq k$, $\sigma^\frown\langle A, \tau, x_0(k), x_0(k+1), \ldots, x_0(n)\rangle \in S$.

If $x_0 \notin X_0$, then there is no $\tau \in \Delta_\sigma$ such that $\tau \subseteq x_0$, so for every natural number $n \geq k$, $\sigma^\frown\langle A, 1, x_0(k), x_0(k+1), \ldots, x_0(n)\rangle \in S$.

So in both cases, there is an infinite branch $x_1$ of $S$ such that $p(x_1) = x_0$.

By the axiom of choice, there is a wellordering $\prec$ of $T_1$.

For every strategy $S$ for player II in $G_1$, we describe a strategy $\phi_{\mathrm{II}}(S)$ for player II in $G_0$ as follows:

Player II follows strategy $S$ until a position $\sigma$ of length $k$ is reached. Let $M$ be the set of all $\tau \in \Delta_\sigma$ such that for some $A \subseteq \Delta_\sigma$, strategy $S$ tells player II to make the move $\tau$ when player I has made the move $A$ at position $\sigma$ in the game $G_1$.

Now player II acts as if player I has played the extra move $\Delta_\sigma \setminus M$. Then, by definition, strategy $S$ does not tell player II to make a move $\tau \in \Delta_\sigma \setminus M$, thus $S$ tells player II to make the extra move 1.

Player II proceeds by following strategy $S$ as if these extra moves are played, until, if ever, a position $\tau \in \Delta_\sigma$ is reached. Since this corresponds to a terminal node $\sigma^\frown\langle \Delta_\sigma \setminus M, 1\rangle^\frown\rho$ of the strategy $S$ such that $\tau = \sigma^\frown\rho$, we have that $\tau \notin \Delta_\sigma \setminus M$, so $\tau \in M$. Using the wellordering $\prec$, player II chooses some $A \subseteq \Delta_\sigma$ such that strategy $S$ tells II to play $\tau$ when player I has made the move $A$ at position $\sigma$ in the game $G_1$. Now player II proceeds by following strategy $S$ as if player I had played the extra move $A$ at position $\sigma$, player II played the move $\tau$ and the moves $\rho(0), \rho(1), \ldots$ were obligatory.

To make this precise, we first define a function $F$. For every strategy $S$ for player II in $G_1$ and every $\sigma \in S$ of length $k$, let $F(S, \sigma, 1) = \sigma^\frown\langle \Delta_\sigma \setminus M, 1\rangle$, where $M = \{\tau \in \Delta_\sigma : \text{for some } A,\ \sigma^\frown\langle A, \tau\rangle \in S\}$, and for every $\tau \in M$, define $F(S, \sigma, \tau)$ as the $\prec$-least element of $S$ of the form $\sigma^\frown\langle A, \tau\rangle$ for some $A$.

For every strategy $S$ for player II in $G_1$, we now define $\phi_{\mathrm{II}}(S) = \{\sigma \in S : \mathrm{length}(\sigma) \leq k\} \cup \{\sigma^\frown\rho : \mathrm{length}(\sigma) = k \text{ and } F(S, \sigma, 1)^\frown\rho \in S\} \cup \{\sigma^\frown\rho : \mathrm{length}(\sigma) = k \text{ and for some } \rho',\ F(S, \sigma, 1)^\frown\rho' \in S \text{ and } \sigma^\frown\rho' \in \Delta_\sigma \text{ and } F(S, \sigma, \sigma^\frown\rho')^\frown\rho \in S\}$.

Let $x_0$ be an infinite branch of $\phi_{\mathrm{II}}(S)$ and put $\sigma = x_0|k$.

If $x_0 \in X_0$, then there is minimal $\tau \in \Delta_\sigma$ such that $\tau \subseteq x_0$, so for every natural number $n \geq k$, $F(S, \sigma, \tau)^\frown\langle x_0(k), x_0(k+1), \ldots, x_0(n)\rangle \in S$.

If $x_0 \notin X_0$, then there is no $\tau \in \Delta_\sigma$ such that $\tau \subseteq x_0$, so for every natural number $n \geq k$, $F(S, \sigma, 1)^\frown\langle x_0(k), x_0(k+1), \ldots, x_0(n)\rangle \in S$.

So in both cases, there is an infinite branch $x_1$ of $S$ such that $p(x_1) = x_0$.

We now show that $\phi_{\mathrm{I}}$ and $\phi_{\mathrm{II}}$ translate winning strategies into winning strategies. Suppose that $S$ is a winning strategy for one of the players $J$ in $G_1$. For every $x_0 \in [\phi_J(S)]$, there is an $x_1 \in [S]$ such that $p(x_1) = x_0$. Since player $J$ wins play $x_1$ in $G_1$ and since $X_1 = p^{-1}X_0$, player $J$ also wins play $x_0$ in $G_0$. So the strategy $\phi_J(S)$ for player $J$ in $G_0$ is winning.

Let $Y_0$ be a subset of $[T_0]$ and let $Y_1$ be $p^{-1}Y_0$. Suppose that $(T_1, P_1, Y_1)$ is determined. Then, by the same argument as above, $(T_0, P_0, Y_0)$ is determined.

## 1E Reducing Borel games to basic open games

In this section we prove, using the axiom of choice, that every Borel game is determined. The idea of the proof is to reduce every Borel game on a tree $T_0$ to a basic open game on a (much larger) tree $T_1$ in such a way that the determinacy of the Borel game follows from the determinacy of the basic open game.

The auxiliary basic open game is played like the original Borel game, but some extra moves are inserted and at some positions the number of possible moves is reduced to 1 (an obligatory move) or even to 0 (a new terminal position). So every position $\sigma$ in the auxiliary game corresponds to some position $\pi(\sigma)$ in the original game by simply leaving out the extra moves. This function $\pi$ from $T_1$ to $T_0$ induces a continuous function from $[T_1]$ to $[T_0]$.

For simplicity we assume that the length of a position in $T_1$ determines whether the moves at that position are to be considered as extra moves. In other words, there is some strictly increasing function $f : \omega \longrightarrow \omega$ such that for every natural number $n$, move $n$ in the original game corresponds to move $f(n)$ in the auxiliary game. If for example $f(n) = 3n+1$ for every $n \in \omega$ and $\sigma = \langle a_0, a_1, a_2, a_3, a_4, a_5\rangle$ is a position in the auxiliary game, then the extra moves are $a_0$, $a_2$, $a_3$, and $a_5$. The corresponding position in the original game is $\pi(\sigma) = \langle a_1, a_4\rangle = \sigma \circ f$. Note that $\text{length}(\pi(\sigma)) = 2 = \{n \in \omega : 3n + 1 < 6\} = f^{-1}\{0, 1, 2, 3, 4, 5\} = f^{-1}6$.

**1.14** Remark Suppose that $f : \omega \longrightarrow \omega$ is strictly increasing. Then for every infinite sequence $x$, the composition $x \circ f$ is an infinite sequence. For every finite sequence $\sigma$ of length $n$, the composition $\sigma \circ f$ is a finite sequence of length $f^{-1}n$, which is the least $m \in \omega$ such that $f(m) \geq n$.

Let $T_0$ and $T_1$ be trees such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Define a function $\pi$ from $T_1$ to $T_0$ by $\pi(\sigma) = \sigma \circ f$. Then $\pi(\langle\rangle) = \langle\rangle$ and for every $\sigma \in T_1$ and every move $a$ in $T_1$ at $\sigma$:

$$\pi(\sigma^\frown\langle a\rangle) = \begin{cases} \pi(\sigma)^\frown\langle a\rangle & \text{if length}(\sigma) \in \text{range}(f), \\ \pi(\sigma) & \text{otherwise.} \end{cases}$$

Define $p : [T_1] \longrightarrow [T_0]$ by $p(x) = \bigcup_{n\in\omega} \pi(x|n) = x \circ f$. Then $p$ is continuous since for every basic open subset $X$ of $T_0$, the inverse image $p^{-1}X$ of $X$ under the function $p$ is a basic open subset of $T_1$.

We will sometimes consider fragments of strategies that only prescribe the moves of a player at positions up to (and including) a certain length $n$. Such a fragment is in fact a strategy for that player in the game that is played like the original game until a position of length $n + 1$ is reached; then the game comes to an end and the other player loses.

**1.15** DEFINITION For every tree $T$ and $n \in \omega$, we denote the tree $\{\sigma \in T : \text{length}(\sigma) \leq n\}$ by $T^{\leq n}$.

Let $G = (T, P, X)$ be a game and let $n \in \omega$.

A **strategy for player I in $G$ up to positions of length $n$** is a strategy for player I in the game $(T^{\leq n+1}, T^{\leq n} \cap P, \emptyset)$.

A **strategy for player II in $G$ up to positions of length $n$** is a strategy for player I in the game $(T, T \setminus P, [T] \setminus X)$ up to positions of length $n$.

Note that for every tree $T$ and every $n \in \omega$, $T^{\leq n}$ is the unique subtree $S$ of $T$ such that for all $\sigma \in S$, if $\text{length}(\sigma) < n$ then the moves in $S$ at $\sigma$ are precisely the moves in $T$ at $\sigma$ and if $\text{length}(\sigma) = n$ then $\sigma$ is a terminal node of $S$.

One easily verifies that for every game $G$ and tree $S$:
$S$ is a strategy for some player in $G$ if and only if for all $n \in \omega$, $S^{\leq n+1}$ is a strategy for that player in $G$ up to positions of length $n$.

The following definitions are based on Martin [1985]. He introduced *coverings*, which connect two games on different trees in such a way that (winning) strategies in one of the games correspond to (winning) strategies in the other game.

**1.16** DEFINITION Let $f : \omega \longrightarrow \omega$ be strictly increasing and consider games $G_0 = (T_0, P_0, X_0)$ and $G_1 = (T_1, P_1, X_1)$ such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Let $J$ be one of the players.

An **$f$-translator of strategies for player $J$ from $G_1$ to $G_0$** is a function $\phi$ that assigns to every strategy $S$ for player $J$ in $G_1$ a strategy $\phi(S)$ for player $J$ in $G_0$ such that for every infinite branch $x_0$ of $\phi(S)$, there is an infinite branch $x_1$ of $S$ such that $x_1 \circ f = x_0$.

We say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ such that:

**(i)** for every $n \in \omega$, domain$(\phi_n)$ is the set of all strategies for player $J$ in $G_1$ up to positions of length $f(n)$ and for every $S \in \text{domain}(\phi_n)$, $\phi_n(S)$ is a strategy for player $J$ in $G_0$ up to positions of length $n$ and for every $m < n$, $\phi_n(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$;

**(ii)** for every strategy $S$ for $J$ in $G_1$ and every $m \in \omega$, $\phi(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$.

Note that if $\phi_0, \phi_1, \ldots$ are functions such that (i) holds, then there is a unique function $\phi$ from the set of all strategies for player $J$ in $G_1$ to the set of all strategies for player $J$ in $G_0$ such that (ii) holds.

**1.17** DEFINITION Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_0 = (T_0, P_0, X_0)$ be a game.

An **$f$-covering of $G_0$** is a game $G_1 = (T_1, P_1, X_1)$ such that:

**(i)** for all $\sigma \in T_1$, $\sigma \circ f \in T_0$;

**(ii)** $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$;

**(iii)** for each player, there is an $f$-translator $\phi$ of strategies for that player from $G_1$ to $G_0$ and there are functions $\phi_0, \phi_1, \ldots$ witnessing that $\phi$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then $T_1^{\leq n+1} = T_0^{\leq n+1}$, $P_1 \cap T_1^{\leq n} = P_0 \cap T_0^{\leq n}$ and for every $S \in \text{domain}(\phi_n)$, $\phi_n(S) = S$.

**1.18** Lemma

**(i)** Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_1$ be an $f$-covering of some game $G_0$. Suppose that $G_1$ is determined. Then $G_0$ is also determined.

**(ii)** Suppose that $(T_1, P_1, X_1)$ is an $f$-covering of $(T_0, P_0, X_0)$ and that $Y_0 \subseteq [T_0]$. Put $Y_1 = \{x \in [T_1] : x \circ f \in Y_0\}$. Then $(T_1, P_1, Y_1)$ is an $f$-covering of $(T_0, P_0, Y_0)$.

**(iii)** Let $G_1$ be an $f$-covering of $G_0$ and let $G_2$ be a $g$-covering of $G_1$. Then $G_2$ is a $(g \circ f)$-covering of $G_0$.

Proof These statements follow directly from the definition of 'covering':

**(i)** Let $G_1$ be an $f$-covering of some game $G_0$. Then player I wins a play $x$ in $G_1$ if and only if he wins play $x \circ f$ in $G_0$. Suppose that one of the players has a winning strategy $S$ in $G_1$. Let $\phi$ be an $f$-translator of strategies for that player from $G_1$ to $G_0$. Then the strategy $\phi(S)$ for that player in $G_0$ is winning, since each $x_0 \in [\phi(S)]$ is of the form $x_1 \circ f$ for some $x_1 \in [S]$.

**(ii)** This is trivial, since the sets $X_0$ and $X_1$ are dummies in Definition 1.16.

**(iii)** To prove this, use the following simple facts:
For every finite or infinite sequence $s$, $s \circ (g \circ f) = (s \circ g) \circ f$.
Suppose that the functions $\phi_0, \phi_1, \ldots$ witness that $\phi$ is a continuous $f$-translator of strategies for some player from $G_1$ to $G_0$ and that the functions $\psi_0, \psi_1, \ldots$ witness that $\psi$ is a continuous $g$-translator of strategies for that player from $G_2$ to $G_1$. Then the functions $\phi_0 \circ \psi_{f(0)}, \phi_1 \circ \psi_{f(1)}, \ldots$ witness that $\phi \circ \psi$ is a continuous $(g \circ f)$-translator of strategies for that player from $G_2$ to $G_0$.
Let $n \in \omega$ such that $g(f(n)) = n$. Then, since $f$ and $g$ are strictly increasing, $f(n) = n$ and $g(n) = n$.

In Definition 1.17, we required that the translators of strategies are continuous. This requirement is essential in the proof of the following lemma.

**1.19** Lemma Let $G_0$ be a game and suppose that for every $n \in \omega$, an $f_n$-covering $G_{n+1}$ of $G_n$ is given.

Assume that for every $n \in \omega$, $\lim_{m\to\omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ exists, that is, there is a (unique) $g_n : \omega \longrightarrow \omega$ such that for all $i \in \omega$, for some $M \geq n$, for all $m \geq M$, $f_m \circ f_{m-1} \circ \cdots \circ f_n(i) = g_n(i)$.

Then AC implies that there is a (unique) game $G$ such that for every $n \in \omega$, $G$ is a $g_n$-covering of $G_n$.

Proof The idea is to construct a game $G = (T, P, X)$ that is played like the games $G_n = (T_n, P_n, X_n)$ for all large $n \in \omega$.

Note that for every $n \in \omega$, $g_n$ is strictly increasing, $g_n = g_{n+1} \circ f_n$ and for all $i \in \omega$, $g_n(i) = i$ if and only if for all $m \geq n$, $f_m(i) = i$.

Let $i \in \omega$ en put $k = g_0(i)$. Then for all large $m$, $f_m \circ \cdots \circ f_0(i) = k$. Thus for all large $m$, $f_m(k) = k$. Since $k \geq i$, this implies that for all large $m$, $f_m(i) = i$ and thus ${T_{m+1}}^{\leq i+1} = {T_m}^{\leq i+1}$ and $P_{m+1} \cap {T_{m+1}}^{\leq i} = P_m \cap {T_m}^{\leq i}$.

This implies that there is a unique tree $T$ and a unique $P \subseteq T$ such that for all $i$ and $n$, if $g_n(i) = i$ then $T^{\leq i+1} = {T_n}^{\leq i+1}$ and $P \cap T^{\leq i} = P_n \cap {T_n}^{\leq i}$.

Let $n \in \omega$ and let $\sigma \in T$. Then for all large $m$, $\sigma \in T_{m+1}$ and $\sigma \circ g_n = \sigma \circ f_m \circ \cdots \circ f_n \in T_n$.

Define $X = \{x \in [T] : x \circ g_0 \in X_0\}$. Let $n \in \omega$. Since $g_0 = g_n \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0$ and $X_n = \{y \in [T_n] : y \circ f_{n-1} \circ \cdots \circ f_1 \circ f_0 \in X_0\}$, we have that $X = \{x \in [T] : x \circ g_n \in X_n\}$. We will prove that the game $G = (T, P, X)$ is a $g_n$-covering of $G_n$.

Choose (using AC) for every $m \in \omega$ an $f_m$-translator $\phi_m$ of strategies for player I from $G_{m+1}$ to $G_m$ and functions $\phi_m^0, \phi_m^1, \ldots$ witnessing that $\phi_m$ is continuous.

Let $n, i \in \omega$ and put $k = g_n(i)$. Consider a strategy $S$ for player I in $G$ up to positions of length $k$. Define a strategy $\psi_n^i(S)$ for player I in in $G_n$ up to positions of length $i$ as follows:
$\psi_n^i(S) = \phi_n^i \circ \phi_{n+1}^{f_n(i)} \circ \cdots \circ \phi_m^{f_{m-1}\circ\cdots\circ f_n(i)}(S)$ for all large $m$. This is well-defined since for all large $m$, $f_{m-1} \circ \cdots \circ f_n(i) = k$ and $f_m(k) = k$, so $S$ is a strategy for player I in $G_{m+1}$ up to positions of length $k$ and $\phi_m^k(S) = S$. Note that if $k = i$ then $\psi_n^i(S) = S$.

Let $S$ be a strategy for player I in $G$. Define, for every $n \in \omega$, a strategy $\psi_n(S)$ for player I in $G_n$ by $\psi_n(S)^{\leq i+1} = \psi_n^i(S^{\leq g_n(i)+1})$ for all $i \in \omega$.

Now let $n \in \omega$ and let $x_n \in [\psi_n(S)]$. One easily verifies that $\psi_n(S) = \phi_n(\psi_{n+1}(S))$, so there is an $x_{n+1} \in [\psi_{n+1}(S)]$ such that $x_{n+1} \circ f_n = x_n$. By repeating this argument we find (using AC) for every $m \geq n$ some $x_{m+1} \in [\psi_{m+1}(S)]$ such that $x_{m+1} \circ f_m = x_m$. For every $i \in \omega$ we have that for all large $m$, $f_m(i) = i$, so $x_m|i = x_{m+1} \circ f_m|i = x_{m+1}|i$ and $x_m|i \in \psi_m(S)^{\leq i+1} = \psi^i_m(S^{\leq g_m(i)+1}) = \psi^i_m(S^{\leq i+1}) = S^{\leq i+1}$. This implies that there is a unique $x \in [S]$ such that for every $i \in \omega$, for all large $m$, $x|i = x_m|i$.

Let $i \in \omega$ and put $k = g_n(i)$. Then for all large $m$, $x \circ g_n|i = (x|k) \circ g_n = (x_m|k) \circ f_m \circ \cdots \circ f_n = x_n|i$. Thus $x \circ g_n = x_n$.

This proves that $\psi_n$ is a $g_n$-translator of strategies for player I from $G$ to $G_n$. The functions $\psi^0_n, \psi^1_n, \ldots$ witness that $\psi_n$ is continuous.

A continuous $g_n$-translator of strategies for player II from $G$ to $G_n$ can be found in the same way. Thus $G$ is a $g_n$-covering of $G_n$.

We now prove that every open or closed game has a basic open covering. By Lemma 1.18(i) and Theorem 1.12 it follows (using AC) that every open or closed game is determined.

**1.20** Lemma Suppose that $G_0$ is an open or closed game. Let $k \in \omega$ and let $f$ be the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k, k+1\}$. (In other words, for all $n < k$, $f(n) = n$ and for all $n \geq k$, $f(n) = n + 2$.)

Then AC implies that there is an $f$-covering $G_1$ of $G_0$ such that the game $G_1$ is basic open.

Proof First suppose that the game $G_0 = (T_0, P_0, X_0)$ is open. Define a basic open game $G_1 = (T_1, P_1, X_1)$ and functions $\phi_{\mathrm{I}}$ and $\phi_{\mathrm{II}}$ as in Section 1D. Then for all $\sigma \in T_1$, $\sigma \circ f \in T_0$.

Let $J$ be a player, so $\phi_J$ is an $f$-translator of strategies for player $J$ from $G_1$ to $G_0$. For each $n \in \omega$, we define a function $\phi^n_J$ just like $\phi_J$ but now on the set of all strategies for player $J$ in $G_1$ up to positions of length $f(n)$. The functions $\phi^0_J, \phi^1_J, \ldots$ witness that $\phi_J$ is continuous.

Let $n \in \omega$ and suppose $f(n) = n$. Then $n < k$, so ${T_1}^{\leq n+1} = {T_0}^{\leq n+1}$, $P_1 \cap {T_1}^{\leq n} = P_0 \cap {T_0}^{\leq n}$, and for every $S \in \mathrm{domain}(\phi^n_J)$, $\phi^n_J(S) = S$.

Thus $G_1$ is a basic open $f$-covering of the open game $G_0$.

Now suppose that the game $G_0 = (T_0, P_0, X_0)$ is closed. Define $Y_0 = [T_0] \setminus X_0$. Then the open game $(T_0, P_0, Y_0)$ has a basic open $f$-covering $(T_1, P_1, Y_1)$. Put $X_1 = [T_1] \setminus Y_1$. Then, by Lemma 1.18(ii), the basic open game $(T_1, P_1, X_1)$ is an $f$-covering of $G_0$.

**1.21** Remark The use of the axiom of choice in the proof above can be avoided by adapting the definition of the function $F$ (that is used in the definition of $\phi_{\mathrm{II}}$) as follows:

Assume that $S$ is a strategy for player II in $G_1$ and let $\sigma \in S$ be of length $k$. Define by transfinite induction for every ordinal $\alpha$ subsets $A_\alpha$ and $D_\alpha$ of $\Delta_\sigma$ as follows: $A_\alpha = \Delta_\sigma \setminus \bigcup_{\beta<\alpha} D_\beta$ and $D_\alpha = \{\tau \in A_\alpha : \sigma^\frown\langle A_\alpha, \tau\rangle \in S\}$. So $D_\alpha$ has at most one element.

For all ordinals $\alpha$ and $\beta$, if $\beta < \alpha$ then $D_\beta \cap D_\alpha = \emptyset$. So there is a least ordinal $\rho$ such that $D_\rho = \emptyset$. Put $M = \bigcup_{\alpha<\rho} D_\alpha$, so $\Delta_\sigma \setminus M = A_\rho$.

Define $F(S, \sigma, 1) = \sigma^\frown\langle A_\rho, 1\rangle$. Suppose that player I makes the move $A_\rho$ at position $\sigma$ in the game $G_1$. Since $D_\rho = \emptyset$, there is no $\tau \in A_\rho$ such that strategy $S$ tells player II to make the move $\tau$. Thus $S$ tells him to make the move 1. In other words, $F(S, \sigma, 1) \in S$.

For every $\tau \in M$, put $F(S, \sigma, \tau) = \sigma^\frown\langle A_\alpha, \tau\rangle$, where $\alpha$ is the unique ordinal such that $\tau \in D_\alpha$. Then, by definition of $D_\alpha$, $F(S, \sigma, \tau) \in S$.

We now prove (using AC) that every Borel game has a basic open covering.

**1.22** Lemma Suppose that $G_0$ is a Borel game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that for infinitely many natural numbers $k$, neither $k$ nor $k+1$ is in the range of $f$.

Then AC implies that there is a basic open $f$-covering $G_1$ of $G_0$.

Proof Define $\Omega$ as the set of all strictly increasing $g : \omega \longrightarrow \omega$ such that for infinitely many natural numbers $k$, neither $k$ nor $k+1$ is in the range of $g$. Let $T$ be a tree. Consider the collection $\mathcal{C}$ of all $X \subseteq [T]$ such that for every strictly increasing $g : \omega \longrightarrow \omega$, for every game $G_0 = (T_0, P_0, X_0)$ such that for all $\sigma \in T_0$, $\sigma \circ g \in T$ and $X_0 = \{x \in [T_0] : x \circ g \in X\}$, and for every $f \in \Omega$, there is a basic open $f$-covering of $G_0$.

Using AC, we will prove that $\mathcal{C}$ contains every basic open subset of $[T]$ and is closed under countable union and countable intersection. This implies that $\mathcal{C}$ contains every Borel subset of $[T]$. Taking for $g$ the identity on $\omega$, we see that for every Borel game $G_0$ on $T$ and every $f \in \Omega$, there is a basic open $f$-covering of $G_0$.

Suppose that $X$ is a basic open subset of $[T]$. Let $g : \omega \longrightarrow \omega$ be strictly increasing and let $G_0 = (T_0, P_0, X_0)$ be a game such that for all $\sigma \in T_0$, $\sigma \circ g \in T$ and $X_0 = \{x \in [T_0] : x \circ g \in X\}$. Then $X_0$ is a basic open subset of $[T_0]$. Now it is easy to construct, for every $f \in \Omega$, a basic open $f$-covering $G_1$ of $G_0$. The idea is to insert some trivial extra move at the right places: Let $T_1$ be the set of all finite sequences $\sigma$ such that $\sigma \circ f \in T_0$ and for all $n < \text{length}(\sigma)$, if $n \notin \text{range}(f)$ then $\sigma(n) = 0$. Put $P_1 = \{\sigma \in T_1 : \sigma \circ f \in P_0\}$ and $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$. Thus the game $G_1 = (T_1, P_1, X_1)$ is played like $G_0$, but at some positions one of the players makes an extra move 0. For each player, a continuous $f$-translator $\phi$ of strategies for that player from $G_1$ to $G_0$ is defined by $\phi(S) = \{\sigma \circ f : \sigma \in S\}$ for every strategy $S$ for that player in $G_1$. Thus the basic open game $G_1$ is an $f$-covering of $G_0$.

This proves that $\mathcal{C}$ contains every basic open subset of $[T]$.

Now suppose that $\langle X_0, X_1, \ldots\rangle$ is an infinite sequence of elements of $\mathcal{C}$. Let $X$ be either $\bigcup_{n\in\omega} X_n$ or $\bigcap_{n\in\omega} X_n$. We will prove that $X \in \mathcal{C}$.

So let $g : \omega \longrightarrow \omega$ be strictly increasing and let $G_0$ be a game of the form $(T_0, P_0, \{x \in [T_0] : x \circ g \in X\})$ such that for all $\sigma \in T_0$, $\sigma \circ g \in T$. Let $f \in \Omega$. We must find a basic open $f$-covering of $G_0$.

Claim There are strictly increasing functions $g_0$ and $h$ from $\omega$ to $\omega$ such that $f = h \circ g_0$, $g_0 \in \Omega$ and for some $k \in \omega$, $\text{range}(h) = \omega \setminus \{k, k+1\}$.

Proof of claim Since $f \in \Omega$, there are natural numbers $i, k$ such that $f(i) < k < k+1 < f(i+1)$. Define $g_0$ and $h$ by

$$g_0(n) = \begin{cases} f(n) & \text{if } n \leq i, \\ f(n) - 2 & \text{if } n > i, \end{cases} \quad \text{and } h(n) = \begin{cases} n & \text{if } n < k, \\ n+2 & \text{if } n \geq k. \end{cases}$$

Then $f = h \circ g_0$ and $g_0 \in \Omega$.

If we find an open or closed $g_0$-covering $G'$ of $G_0$, then, by Lemma 1.20 and AC, $G'$ has some basic open $h$-covering $G$. By Lemma 1.18(iii), $G$ is an $(h \circ g_0)$-covering of $G_0$, so $G$ is a basic open $f$-covering of $G_0$.

Claim There are $f_0, g_1, f_1, g_2, f_2, \ldots \in \Omega$ such that for all $n \in \omega$, $g_n = \lim_{m \to \omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$.

Proof of claim The function $g_0 \in \Omega$ is already defined. Let $n \in \omega$ and suppose that $g_n \in \Omega$ is defined. Choose natural numbers $k_0, k_1, \ldots$ such that for all $i \in \omega$, $k_i \notin \text{range}(g_n)$ and $k_i + 1 \notin \text{range}(g_n)$ and $k_i + 1 < k_{i+1}$. Let $g_{n+1}$ be the unique element of $\Omega$ whose range is $\omega \setminus \{k_1, k_1+1, k_3, k_3+1, k_5, \ldots\}$. Note that $k_1$, the least element of $\omega \setminus \text{range}(g_{n+1})$, is larger than the least element of $\omega \setminus \text{range}(g_n)$.

Since $\text{range}(g_n) \subseteq \text{range}(g_{n+1})$, there is, for every $a \in \omega$, a unique $b \in \omega$ such that $g_{n+1}(b) = g_n(a)$. This defines a function $f_n$ such that $g_n = g_{n+1} \circ f_n$. To see that $f_n \in \Omega$, let $i$ be even. Then $\{k_i, k_i + 1\} \subseteq \text{range}(g_{n+1}) \setminus \text{range}(g_n)$. So there is a $b \in \omega$ such that $g_{n+1}(b) = k_i$, $g_{n+1}(b+1) = k_i + 1$, and neither $b$ nor $b+1$ is in the range of $f_n$.

Let $n \in \omega$. To see that $g_n = \lim_{m \to \omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$, let $i \in \omega$ and define $k = g_n(i)$. Choose $m \in \omega$ so large that the least element of $\omega \setminus \text{range}(g_{m+1})$ is larger than $k$. Then $g_{m+1}(k) = k$. But also $g_{m+1} \circ f_m \circ \cdots \circ f_{n+1} \circ f_n(i) = g_n(i) = k$. Thus, since $g_{m+1}$ is injective, $f_m \circ f_{m-1} \circ \cdots \circ f_n(i) = k$.

Using AC, we construct, inductively, for each $n \in \omega$, a game $(T_n, P_n, Y_n)$ such that for all $\sigma \in T_n$, $\sigma \circ f_{n-1} \circ \cdots \circ f_0 \circ g \in T$, as follows:

We already have $T_0$ and $P_0$. Now let $n \in \omega$ and suppose that we have chosen $T_n$ and $P_n$. Put $Y_n = \{x \in [T_n] : x \circ f_{n-1} \circ \cdots \circ f_0 \circ g \in X_n\}$. Since $X_n \in \mathcal{C}$ and $f_n \in \Omega$, we can choose (using AC) a basic open $f_n$-covering $(T_{n+1}, P_{n+1}, Z_n)$ of the game $(T_n, P_n, Y_n)$.

For every $n \in \omega$, define $G_{n+1}$ as the game $(T_{n+1}, P_{n+1}, \{x \in [T_{n+1}] : x \circ f_n \circ f_{n-1} \circ \cdots \circ f_0 \circ g \in X\})$. By Lemma 1.18(ii), $G_{n+1}$ is an $f_n$-covering of $G_n$.

By Lemma 1.19 and AC, there is a (unique) game $G' = (T', P', X')$ such that for every $n \in \omega$, $G'$ is a $g_n$-covering of $G_n$.

For every $n \in \omega$, put $X'_n = \{x \in [T'] : x \circ g_0 \circ g \in X_n\}$. Since $g_0 \circ g = g_{n+1} \circ f_n \circ f_{n-1} \circ \cdots \circ f_0 \circ g$, we have that $X'_n = \{x \in [T'] : x \circ g_{n+1} \in Z_n\}$, so $X'_n$ is a basic open subset of $[T']$.

Since $X'$ is either $\bigcup_{n\in\omega} X'_n$ or $\bigcap_{n\in\omega} X'_n$, we conclude that the game $G'$ is either open or closed.

**1.23** Remark In the proof above, we defined, for a given tree $T$, a collection $\mathcal{C}$ and proved that it contains every Borel subset of $[T]$. Instead of this, we can use the Borel rank (see Remark 1.9) and prove by transfinite induction for each ordinal $\alpha$ the following: For every Borel game $G = (T, P, X)$ such that $X$ has Borel rank $\alpha$, and for every $f \in \Omega$, $G$ has a basic open $f$-covering.

If $\alpha = 0$ then $X$ is a basic open subset of $[T]$ and we can define a basic open $f$-covering of $G$ as in the proof above.

If $\alpha > 0$ then there are Borel subsets $X_0, X_1, \ldots$ of $[T]$, each of Borel rank less than $\alpha$, such that $X$ is either $\bigcup_{n\in\omega} X_n$ or $\bigcap_{n\in\omega} X_n$. We construct, inductively, for every $n \in \omega$, a game $G_n$ as in the proof above, where we let $g$ be the identity on $\omega$ and $T_0 = T$. In order to see that the auxiliary game $(T_n, P_n, Y_n)$ has a basic open $f_n$-covering, we use the induction hypothesis and the fact that $Y_n$ is a Borel subset of $[T_n]$ of Borel rank less than $\alpha$.

**1.24** Theorem AC implies that every Borel game is determined.

Proof Suppose that AC holds and let $G_0$ be a Borel game. Define $f : \omega \longrightarrow \omega$ by $f(n) = 3n$. Then for all natural numbers $n$, neither $3n+1$ nor $3n+2$ is in the range of $f$.

Thus, by Lemma 1.22 and AC, there is an $f$-covering $G_1$ of $G_0$ such that the game $G_1$ is basic open. By Theorem 1.12 and AC, $G_1$ is determined. So, by Lemma 1.18(i), the game $G_0$ is determined.

# 2 The role of the axiom of choice

In this chapter we will see that we cannot prove Borel determinacy without using AC, unless we reformulate the concepts 'Borel' and 'determinacy'.

## 2A Pseudostrategies

**2.1** PROPOSITION The following statements are equivalent:

- **(i)** AC;
- **(ii)** every Borel game is determined;
- **(iii)** every trivial game is determined.

PROOF By Theorem 1.24, (i) implies (ii). Since every trivial game is a Borel game, (ii) implies (iii).

That (iii) implies (i) can be seen as follows: Let $\mathcal{C}$ be a collection of non-empty sets. Consider the game $G$ in which player I starts by choosing some $A \in \mathcal{C}$; then player II chooses some $a \in A$ and finally player I loses the game.

In other words, $G$ is the trivial game $(T, P, \emptyset)$, where $T = \{\langle\rangle\} \cup \{\langle A\rangle : A \in \mathcal{C}\} \cup \{\langle A, a\rangle : A \in \mathcal{C} \text{ and } a \in A\}$ and $P$ is the set of elements of $T$ of even length.

Now suppose that this trivial game is determined. It is clear that player I does not have a strategy in $G$ and that every strategy for player II in $G$ corresponds to a choice function on $\mathcal{C}$.

Thus, if every trivial game is determined, then every collection of non-empty sets has a choice function.

Since the axiom of choice is independent of ZF, this shows that Borel determinacy is unprovable in ZF (if ZF is consistent).

Nevertheless, it is clear that player II can win game $G$ in the proof of Proposition 2.1 easily: After player I's move $A$, player II may choose *any* $a \in A$ and then player II has won. This is an example of a *pseudostrategy.* A pseudostrategy for some player need not prescribe his moves completely; it just has to indicate *at least* one move at positions where it is that player's turn.

A pseudostrategy for player I in a game on a tree $T$ is also called a I-*imposed subtree* of $T$ (see Davis [1964]), a *multiple-valued strategy*, or a *quasistrategy* for player I (see Moschovakis [1980], page 446).

To define the concept of 'pseudostrategy', we only have to weaken the condition 'exactly one' in Definition 1.6.

**2.2** DEFINITION Let $G = (T, P, X)$ be a game.

A **pseudostrategy for player I** in $G$ is a tree $S \subseteq T$ such that for every $\sigma \in S \cap P$ there is a move in $S$ at $\sigma$ and such that for every $\sigma \in S \setminus P$, every move in $T$ at $\sigma$ is a move in $S$ at $\sigma$.

A **winning pseudostrategy for player I** in $G$ is a pseudostrategy $S$ for player I in $G$ such that $[S] \subseteq X$.

A (**winning**) **pseudostrategy for player II** in $G$ is a (winning) pseudostrategy for player I in the game $(T, T \setminus P, [T] \setminus X)$.

The game $G$ is **pseudodetermined** if there is a winning pseudostrategy for player I or a winning pseudostrategy for player II in $G$.

Since every strategy is a pseudostrategy, every determined game is pseudodetermined.

**2.3** Proposition The following statements are equivalent:

- **(i)** every pseudodetermined game is determined;
- **(ii)** AC.

Proof Suppose that every pseudodetermined game is determined. Let $\mathcal{C}$ be a collection of non-empty sets. Let $G = (T, P, \emptyset)$ be as in the proof of Proposition 2.1. Then the tree $T$ itself is a winning pseudostrategy for player II in $G$, since there is a move in $T$ at every position of length 1. Since $G$ is pseudodetermined, it is also determined. This implies, as we have seen in the proof of Proposition 2.1, that there is a choice function on $\mathcal{C}$.

Now assume AC and let $G = (T, P, X)$ be a pseudodetermined game. Let $S$ be a winning pseudostrategy for some player in $G$, say for player I (the other case is similar). Then for every $\sigma \in S \cap P$, the set $M_\sigma$ of all moves in $S$ at $\sigma$ is non-empty, so, by the axiom of choice, there is a choice function $f$ on $\{M_\sigma : \sigma \in S \cap P\}$. Let $S' = \{\sigma \in S :$ for all $n <$ length$(\sigma)$, if $\sigma|n \in P$ then $\sigma(n) = f(M_{\sigma|n})\}$. In other words, $S'$ is the unique subtree of $S$ such that for all $\sigma \in S'$, if $\sigma \in P$ then $f(M_\sigma)$ is the only move in $S'$ at $\sigma$ and if $\sigma \notin P$ then the moves in $S'$ at $\sigma$ are the same as the moves in $S$ at $\sigma$. So $S'$ is a winning strategy for $P$ in $G$ and thus $G$ is determined.

So the axiom of choice implies that the concepts 'determinacy' and 'pseudodeterminacy' are equivalent. On the other hand, it is consistent with ZF that there is an infinite subset $M$ of $\mathbb{R}$ that has no countable subset. Assuming the existence of such a set $M$, S.H. Hechler [1974] constructed a game on the tree ${}^{<\omega}M$ in which *both* players have a winning pseudostrategy but no winning strategy.

Note that if the tree $T$ in the second part of the proof of Proposition 2.3 can be wellordered, then we can find a choice function $f$ without using AC.

In particular, a game on a countable tree is determined if and only if it is pseudodetermined.

We proved in Section 1A that not every game is determined. A similar proof shows that not every game is pseudodetermined:

Case 1: The set ${}^{\omega}2$ can be wellordered. Then we can construct an undetermined game on a countable tree. So this game is not pseudodetermined.

Case 2: The set ${}^{\omega}2$ cannot be wellordered. Then the game $G$ in Example 1.7 is undetermined. Since player I makes only the first move, every pseudostrategy for player I in $G$ contains a strategy. Since player II only chooses elements of $\{0, 1\}$, every pseudostrategy for player II in $G$ contains a strategy. So the game $G$ is not pseudodetermined.

## 2B Pseudodeterminacy of basic open games

In this section we prove, without using AC, that all trivial and all basic open games are pseudodetermined.

**2.4** Theorem Every trivial game is pseudodetermined.

Proof We will *define* for every trivial game $G$ a winning pseudostrategy $S$ for one of the players. By symmetry, we only have to consider games in which player I's winning set is empty. Let $G = (T, P, \emptyset)$ be such a game.

Define, by transfinite induction, for every ordinal $\alpha$ a subset $A_\alpha$ of $T$ as follows:

$A_\alpha = \{\sigma \in P : \text{for some move } a \text{ in } T \text{ at } \sigma,\ \sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta\} \cup \{\sigma \in T \setminus P : \text{for every move } a \text{ in } T \text{ at } \sigma,\ \sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta\}$.

(Note that $A_0$ is the set of all terminal positions at which it is player II's turn. Thus player I wins immediately at positions in $A_0$. In general, $A_\alpha$ consists of all positions from which player I can win the game in at most $\alpha$ 'steps'. In fact, for every ordinal $\alpha$ and for every $\sigma \in T$, $\sigma \in A_\alpha$ if and only if player I has a pseudostrategy in the game that is played as follows:

The player whose turn it is at position $\sigma$ in $G$ starts by choosing a move $a_0$ in $T$ at $\sigma$. Then player I chooses an ordinal $\alpha_0 < \alpha$. Now the player whose turn it is at position $\sigma^\frown\langle a_0\rangle$ in $G$ chooses a move $a_1$ in $T$ at $\sigma^\frown\langle a_0\rangle$ and player I chooses an ordinal $\alpha_1 < \alpha_0$, and so on.)

Let $B = \bigcup_\alpha A_\alpha$ and define, for each $\sigma \in B$, $\rho(\sigma)$ as the least ordinal $\alpha$ such that $\sigma \in A_\alpha$.

Case 1: $\langle\rangle \in B$.

Define $S = \{\sigma \in B : \text{for all } n < \text{length}(\sigma),\ \sigma|n \in B$ and

$\rho(\sigma|n) > \rho(\sigma|(n+1))\}$. Since there is no strictly decreasing infinite sequence of ordinals, $S$ is a tree without infinite branches. Let $\sigma \in S$. Since $\sigma \in A_{\rho(\sigma)}$, the following holds: If $\sigma \in P$ then for some move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in S$; if $\sigma \notin P$ then for every move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in S$. This proves that $S$ is a winning pseudostrategy for player I in $G$.

Case 2: $\langle\rangle \notin B$.

Define $S = \{\sigma \in T : \text{for all } n \leq \text{length}(\sigma), \sigma|n \notin B\}$. Let $\sigma \in S$.

If $\sigma \notin P$, then for some move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in S$, since otherwise we would have that for every move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in B$. But then there is an ordinal $\alpha$ such that $\alpha > \rho(\sigma^\frown\langle a\rangle)$ for every move $a$ in $T$ at $\sigma$, so $\sigma \in A_\alpha$. This contradicts the fact that $\sigma \notin B$.

If $\sigma \in P$, then for every move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in S$, since otherwise we would have that for some move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in B$. But then $\sigma \in A_\alpha$, where $\alpha = \rho(\sigma^\frown\langle a\rangle) + 1$. This contradicts the fact that $\sigma \notin B$.

So $S$ is a pseudostrategy for player II in $G$. This pseudostrategy is winning since player II's winning set is $[T]$.

So in both cases the trivial game $G$ is pseudodetermined.

**2.5** Theorem Every basic open game is pseudodetermined.

Proof The idea of the proof is to 'split' each basic open game into some trivial games. Let $(T, P, X)$ be a basic open game. Let $n \in \omega$ such that for all $x \in X$, for all $y \in [T]$, if $x|n = y|n$ then $y \in X$.

Let $\sigma \in T$ of length $n$. Then the game $(T^{\text{via}\,\sigma}, P \cap T^{\text{via}\,\sigma}, X \cap [T^{\text{via}\,\sigma}])$ is trivial. By (the proof of) Theorem 2.4, we can find a player $W_\sigma$ and a winning pseudostrategy $S_\sigma$ for player $W_\sigma$ in this game.

Now consider the trivial game $(T^{\leq n}, Q, \emptyset)$, where $Q = \{\sigma \in P : \text{length}(\sigma) < n\} \cup \{\sigma \in T : \text{length}(\sigma) = n \text{ and } W_\sigma = \text{II}\}$. This game is played like the game $(T, P, X)$, until a position $\sigma$ of length $n$ is reached; then player $W_\sigma$ wins.

By Theorem 2.4, we can find a player $W$ and a winning pseudostrategy $S$ for player $W$ in this trivial game. Note that every $\sigma \in S$ of length $n$ is a terminal node of $T^{\leq n}$, so $W_\sigma = W$. Now one easily verifies that the tree $S \cup \bigcup_{\sigma \in S, \text{length}(\sigma) = n} S_\sigma$ is a winning pseudostrategy for player $W$ in the basic open game $(T, P, X)$.

Note that by Proposition 2.3, Theorem 2.4 implies Theorem 1.11 and Theorem 2.5 implies Theorem 1.12.

## 2C Wellfounded trees and the principle of dependent choices

The **principle of dependent choices** (DC) is the following consequence of AC: For every set $A$, every $x \in A$, and every relation $\prec$ on $A$ such that for all $y \in A$, for some $z \in A$, $y \prec z$, there is an infinite sequence $s : \omega \longrightarrow A$ such that $s(0) = x$ and for all $n \in \omega$, $s(n) \prec s(n+1)$.

**2.6** PROPOSITION The following statements are equivalent:

- **(i)** DC;
- **(ii)** every tree without terminal nodes has an infinite branch;
- **(iii)** there is no game in which both players have a winning pseudostrategy.

PROOF We first prove that (i) and (ii) are equivalent.

Suppose that DC holds. Let $T$ be a tree without terminal nodes. Let $\prec$ be the relation on $T$ defined by: $\sigma \prec \tau$ if and only if for some $a$, $\sigma^\frown\langle a\rangle = \tau$. Then, by DC, there is an $s : \omega \longrightarrow T$ such that $s(0) = \langle\rangle$ and for all $n \in \omega$, $s(n) \prec s(n+1)$. Thus $\bigcup_{n\in\omega} s(n)$ is an infinite branch of $T$.

Now suppose that every tree without terminal nodes has an infinite branch. Let $A$ be a set, $x \in A$, and let $\prec$ be a relation on $A$ such that for all $y \in A$, for some $z \in A$, $y \prec z$. Let $T = \{\langle\rangle\} \cup \{\langle x, a_1, a_2, \dots, a_n\rangle : n \in \omega \text{ and } x \prec a_1 \prec a_2 \prec \dots \prec a_n\}$. Then $T$ is a tree without terminal nodes, so it has an infinite branch $s$. Now $s : \omega \longrightarrow A$, $s(0) = x$, and for all $n \in \omega$, $s(n) \prec s(n+1)$. Thus DC holds.

We now prove that (ii) and (iii) are equivalent.

Suppose that every tree without terminal nodes has an infinite branch. Let $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ be pseudostrategies for player I and player II in some game $G$. Then $S_{\mathrm{I}} \cap S_{\mathrm{II}}$ is a tree without terminal nodes, so it has an infinite branch $x$. Since $x \in [S_{\mathrm{I}}] \cap [S_{\mathrm{II}}]$, it is impossible that both $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ are winning pseudostrategies in $G$.

Now suppose that there is a tree $T$ without terminal nodes that has no infinite branches. Let $G$ be any game on $T$. Then $T$ itself is a winning pseudostrategy for both players in $G$.

**2.7** DEFINITION A **wellfounded tree** is a tree $T$ such that every subtree of $T$ has a terminal node.

Note that for every tree $T$ and every $x \in [T]$, the set $\{x|n : n \in \omega\}$ is a subtree of $T$ without terminal nodes. So wellfounded trees do not have infinite branches.

**2.8** PROPOSITION

**(i)** For every countable tree $T$, $T$ is wellfounded if and only if $[T] = \emptyset$.

**(ii)** DC if and only if every tree without infinite branches is wellfounded.

PROOF Every countable tree can be wellordered. Using a wellordering of a tree $S$ without terminal nodes, one easily constructs an infinite branch of $S$. This proves (i).

To prove (ii), note that the following statements are equivalent:

- every tree $T$ without infinite branches is wellfounded;
- there is no tree $T$ without infinite branches that has a subtree $S$ without terminal nodes;
- there is no tree $S$ without terminal nodes such that $[S] = \emptyset$.

By Proposition 2.6, the last statement is equivalent to DC.

The following characterization of wellfounded trees is well-known.

**2.9** PROPOSITION Let $T$ be a tree. Then $T$ is wellfounded if and only if there is a function $\rho$ from $T$ to the class of all ordinals, such that for every $\sigma \in T$ and every move $a$ in $T$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$.

PROOF Suppose that $\rho$ is a function from $T$ to the class of all ordinals, such that for every $\sigma \in T$ and every move $a$ in $T$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$. Let $S$ be a subtree of $T$. Then $\{\rho(\sigma) : \sigma \in S\}$ is a non-empty set of ordinals, so it has a least element $\alpha$. Choose $\sigma \in S$ such that $\rho(\sigma) = \alpha$. Then for every move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \notin S$, so $\sigma$ is a terminal node of $S$. This proves that $T$ is a wellfounded tree.

Now let $T$ be a tree and let $G$ be the trivial game $(T, \emptyset, \emptyset)$ (so in this game on $T$, player II has to make all moves and wins every play). Now the only pseudostrategy for player I in $G$ is the tree $T$ itself and the pseudostrategies for player II in $G$ are precisely the subtrees of $T$ without terminal nodes. Suppose that $T$ is a wellfounded tree. Then player II has no (winning) pseudostrategy in $G$, so, by Theorem 2.4, $T$ is a pseudostrategy for player I in $G$. Now let $\rho$ be as in the proof of Theorem 2.4.

**2.10** REMARK Let $T$ be a tree and $P \subseteq T$. Then one easily verifies, using the proof of Theorem 2.4, that exactly one of the following statements holds:

- player I has a wellfounded pseudostrategy in $(G, P, \emptyset)$;
- player II has a pseudostrategy in $(G, P, \emptyset)$.

Ordinals can be used to 'measure' the wellfoundedness of a tree.

**2.11** Definition The **tree rank** of a wellfounded tree $T$ is the least ordinal $\alpha$ for which there exists a function $\rho$ from $T$ to the class of all ordinals, such that $\rho(\langle\rangle) = \alpha$ and for every $\sigma \in T$ and every move $a$ in $T$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$.

We can give *proofs* and *definitions by induction* on a wellfounded tree, in the usual way. For instance, we can define for every wellfounded tree $T$ the **rank function** of $T$ as the unique function $\rho$ on $T$ such that for every $\sigma \in T$, $\rho(\sigma)$ is the least ordinal that is larger than all elements of $\{\rho(\sigma^\frown\langle a\rangle) : a \text{ is a move in } T \text{ at } \sigma\}$.

This definition is justified in the usual way by proving by induction on $T$ that every $\tau \in T$ has the following property: There is a unique function $\rho$ on $\{\sigma \in T : \tau \subseteq \sigma\}$ such that for every $\sigma \in \text{domain}(\rho)$, $\rho(\sigma)$ is the least ordinal that is larger than all elements of $\{\rho(\sigma^\frown\langle a\rangle) : a \text{ is a move in } T \text{ at } \sigma\}$.

In such a proof by induction, one shows that for all $\tau \in T$, if for every move $a$ in $T$ at $\tau$, $\tau^\frown\langle a\rangle$ has a certain property $P$, then $\tau$ also has that property. Now one may conclude that every $\tau \in T$ has property $P$, since otherwise $\{\sigma \in T : \text{some } \tau \in T \text{ such that } \sigma \subseteq \tau \text{ does not have property } P\}$ would be a subtree of $T$ without terminal nodes.

## 2D Borel codes and the countable axiom of choice

We have seen that we cannot prove in ZF that every Borel game is determined. But can we prove that every Borel game is pseudodetermined? The answer is no. It is even unprovable in ZF that every Borel game on the countable tree ${}^{<\omega}\omega$ is determined. This follows from the next proposition and the fact that it is consistent with ZF that $\mathbb{R}$ is a countable union of countable sets (see Jech [1973], page 142).

**2.12** Proposition Suppose that ${}^{\omega}\omega$ is a countable union of countable sets. Then there is an undetermined Borel game on the tree ${}^{<\omega}\omega$.

Proof Suppose that ${}^{\omega}\omega = \bigcup_{n\in\omega} A_n$ for some infinite sequence $\langle A_0, A_1, \ldots\rangle$ of countable sets. Then every subset $X$ of ${}^{\omega}\omega$ is Borel (in fact of Borel rank at most 3), since $X = \bigcup_{n\in\omega}(X \cap A_n)$ and for every $n \in \omega$, $X \cap A_n$ is a countable union of finite (and therefore closed) sets. So every game on ${}^{<\omega}\omega$ is a Borel game.

Consider the game in which player I starts by choosing a natural number $n$ and then player II tries to code a bijection $f$ from $\omega$ to $A_n$ by playing the natural numbers $f(i)(j)$ for $i, j \in \omega$ in a standard order. In other words, let $G$ be the game $({}^{<\omega}\omega, \{\langle\rangle\}, X)$, where for all $x \in {}^{\omega}\omega$, $x \in X$ if and only if the function $f : \omega \longrightarrow {}^{\omega}\omega$ defined by $f(i)(j) = x(2^i \cdot (2j+1))$ for all natural numbers $i$ and $j$, is not a bijection from $\omega$ to $A_{x(0)}$.

Player I does not have a winning strategy in $G$: After player I's move $n \in \omega$, player II can code some bijection from $\omega$ to the countable set $A_n$.

But player II does not have a winning strategy in $G$ either: Since $\bigcup_{n\in\omega} A_n$ is uncountable, there is no infinite sequence $\langle f_0, f_1, \ldots\rangle$ such that for every $n \in \omega$, $f_n$ is a bijection from $\omega$ to $A_n$.

So $G$ is an undetermined Borel game on ${}^{<\omega}\omega$.

We will prove in ZF that a Borel game $(T, P, X)$ is pseudodetermined if $X$ has a *Borel code*.

A Borel code tells us how a Borel set is constructed from basic open sets by means of the operations of countable union and countable intersection. There are many ways of formalizing this concept. Our specific choice may seem somewhat awkward, especially in its treatment of the codes of basic open sets, but we will see in Chapter 6 that it easily extends to a definition of codes for so-called 'quasi-Borel sets'.

To increase readability, we denote by $\bot, \top, \vee, \wedge$, and $\diamond$ the five different sets 0, 1, 2, 3, and 4, respectively.

**2.13** DEFINITION Let $T$ be a tree. A **Borel code** with respect to $T$ is a function $c$ from a wellfounded tree $C$ to the set $\{\bot, \top, \vee, \wedge, \diamond\}$ such that for every $\gamma \in C$:

- if $c(\gamma) \in \{\bot, \top\}$, then $\gamma$ is a terminal node of $C$;
- if $c(\gamma) \in \{\vee, \wedge\}$, then the moves in $C$ at $\gamma$ are the natural numbers;
- if $c(\gamma) = \diamond$ then for some $n \in \omega$, the moves in $C$ at $\gamma$ are the elements of $T$ of length $n$, and for every move $\sigma$ in $C$ at $\gamma$, $c(\gamma^\frown\langle\sigma\rangle) \in \{\bot, \top\}$.

We define, by induction on the wellfounded tree $C$, for every $\gamma \in C$ a (Borel) subset $X_\gamma$ of $[T]$ as follows:

- if $c(\gamma)$ is $\bot$ or $\top$ then $X_\gamma$ is the trivial set $\emptyset$ or $[T]$, respectively;
- if $c(\gamma)$ is $\vee$ or $\wedge$ then $X_\gamma$ is $\bigcup_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$ or $\bigcap_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$, respectively;
- if $c(\gamma) = \diamond$ and for some $n \in \omega$, the moves in $C$ at $\gamma$ are the elements of $T$ of length $n$, then $X_\gamma$ is the basic open set $\{x \in [T] : x \in X_{\gamma^\frown\langle x|n\rangle}\}$.

We say that $c$ **is a Borel code for** $X_{\langle\rangle}$.

A **coded Borel** subset of $[T]$ is a set $X$ such that there is a Borel code (with respect to $T$) for $X$.

Note that if $c$ is a Borel code with respect to $T$ for $X$, then $\{(\bot,\top),(\top,\bot),(\vee,\wedge),(\wedge,\vee),(\Diamond,\Diamond)\} \circ c$ is a Borel code for $[T] \setminus X$.

**2.14** Remark Let $c : C \longrightarrow \{\bot,\top,\vee,\wedge,\Diamond\}$ be a Borel code with respect to a tree $T$. For every $x \in [T]$, define $C_x$ as the wellfounded tree $\{\gamma \in C : \text{for all } i < \text{length}(\gamma), \text{ if } c(\gamma|i) = \Diamond \text{ then } \gamma(i) \subseteq x\}$, and let $G_x$ be the game $(C_x, \{\gamma \in C_x : c(\gamma) \in \{\bot,\vee\}\}, \emptyset)$. In other words, $G_x$ is the game that is played as follows:

Each position $\gamma$ belongs to $C$. If $c(\gamma)$ is $\vee$ or $\wedge$, then player I or II, respectively, chooses a natural number; if $c(\gamma) = \Diamond$, then player II 'chooses' the unique move $\sigma$ in $C$ at $\gamma$ such that $\sigma \subseteq x$; if $c(\gamma)$ is $\bot$ or $\top$, then player I or II, respectively, has lost the game.

Define the sets $X_\gamma$ ($\gamma \in C$) as before. One easily verifies that for every $x \in X_{\langle\rangle}$, the tree $\{\gamma \in C_x : \text{for all } n \leq \text{length}(\gamma),\ x \in X_{\gamma|n}\}$ is a pseudostrategy for player I in $G_x$ and for every $x \in [T] \setminus X_{\langle\rangle}$, the tree $\{\gamma \in C_x : \text{for all } n \leq \text{length}(\gamma),\ x \notin X_{\gamma|n}\}$ is a pseudostrategy for player II in $G_x$.

Since the tree $C_x$ is wellfounded and at most countable, this implies that the Borel set $X_{\langle\rangle}$ coded by $c$ could also have been defined as the set $X = \{x \in [T] : \text{player I has a winning strategy in } G_x\}$.

One easily verifies that for every tree $T$ and every set $X$: $X$ is a coded Borel subset of $[T]$ if and only if there is a (finite or) countable ordinal $\alpha$ and, for every ordinal $\beta \leq \alpha$, a subset $X_\beta$ of $[T]$ such that:

- $X = X_\alpha$;
- for every $\beta \leq \alpha$, either $X_\beta$ is a basic open subset of $[T]$, or for some infinite sequence $f : \omega \longrightarrow \beta$ of ordinals smaller than $\beta$, either $X_\beta = \bigcup_{n\in\omega} X_{f(n)}$ or $X_\beta = \bigcap_{n\in\omega} X_{f(n)}$.

The **countable axiom of choice** (CAC) is the statement that every countable collection of non-empty sets has a choice function. The following well-known facts are easily verified:

- AC implies DC;
- DC implies CAC;
- CAC implies that the union of a countable collection of countable sets is countable.

**2.15** Proposition CAC implies that for every tree $T$, every Borel subset of $[T]$ is coded Borel.

Proof Assume CAC. Let $T$ be a tree. By the definition of 'Borel subset', it is enough to show that the collection of all coded Borel subsets of $[T]$ contains all basic open subsets of $[T]$ and is closed under the operations of countable union and countable intersection.

Let $X$ be a basic open subset of $[T]$. Let $n \in \omega$ such that for all $x \in X$, for all $y \in [T]$, if $x|n = y|n$ then $y \in X$. Let $C$ be the wellfounded tree $\{\langle\rangle\} \cup \{\langle\sigma\rangle : \sigma \in T \text{ and length}(\sigma) = n\}$. Define a function $c$ on $C$ as follows:

- $c(\langle\rangle) = \diamond$;
- for every $\sigma \in T$ of length $n$, $c(\langle\sigma\rangle) = \top$ if for some $x \in X$, $x|n = \sigma$, and $c(\langle\sigma\rangle) = \bot$ otherwise.

Then $c$ is a Borel code for $X$. Thus every basic open subset of $[T]$ is coded Borel.

Let $\langle X_0, X_1, \ldots\rangle$ be an infinite sequence of coded Borel subsets of $[T]$. By CAC there is an infinite sequence $\langle c_0, c_1, \ldots\rangle$ such that for every $n \in \omega$, $c_n$ is a Borel code for $X_n$. Let $C$ be the wellfounded tree $\{\langle\rangle\} \cup \{\langle n\rangle^\frown\gamma : n \in \omega \text{ and } \gamma \in \text{domain}(c_n)\}$. Define functions $c$ and $c'$ on $C$ as follows:

- $c(\langle\rangle) = \vee$ and $c'(\langle\rangle) = \wedge$;
- for all $n \in \omega$, for all $\gamma \in \text{domain}(c_n)$, $c(\langle n\rangle^\frown\gamma) = c'(\langle n\rangle^\frown\gamma) = c_n(\gamma)$.

Then $c$ and $c'$ are Borel codes for $\bigcup_{n\in\omega} X_n$ and $\bigcap_{n\in\omega} X_n$, respectively.

So the countable axiom of choice implies that the concepts 'Borel' and 'coded Borel' are equivalent.

Main Theorem Every coded Borel game is pseudodetermined.

In the following chapters we will give several proofs of this result.

## 2E Borel games on countable trees

This section is an elaboration of a remark in Martin [1975], page 368 (see also Martin [1985], page 307). We prove, using CAC, that every Borel game $(T, P, X)$ on a countable tree $T$ is determined. CAC is only needed to get a Borel code for $X$. The proof uses some standard metamathematics concerning classes and relative constructability (see for example Moschovakis [1980], page 531). The idea of the proof is to restrict the universe to some transitive class $M$ in which not only each axiom of ZF holds, but also the axiom of choice. Then, by Theorem 1.24, in $M$ every Borel game is determined. In order to prove that such a game is also determined in the real universe, we need some absoluteness properties.

The *Suslin Theorem* (see Moschovakis [1980], pages 90 and 114) was announced by M. Suslin [1917] and proved by N. Lusin and W. Sierpiński [1918]. It gives a nice characterization of Borel sets, but uses CAC. The following reformulation of the Suslin Theorem is provable in ZF:

**2.16** THEOREM Let $T$ be a countable tree and let $X \subseteq [T]$. Then: $X$ is coded Borel if and only if there are finite or countable trees $A$ and $B$ and strictly increasing functions $f$ and $g$ from $\omega$ to $\omega$ such that $X = \{a \circ f : a \in [A]\}$ and $[T] \setminus X = \{b \circ g : b \in [B]\}$.

PROOF Suppose that $c$ is a Borel code for $X$. Let $C = \text{domain}(c)$ and define for every $\gamma \in C$ a the Borel subset $X_\gamma$ of $[T]$ as in Definition 2.13.

Let $f : \omega \longrightarrow \omega$ be defined by $f(n) = 2n + 1$ for every $n \in \omega$. Define a bijection $\pi : \omega \times \omega \longrightarrow \omega$ by $\pi(n, m) = 2^n \cdot (2m + 1) - 1$ for all $n, m \in \omega$. Let $D$ be the finite or countable tree consisting of all finite sequences $\sigma$ such that $\sigma \circ f \in T$ and for every even $i < \text{length}(\sigma)$, $\sigma(i) \in \omega$.

Define, by induction on the wellfounded tree $C$, for every $\gamma \in C$ a subtree $A_\gamma$ of $D$ such that $X_\gamma = \{a \circ f : a \in [A_\gamma]\}$ as follows:

- if $c(\gamma) = \bot$ then $A_\gamma = \{\langle\rangle\}$, so $\{a \circ f : a \in [A_\gamma]\} = \emptyset = X_\gamma$;
- if $c(\gamma) = \top$ then $A_\gamma = \{\langle a_0, x_0, a_1, x_1, a_2, \ldots\rangle \in D : a_0 = a_1 = a_2 = \cdots = 0\}$, so $\{a \circ f : a \in [A_\gamma]\} = [T] = X_\gamma$;
- if $c(\gamma) = \vee$ then $A_\gamma = \{\langle n, x_0, a_0, x_1, a_1, x_2, \ldots\rangle \in D : \langle a_0, x_0, a_1, x_1, \ldots\rangle \in A_{\gamma\frown\langle n\rangle}\}$, so $\{a \circ f : a \in [A_\gamma]\} = \bigcup_{n\in\omega}\{a \circ f : a \in [A_{\gamma\frown\langle n\rangle}]\} = \bigcup_{n\in\omega} X_{\gamma\frown\langle n\rangle} = X_\gamma$;
- if $c(\gamma) = \wedge$ then $A_\gamma = \{\langle a_0, x_0, a_1, x_1, a_2, \ldots\rangle \in D :$ for all $n \in \omega$, $\langle a_{\pi(n,0)}, x_0, a_{\pi(n,1)}, x_1, a_{\pi(n,2)}, \ldots\rangle \in A_{\gamma\frown\langle n\rangle}\}$; for each $n \in \omega$ and $x \in X_{\gamma\frown\langle n\rangle}$, we can *define* an $a \in [A_{\gamma\frown\langle n\rangle}]$ such that $a \circ f = x$; therefore $\{a \circ f : a \in [A_\gamma]\} = \bigcap_{n\in\omega}\{a \circ f : a \in [A_{\gamma\frown\langle n\rangle}]\} = \bigcap_{n\in\omega} X_{\gamma\frown\langle n\rangle} = X_\gamma$;
- if $c(\gamma) = \Diamond$ and the moves in $C$ at $\gamma$ are the elements of $T$ of length $n$, then $A_\gamma = \{\sigma \in D : \text{if length}(\sigma \circ f) \geq n \text{ then } \sigma \in A_{\gamma\frown\langle\sigma\circ f|n\rangle}\}$, so $\{a \circ f : a \in [A_\gamma]\} = \{a \circ f : a \in [A_{\gamma\frown\langle a\circ f|n\rangle}]\} = \{x \in [T] : x \in X_{\gamma\frown\langle x|n\rangle}\} = X_\gamma$.

Thus $A_{\langle\rangle}$ is a finite or countable tree such that $X = \{a \circ f : a \in [A_{\langle\rangle}]\}$.

Since $[T] \setminus X$ also has a Borel code, we can construct a finite or countable tree $B$ and a strictly increasing function $g$ from $\omega$ to $\omega$ such that $X = \{b \circ g : b \in [B]\}$.

Before proving the other direction, let us mention that by Remark 2.14, a Borel code for $X$ induces, for every $x \in [T]$, some game $G_x$ on a finite or countable, wellfounded tree such that $x \in X$ if and only if player I has a (winning) strategy in $G_x$. We have constructed the tree $A_{\langle\rangle}$ in such a way that there is, for each $\langle x_0, x_1, \ldots\rangle \in [T]$, a bijection from $\{\langle a_0, a_1, \ldots\rangle \in {}^\omega\omega : \langle a_0, x_0, a_1, x_1, \ldots\rangle \in A_{\langle\rangle}\}$ to the set of all strategies for player I in the game $G_{\langle x_0, x_1, \ldots\rangle}$.

The idea of the proof in the other direction is to construct, given that $X = \{a \circ f : a \in [A]\}$ and $[T] \setminus X = \{b \circ g : b \in [B]\}$, a Borel code for $X$ such that, for every $x \in [T]$, the corresponding game $G_x$ is similar to the game that is played as follows:

Player I starts by choosing a $\sigma_0 \in A$ such that $\sigma_0 \circ f = \langle x(0)\rangle$. Then player II chooses a $\sigma_1 \in B$ such that $\sigma_1 \circ g = \langle x(0), x(1)\rangle$. Now player I chooses a $\sigma_2 \in A$ such that $\sigma_0 \subseteq \sigma_2$ and $\sigma_2 \circ f = \langle x(0), x(1), x(2)\rangle$, and so on.

Since player I tries to prove that $x \in X$ by constructing an infinite branch $a$ of $A$ such that $a \circ f = x$ and player II tries to prove that $x \in [T] \setminus X$, this game will end after a finite number of moves.

Proof (continued) Suppose that $A$ and $B$ are finite or countable trees and that $f$ and $g$ are strictly increasing functions from $\omega$ to $\omega$ such that $X = \{a \circ f : a \in [A]\}$ and $[T] \setminus X = \{b \circ g : b \in [B]\}$. Choose a surjection $F$ from $\omega$ onto $A$ and a surjection $G$ from $\omega$ onto $B$.

Let $D$ be the tree $\{\gamma \in {}^{<\omega}\omega : F(\gamma(0)) \subseteq F(\gamma(2)) \subseteq \ldots$ and $G(\gamma(1)) \subseteq G(\gamma(3)) \subseteq \ldots$ and for some $\sigma \in T$, $\text{length}(\sigma) = \text{length}(\gamma)$ and $F(\gamma(0)) \circ f = \sigma|1, G(\gamma(1)) \circ g = \sigma|2, F(\gamma(2)) \circ f = \sigma|3, G(\gamma(3)) \circ g = \sigma|4, \ldots\}$.

Suppose that $d$ is an infinite branch of $D$. Let $a = \bigcup_{n\in\omega} F(d(2n))$ and $b = \bigcup_{n\in\omega} G(d(2n+1))$. Then $a \in [A]$, $b \in [B]$, and $a \circ f = b \circ g$. This is impossible, so $[D] = \emptyset$ and thus, since $D$ is finite or countable, the tree $D$ is wellfounded.

Let $C$ be the tree $\{\langle\rangle\} \cup \{\gamma^\frown\langle n\rangle : \gamma \in D \text{ and } n \in \omega\} \cup \{\gamma^\frown\langle n, \sigma\rangle : \gamma \in D$ and $n \in \omega$ and $\gamma^\frown\langle n\rangle \notin D$ and $\sigma \in T$ and $\text{length}(\sigma) = \text{length}(\gamma)\}$. Then $C$ is also wellfounded.

Let $c$ be the function on $C$ defined as follows:

- for every $\gamma \in D$, $c(\gamma) = \vee$ if the length of $\gamma$ is even, and $c(\gamma) = \wedge$ otherwise;
- for every $\gamma \in D$ and $n \in \omega$ such that $\gamma^\frown\langle n\rangle \notin D$, $c(\gamma^\frown\langle n\rangle) = \Diamond$;
- for every $\gamma \in D$ and $n \in \omega$ such that $\gamma^\frown\langle n\rangle \notin D$ and for every $\sigma \in T$ of the same length as $\gamma$, $c(\gamma^\frown\langle n, \sigma\rangle) = \top$ if the least element of $\omega \setminus \{i < \text{length}(\gamma) : (i \text{ is even and } F(\gamma(i)) \circ f \subseteq \sigma) \text{ or } (i \text{ is odd and } G(\gamma(i)) \circ g \subseteq \sigma)\}$ is odd, and $c(\gamma^\frown\langle n, \sigma\rangle) = \bot$ otherwise.

Then one easily verifies that $c$ is a Borel code. For every $\gamma \in C$, define a Borel subset $X_\gamma$ of $[T]$ as in Definition 2.13. We want to show that $X = X_{\langle\rangle}$.

Let $x \in X$. Then there is an $a \in [A]$ such that $a \circ f = x$. Now one easily proves, by induction on the wellfounded tree $D$, that for every $\gamma \in D$, if for every even $i < \text{length}(\gamma)$, $F(\gamma(i)) \subseteq a$, then $x \in X_\gamma$. Thus $x \in X_{\langle\rangle}$.

Now let $x \in [T] \setminus X$. Then there is a $b \in [B]$ such that $b \circ g = x$. One easily proves, by induction on the wellfounded tree $D$, that for every $\gamma \in D$, if for every odd $i < \text{length}(\gamma)$, $G(\gamma(i)) \subseteq b$, then $x \notin X_\gamma$. Thus $x \notin X_{\langle\rangle}$.

Let $M$ be a *class*, i.e. for some formula $\phi$ in the language of set theory, the expression $x \in M$ is just an abbreviation of $\phi(x)$. $M$ is called *transitive* if for all $x \in M$, for all $y \in x$, $y \in M$.

$M$ is called a *class model* of ZF if for every axiom $\psi$ of ZF, $M \models \psi$, i.e. if we replace all quantifiers $\forall x$ and $\exists x$ in the formula $\psi$ by $\forall x \in M$ and $\exists x \in M$, then the resulting formula holds.

Suppose that $M$ is a transitive class model of ZF and $T \in M$ such that $M \models (T \text{ is a tree})$. Then $T$ is a tree. But if $M \models ([T] = \emptyset)$, then not necessarily $[T] = \emptyset$, since $T$ may have infinite branches $x$ such that $x \notin M$. Note that we chose our definition of Borel code in such a way that, for each $c \in M$, $M \models (c \text{ is a Borel code with respect to } T)$ if and only if $c$ is a Borel code with respect to $T$. This is a consequence of the following lemma.

**2.17** Lemma Let $M$ be a transitive class model of ZF and let $T \in M$. Then $M \models (T \text{ is a wellfounded tree})$ if and only if $T$ is a wellfounded tree.

Proof Suppose that $T$ is a wellfounded tree. Then one easily verifies that $M \models (T \text{ is a tree})$ and for every $S \in M$, $M \models (\text{if } S \text{ is a subtree of } T \text{ then } S \text{ has a terminal node})$. So $M \models (T \text{ is a wellfounded tree})$.

Now suppose that $M \models (T \text{ is a wellfounded tree})$. Then one easily verifies that $T$ is a tree and, since Proposition 2.9 holds in $M$, there is a function $\rho \in M$ from $T$ to the class of all ordinals, such that for every $\sigma \in T$ and every move $a$ in $T$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$. So, by Proposition 2.9, $T$ is a wellfounded tree.

**2.18** Theorem Every coded Borel game on a countable tree is determined.

Proof Let $(T, P, X)$ be a coded Borel game such that the tree $T$ is countable. Then $T$ is isomorphic to a subtree of ${}^{<\omega}\omega$ and we may assume, without loss of generality, that $T \subseteq {}^{<\omega}\omega$.

By Theorem 2.16, there are finite or countable trees $A$ and $B$ and strictly increasing functions $f$ and $g$ from $\omega$ to $\omega$ such that $X = \{a \circ f : a \in [A]\}$ and $[T] \setminus X = \{b \circ g : b \in [B]\}$.

We may assume that also $A$ and $B$ are subtrees of ${}^{<\omega}\omega$.

Let $M = \mathrm{L}[T, P, A, B, f, g]$, the class of all sets constructible from $T$, $P$, $A$, $B$, $f$, and $g$. Then $M$ is the smallest transitive class model of ZF containing the sets $T, P, A, B, f, g$, and all ordinals. Furthermore we have that $M \models$AC, i.e. the axiom of choice holds in $M$. Since Theorem 1.24 holds in $M$, we have that $M \models$(every Borel game is determined).

Let $Y$ and $Z$ be the elements of $M$ for which $M \models (Y = \{a \circ f : a \in [A]\}$ and $Z = \{b \circ g : b \in [B]\})$.

Claim $Y = X \cap M$.

Proof of claim For every $y \in Y$, $y \in M$ and $y = a \circ f$ for some $a \in [A] \cap M$, so $y \in X$.

Now let $y \in X \cap M$. Let $C \in M$ be the tree $\{\sigma \in A : \sigma \circ f \subseteq y\}$. Since $y \in X$, there is an $a \in [A]$ such that $a \circ f = y$, so $a \in [C]$. Thus $C$ is not a wellfounded tree. By Lemma 2.17, $M \models$ ($C$ is not a wellfounded tree). Since $M \models$ ($C$ is countable), we have that $M \models$ ($C$ has an infinite branch), so $y \in Y$.

A similar proof shows that $Z = ([T] \setminus X) \cap M$. Thus $M \models (Z = [T] \setminus Y)$. Since Theorem 2.16 holds in $M$, $M \models$ ($Y$ is a coded Borel subset of $[T]$), so $M \models$(the Borel game $(T, P, Y)$ is determined).

Suppose that there is an $S \in M$ such that $M \models$ ($S$ is a winning strategy for player I in $(T, P, Y)$). (The other case is similar.) Then one easily verifies that $S$ is a strategy for player I in $(T, P, X)$. To see that $S$ is a winning strategy, we have to prove that $S \cap ([T] \setminus X) = \emptyset$ or, equivalently, that the tree $C = \{\sigma \in B : \sigma \circ g \in S\}$ has no infinite branches.

Since $C \in M$ and $M \models$ ($S$ is a winning strategy for player I), we know that $M \models$ ($C$ has no infinite branches). Now $M \models$ ($C$ is finite or countable), so $M \models$ ($C$ is a wellfounded tree). Thus, by Lemma 2.17, $C$ is a wellfounded tree, so $C$ has no infinite branches.

# 3 A proof of Borel pseudodeterminacy using DC

The principle of dependent choices implies that every Borel game has a basic open covering. This follows from Remark 1.21 and a careful inspection of the proof of Lemma 1.22. But DC is weaker than AC, so, by Proposition 2.1, it does not imply that every Borel game is determined.

In this chapter we prove, using DC, that every Borel game is pseudodetermined. We have already proved that every basic open game is pseudodetermined (Theorem 2.5), so we could try to adapt the proof of Lemma 1.22 by replacing 'strategy' by 'pseudostrategy' at all relevant places.

But in the description of $\phi_{\mathrm{I}}(S)$ in Section 1D, it is essential that, for every $\sigma \in S$ of length $k$, there is only one move $A$ in $S$ at $\sigma$. We use this to prove that each $x_0 \in [\phi_{\mathrm{I}}(S)]$ corresponds to an $x_1 \in [S]$. If $S$ is a pseudostrategy for player I in $G_1$, then there may be more than one move in $S$ at $\sigma$, so we would like player I to choose one of these extra moves $A$ in $S$ at position $\sigma$ (without actually playing it) and to proceed as described in Section 1D.

Therefore we introduce a new concept.

## 3A Tactics

A *tactic* for some player in a game is like a pseudostrategy but has auxiliary moves: A player who follows a tactic not only makes moves in the game; he also makes extra moves that serve as an aid to memory.

**3.1** Definition Let $(T, P, X)$ be a game and let $s : \omega \longrightarrow \omega$ be strictly increasing.

An **$s$-tactic for player I in** $(T, P, X)$ is a tree $S$ such that for every $\sigma \in S$, $\sigma \circ s \in T$ and:

- if $\text{length}(\sigma) \notin \text{range}(s)$ or $\sigma \circ s \in P$ then there is a move in $S$ at $\sigma$;
- if $\text{length}(\sigma) \in \text{range}(s)$ and $\sigma \circ s \notin P$ then the moves in $S$ at $\sigma$ are precisely the moves in $T$ at $\sigma \circ s$.

A **winning $s$-tactic for player I in** $(T, P, X)$ is an $s$-tactic $S$ for player I in $(T, P, X)$ such that for all $x \in [S]$, $x \circ s \in X$.

A (**winning**) **$s$-tactic for player II in** $(T, P, X)$ is (winning) $s$-tactic for player I in the game $(T, T \setminus P, [T] \setminus X)$.

The **extra moves** of an $s$-tactic $S$ are the sets of the form $\sigma(n)$ for some $\sigma \in S$ and some $n < \text{length}(\sigma)$ such that $n \notin \text{range}(s)$.

A (**winning**) **tactic** for some player $J$ in $(T, P, X)$ is just a (winning) $t$-tactic for $J$ in $(T, P, X)$, for some strictly increasing $t : \omega \longrightarrow \omega$.

**3.2** EXAMPLE Let $G$ be the game described in Example 1.7. We have seen that player II has a winning (pseudo)strategy in $G$ if and only if there is a choice function on the set $\mathcal{C} = \{A \subseteq {}^{\omega}2 : A \neq \emptyset\}$. But player II can easily win $G$ as follows: After player I's move $A$, player II makes an auxiliary move by choosing an $x \in A$, and then he plays the moves $x(0), x(1), \ldots$.

We give a precise definition of this tactic. Define $s$ as the strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{1\}$; in other words, $s(0) = 0$ and for all $n \in \omega$, $s(n+1) = n+2$. Let $S$ be the tree $\{\langle\rangle\} \cup \{\langle A\rangle : A \in \mathcal{C}\} \cup \{\langle A, x, x(0), x(1), \ldots, x(n-1)\rangle : x \in A \subseteq {}^{\omega}2$ and $n \in \omega\}$.

Then $S$ is a winning $s$-tactic for player II in $G$ whose extra moves are elements of ${}^{\omega}2$.

A more general example is the following. Let $G = (T, P, X)$ be a game such that $\langle\rangle \in P$ and let $s : \omega \longrightarrow \omega$ be strictly increasing. Suppose that for each move $a$ in $T$ at $\langle\rangle$, player II has a winning $s$-tactic in the game $G_a = (\{\tau : \langle a\rangle^\frown\tau \in T\}, \{\tau : \langle a\rangle^\frown\tau \in P\}, \{x : \langle a\rangle^\frown x \in X\})$ (the part of $G$ that is played if player I's first move is $a$). Then player II can win $G$ as follows: After player I's first move $a$, player II makes an auxiliary move by choosing a winning $s$-tactic $S$ in $G_a$ of minimal (set theoretical) rank, and then he continues as if he is following $S$.

In other words, $T^{\leq 1} \cup \{\langle a, S\rangle^\frown\sigma : \langle a\rangle \in T$ and $S$ is a winning $s$-tactic for II in $G_a$ of minimal rank and $\sigma \in S\}$ is a winning $t$-tactic for II in $G$, where $t : \omega \longrightarrow \omega$ is defined by $t(0) = 0$ and for all $n \in \omega$, $t(n+1) = s(n) + 2$.

Let $(T, P, X)$ be a game and let $s : \omega \longrightarrow \omega$ be strictly increasing. Apart from some trivial cases, the collection of $s$-tactics for player I in $(T, P, X)$ is not a set since there is no restriction on the extra moves. Now let $E$ be a set. Then one easily verifies that the (winning) $s$-tactics for player I in $(T, P, X)$ whose extra moves are elements of $E$, are precisely the (winning) pseudostrategies for player I in the game $(T', P', X')$, where:

- $T'$ is the set of all finite sequences $\sigma$ such that $\sigma \circ s \in T$ and for all $n < \mathrm{length}(\sigma)$, if $n \notin \mathrm{range}(s)$ then $\sigma(n) \in E$;
- $P' = \{\sigma \in T' : \mathrm{length}(\sigma) \notin \mathrm{range}(s)$ or $\sigma \circ s \in P\}$;
- $X' = \{x \in [T] : x \circ s \in X\}$.

In particular, if $s$ is the identity on $\omega$, then the (winning) $s$-tactics for some player in a game $G$ are precisely the (winning) pseudostrategies for that player in $G$.

Note that if $S$ is a winning $t$-tactic for player I in $(T', P', X')$ whose extra moves are elements of some set $E'$, then $S$ is a $(t \circ s)$-tactic for I in $(T, P, X)$ with extra moves in $E' \cup E$.

If player II has a winning strategy, pseudostrategy or tactic in $(T, P, X)$, then he can win the game $(T', P', X')$ in the same way, by ignoring the extra moves of player I. By exchanging the roles of the players, we see that each tactic for II in $(T', P', X')$ corresponds to a pseudostrategy for II in some game $(T'', P'', X'')$ in which player II makes extra moves, and each pseudostrategy for I in $(T', P', X')$ corresponds to a pseudostrategy for I in $(T'', P'', X'')$. Thus, if both players have a winning tactic in $(T, P, X)$, then both players have a winning pseudostrategy in some other game.

Now assume that $E$ has some element $e$. Then player I can win $(T', P', X')$ if he has a winning strategy, pseudostrategy or tactic in $(T, P, X)$: He can choose $e$ for his extra moves. For player II, the two games are even 'equivalent': If he has a winning strategy, pseudostrategy or tactic in $(T', P', X')$, then he can win $(T, P, X)$ in the same way, by acting as if player I plays extra moves $e$.

**3.3** Proposition The following statements are equivalent:

- **(i)** not DC;
- **(ii)** in each game on a tree without terminal nodes, both players have a winning tactic;
- **(iii)** there is a game in which both players have a winning tactic.

Proof Suppose that DC does not hold. Then, by Proposition 2.6, there is a tree $D$ without terminal nodes such that $[D] = \emptyset$. Let $G$ be a game on a tree $T$ without terminal nodes. Define strictly increasing functions $t$ and $d$ from $\omega$ to $\omega$ by $t(n) = 2n$ and $d(n) = 2n + 1$ for every $n \in \omega$. Define $S$ as the set of all finite sequences $\sigma$ such that $\sigma \circ t \in T$ and $\sigma \circ d \in D$. Since $T$ and $D$ have no terminal nodes, $S$ has no terminal nodes, so $S$ is an $t$-tactic for both players in $G$. Since $[D] = \emptyset$, $[S] = \emptyset$, so $S$ is winning. (In other words, by making extra moves $e_0, e_1, \ldots$ such that for every $n \in \omega$, $\langle e_0, e_1, \ldots, e_{n-1} \rangle \in D$, a player cannot lose since there is no infinite sequence of extra moves!)

This proves that (i) implies (ii). Since there is a game on a tree without terminal nodes, (ii) implies (iii). From the remarks above it follows that (iii) implies that there is a game in which both players have a winning pseudostrategy, so, by Proposition 2.6, DC does not hold. The following proof is more direct.

Suppose that player I has a winning $s_{\mathrm{I}}$-tactic $S_{\mathrm{I}}$ and player II has a winning $s_{\mathrm{II}}$-tactic $S_{\mathrm{II}}$ in some game $G$. Then these tactics can be combined as follows: Before the first move $a_0$ in $G$ is made by one of the players, player I, who follows $S_{\mathrm{I}}$, plays $s_{\mathrm{I}}(0)$ extra moves and player II, who follows $S_{\mathrm{II}}$, plays his $s_{\mathrm{II}}(0)$ extra moves. Before the second move $a_1$ in $G$ is made, player I plays $s_{\mathrm{I}}(1)-s_{\mathrm{I}}(0)-1$ extra moves according to $S_{\mathrm{I}}$ and player II plays his next $s_{\mathrm{II}}(1)-s_{\mathrm{II}}(0)-1$ extra moves according to $S_{\mathrm{II}}$; and so on. But then, after infinitely many moves, both players would have won the game.

To make this precise, define strictly increasing functions $f_{\mathrm{I}}$, $f_{\mathrm{II}}$, and $f$ from $\omega$ to $\omega$ as follows:

$$f_{\mathrm{I}}(n) = \begin{cases} n & \text{if } n < s_{\mathrm{I}}(0), \\ n + s_{\mathrm{II}}(m) - m & \text{if } m = \max\{k \in \omega : s_{\mathrm{I}}(k) \leq n\}; \end{cases}$$

$$f_{\mathrm{II}}(n) = n + s_{\mathrm{I}}(m) - m, \text{ where } m = \min\{k \in \omega : n \leq s_{\mathrm{II}}(k)\};$$

$$f(n) = s_{\mathrm{I}}(n) + s_{\mathrm{II}}(n) - n.$$

One easily verifies that $f_{\mathrm{II}} \circ s_{\mathrm{II}} = f_{\mathrm{I}} \circ s_{\mathrm{I}} = f$, $\mathrm{range}(f_{\mathrm{II}}) \cap \mathrm{range}(f_{\mathrm{I}}) = \mathrm{range}(f)$, and $\mathrm{range}(f_{\mathrm{II}}) \cup \mathrm{range}(f_{\mathrm{I}}) = \omega$. Define $D$ as the set of all finite sequences $\sigma$ such that $\sigma \circ f_{\mathrm{I}} \in S_{\mathrm{I}}$ and $\sigma \circ f_{\mathrm{II}} \in S_{\mathrm{II}}$.

Let $\sigma \in D$. If $\mathrm{length}(\sigma) \in \mathrm{range}(f_{\mathrm{I}}) \setminus \mathrm{range}(f_{\mathrm{II}})$, then there is a move $a$ in $S_{\mathrm{I}}$ at $\sigma \circ f_{\mathrm{I}}$, so $\sigma^\frown\langle a\rangle \in D$. If $\mathrm{length}(\sigma) \in \mathrm{range}(f)$ and it is player I's turn at position $\sigma \circ f$ in $G$, then there is a move in $S_{\mathrm{I}}$ at $\sigma \circ f_{\mathrm{I}}$ and this is also a move in $S_{\mathrm{II}}$ at $\sigma \circ f_{\mathrm{II}}$. The same holds with I and II interchanged. Therefore $D$ is a tree without terminal nodes. Since both tactics are winning, $D$ has no infinite branch: If $x \in [D]$ then $x \circ f_{\mathrm{I}} \in [S_{\mathrm{I}}]$, so $x \circ f_{\mathrm{I}} \circ s_{\mathrm{I}} \in X$; but also $x \circ f_{\mathrm{II}} \in [S_{\mathrm{II}}]$, so $x \circ f_{\mathrm{II}} \circ s_{\mathrm{II}} \notin X$.

By Proposition 2.6, this contradicts DC.

We have seen in Proposition 2.3 that AC implies that each pseudodetermined game is determined. We noted that AC is not needed if it is a game on a countable tree.

**3.4** Proposition Suppose that some player has a winning tactic in a game $(T, P, X)$. Then AC implies that this player also has a winning strategy in $(T, P, X)$. If $T$ is countable, then DC is sufficient.

Proof Let $S$ be a winning $s$-tactic for some player, say I, in $(T, P, X)$. If AC holds, then we find, just as in the proof of Proposition 2.3, an $s$-tactic $S' \subseteq S$ such that there is exactly one move in $S'$ at each position in $S'$ at which player I has to make a move in $(T, P, X)$ or an extra move. Now one easily verifies that for all $\sigma, \sigma' \in S'$, if $\sigma \circ s = \sigma' \circ s$ then $\sigma \subseteq \sigma'$ or $\sigma' \subseteq \sigma$, and thus $\{\sigma \circ s : \sigma \in S'\}$ is a winning strategy for I in $(T, P, X)$.

Now suppose that $T$ is countable and DC holds. Let $S' \subseteq S$ be an $s$-tactic for player I in $(T, P, X)$ and let $\sigma \in S'$. Then one easily verifies that the following statements are equivalent:

- for every $s$-tactic $S'' \subseteq S'$ for I in $(T, P, X)$, $\sigma \in S''$;

- for all $n < \text{length}(\sigma)$, if $n \notin \text{range}(s)$ or $(\sigma|n) \circ s \in P$ then $\sigma(n)$ is the only move in $S'$ at $\sigma|n$.

Let us call such a $\sigma$ *indispensable for* $S'$.

We can easily find, for every $\sigma \in S'$, an $s$-tactic $S'' \subseteq S'$ for I in $(T, P, X)$ such that $\sigma$ is indispensable for $S''$: Define $S''$ as the set of all $\tau \in S'$ such that for all $\rho$, $a$, and $b$, if $\rho^\frown\langle a\rangle \subseteq \sigma$ and $\rho^\frown\langle b\rangle \subseteq \tau$ and either $\text{length}(\rho) \notin \text{range}(s)$ or $\rho \circ s \in P$, then $a = b$.

Note that for every $\sigma' \in S''$ such that $\sigma' \circ s = \sigma \circ s$, either $\sigma' \subseteq \sigma$ (so $\sigma'$ is also indispensable for $S''$) or $\sigma \subseteq \sigma'$.

Choose $\tau_0, \tau_1, \ldots$ such that $T = \{\tau_n : n \in \omega\}$. Using DC, we find $s$-tactics $S_0, S_1, \ldots$ for I in $(T, P, X)$ such that $S_0 = S$ and for every $n \in \omega$, $S_{n+1} \subseteq S_n$ and each $\sigma \in S_{n+1}$ with $\sigma \circ s = \tau_n$ is indispensable for $S_{n+1}$: If there is no $\sigma \in S_n$ such that $\sigma \circ s = \tau_n$, then we can take $S_{n+1} = S_n$; if $\sigma \in S_n$ of maximal length such that $\sigma \circ s = \tau_n$, then we can take $S_{n+1} = S''$ as above (with $S' = S_n$).

Now put $S' = \bigcap_{n\in\omega} S_n$. Then one easily verifies that $S' \subseteq S$ is an $s$-tactic for player I in $(T, P, X)$ and each $\sigma \in S'$ is indispensable for $S'$. (In other words, $S' \subseteq S$ is a minimal $s$-tactic for player I in $(T, P, X)$.) Therefore $\{\sigma \circ s : \sigma \in S'\}$ is a winning strategy for I in $(T, P, X)$.

Using Proposition 3.3 and the fact that DC is not provable in ZF, it is not difficult to see that it is consistent with ZF that in each game, *at least one* of the players has a winning tactic. But I do not know whether the following is consistent with ZF: In each game, *exactly one* of the players has a winning tactic. By Propositions 3.3 and 3.4, this statement implies both DC and AD, the statement that each game on the tree ${}^{<\omega}2$ is determined.

## 3B Reducing open games to basic open games

Let $G_0 = (T_0, P_0, X_0)$ be an open game and let $k \in \omega$. Define a basic open game $G_1 = (T_1, P_1, X_1)$, sets $\Delta_\sigma$, and a function $p : [T_1] \longrightarrow [T_0]$ as in Section 1D. In that section, we described translators of strategies for both players from $G_1$ to $G_0$. Now we translate tactics for player I from $G_1$ to $G_0$.

So let $s_1 : \omega \longrightarrow \omega$ be strictly increasing and let $S_1$ be an $s_1$-tactic for player I in $G_1$. We are going to describe an $s_0$-tactic $S_0$ for player I in $G_0$, where $s_0 : \omega \longrightarrow \omega$ is defined inductively by

$$s_0(n) = \begin{cases} s_1(n) & \text{if } n < k, \\ s_1(n+2) & \text{if } n = k = 0, \\ s_0(n-1) + 1 + s_1(n+2) & \text{if } n \geq k \text{ and } n > 0. \end{cases}$$

Let $a_0, a_1, \ldots$ denote the moves that are going to be played in $G_0$. For every $n \in \omega$, let $\sigma_n$ denote the finite sequence of extra moves that player I is going to play immediately before I or II has to play move $a_n$. For $n \geq k$, the definition of $s_0$ expresses that length$(\sigma_n) = s_1(n+2)$. In fact, player I will choose $\sigma_n \in S_1$ in such a way that if it is player II's turn at position $\langle a_0, a_1, \ldots, a_{n-1}\rangle$ in $G_0$ and player II makes the move $a_n$, then $\sigma_n{}^\frown\langle a_n\rangle \in S_1$ and if it is player I's turn, then player I can (and does) make a move $a_n$ such that $\sigma_n{}^\frown\langle a_n\rangle \in S_1$.

But first player I follows the $s_1$-tactic $S_1$ (making extra moves when $S_1$ tells him to do so) until a position $\langle a_0, a_1, \ldots, a_{k-1}\rangle$ in $G_0$ is reached.
So $\sigma_0{}^\frown\langle a_0\rangle^\frown\cdots{}^\frown\sigma_{k-1}{}^\frown\langle a_{k-1}\rangle \in S_1$. For each $n \geq k$, we require that $\sigma_n \in S_1$. Thus each extra move of the $s_0$-tactic $S_0$ will be either an extra move of the original $s_1$-tactic $S_1$ or some move in the auxiliary game $G_1$.

If $\langle a_0, a_1, \ldots, a_{k-1}\rangle \notin \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$, then player I chooses $\sigma_k$ such that $\sigma_0{}^\frown\langle a_0\rangle^\frown\cdots{}^\frown\sigma_{k-1}{}^\frown\langle a_{k-1}\rangle \subsetneq \sigma_k$ and $\sigma_k(s_1(k+1)) = 1$ (in other words, player I follows $S_1$ as if in $G_1$, after player I's extra move $A = \sigma_k(s_1(k))$, player II played the extra move 1).

For every $n > k$, as long as $\langle a_0, a_1, \ldots, a_{n-1}\rangle \notin \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$, player I chooses $\sigma_n$ such that $\sigma_{n-1}{}^\frown\langle a_{n-1}\rangle \subseteq \sigma_n$ (in other words, player I continues to follow $S_1$).

Suppose that for some $n \geq k$, $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$. Let $N$ be the least such $n$ and put $\tau = \langle a_0, a_1, \ldots, a_{N-1}\rangle$. Then there is some terminal node $\rho$ of $S_1$ of length $s_1(N+2)$ such that for some $A \subseteq \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$, $\rho \circ s_1 = \langle a_0, a_1, \ldots, a_{k-1}, A, 1, a_k, \ldots, a_{N-1}\rangle$. Since $S_1$ is an $s_1$-tactic for player I, it must be player II's turn at position $\rho \circ s_1$ in $G_1$, so $\tau \in A$. Player I can now choose $\sigma_N$ such that $\sigma_0{}^\frown\langle a_0\rangle^\frown\cdots{}^\frown\sigma_{k-1}{}^\frown\langle a_{k-1}\rangle \subsetneq \sigma_N$ and $\sigma_N(s_1(k+1)) = \tau$ (in other words, player I follows $S_1$ as if in $G_1$, after player I's extra move $A$, player II played the extra move $\tau$ and as if the moves $a_k, \ldots, a_{N-1}$ were obligatory).

For every $n > N$, player I chooses $\sigma_n$ such that $\sigma_{n-1}{}^\frown\langle a_{n-1}\rangle \subseteq \sigma_n$.

We now give a more formal definition of $S_0$.
Put $l = \begin{cases} 0 & \text{if } k = 0, \\ s_1(k-1)+1 & \text{if } k > 0. \end{cases}$ Then $S_0$ is the set of all finite sequences $\sigma$ such that either $\sigma \in S_1{}^{\leq l}$ or, if we write $\sigma \circ s_0 = \langle a_0, a_1, \ldots, a_{m-1}\rangle$, then $m \geq k$ and there is, for every $n$ such that $k \leq n \leq m$, a $\sigma_n \in S_1$ of length $s_1(n+2)$, such that $\sigma \subseteq (\sigma|l)^\frown\sigma_k{}^\frown\langle a_k\rangle^\frown\cdots{}^\frown\sigma_{m-1}{}^\frown\langle a_{m-1}\rangle^\frown\sigma_m$ and:

- if for each $n$ such that $k \leq n \leq m$, $\langle a_0, a_1, \ldots, a_{n-1}\rangle \notin \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$, then $\sigma_k(s_1(k+1)) = 1$ and $\sigma|l \subseteq \sigma_k{}^\frown\langle a_k\rangle \subseteq \cdots \subseteq \sigma_{m-1}{}^\frown\langle a_{m-1}\rangle \subseteq \sigma_m$;

- if $N$ is the least $n$ such that $k \leq n \leq m$ and $\langle a_0, a_1, \ldots, a_{n-1}\rangle \in \Delta_{\langle a_0,a_1,\ldots,a_{k-1}\rangle}$, then $\sigma_N(s_1(k+1)) = \langle a_0, a_1, \ldots, a_{N-1}\rangle$ and $\sigma|l \subseteq \sigma_N{}^\frown\langle a_N\rangle \subseteq \cdots \subseteq \sigma_{m-1}{}^\frown\langle a_{m-1}\rangle \subseteq \sigma_m$ and if $k < N$ then $\sigma_k(s_1(k+1)) = 1$ and $\sigma|l \subseteq \sigma_k{}^\frown\langle a_k\rangle \subseteq \cdots \subseteq \sigma_{N-1}{}^\frown\langle a_{N-1}\rangle \in S_1$.

Just as in Section 1D, we want to show that if for some $Y_0 \subseteq [T_0]$, $S_1$ is a winning $s_1$-tactic for player I in the game $(T_1, P_1, p^{-1}Y_0)$, then $S_0$ is a winning

$s_0$-tactic for player I in the game $(T_0, P_0, Y_0)$. To see this, let $x_0 \in [S_0]$. It suffices to find some infinite branch $x_1$ of $S_1$ such that $p(x_1 \circ s_1) = x_0 \circ s_0$. Write $\langle a_0, a_1, \ldots \rangle$ for the infinite branch $x_0 \circ s_0$ of $T_0$. There are unique $\sigma_k, \sigma_{k+1}, \ldots \in S_1$ such that for every $n \geq k$, length$(\sigma_n) = s_1(n+2)$, and $x_0 = (x_0|l)^\frown \sigma_k{}^\frown \langle a_k \rangle^\frown \sigma_{k+1}{}^\frown \langle a_{k+1} \rangle^\frown \cdots$. We distinguish two cases.

If $\langle a_0, a_1, \ldots \rangle \notin X_0$, then there is no $n \geq k$ such that $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \Delta_{\langle a_0, a_1, \ldots, a_{k-1} \rangle}$, so $x_0|l \subseteq \sigma_k{}^\frown \langle a_k \rangle \subseteq \sigma_{k+1}{}^\frown \langle a_{k+1} \rangle \subseteq \cdots$. Thus, if we put $x_1 = \bigcup_{n \geq k} \sigma_n$, then $x_1 \in [S_1]$ and $p(x_1 \circ s_1) = \langle a_0, a_1, \ldots \rangle$.

If $\langle a_0, a_1, \ldots \rangle \in X_0$ and $N$ is the least $n \geq k$ such that $\langle a_0, a_1, \ldots, a_{n-1} \rangle \in \Delta_{\langle a_0, a_1, \ldots, a_{k-1} \rangle}$, then $x_0|l \subseteq \sigma_N{}^\frown \langle a_N \rangle \subseteq \sigma_{N+1}{}^\frown \langle a_{N+1} \rangle \subseteq \cdots$. Thus, if we put $x_1 = \bigcup_{n \geq N} \sigma_n$, then $x_1 \in [S_1]$ and $p(x_1 \circ s_1) = \langle a_0, a_1, \ldots \rangle$.

Note that in both cases, for all $n \in \omega$, if $n < k$ then $x_1|(s_1(n)) = x_0|(s_0(n))$.

## 3C Reducing Borel games to basic open games

In Chapter 1 we proved, using AC, that in every Borel game player I or player II has a winning strategy. In a similar way, we now prove, using DC, that in every Borel game either player I has a winning tactic or player II has a winning pseudostrategy.

We have to adjust some definitions since the role of strategies is taken over by pseudostrategies and tactics. The following definition is similar to Definition 1.15.

**3.5** Definition Let $G = (T, P, X)$ be a game. Let $n \in \omega$ and let $\sigma : n+1 \longrightarrow \omega$ be strictly increasing.
A **$\sigma$-tactic for player I in** $G$ is an $s$-tactic for player I in the game $(T^{\leq n+1}, T^{\leq n} \cap P, \emptyset)$, where $s$ is any strictly increasing function from $\omega$ to $\omega$ such that $\sigma \subseteq s$ and $s(n+1) = \sigma(n) + 1$.

A **$\sigma$-tactic for player II in** $G$ is a $\sigma$-tactic for player I in the game $(T, T \setminus P, [T] \setminus X)$.

Let $J$ be a player. A **pseudostrategy for player** $J$ **in** $G$ **up to positions of length** $n$ is a $\tau$-tactic for player $J$ in $G$, where $\tau$ is the identity on $n+1$.

One easily verifies that for every game $G$, tree $S$, and strictly increasing function $s : \omega \longrightarrow \omega$:
$S$ is an $s$-tactic for some player in $G$ if and only if for all $n \in \omega$, $S^{\leq s(n)+1}$ is an $s|(n+1)$-tactic for that player in $G$.

**3.6** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing and consider games $G_0 = (T_0, P_0, X_0)$ and $G_1 = (T_1, P_1, X_1)$ such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Let $J$ be one of the players.

(**Continuous**) **$f$-translators of pseudostrategies for player $J$ from $G_1$ to $G_0$** are defined as in Definition 1.16, with pseudostrategies instead of strategies.

Let $E$ be a set. An **$f$-translator of tactics for player $J$ with extra moves in $E$, from $G_1$ to $G_0$** is a function $\phi$ that assigns to every pair $(s_1, S_1)$ for which $s_1 : \omega \longrightarrow \omega$ is strictly increasing and $S_1$ is an $s_1$-tactic for player $J$ in $G_1$ whose extra moves are in $E$, a pair $(s_0, S_0)$ such that $s_0 : \omega \longrightarrow \omega$ is strictly increasing, $S_0$ is an $s_0$-tactic for player $J$ in $G_0$ whose extra moves are in $E$, and for every infinite branch $x_0$ of $S_0$, there is an infinite branch $x_1$ of $S_1$ such that $x_1 \circ s_1 \circ f = x_0 \circ s_0$ and for all $n \in \omega$, if $f(n) = n$ then $x_1|(s_1(n)) = x_0|(s_0(n))$.

We say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ such that:

**(i)** for every $n \in \omega$, $\phi_n$ is a function that assigns to every pair $(\sigma_1, S_1)$ for which $\sigma_1 : f(n) + 1 \longrightarrow \omega$ is strictly increasing and $S_1$ is a $\sigma_1$-tactic for player $J$ in $G_1$ whose extra moves are in $E$, a pair $(\sigma_0, S_0)$ such that $\sigma_0 : n + 1 \longrightarrow \omega$ is strictly increasing and $S_0$ is a $\sigma_0$-tactic for player $J$ in $G_0$ whose extra moves are in $E$, and for every $m < n$, $(\sigma_0|(m+1), S_0{}^{\leq \sigma_0(m)+1}) = \phi_m(\sigma_1|(f(m)+1), S_1{}^{\leq \sigma_1(f(m))+1})$;

**(ii)** for every $m \in \omega$ and $(s_1, S_1) \in \text{domain}(\phi)$, if $\phi((s_1, S_1)) = (s_0, S_0)$ then $(s_0|(m + 1), S_0{}^{\leq s_0(m)+1}) = \phi_m(s_1|(f(m) + 1), S_1{}^{\leq s_1(f(m))+1})$.

Note that if $\phi_0, \phi_1, \ldots$ are functions such that (i) holds, then there is a unique function $\phi$ from $\{(s_1, S_1) : S_1$ is an $s_1$-tactic for player $J$ in $G_1$ whose extra moves are in $E\}$ to $\{(s_0, S_0) : S_0$ is an $s_0$-tactic for player $J$ in $G_0$ whose extra moves are in $E\}$ such that (ii) holds.

The following definition is similar to Definition 1.17.

**3.7** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_0 = (T_0, P_0, X_0)$ be a game.

A **tactical $f$-covering of $G_0$** is a game $G_1 = (T_1, P_1, X_1)$ such that:

**(i)** $T_0 = \{\sigma \circ f : \sigma \in T_1\}$;

**(ii)** $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$;

**(iii)** there is an $f$-translator $\phi$ of pseudostrategies for player II from $G_1$ to $G_0$ and there are functions $\phi_0, \phi_1, \ldots$ witnessing that $\phi$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then $T_1{}^{\leq n+1} = T_0{}^{\leq n+1}$, $P_1 \cap T_1{}^{\leq n} = P_0 \cap T_0{}^{\leq n}$ and for every $S \in \text{domain}(\phi_n)$, $\phi_n(S) = S$;

**(iv)** for every set $E$ such that $T_1 \subseteq {}^{<\omega}E$, there is an $f$-translator of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$ and there are functions $\phi_0, \phi_1, \ldots$ witnessing that $\phi$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then for every $(\sigma, S) \in \text{domain}(\phi_n)$, $\phi_n((\sigma, S)) = (\sigma, S)$.

The following lemmas correspond to similar lemmas in Chapter 1.

**3.8** LEMMA

**(i)** Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_1$ be a tactical $f$-covering of some game $G_0$. Suppose that in $G_1$, player I has a winning tactic or player II has a winning pseudostrategy. Then also in $G_0$, player I has a winning tactic or player II has a winning pseudostrategy.

**(ii)** Suppose that $(T_1, P_1, X_1)$ is a tactical $f$-covering of $(T_0, P_0, X_0)$ and $Y_0 \subseteq [T_0]$. Put $Y_1 = \{x \in [T_1] : x \circ f \in Y_0\}$. Then $(T_1, P_1, Y_1)$ is a tactical $f$-covering of $(T_0, P_0, Y_0)$.

**(iii)** Let $G_1$ be a tactical $f$-covering of $G_0$ and let $G_2$ be a tactical $g$-covering of $G_1$. Then $G_2$ is a tactical $(g \circ f)$-covering of $G_0$.

PROOF These statements follow directly from the definition of 'tactical covering':

**(i)** Let $G_1$ be a tactical $f$-covering of some game $G_0$.
First suppose that player II has a winning pseudostrategy $S$ in $G_1$. Let $\phi$ be an $f$-translator of pseudostrategies for player II from $G_1$ to $G_0$. Then we see, just as in the proof of Lemma 1.18(i), that $\phi(S)$ is a winning pseudostrategy for player II in $G_0$.
Now suppose that player I has a winning $s_1$-tactic $S_1$ in $G_1$. Let $T_1$ be the tree on which $G_1$ is a game and put $E = \{a : a \text{ is a move in } T_1 \text{ at some } \sigma \in T_1 \text{ or } a \text{ is a move in } S_1 \text{ at some } \sigma \in S_1 \text{ whose length is not in range}(s_1)\}$. Then the extra moves of $S_1$ are in $E$ and $T_1 \subseteq {}^{<\omega}E$, so there is an $f$-translator $\phi$ of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$. Put $(s_0, S_0) = \phi((s_1, S_1))$. For every $x_0 \in [S_0]$, there is an $x_1 \in [S_1]$ such that $x_1 \circ s_1 \circ f = x_0 \circ s_0$. Since $S_1$ is winning, $x_1 \circ s_1$ is in the winning set for player I in $G_1$, so $x_1 \circ s_1 \circ f$ is in the winning set for player I in $G_0$. This shows that the $s_0$-tactic $S_0$ for player I in $G_0$ is winning.

**(ii)** This is trivial, since the sets $X_0$ and $X_1$ are dummies in Definition 3.6.

**(iii)** To prove this, use the same simple facts as in the proof of Lemma 1.18(iii), but with 'pseudostrategies for player II' and 'tactics for player I with extra moves in some set $E$' instead of 'strategies for some player'.

**3.9** Lemma Let $G_0$ be a game and suppose that for every $n \in \omega$, a tactical $f_n$-covering $G_{n+1}$ of $G_n$ is given. Assume that for every $n \in \omega$, $\lim_{m\to\omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ exists and is equal to $g_n$.

Then DC implies that there is a (unique) game $G$ such that for every $n \in \omega$, $G$ is a tactical $g_n$-covering of $G_n$.

Proof Suppose that DC holds, so CAC holds as well. Define a game $G = (T, P, X)$ just as in the proof of Lemma 1.19 and note that for all $n \in \omega$, $T_n = \{\sigma \circ g_n : \sigma \in T\}$. By CAC, we can choose, for every $m \in \omega$, an $f_m$-translator of pseudostrategies for player II from $G_{m+1}$ to $G_m$ and functions witnessing its continuity. Using DC, we find for every $n \in \omega$ a continuous $g_n$-translator of pseudostrategies for player II from $G$ to $G_n$ in the same way as in the proof of Lemma 1.19.

Let $E$ be a set such that $T \subseteq {}^{<\omega}E$. Then for all $m \in \omega$, $T_{m+1} \subseteq {}^{<\omega}E$, so we can choose, using CAC, for every $m \in \omega$, an $f_m$-translator $\phi_m$ of tactics for player I with extra moves in $E$, from $G_{m+1}$ to $G_m$ and functions $\phi^0_m, \phi^1_m, \ldots$ witnessing that $\phi_m$ is continuous, such that for every $i \in \omega$, if $f_m(i) = i$, then for every $(\sigma, S) \in \text{domain}(\phi^i_m)$, $\phi^i_m((\sigma, S)) = (\sigma, S)$.

Let $n \in \omega$. Just as in the proof of Lemma 1.19, we define, for every $i \in \omega$, a function $\psi^i_n$ by $\psi^i_n = \phi^i_n \circ \phi^{f_n(i)}_{n+1} \circ \cdots \circ \phi^{f_{m-1}\circ\cdots\circ f_n(i)}_m$ for all large $m$. We must show that the function $\psi_n$ that is induced by these functions, is a continuous $g_n$-translator of tactics for player I with extra moves in $E$, from $G$ to $G_n$.

So let $s : \omega \longrightarrow \omega$ be strictly increasing and let $S$ be an $s$-tactic for player I in $G$ whose extra moves are in $E$. For every $m \geq n$, put $(s_m, S_m) = \psi_m((s, S))$, so $(s_m, S_m) = \phi_m((s_{m+1}, S_{m+1}))$. Note that for every $i \in \omega$, if $g_m(i) = i$ then $s_m|(i+1) = s|(i+1)$ and $S_m^{\leq s_m(i)+1} = S^{\leq s(i)+1}$, since $\psi^i_m$ is the identity on its domain.

Now let $x_n \in [S_n]$. We must find an $x \in [S]$ such that $x \circ s \circ g_n = x_n \circ s_n$ and for all $i \in \omega$, if $g_n(i) = i$ then $x|s(i) = x_n|s_n(i)$.

Using DC, we find, for every $m \geq n$, an $x_{m+1} \in [S_{m+1}]$ such that $x_{m+1} \circ s_{m+1} \circ f_m = x_m \circ s_m$ and for all $i \in \omega$, if $f_m(i) = i$ then $x_{m+1}|s_{m+1}(i) = x_m|s_m(i)$.

Since for all $i \in \omega$, for all large $m$, $g_m(i) = f_m(i) = i$, there is a unique infinite sequence $x$ such that for every $i \in \omega$, for all large $m$, $x|s(i) = x_m|s_m(i)$ and we have that $x \in [S]$.

Note that for every $i \in \omega$, if $g_n(i) = i$ then for every $m \geq n$, $f_m(i) = i$, so $x|s(i) = x_n|s_n(i)$. In order to see that $x \circ s \circ g_n = x_n \circ s_n$, let $i \in \omega$ and put $k = g_n(i)$. Let $m \in \omega$ be so large that $x(s(k)) = x_m(s(k))$ and $k = f_{m-1} \circ \cdots \circ f_{n+1} \circ f_n(i)$. Then $g_m(k) = k$, so $s(k) = s_m(k)$. Since $x_n \circ s_n = x_{n+1} \circ s_{n+1} \circ f_n = \ldots = x_m \circ s_m \circ f_{m-1} \circ \cdots \circ f_{n+1} \circ f_n$, we have that $x_n \circ s_n(i) = x_m \circ s_m(k) = x_m(s(k)) = x \circ s \circ g_n(i)$.

**3.10** LEMMA Let $G_0$ be an open or closed game. Let $k \in \omega$ and define $f$ as the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k, k+1\}$. Then there is a tactical $f$-covering $G_1$ of $G_0$ such that the game $G_1$ is basic open.

PROOF By Lemma 3.8(ii), it is enough to consider the case that the game $G_0 = (T_0, P_0, X_0)$ is open. Define a basic open game $G_1 = (T_1, P_1, X_1)$ as in Section 1D.

Define, for every pseudostrategy $S$ for player II in $G_1$, a pseudostrategy $\phi_{\mathrm{II}}(S)$ for player II in $G_0$ just as in Section 1D, after adapting the definition of the function $F$ as described in Remark 1.21 in order to avoid the use of the axiom of choice. Since we are dealing with pseudostrategies instead of strategies, the sets $D_\alpha$ in that remark may now have more than one element, but this makes no essential difference. In the same way as in Lemma 1.20 we prove that $\phi_{\mathrm{II}}$ is a continuous $f$-translator of pseudostrategies for player II from $G_1$ to $G_0$.

Let $E$ be a set such that $T_1 \subseteq {}^{<\omega}E$. We define an $f$-translator of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$ as follows: For every strictly increasing $s_1 : \omega \longrightarrow \omega$ and every $s_1$-tactic $S_1$ for player I in $G_1$ whose extra moves are in $E$, let $\phi_{\mathrm{I}}((s_1, S_1))$ be the pair $(s_0, S_0)$ as defined in Section 3B.

Define, for every $n \in \omega$, a function $\phi_{\mathrm{I}}^n$ just like $\phi_{\mathrm{I}}$ but now on the set of all pairs $(\sigma_1, S_1)$ for which $\sigma_1 : f(n)+1 \longrightarrow \omega$ is strictly increasing and $S_1$ is a $\sigma_1$-tactic for player I in $G_1$ whose extra moves are in $E$. Then for every $n < k$, $\phi_{\mathrm{I}}^n$ is the identity on its domain and the functions $\phi_{\mathrm{I}}^0, \phi_{\mathrm{I}}^1, \ldots$ witness that $\phi_{\mathrm{I}}$ is continuous.

Thus the basic open game $G_1$ is a tactical $f$-covering of the open game $G_0$.

**3.11** LEMMA Let $G_0$ be a Borel game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that for infinitely many natural numbers $k$, neither $k$ nor $k+1$ is in the range of $f$.

Then DC implies that there is a basic open tactical $f$-covering $G_1$ of $G_0$.

PROOF Let $G_0 = (T_0, P_0, X_0)$ be a basic open game and let $f : \omega \longrightarrow \omega$ be strictly increasing. In the proof of Lemma 1.22, we constructed, for every basic open game $G_0$ and every $f \in \Omega$, a basic open $f$-covering $G_1 = (T_1, P_1, X_1)$ of $G_0$ by inserting some trivial extra move at the right places. A continuous $f$-translator $\phi_{\mathrm{II}}$ of pseudostrategies for player II from $G_1$ to $G_0$ can easily been defined by $\phi_{\mathrm{II}}(S) = \{\sigma \circ f : \sigma \in S\}$ for every pseudostrategy $S$ for II in $G_1$. Let $E$ be a set such that $T_1 \subseteq {}^{<\omega}E$. Then a continuous $f$-translator $\phi_{\mathrm{I}}$ of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$ can easily be defined by $\phi_{\mathrm{I}}((s,S)) = (s\circ f, S)$ for every strictly increasing $s : \omega \longrightarrow \omega$ and every $s$-tactic $S$ for player I in $G_1$: Each move 0 in the $s$-tactic $S$ that corresponds to an inserted trivial move in the game $G_1$, is considered to be an extra move of the $(s \circ f)$-tactic $S$ for player I in $G_0$.

This shows that $G_1$ is a tactical $f$-covering of $G_0$.

Now the rest of the proof is the same as the proof of Lemma 1.22, with 'tactical covering' instead of 'covering', DC instead of AC, and Lemmas 3.10, 3.8(iii), 3.8(ii), and 3.9 instead of Lemmas 1.20, 1.18(iii), 1.18(ii), and 1.19, respectively.

**3.12** THEOREM DC implies that every Borel game is pseudodetermined.

PROOF Suppose that DC holds and let $(T, P, X)$ be a Borel game. Define $f : \omega \longrightarrow \omega$ by $f(n) = 3n$. Then for all natural numbers $n$, neither $3n + 1$ nor $3n + 2$ is in the range of $f$.

Thus, by DC and Lemma 3.11, there is a basic open tactical $f$-covering $G$ of $(T, P, X)$. By Theorem 2.5, the basic open game $G$ is pseudodetermined. Since every pseudostrategy is a tactic, player I has a winning tactic or player II has a winning pseudostrategy in $G$. So, by Lemma 3.8(i), player I has a winning tactic or player II has a winning pseudostrategy in the Borel game $(T, P, X)$.

Since $(T, T \setminus P, [T] \setminus X)$ is a Borel game as well, we also have that player I has a winning tactic or player II has a winning pseudostrategy in $(T, T \setminus P, [T] \setminus X)$. This implies that in $(T, P, X)$, player II has a winning tactic or player I has a winning pseudostrategy.

So either both players have a winning tactic in $(T, P, X)$ or at least one of the players has a winning pseudostrategy in $(T, P, X)$. The first possibility is ruled out by Proposition 3.3 and DC. Thus the Borel game $(T, P, X)$ is pseudodetermined.

# 4 A proof of Borel pseudodeterminacy using CAC

In this chapter we prove, using the countable axiom of choice, that every Borel game is pseudodetermined.

We only use CAC to get a Borel code for the winning set for player I. Using such a code, we will define a basic open game similar to the auxiliary game in Chapters 1 and 3. Instead of trying to find translators of *all* pseudostrategies from that basic open game to the Borel game, we will only translate pseudostrategies that satisfy some 'reasonable' condition. Unlike the condition of being a strategy, this condition is so weak that we can prove, without using the axiom of choice, that in each basic open game one of the players has a winning pseudostrategy that satisfies this condition.

## 4A Preference relations and standard games

Player I may prefer starting a game $(T, P, X)$ at some position $\sigma$ to starting it at another position $\sigma'$, for example if the part of the game from $\sigma$ on is the same as the part from $\sigma'$ on, except that some of player II's moves are restricted. It is clear that if player I can win the game, starting at position $\sigma'$, then he can also win the game, starting at position $\sigma$. Of course, player II will prefer starting at $\sigma'$ to starting at $\sigma$: In the part of the game from $\sigma'$ on, he has more moves to choose from than in the part from $\sigma$ on.

This idea of comparing positions in a game leads to the following definition.

**4.1** Definition Let $G = (T, P, X)$ be a game. A **preference relation** in $G$ is a relation $R$ on $T$ such that for all $\sigma, \sigma'$, if $\sigma \mathrel{R} \sigma'$ then either it is player I's turn at both $\sigma$ and $\sigma'$ and for each move $a'$ in $T$ at $\sigma'$ there is a move $a$ in $T$ at $\sigma$ such that $\sigma^\frown\langle a\rangle \mathrel{R} \sigma'^\frown\langle a'\rangle$, or it is player II's turn at both $\sigma$ and $\sigma'$ and for each move $a$ in $T$ at $\sigma$ there is a move $a'$ in $T$ at $\sigma'$ such that $\sigma^\frown\langle a\rangle \mathrel{R} \sigma'^\frown\langle a'\rangle$.

Note that the inverse of a preference relation in a game $(T, P, X)$ is a preference relation in the game $(T, T \setminus P, [T] \setminus X)$. Also note that the identity on $T$ is a preference relation and that the set of preference relations in a given game is closed under composition and arbitrary unions. Thus, if $R$ is a preference relation, then the smallest reflexive, transitive relation $R'$ on $T$ for which $R \subseteq R'$, is also a preference relation.

**4.2** Remark Let $R$ be a preference relation in some game $(T, P, X)$. Suppose that for all $x, x' \in [T]$ and $m, m' \in \omega$, if $x' \in X$ and for all $n \in \omega$, $x|(m+n) \mathrel{R} x'|(m'+n)$, then $x \in X$.

For every $\sigma \in T$, let $G_\sigma$ be the part of $(T, P, X)$ that is played after position $\sigma$ is reached. In other words, $G_\sigma = (\{\rho : \sigma^\frown\rho \in T\}, \{\rho : \sigma^\frown\rho \in P\}, \{r : \sigma^\frown r \in X\})$. Suppose that $\sigma \mathrel{R} \sigma'$. Then one easily verifies that if player I has a winning tactic in $G_{\sigma'}$ then he also has one in $G_\sigma$, and if player II has a winning tactic in $G_\sigma$ then he also has one in $G_{\sigma'}$. The idea is to act as if both games are played simultaneously such that at each moment the relation $R$ holds between the two corresponding positions in $T$.

The same holds when we replace 'winning tactic' by 'pseudostrategy' or 'wellfounded pseudostrategy of rank at most $\alpha$' for some ordinal $\alpha$. Thus if $\sigma \mathrel{R} \sigma'$, then, informally speaking, player I will prefer $G_\sigma$ to $G_{\sigma'}$ and player II will prefer $G_{\sigma'}$ to $G_\sigma$.

For convenience, we consider games in which the players alternately make a move and player I starts. In such games, it is player I's turn at positions of even length.

**4.3** Definition Let $T$ be a tree. We denote the set of all elements of $T$ of even length by $T^{\text{even}}$. A **standard game on** $T$ is a game of the form $(T, T^{\text{even}}, X)$.

For every game $(T, P, X)$, we can easily find an 'equivalent' standard game which is played as follows: If $\langle\rangle \in P$ then player I starts by choosing a move $a_0$ in $T$ at $\langle\rangle$ and player II makes an obligatory move 0; if $\langle\rangle \notin P$ then player I starts by making an obligatory move 0 and player II chooses a move $a_0$ in $T$ at $\langle\rangle$. The players continue in this way. So each position of length $2n$ in this standard game corresponds to some position of length $n$ in the game $(T, P, X)$ and each play in this standard game corresponds to some play in the game $(T, P, X)$. One easily verifies that this standard game is pseudodetermined or (coded) Borel if and only if the game $(T, P, X)$ is pseudodetermined or (coded) Borel, respectively.

Given a preference relation $R$ in some standard game, we will define an $R$-pseudostrategy as a pseudostrategy $S$ such that if certain positions are in $S$, then also certain 'easier' positions are in $S$. In fact, for each $\sigma \in S^{\text{even}}$, we only look at the positions of the form $\sigma^\frown\langle a, b\rangle$ and at the relation $R$ between these positions.

If $S$ is a pseudostrategy for player II, then we simply require that each position $\sigma^\frown\langle a', b'\rangle$ that is 'easier' than some $\sigma^\frown\langle a, b\rangle \in S$, is also in $S$.

If $S$ is a pseudostrategy for player I and $\sigma^\frown\langle a, b\rangle \in S$, then each position of the form $\sigma^\frown\langle a, b'\rangle$ is in $S$. Therefore we require that, if for some $a$, *each* position of the form $\sigma^\frown\langle a, b\rangle$ is 'easier' than some position in $S$, then each position of this form is in $S$.

**4.4** DEFINITION Let $R$ be a preference relation in some standard game $G = (T, T^{\text{even}}, X)$. An $R$**-pseudostrategy for I** in $G$ is a pseudostrategy $S$ for player I in $G$ such that for every $\sigma \in S^{\text{even}}$ and for every move $a$ in $T$ at $\sigma$, if for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$ there are $a', b'$ such that $\sigma^\frown\langle a', b'\rangle \in S$ and $\sigma^\frown\langle a, b\rangle \mathrel{R} \sigma^\frown\langle a', b'\rangle$, then for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, $\sigma^\frown\langle a, b\rangle \in S$.

An $R$**-pseudostrategy for II** in $G$ is a pseudostrategy $S$ for player II in $G$ such that for every $\sigma \in S^{\text{even}}$ and for all $a', b'$ such that $\sigma^\frown\langle a', b'\rangle \in T$, if there are $a, b$ such that $\sigma^\frown\langle a, b\rangle \in S$ and $\sigma^\frown\langle a, b\rangle \mathrel{R} \sigma^\frown\langle a', b'\rangle$, then $\sigma^\frown\langle a', b'\rangle \in S$.

We now show, assuming the axiom of choice, that the conditions in the definition are 'reasonable' if $X$ and $R$ are related in a certain way.

**4.5** PROPOSITION Let $R$ be a preference relation in some standard game $G = (T, T^{\text{even}}, X)$. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that for all $x, x' \in [T]$, if $x \circ f = x' \circ f$ and $x' \in X$ then $x \in X$. Suppose that for all $\sigma, \sigma'$, if $\sigma \mathrel{R} \sigma'$ then $\sigma \circ f = \sigma' \circ f$. Then AC implies that each player that has a winning pseudostrategy in $G$ also has a winning $R$-pseudostrategy in $G$.

PROOF We only use AC to find a wellordering $\prec$ of $T$ (so AC is not needed if $T$ is countable). Note that if $R'$ is the smallest reflexive and transitive relation on $T$ such that $R \subseteq R'$, then each $R'$-pseudostrategy is an $R$-pseudostrategy and for all $\sigma, \sigma'$, if $\sigma \mathrel{R'} \sigma'$ then $\sigma \circ f = \sigma' \circ f$. So we may assume that $R$ is reflexive and transitive.

First suppose that player I has a winning pseudostrategy $S$ in $G$. Define a pseudostrategy $S''$ and a function $\pi : S''^{\text{even}} \longrightarrow S^{\text{even}}$ as follows:

Put $\pi(\langle\rangle) = \langle\rangle$. Now let $\sigma \in S''^{\text{even}}$ and suppose that $\pi(\sigma) \in S$ has been defined such that $\sigma \mathrel{R} \pi(\sigma)$.

We let the moves in $S''$ at $\sigma$ be the moves $a$ in $T$ at $\sigma$ such that for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, there are $a', b'$ such that $\pi(\sigma)^\frown\langle a', b'\rangle \in S$ and $\sigma^\frown\langle a, b\rangle \mathrel{R} \sigma^\frown\langle a', b'\rangle$.

For each such $a$, we let the moves in $S''$ at $\sigma^\frown\langle a\rangle$ be all moves $b$ in $T$ at $\sigma^\frown\langle a\rangle$, and for each such $b$, we define $\pi(\sigma^\frown\langle a, b\rangle)$ as the $\prec$-least element of $S$ of the form $\pi(\sigma)^\frown\langle a', b'\rangle$ such that $\sigma^\frown\langle a, b\rangle \mathrel{R} \sigma^\frown\langle a', b'\rangle$.

To see that $S''$ is a pseudostrategy for player I in $G$, let $\sigma \in S''^{\text{even}}$. Then there is a move $a'$ in $S$ at $\pi(\sigma)$. Since $\sigma \mathrel{R} \pi(\sigma)$, there is a move $a$ in $T$ at $\sigma$ such that $\sigma^\frown\langle a\rangle \mathrel{R} \sigma^\frown\langle a'\rangle$. Now $a$ is a move in $S''$ at $\sigma$.

That $S''$ is an $R$-pseudostrategy follows from the transitivity of the relation $R$. $S''$ is winning, since for every $x \in [S'']$, there is some $x' \in [S]$ such that $x \circ f = x' \circ f$: For every even $n$, put $x'|n = \pi(x|n)$.

Now suppose that player II has a winning pseudostrategy $S$ in $G$. Then we define a winning $R$-pseudostrategy $S''$ for II in $G$ and a function $\pi$ in a similar way:

Put $\pi(\langle\rangle) = \langle\rangle$. Now let $\sigma \in S''^{\,\text{even}}$ and suppose that $\pi(\sigma) \in S$ has been defined such that $\pi(\sigma)\, R\, \sigma$.

We let the moves in $S''$ at $\sigma$ be all moves $a'$ in $T$ at $\sigma$. For each such $a'$, we let the moves in $S''$ at $\sigma^\frown\langle a'\rangle$ be the moves $b'$ in $T$ at $\sigma^\frown\langle a'\rangle$ for which there are $a, b$ such that $\pi(\sigma)^\frown\langle a, b\rangle \in S$ and $\pi(\sigma)^\frown\langle a, b\rangle\, R\, \sigma^\frown\langle a', b'\rangle$. For such $b'$, we define $\pi(\sigma^\frown\langle a', b'\rangle)$ as the $\prec$-least element of $S$ of the form $\pi(\sigma)^\frown\langle a, b\rangle$ with $\pi(\sigma)^\frown\langle a, b\rangle\, R\, \sigma^\frown\langle a', b'\rangle$.

To see that $S''$ is a pseudostrategy for player II in $G$, let $\sigma \in S''^{\,\text{even}}$ and let $a'$ be a move in $S''$ at $\sigma$. Since $\pi(\sigma)\, R\, \sigma$, there is a move $a$ in $S$ at $\pi(\sigma)$ such that $\pi(\sigma)^\frown\langle a\rangle\, R\, \sigma^\frown\langle a'\rangle$. Since $S$ is a pseudostrategy for player II, there is a move $b$ in $S$ at $\pi(\sigma)^\frown\langle a\rangle$. So, for some move $b'$ in $T$ at $\sigma^\frown\langle a'\rangle$, $\pi(\sigma)^\frown\langle a, b\rangle\, R\, \sigma^\frown\langle a', b'\rangle$ and thus $b'$ is a move in $S''$ at $\sigma^\frown\langle a'\rangle$.

As before, we see that $S''$ is winning.

## 4B Basic open standard games

In this section we consider basic open standard games. Just as in previous chapters, we first consider trivial games.

**4.6** Theorem Let $R$ be a preference relation in some trivial standard game $G$. Then one of the players has a winning $R$-pseudostrategy in $G$.

Proof First suppose that $G = (T, T^{\text{even}}, \emptyset)$ for some tree $T$. Define for every ordinal $\alpha$ a subset $A_\alpha$ of $T$ as in the proof of Theorem 2.4. So for every $\sigma \in T$ of even (odd) length, $\sigma \in A_\alpha$ if and only if for some (every) move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta$.

One easily proves, by transfinite induction, that for each ordinal $\alpha$ and all $\sigma, \sigma'$ such that $\sigma\, R\, \sigma'$, if $\sigma' \in A_\alpha$ then $\sigma \in A_\alpha$.

Thus, if we put $B = \bigcup_\alpha A_\alpha$, then for all $\sigma, \sigma'$ such that $\sigma\, R\, \sigma'$, if $\sigma \notin B$ then $\sigma' \notin B$.

For each $\sigma \in B$, define $\rho(\sigma)$ as the least ordinal $\alpha$ such that $\sigma \in A_\alpha$. We now define a winning $R$-pseudostrategy for one of the players in $G$.

Case 1: $\langle\rangle \in B$.

Let $S$ be the unique pseudostrategy for player I in $G$ such that for every $\sigma \in S^{\text{even}}$, $\sigma \in B$ and the moves in $S$ at $\sigma$ are the moves $a$ in $T$ at $\sigma$ such that for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, $\sigma^\frown\langle a, b\rangle \in \bigcup_{\beta<\rho(\sigma)} A_\beta$. (Since $\sigma \in A_{\rho(\sigma)}$, such an $a$ exists.)

Since there is no strictly decreasing infinite sequence of ordinals, $S$ is winning, in fact wellfounded.

To see that $S$ is an $R$-pseudostrategy for player I in $G$, let $\sigma \in S^{\text{even}}$ and let $a$ be a move in $T$ at $\sigma$ such that for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, there are $a', b'$ such that $\sigma^\frown\langle a', b'\rangle \in S$ and $\sigma^\frown\langle a, b\rangle \, R \, \sigma^\frown\langle a', b'\rangle$. For such $b$, $a'$, and $b'$, $\sigma^\frown\langle a', b'\rangle \in \bigcup_{\beta<\rho(\sigma)} A_\beta$ and thus $\sigma^\frown\langle a, b\rangle \in \bigcup_{\beta<\rho(\sigma)} A_\beta$. This proves that $a$ is a move in $S$ at $\sigma$.

Case 2: $\langle\rangle \notin B$.

Let $\sigma \in T^{\text{even}} \setminus B$. Then for every move $a$ in $T$ at $\sigma$, for some move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, $\sigma^\frown\langle a, b\rangle \notin B$; otherwise there would be an ordinal $\alpha$ such that $\sigma \in A_\alpha$.

So there is a unique pseudostrategy $S$ for player II in $G$ such that for every $\sigma \in S^{\text{even}}$, $\sigma \notin B$ and for every move $a$ in $T$ at $\sigma$, the moves in $S$ at $\sigma^\frown\langle a\rangle$ are the sets $b$ such that $\sigma^\frown\langle a, b\rangle \in T^{\text{even}} \setminus B$.

Since the winning set for player II in $G$ is $[T]$, $S$ is winning. To see that $S$ is an $R$-pseudostrategy for player II in $G$, let $\sigma \in S^{\text{even}}$. Let $a', b'$ be such that $\sigma^\frown\langle a', b'\rangle \in T$ and, for some $a, b$, $\sigma^\frown\langle a, b\rangle \in S$ and $\sigma^\frown\langle a, b\rangle \, R \, \sigma^\frown\langle a', b'\rangle$. Then $\sigma^\frown\langle a, b\rangle \notin B$, so $\sigma^\frown\langle a', b'\rangle \notin B$. Thus $\sigma^\frown\langle a', b'\rangle \in S$.

Now suppose that $G = (T, T^{\text{even}}, [T])$ for some tree $T$. Define for every ordinal $\alpha$ a subset $A_\alpha$ of $T$ as in the proof of Theorem 2.4, but now for the game $(T, T \setminus T^{\text{even}}, \emptyset)$. So for every $\sigma \in T$ of even (odd) length, $\sigma \in A_\alpha$ if and only if for every (some) move $a$ in $T$ at $\sigma$, $\sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta$.

We now have that for each ordinal $\alpha$ and all $\sigma, \sigma'$ such that $\sigma \, R \, \sigma'$, if $\sigma \in A_\alpha$ then $\sigma' \in A_\alpha$. So, if we put $B = \bigcup_\alpha A_\alpha$, then for all $\sigma, \sigma'$ such that $\sigma \, R \, \sigma'$, if $\sigma' \notin B$ then $\sigma \notin B$. Define a function $\rho$ on $B$ as before.

Case 1′: $\langle\rangle \in B$.

Let $S$ be the unique pseudostrategy for player II in $G$ such that for every $\sigma \in S^{\text{even}}$, $\sigma \in B$ and for every move $a$ in $T$ at $\sigma$, the moves in $S$ at $\sigma^\frown\langle a\rangle$ are the sets $b$ such that $\sigma^\frown\langle a, b\rangle \in \bigcup_{\beta<\rho(\sigma)} A_\beta$. (Since $\sigma \in A_{\rho(\sigma)}$, such a $b$ exists.)

Since there is no strictly decreasing infinite sequence of ordinals, $S$ is winning, in fact wellfounded. $S$ is an $R$-pseudostrategy for player II in $G$, by the same argument as in Case 2, with '$\in \bigcup_{\beta<\rho(\sigma)} A_\beta$' instead of '$\notin B$'.

Case 2′: $\langle\rangle \notin B$.

Let $\sigma \in T^{\text{even}} \setminus B$. Then for some move $a$ in $T$ at $\sigma$, for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, $\sigma^\frown\langle a, b\rangle \notin B$; otherwise there would be an ordinal $\alpha$ such that $\sigma \in A_\alpha$.

So there is a unique pseudostrategy $S$ for player I in $G$ such that for every $\sigma \in S^{\text{even}}$, $\sigma \notin B$ and the moves in $S$ at $\sigma$ are the moves $a$ in $T$ at $\sigma$ such that for every move $b$ in $T$ at $\sigma^\frown\langle a\rangle$, $\sigma^\frown\langle a,b\rangle \notin B$.

Since the winning set for player I in $G$ is $[T]$, $S$ is winning. $S$ is an $R$-pseudostrategy for player I in $G$, by the same argument as in Case 1, with '$\notin B$' instead of '$\in \bigcup_{\beta<\rho(\sigma)} A_\beta$'.

**4.7** Theorem Let $R$ be a preference relation in some basic open standard game $G = (T, T^{\text{even}}, X)$. Suppose that there is some even $n$ and some set $\Delta$ of elements of $T$ of length $n$ such that for all $x \in [T]$, $x \in X$ if and only if $x|n \in \Delta$, and such that for all $\tau, \tau'$ of length $n$, if $\tau \, R \, \tau'$ then $\tau \in \Delta$ if and only if $\tau' \in \Delta$.

Then one of the players has a winning $R$-pseudostrategy in $G$.

Proof Let $\tau \in T$ of length $n$. Define a trivial standard game $G_\tau = (T^{\text{via}\,\tau}, (T^{\text{via}\,\tau})^{\text{even}}, X \cap [T^{\text{via}\,\tau}])$. Let $R_\tau$ be the preference relation $R \cap (T^{\text{via}\,\tau} \times T^{\text{via}\,\tau})$ in $G_\tau$. By (the proof of) Theorem 4.6, we find a winning $R_\tau$-pseudostrategy $S_\tau$ for some player $W_\tau$ in $G_\tau$. In fact, if $\tau \in \Delta$, then $W_\tau = \text{I}$ if and only if I has a pseudostrategy in $G_\tau$, and if $\tau \notin \Delta$, then $W_\tau = \text{I}$ if and only if I has a wellfounded pseudostrategy in $G_\tau$.

Now consider the standard game $G' = (T', T'^{\text{even}}, \emptyset)$, where $T'$ is the tree $T^{\leq n} \cup \{\tau^\frown\langle 0\rangle : \tau \in T \text{ and length}(\tau) = n \text{ and } W_\tau = \text{I}\}$. So this game is played like $G$, until a position $\tau$ of length $n$ is reached. If $W_\tau = \text{I}$, then player I makes an obligatory move 0 and wins. If $W_\tau = \text{II}$, then player II wins immediately.

Let $R'$ be the relation $(R \cap (T^{\leq n} \times T^{\leq n})) \cup \{(\tau^\frown\langle 0\rangle, \tau'^\frown\langle 0\rangle) \in T' \times T' : \text{length}(\tau) = \text{length}(\tau') = n \text{ and } \tau \, R \, \tau'\}$. Note that for all $\tau, \tau' \in T$ of length $n$, if $\tau \, R \, \tau'$ and $W_{\tau'} = \text{I}$ then $W_\tau = \text{I}$. Therefore $R'$ is a preference relation in $G'$.

By Theorem 4.6, we find a winning $R'$-pseudostrategy $S$ for some player $W$ in the trivial game $G'$.

Now let $\tau \in S$ of length $n$. Then either $\tau^\frown\langle 0\rangle$ (if $W_\tau = \text{I}$) or $\tau$ (if $W_\tau = \text{II}$) is a terminal node of $S$. In both cases, $W_\tau = W$. This implies that the tree $S^{\leq n} \cup \bigcup_{\tau \in S, \text{length}(\tau)=n} S_\tau$ is a winning $R$-pseudostrategy for player $W$ in the basic open game $(T, T^{\text{even}}, X)$.

Note that the empty set is a preference relation in every game $G$ and that an $\emptyset$-pseudostrategy in $G$ is just a pseudostrategy in $G$. Therefore Theorem 4.7 implies that every basic open standard game is pseudodetermined. Since every basic open game is 'equivalent' to some basic open standard game, this gives another proof of Theorem 2.5.

## 4C Reducing open preferential games to basic open preferential games

We will consider preference relations in standard games with some nice properties.

**4.8** Definition A **preferential game** is a triple $P = (T, R, X)$ where $T$ is a tree, $X \subseteq [T]$ and $R$ is a reflexive, transitive preference relation in the standard game $(T, T^{\text{even}}, X)$ such that for all $\sigma$ and $\sigma'$, if $\sigma\, R\, \sigma'$ then $\text{length}(\sigma) = \text{length}(\sigma')$ and for all even $n < \text{length}(\sigma)$, $\sigma|n\, R\, \sigma'|n$.

We say that $P$ is **open** if there is some $\Delta \subseteq T^{\text{even}}$ such that $X = \{x \in [T] : \text{for some } \tau \in \Delta,\ \tau \subseteq x\}$ and such that for all $\tau$ and $\tau'$, if $\tau\, R\, \tau'$ then $\tau \in \Delta$ if and only if $\tau' \in \Delta$.

If $\Delta$ can be chosen such that all its elements have the same length, then we say that $P$ is **basic open**.

We say that $P$ is **closed** if $(T, R, [T] \setminus X)$ is an open preferential game.

We say that $P$ is **coded Borel** if there is a Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ for $X$ (with respect to $T$) such that for all $\gamma \in C$, if $c(\gamma) = \diamond$ then for all moves $\tau$ and $\tau'$ in $C$ at $\gamma$, $\tau$ has even length and if $\tau\, R\, \tau'$ then $c(\gamma^\frown\langle\tau\rangle) = c(\gamma^\frown\langle\tau'\rangle)$.

**4.9** Example Let $(T_0, {T_0}^{\text{even}}, X_0)$ be an open standard game and define a relation $R_0$ on $T_0$ by: $\sigma\, R_0\, \sigma'$ if and only if $\sigma = \sigma'$. Then $(T_0, R_0, X_0)$ is an open preferential game.

Note that for every basic open preferential game $(T, R, X)$, the preferential game $(T, R, [T]\setminus X)$ is also basic open, and that every (basic) open preferential game is coded Borel. Also note that if $R$ is a reflexive, transitive preference relation in some standard game $(T, T^{\text{even}}, X)$, then one easily finds an $R'$ such that $(T, R', X)$ is a preferential game and such that the $R'$-pseudostrategies are the same as the $R$-pseudostrategies in $(T, T^{\text{even}}, X)$.

If $(T, R, X)$ is a basic open preferential game, then, by Theorem 4.7, one of the players has a winning $R$-pseudostrategy in $(T, T^{\text{even}}, X)$.

We will reduce each coded Borel preferential game to some basic open preferential game. We now describe the basic step, the reduction of an open preferential game to a basic open one.

So let $(T_0, R_0, X_0)$ be an open preferential game and let $\Delta$ be as in Definition 4.8. We may assume, without loss of generality, that for all $\sigma \in \Delta$, for all $\tau \in {T_0}^{\text{even}}$, if $\sigma \subseteq \tau$ then $\tau \in \Delta$. Put $G_0 = (T_0, {T_0}^{\text{even}}, X_0)$. Let $k$ be even. Define $f : \omega \longrightarrow \omega$ by $f(n) = \begin{cases} n & \text{if } n < k, \\ n+2 & \text{if } n \geq k. \end{cases}$
We construct a basic open preferential game $(T_1, R_1, X_1)$ such that for all $\sigma \in {T_1}^{\text{even}}$, $\sigma \circ f \in {T_0}^{\text{even}}$ and for all $\sigma'$, if $\sigma\, R_1\, \sigma'$ then $\sigma \circ f\, R_0\, \sigma' \circ f$.

Let us denote, for every $A \subseteq \Delta$ and every $\sigma \in T_0$ of length $k$, the set $\{\tau \in \Delta : \sigma \subseteq \tau$ and for some $\tau' \in A$, $\tau\, R_0\, \tau'\}$ by $A_\sigma$. In particular, since $R_0$ is reflexive, $\Delta_\sigma = \{\tau \in \Delta : \sigma \subseteq \tau\}$.

The definition of the basic open game $G_1 = (T_1, {T_1}^{\text{even}}, X_1)$ is the same as in Section 1D, with the following adjustments:

- The extra move of player I at a position $\sigma$ of length $k$ has to be a subset $A$ of $\Delta_\sigma$ such that for all $\tau, \tau' \in \Delta_\sigma$, if $\tau\, R_0\, \tau'$ and $\tau' \in A$ then $\tau \in A$. (Since $R_0$ is reflexive and transitive, these sets $A$ are the same as the sets of the form $B_\sigma$ for some $B \subseteq \Delta$.)
- If a position $\sigma^\frown\langle A, 1\rangle^\frown\rho$ is reached such that $\sigma^\frown\rho \in A$, then player I makes an obligatory move 0 and wins. (If $\sigma^\frown\rho \in \Delta_\sigma \setminus A$, then player II wins immediately.)

The reason for the second adjustment is that we only consider standard games. Note that if $R_0$ is as in Example 4.9, then $G_1$ is the same as in Section 1D, except for the second adjustment, and for every $A \subseteq \Delta$ and every $\sigma \in T_0$ of length $k$, $A_\sigma = A \cap \Delta_\sigma$.

Each element $\pi$ of $T_1$ has exactly one of the following forms:

**(i)** $\sigma$, where $\sigma \in {T_0}^{\leq k}$;

**(ii)** $\sigma^\frown\langle A\rangle$, where $\sigma \in T_0$ of length $k$ and $A_\sigma \subseteq A \subseteq \Delta_\sigma$;

**(iii)** $\sigma^\frown\langle A, 1\rangle^\frown\rho$, where $\sigma \in T_0$ of length $k$, $A_\sigma \subseteq A \subseteq \Delta_\sigma$, and $\sigma^\frown\rho \in T_0$ such that for all $n < \text{length}(\rho)$, $\sigma^\frown(\rho|n) \notin \Delta$;

**(iv)** $\sigma^\frown\langle A, 1\rangle^\frown\rho^\frown\langle 0\rangle$, where $\sigma \in T_0$ of length $k$, $A_\sigma \subseteq A \subseteq \Delta_\sigma$, and $\sigma^\frown\rho \in A$ such that for all $n < \text{length}(\rho)$, $\sigma^\frown(\rho|n) \notin \Delta$;

**(v)** $\sigma^\frown\langle A, \tau\rangle^\frown\rho$, where $\sigma \in T_0$ of length $k$, $A_\sigma \subseteq A \subseteq \Delta_\sigma$, $\tau \in A$, and $\sigma^\frown\rho \in {T_0}^{\text{via}\,\tau}$.

We put $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$, so $X_1 = \{x \in [T_1] : x|(k+2) \in \Delta'\}$, where $\Delta' = \{\tau' \in T_1 : \text{length}(\tau') = k+2 \text{ and } \tau'(k+1) \neq 1\}$.

We will define a preference relation $R_1$ in $G_1$ such that $R_1$-pseudostrategies in $G_1$ can be translated into $R_0$-pseudostrategies in $G_0$. First we define a relation $R$ on $T_1$ as follows, using the notation above: For all $\pi, \pi' \in T_1$, $\pi\, R\, \pi'$ if and only if $\pi$ and $\pi'$ have the same form and, respectively:

**(i)** $\sigma\, R_0\, \sigma'$;

**(ii)** $\sigma\, R_0\, \sigma'$ and $A = {A'}_\sigma$;

**(iii)** $\sigma^\frown\rho\, R_0\, \sigma'^\frown\rho'$ and ${A'}_\sigma \subseteq A$;

**(iv)** $\sigma^\frown\rho\, R_0\, \sigma'^\frown\rho'$ and ${A'}_\sigma \subseteq A$;

**(v)** $\tau\, R_0\, \tau'$ and $\text{length}(\rho) = \text{length}(\rho')$ and if $\tau \subseteq \sigma^\frown\rho$ then $\sigma^\frown\rho\, R_0\, \sigma'^\frown\rho'$.

Note that if $R_0$ is as in Example 4.9 and $\pi$ and $\pi'$ are for example both of form (iii), then $\pi \, R \, \pi'$ if and only if $\sigma^\frown \rho = \sigma'^\frown \rho'$ and $A' \subseteq A$. The interpretation is that player I prefers a position of this form to a position with a smaller set $A$, since player II will give up at more positions.

Since $(T_0, R_0, X_0)$ is a preferential game, the following holds for all $\pi, \pi'$ such that $\pi \, R \, \pi'$:

- length$(\pi)$ = length$(\pi')$;
- for all even $n <$ length$(\pi)$, $\pi|n \, R \, \pi'|n$;
- if length$(\pi)$ is even, then $\pi \circ f \, R_0 \, \pi' \circ f$.

One also easily verifies that $R$ is reflexive (but $R$ need not be transitive). We now define $R_1$ as the smallest transitive relation for which $R \subseteq R_1$. So $\pi \, R_1 \, \pi'$ if and only if for some $n \in \omega$, for some $\pi_0, \ldots, \pi_{n-1}$, $\pi \, R \, \pi_0 \, R \, \ldots \, R \, \pi_{n-1} \, R \, \pi'$.

In order to see that $(T_1, R_1, X_1)$ is a preferential game, it is enough to verify that $R$ is a preference relation in $G_1$.

So suppose that $\pi \, R \, \pi'$. We must show that if length$(\pi)$ is even, then for each move $a'$ in $T_1$ at $\pi'$ there is a move $a$ in $T_1$ at $\pi$ such that $\pi^\frown\langle a\rangle \, R \, \pi'^\frown\langle a'\rangle$, and if length$(\pi)$ is odd, then for each move $a$ in $T_1$ at $\pi$ there is a move $a'$ in $T_1$ at $\pi'$ such that $\pi^\frown\langle a\rangle \, R \, \pi'^\frown\langle a'\rangle$.

Case 1: length$(\pi) = k$.
Then, for every move $A'$ in $T_1$ at $\pi'$, the set $A = A'_\pi$ is a move in $T_1$ at $\pi$ and $\pi^\frown\langle A\rangle \, R \, \pi'^\frown\langle A'\rangle$.

Case 2: $\pi$ and $\pi'$ are of form (ii) as described above.
Then each move in $T_1$ at $\pi$ is either 1 or an element of $A$. Clearly, 1 is also a move in $T_1$ at $\pi'$ and $\pi^\frown\langle 1\rangle \, R \, \pi'^\frown\langle 1\rangle$. Furthermore, for every $\tau \in A = A'_\sigma$ there is a $\tau' \in A'$ such that $\pi^\frown\langle \tau\rangle \, R \, \pi'^\frown\langle \tau'\rangle$.

Case 3: $\pi$ and $\pi'$ are of form (iii) and $\sigma^\frown\rho \in \Delta_\sigma$.
Then also $\sigma'^\frown\rho' \in \Delta_{\sigma'}$ since $\sigma^\frown\rho \, R_0 \, \sigma'^\frown\rho'$. Suppose that there is a move in $T_1$ at $\pi'$. Then this move is 0 and $\sigma'^\frown\rho' \in A'$. So $\sigma^\frown\rho \in A'_\sigma \subseteq A$ and thus 0 is a move in $T_1$ at $\pi$ and $\pi^\frown\langle 0\rangle \, R \, \pi'^\frown\langle 0\rangle$.

Case 4: $\pi$ and $\pi'$ are of form (iv).
Then there is no move in $T_1$ at $\pi$.

Case 5: $\pi$ and $\pi'$ are of form (v) and length$(\sigma^\frown\rho) <$ length$(\tau)$.
Then, for the unique move $a$ in $T_1$ at $\pi$ and the unique move $a'$ in $T_1$ at $\pi'$, we have that $\pi^\frown\langle a\rangle \, R \, \pi'^\frown\langle a'\rangle$.

Case 6: all other cases.
Then $\pi \circ f \, R_0 \, \pi' \circ f$. The moves $a$ in $T_1$ at $\pi$ are the same as the moves in $T_0$ at $\pi \circ f$ and the moves $a'$ in $T_1$ at $\pi'$ are the same as the moves

in $T_0$ at $\pi' \circ f$. For all such $a$ and $a'$, we have that $\pi^\frown\langle a\rangle \, R \, \pi'^\frown\langle a'\rangle$ if and only if $(\pi \circ f)^\frown\langle a\rangle \, R_0 \, (\pi' \circ f)^\frown\langle a'\rangle$.

The preferential game $(T_1, R_1, X_1)$ is basic open, since for all $\pi, \pi'$, if $\pi \, R \, \pi'$ then $\pi \in \Delta'$ if and only if $\pi' \in \Delta'$.

For every $R_1$-pseudostrategy $S$ for player I in $G_1$, we describe an $R_0$-pseudostrategy $\phi_{\mathrm{I}}(S)$ for player I in $G_0$ as follows:

Player I follows $S$ until a position $\sigma$ of length $k$ is reached. Define $A$ as the union of all moves $A'$ in $S$ at $\sigma$.

Claim $A$ is a move in $S$ at $\sigma$.

Proof of Claim $A$ is a move in $T$ at $\sigma$, since $A \subseteq \Delta_\sigma$ and for all $\tau, \tau' \in \Delta_\sigma$, if $\tau \, R_0 \, \tau'$ and $\tau' \in A$ then for some $A' \subseteq A$, $\tau \in A'$. There is a move $A'$ in $S$ at $\sigma$, so $A' = A'_\sigma \subseteq A$ and thus $\sigma^\frown\langle A, 1\rangle \, R_1 \, \sigma^\frown\langle A', 1\rangle$. For every $\tau \in A$, there is a move $A'$ in $S$ at $\sigma$ such that $\tau \in A'$ and thus $\sigma^\frown\langle A, \tau\rangle \, R_1 \, \sigma^\frown\langle A', \tau\rangle$. Therefore, since $S$ is an $R_1$-pseudostrategy for player I, $A$ is a move in $S$ at $\sigma$.

Now player I proceeds by following $S$ as if he has played this extra move $A$ in the game $G_1$ and as if player II has played the extra move 1, until, if ever, a position $\tau \in \Delta_\sigma$ is reached. Since this corresponds to an element $\sigma^\frown\langle A, 1\rangle^\frown\rho$ of the pseudostrategy $S$ for player I in $G_1$ such that $\tau = \sigma^\frown\rho$, we have that $\tau \in A$. Now player I proceeds by following $S$ as if player II had played the extra move $\tau$ at position $\sigma^\frown\langle A\rangle$ and as if the moves $\rho(0), \rho(1), \ldots$ were obligatory.

This is an $R_0$-pseudostrategy for player I in $G_0$, since for each $\sigma_0 \in \phi_{\mathrm{I}}(S)^{\text{even}}$, there is some $\sigma_1 \in S^{\text{even}}$ such that for all $a, b$, $\sigma_0{}^\frown\langle a, b\rangle \in \phi_{\mathrm{I}}(S)$ if and only if $\sigma_1{}^\frown\langle a, b\rangle \in S$ and $\sigma_0{}^\frown\langle a, b\rangle \in T_0$ if and only if $\sigma_1{}^\frown\langle a, b\rangle \in T_1$, and such that for all $a, b, a', b'$, $\sigma_0{}^\frown\langle a, b\rangle \, R_0 \, \sigma_0{}^\frown\langle a', b'\rangle$ if and only if $\sigma_1{}^\frown\langle a, b\rangle \, R_1 \, \sigma_1{}^\frown\langle a', b'\rangle$.

For every $R_1$-pseudostrategy $S$ for player II in $G_1$, we describe an $R_0$-pseudostrategy $\phi_{\mathrm{II}}(S)$ for player II in $G_0$ as follows:

Player II follows $S$ until a position $\sigma$ of length $k$ is reached. Put $A = \{\tau \in \Delta_\sigma : \sigma^\frown\langle \Delta_\sigma, \tau\rangle \notin S\}$.

Claim $\sigma^\frown\langle A, 1\rangle \in S$.

Proof of Claim $A$ is a move in $T$ at $\sigma$, since $A \subseteq \Delta_\sigma$ and for all $\tau, \tau' \in \Delta_\sigma$, if $\tau \, R_0 \, \tau'$ and $\sigma^\frown\langle \Delta_\sigma, \tau\rangle \in S$ then $\sigma^\frown\langle \Delta_\sigma, \tau'\rangle \in S$. Thus $\sigma^\frown\langle A\rangle \in S$. There is a move in the $R_1$-pseudostrategy $S$ for player II at $\sigma^\frown\langle A\rangle$. This move must be 1, since for every $\tau \in A$, $\sigma^\frown\langle A, \tau\rangle \, R_1 \, \sigma^\frown\langle \Delta_\sigma, \tau\rangle$ and, by definition, $\sigma^\frown\langle \Delta_\sigma, \tau\rangle \notin S$.

Now player II proceeds by following $S$ as if player I has played the extra move $A$ and as if player II has played the extra move 1 in $G_1$, until, if ever, a position $\tau \in \Delta_\sigma$ is reached. Since this corresponds to a terminal node $\sigma^\frown\langle A, 1\rangle^\frown\rho$ of $S$ such that $\tau = \sigma^\frown\rho$, we have that $\tau \notin A$, so $\sigma^\frown\langle\Delta_\sigma, \tau\rangle \in S$. Now player II proceeds by following $S$ as if player I had played the extra move $\Delta_\sigma$ at position $\sigma$, player II played the move $\tau$ and the moves $\rho(0), \rho(1), \ldots$ were obligatory.

## 4D Reducing coded Borel preferential games to basic open preferential games

The following definition corresponds to Definition 1.15.

**4.10** Definition Let $R$ be a preference relation in some standard game $G = (T, T^{\text{even}}, X)$. Let $n \in \omega$ and let $J$ be player I if $n$ is even and player II if $n$ is odd.

An **$R$-pseudostrategy for $J$ in $G$ up to positions of length $n$** is an $R_n$-pseudostrategy for $J$ in $G_n$, where $G_n$ is the standard game $(T^{\leq n+1}, (T^{\leq n+1})^{\text{even}}, \emptyset)$ and $R_n$ is the preference relation $R \cap (T^{\leq n+1} \times T^{\leq n+1})$ in $G_n$.

Note that at positions of length $n+1$ in $G_n$, player $J$ wins since the other player can make no move. Using this, one easily verifies that for every tree $S$: $S$ is an $R$-pseudostrategy for player I (II) in $G$ if and only if for each even (odd) $n$, $S^{\leq n+1}$ is an $R$-pseudostrategy for I (II) in $G$ up to positions of length $n$.

The following definition is similar to Definition 1.16.

**4.11** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing. Let $(T_0, R_0, X_0)$ and $(T_1, R_1, X_1)$ be preferential games such that for all $\sigma \in {T_1}^{\text{even}}$, $\sigma \circ f \in {T_0}^{\text{even}}$. Put $G_0 = (T_0, {T_0}^{\text{even}}, X_0)$ and $G_1 = (T_1, {T_1}^{\text{even}}, X_1)$. Let $J$ be one of the players.

A **preferential $f$-translator of pseudostrategies for player $J$ from $(T_1, R_1, X_1)$ to $(T_0, R_0, X_0)$** is a function $\phi$ that assigns to every $R_1$-pseudostrategy $S$ for player $J$ in $G_1$ an $R_0$-pseudostrategy $\phi(S)$ for player $J$ in $G_0$ such that for every infinite branch $x_0$ of $\phi(S)$, there is an infinite branch $x_1$ of $S$ such that $x_1 \circ f = x_0$.

For $J =$ I we say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ such that:

- **(i)** for every even $n$, $f(n)$ is even and domain$(\phi_n)$ is the set of all $R_1$-pseudostrategies for player $J$ in $G_1$ up to positions of length $f(n)$ and for every $S \in$ domain$(\phi_n)$, $\phi_n(S)$ is an $R_0$-pseudostrategy for player $J$ in $G_0$ up to positions of length $n$ and for every even $m < n$, $\phi_n(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$;

**(ii)** for every $R_1$-pseudostrategy $S$ for $J$ in $G_1$ and every even $m$, $\phi(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$.

For $J = \mathrm{II}$ we say that $\phi$ is **continuous** if the same holds with 'odd' instead of 'even'.

Note that if $\phi_0, \phi_1, \ldots$ are functions such that (i) holds, then there is a unique function $\phi$ from the set of all $R_1$-pseudostrategies for player $J$ in $G_1$ to the set of all $R_0$-pseudostrategies for player $J$ in $G_0$ such that (ii) holds.

We will construct, for every coded Borel standard game, a basic open 'covering'. In order to do this without using the axiom of choice, we adjust Definition 1.17.

**4.12** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $P_0 = (T_0, R_0, X_0)$ be a preferential game.

A **preferential $f$-covering of** $P_0$ is a triple $(P_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$, where:

**(i)** $P_1$ is a preferential game $(T_1, R_1, X_1)$ such that for all $\sigma \in {T_1}^{\text{even}}$, $\sigma \circ f \in {T_0}^{\text{even}}$ and for all $\sigma'$, if $\sigma \, R_1 \, \sigma'$ then $\sigma \circ f \; R_0 \; \sigma' \circ f$, and such that $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$;

**(ii)** for each player $J$, $\Phi_J$ is of the form $(\phi, F, \langle \phi_0, \phi_1, \ldots \rangle)$, where $\phi$ is a preferential $f$-translator of pseudostrategies for player $J$ from $P_1$ to $P_0$, $F$ is a function that assigns to every $S \in \mathrm{domain}(\phi)$ a function $F(S) : [\phi(S)] \longrightarrow [S]$ such that for all $x \in [\phi(S)]$, $F(S)(x) \circ f = x$, and $\phi_0, \phi_1, \ldots$ are functions that witness that $\phi$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then ${T_1}^{\leq n+1} = {T_0}^{\leq n+1}$, $R_1 \cap ({T_1}^{\leq n+1} \times {T_1}^{\leq n+1}) = R_0 \cap ({T_0}^{\leq n+1} \times {T_0}^{\leq n+1})$, and for every $S \in \mathrm{domain}(\phi_n)$, $\phi_n(S) = S$.

This preferential $f$-covering is called **basic open** if $P_1$ is a basic open preferential game.

**4.13** Lemma

**(i)** Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $((T_1, R_1, X_1), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ be a preferential $f$-covering of some preferential game $(T_0, R_0, X_0)$. Suppose that one of the players has a winning $R_1$-pseudostrategy in $(T_1, {T_1}^{\text{even}}, X_1)$. Then that player also has a winning $R_0$-pseudostrategy in $(T_0, {T_0}^{\text{even}}, X_0)$.

**(ii)** Suppose that $((T_1, R_1, X_1), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ is a preferential $f$-covering of some preferential game $(T_0, R_0, X_0)$. Let $Y_0 \subseteq [T_0]$ and define $Y_1 = \{x \in [T_1] : x \circ f \in Y_0\}$. Then $((T_1, R_1, Y_1), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ is a preferential $f$-covering of $(T_0, R_0, Y_0)$.

**(iii)** Let $(P_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ be a preferential $f$-covering of $P_0$. Let $(P_2, \Psi_{\mathrm{I}}, \Psi_{\mathrm{II}})$ be a preferential $g$-covering of $P_1$. Then $(P_2, \Phi_{\mathrm{I}} \Box \Psi_{\mathrm{I}}, \Phi_{\mathrm{II}} \Box \Psi_{\mathrm{II}})$ is a preferential $(g \circ f)$-covering of $P_0$, where for every player $J$, if we write $\Phi_J = (\phi, F, \langle \phi_0, \phi_1, \ldots \rangle)$ and $\Psi_J = (\psi, G, \langle \psi_0, \psi_1, \ldots \rangle)$, and if we define, for every $S \in \mathrm{domain}(G)$, $H(S) = G(S) \circ F(\psi(S))$, then $\Phi_J \Box \Psi_J = (\phi \circ \psi, H, \langle \phi_0 \circ \psi_{f(0)}, \phi_1 \circ \psi_{f(1)}, \ldots \rangle)$.

Proof Just as in Lemma 1.18, these statements follow directly from the definition of 'preferential covering'.

**4.14** Lemma Let $P_0$ be a preferential game and suppose that for every $n \in \omega$, $(P_{n+1}, \Phi^n_{\mathrm{I}}, \Phi^n_{\mathrm{II}})$ is a preferential $f_n$-covering of $P_n$.

Assume that for every $n \in \omega$, $\lim_{m \to \omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ exists and is equal to $g_n$.

Then there is a (unique) preferential game $P$ and, for every $n \in \omega$, a (unique) preferential $g_n$-covering $(P, \Psi^n_{\mathrm{I}}, \Psi^n_{\mathrm{II}})$ of $P_n$ such that for each player $J$, $\Psi^n_J = \Phi^n_J \Box \Psi^{n+1}_J$ (as defined in Lemma 4.13(iii)).

Proof The proof is similar to that of Lemma 1.19:

For every $n \in \omega$, write $P_n = (T_n, R_n, X_n)$. Then $P = (T, R, X)$, where for every $i \in \omega$, for all large $m$, $T^{\leq i+1} = {T_m}^{\leq i+1}$ and $R \cap (T^{\leq i+1} \times T^{\leq i+1}) = R_m \cap ({T_m}^{\leq i+1} \times {T_m}^{\leq i+1})$, and for every $n \in \omega$, $X = \{x \in [T] : x \circ g_n \in X_n\}$.

Let $J$ be a player. For each $n \in \omega$, write $\Phi^n_J = (\phi_n, F_n, \langle \phi^0_n, \phi^1_n, \ldots \rangle)$. Then, for every $n \in \omega$, $\Psi^n_J = (\psi_n, H_n, \langle \psi^0_n, \psi^1_n, \ldots \rangle)$, where for every $i \in \omega$, for all large $m$, $\psi^i_n = \phi^i_n \circ \phi^{f_n(i)}_{n+1} \circ \cdots \circ \phi^{f_{m-1} \circ \cdots \circ f_n(i)}_m$, and for every $R$-pseudostrategy $S$ for player $J$ in $(T, T^{\mathrm{even}}, X)$ and every $x_n \in [\psi_n(S)]$, if we define inductively for every $m \geq n$, $x_{m+1} = F_m(\psi_{m+1}(S))(x_m)$, then $H(S)(x_n) = \lim_{m \to \omega} x_m$.

**4.15** Lemma Let $P_0$ be an open or closed preferential game. Let $k$ be even and define $f$ as the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k, k+1\}$.

Then $P_0$ has a basic open preferential $f$-covering $(P_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$.

Proof By Lemma 4.13(ii), we only have to consider the case that the preferential game $P_0 = (T_0, R_0, X_0)$ is open. Define a basic open preferential game $P_1 = (T_1, R_1, X_1)$ and $\phi_{\mathrm{I}}, \phi_{\mathrm{II}}$ as in Section 4C.

Just as in Section 1D, one easily verifies that $\phi_{\mathrm{I}}$ is a preferential $f$-translator of pseudostrategies for player I from $P_1$ to $P_0$ and one easily defines, for every $R_1$-pseudostrategy $S$ for I in $(T_1, T_1^{\mathrm{even}}, X_1)$, a function $F_{\mathrm{I}}(S) : [\phi(S)] \longrightarrow [S]$ such that for all $x \in [\phi(S)]$, $F_{\mathrm{I}}(S)(x) \circ f = x$.

For every odd $n$, put $\phi^n_{\mathrm{I}} = \emptyset$ and for every even $n$, define a function $\phi^n_{\mathrm{I}}$ just like $\phi_{\mathrm{I}}$ but now from the set of all $R_1$-pseudostrategies for player

I in $(T_1, T_1^{\text{even}}, X_1)$ up to positions of length $f(n)$ to the set of all $R_0$-pseudostrategies for player I in $(T_0, T_0^{\text{even}}, X_0)$ up to positions of length $n$. Then the functions $\phi_{\text{I}}^0, \phi_{\text{I}}^1, \ldots$ witness that $\phi_{\text{I}}$ is continuous.

Now put $\Phi_{\text{I}} = (\phi_{\text{I}}, F_{\text{I}}, \langle \phi_{\text{I}}^0, \phi_{\text{I}}^1, \ldots \rangle)$ and define $\Phi_{\text{II}}$ in a similar way. Then $(P_1, \Phi_{\text{I}}, \Phi_{\text{II}})$ is a basic open preferential $f$-covering of $P_0$.

We now construct, for every coded Borel preferential game, a basic open preferential covering.

**4.16** LEMMA Let $(T, R, X)$ be a coded Borel preferential game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that $\omega \setminus \text{range}(f)$ is infinite and for every even $n$, $f(n)$ is even and $f(n+1) = f(n) + 1$.

Then $(T, R, X)$ has a basic open preferential $f$-covering.

PROOF Choose a Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ for $X$ as in Definition 4.8. We will *define* a basic open preferential $f$-covering of $(T, R, X)$ by transfinite induction to the rank of the wellfounded tree $C$.

First suppose that $c(\langle\rangle) \in \{\bot, \top, \diamond\}$. Then $(T, R, X)$ is a basic open preferential game. Just as in the proof of Lemma 1.22, we easily find a basic open preferential $f$-covering $((T_1, R_1, X_1), \Phi_{\text{I}}, \Phi_{\text{II}})$ of $(T, R, X)$ by inserting some trivial extra move 0 at the right places:

Let $T_1$ be the set of all finite sequences $\sigma$ such that $\sigma \circ f \in T$ and for all $n < \text{length}(\sigma)$, if $n \notin \text{range}(f)$ then $\sigma(n) = 0$. Put $X_1 = \{x \in [T_1] : x \circ f \in X\}$ and define a preference relation $R_1$ in $(T_1, {T_1}^{\text{even}}, X_1)$ as follows: For all $\sigma, \sigma' \in T_1$, $\sigma \, R_1 \, \sigma'$ if and only if $\text{length}(\sigma) = \text{length}(\sigma')$ and $\sigma \circ f \; R \; \sigma' \circ f$. For every $R_1$-pseudostrategy $S$ for player I in $(T_1, {T_1}^{\text{even}}, X_1)$, let $\phi_{\text{I}}(S)$ be the $R$-pseudostrategy $\{\sigma \circ f : \sigma \in S\}$ for player I in $(T, T^{\text{even}}, X)$. The definitions of $\Phi_{\text{I}} = (\phi_{\text{I}}, F_{\text{I}}, \langle \phi_{\text{I}}^0, \phi_{\text{I}}^1, \ldots \rangle)$ and $\Phi_{\text{II}}$ are straightforward.

Now suppose that $c(\langle\rangle) \in \{\vee, \wedge\}$. For every $n \in \omega$, let $C_n$ be the wellfounded tree $\{\gamma : \langle n \rangle^\frown \gamma \in C\}$ and define a function $c_n$ on $C_n$ by $c_n(\gamma) = c(\langle n \rangle^\frown \gamma)$. Then $c_n$ is a Borel code of some $X_n \subseteq [T]$. Note that either $X = \bigcup_{n \in \omega} X_n$ or $X = \bigcap_{n \in \omega} X_n$.

Just as in the proof of Lemma 1.22, we can define strictly increasing functions $g_0$ and $h$ from $\omega$ to $\omega$ such that $f = h \circ g_0$ and for some even $k$, $\text{range}(h) = \omega \setminus \{k, k+1\}$. We also find strictly increasing functions $f_0, g_1, f_1, g_2, f_2, \ldots$ such that for all $n \in \omega$, $g_n = \lim_{m \to \omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ and such that for each $m$, $\omega \setminus \text{range}(f_m)$ is infinite and for every even $i$, $f_m(i)$ is even and $f_m(i+1) = f_m(i) + 1$.

Define, inductively, for each $n \in \omega$, a preferential game $(T_n, R_n, Y_n)$ such that for all $\sigma \in T_n$, $\sigma \circ f_{n-1} \circ \cdots \circ f_0 \in T$, and such that for all $\tau, \tau' \in {T_n}^{\text{even}}$, if $\tau \, R_n \, \tau'$ then $\tau \circ f_{n-1} \circ \cdots \circ f_0 \; R \; \tau' \circ f_{n-1} \circ \cdots \circ f_0$, as follows:

We put $T_0 = T$ and $R_0 = R$. Now let $n \in \omega$ and suppose that we have chosen $T_n$ and $R_n$. Put $Y_n = \{x \in [T_n] : x \circ f_{n-1} \circ \cdots \circ f_0 \in X_n\}$. Using the Borel code $c_n$ for $X_n$, one easily finds a Borel code $d_n : D_n \longrightarrow \{\bot, \top, \vee, \wedge, \Diamond\}$ for the Borel subset $Y_n$ of $[T_n]$ such that for all $\gamma \in D_n$, if $d_n(\gamma) = \Diamond$ then $\gamma \in C_n$ and for every move $\tau$ in $D_n$ at $\gamma$, we have that $\tau \in {T_n}^{\text{even}}$, $\tau \circ f_{n-1} \circ \cdots \circ f_0$ is a move in $C_n$ at $\gamma$, and $d_n(\gamma^\frown\langle\tau\rangle) = c_n(\gamma^\frown\langle\tau \circ f_{n-1} \circ \cdots \circ f_0\rangle)$. Using this, one easily verifies that the preferential game $(T_n, R_n, Y_n)$ is coded Borel. Since the rank of the wellfounded tree $D_n$ is less than the rank of $C$, we already know how to define a basic open preferential $f_n$-covering $((T_{n+1}, R_{n+1}, Z_n), \Phi_{\text{I}}^n, \Phi_{\text{II}}^n)$ of $(T_n, R_n, Y_n)$.

For every $n \in \omega$, let $P_n$ be the preferential game $(T_n, R_n, \{x \in [T_n] : x \circ f_{n-1} \circ \cdots \circ f_0 \in X\})$. So $P_0 = (T, R, X)$ and, by Lemma 4.13(ii), for every $n$, $(P_{n+1}, \Phi_{\text{I}}^n, \Phi_{\text{II}}^n)$ is a preferential $f_n$-covering of $P_n$.

By Lemma 4.14, we find a preferential game $P' = (T', R', X')$ and, for every $n \in \omega$, a preferential $g_n$-covering $(P', \Psi_{\text{I}}^n, \Psi_{\text{II}}^n)$ of $P_n$.

For every $n \in \omega$, put $X'_n = \{x \in [T'] : x \circ g_{n+1} \in Z_n\}$. Then the preferential game $(T', R', X'_n)$ is basic open since for all $\tau, \tau' \in {T'}^{\text{even}}$, if $\tau \, R' \, \tau'$ then $\tau \circ g_{n+1} \, R_{n+1} \, \tau' \circ g_{n+1}$.

Since $X'$ is either $\bigcup_{n\in\omega} X'_n$ or $\bigcap_{n\in\omega} X'_n$, we conclude that the preferential game $P'$ is either open or closed. So, by Lemma 4.15, $P'$ has some basic open preferential $h$-covering. Using Lemma 4.13(iii), we find a basic open preferential $(h \circ g_0)$-covering of $P_0$. In other words, the coded Borel preferential game $(T, R, X)$ has a basic open preferential $f$-covering.

**4.17** Theorem Every coded Borel standard game is pseudodetermined.

Proof Let $(T_0, {T_0}^{\text{even}}, X_0)$ be a coded Borel standard game. Define a preference relation $R_0$ in this game by $\sigma \, R_0 \, \sigma'$ if and only if $\sigma = \sigma'$, for all $\sigma, \sigma' \in T_0$. Then $(T_0, R_0, X_0)$ is a coded Borel preferential game. Define $f : \omega \longrightarrow \omega$ by $f(2n) = 4n$ and $f(2n+1) = 4n+1$ for every $n \in \omega$.

By Lemma 4.16, $(T_0, R_0, X_0)$ has a basic open preferential $f$-covering $((T_1, R_1, X_1), \Phi_{\text{I}}, \Phi_{\text{II}})$. By Theorem 4.7, one of the players has a winning $R_1$-pseudostrategy in the basic open standard game $(T_1, {T_1}^{\text{even}}, X_1)$. By Lemma 4.13(i), that player also has a winning $R_0$-pseudostrategy in $(T_0, {T_0}^{\text{even}}, X_0)$. So the game $(T_0, {T_0}^{\text{even}}, X_0)$ is pseudodetermined.

# 5 Another proof of Borel pseudodeterminacy using CAC

In the auxiliary basic open game $G_1$ that we constructed in Section 1D, the players play different roles: The first extra move, a subset $A$ of some given set $\Delta_\sigma$, is chosen by player I and the second extra move, either 1 or some $\tau \in A$, is chosen by player II.

Nonetheless, strategies for both players were translated from $G_1$ to $G_0$ in a similar way. Each player plays the original game, using a strategy $S$ in the auxiliary game, as follows: At each position $\sigma$ of length $k$, he plays on as if he is at some position $\sigma^\frown\langle A, 1\rangle \in S$, until, if ever, a position $\tau \in \Delta_\sigma$ is reached at which the other player is assumed to give up. Then he plays on as if some obligatory moves have been played at some position $\sigma^\frown\langle A', \tau\rangle \in S$.

In this chapter we generalize the concept of a game in such a way that the two extra moves in $G_1$ can be combined: At each position $\sigma$ of length $k$ in the tree $T$ on which the auxiliary basic open generalized game is played, there is one extra move, either $(0, \tau)$ for some $\tau \in \Delta_\sigma$ or $(1, A)$ for some $A \subseteq \Delta_\sigma$. But neither I nor II makes this move by himself. In other words, for a pseudostrategy $S$ such that $\sigma \in S$, we neither require that the set of all moves in $S$ at $\sigma$ is non-empty nor that it is the set of all moves in $T$ at $\sigma$. Instead of this, we put $A = \{\tau : (0, \tau) \text{ is a move in } S \text{ at } \sigma\}$ and we require that $(1, A)$ or $(1, \Delta_\sigma \setminus A)$, for player I or II, respectively, is a move in $S$ at $\sigma$.

The definitions of pseudostrategies for player I and player II in a generalized game on some tree $T$ will be each other's 'dual' in a certain sense. Using this duality, we prove that every trivial generalized game is pseudodetermined. Then we prove, using the countable axiom of choice, that every Borel generalized game is pseudodetermined. Just as in Chapter 4, we only use CAC to get a Borel code. In this chapter, the players play essentially the same role. This makes some proofs much shorter than the corresponding ones in Chapter 4.

## 5A Generalized games

**5.1** Definition A **generalized game** $G$ is a triple $(T, p, X)$ where $T$ is a tree, $p$ is a function that assigns to every $\sigma \in T$ a set of sets of moves in $T$ at $\sigma$, and $X$ is a subset of $[T]$.

A **pseudostrategy for player I** in $G$ is a tree $S \subseteq T$ such that for all $\sigma \in S$, $\{a : \sigma^\frown\langle a\rangle \in S\} \in p(\sigma)$. $S$ is a **winning** pseudostrategy for player I in $G$ if $[S] \subseteq X$.

A **pseudostrategy for player II** in $G$ is a tree $S \subseteq T$ such that for all $\sigma \in S$, $\{a : \sigma^\frown\langle a\rangle \in T \setminus S\} \notin p(\sigma)$. $S$ is a **winning** pseudostrategy for player II in $G$ if $[S] \subseteq [T] \setminus X$.

> $G$ is **pseudodetermined** if there is a winning pseudostrategy for player I or a winning pseudostrategy for player II in $G$.

In other words, if for every tree $T$ and $\sigma \in T$, the set of all moves in $T$ at $\sigma$ is denoted by $T/\sigma$, then a generalized game is a triple $(T, p, X)$ where $T$ is a tree, $p$ is a function on $T$ such that for all $\sigma \in T$, $p(\sigma) \subseteq \{M : M \subseteq T/\sigma\}$, and $X \subseteq [T]$. A pseudostrategy for player I in $(T, p, X)$ is a tree $S \subseteq T$ such that for all $\sigma \in S$, $S/\sigma \in p(\sigma)$ and it is winning if $[S] \subseteq X$. The definitions of pseudostrategies for player I and player II are each other's 'dual' in the following sense: A (winning) pseudostrategy for some player in $(T, p, X)$ is a (winning) pseudostrategy for the other player in the generalized game $(T, q, [T] \setminus X)$, where $q$ is the function on $T$ defined by $q(\sigma) = \{M \subseteq T/\sigma : (T/\sigma) \setminus M \notin p(\sigma)\}$.

We now describe three types of positions.

**5.2** Definition A **position of type I** in a generalized game $(T, p, X)$ is a $\sigma \in T$ for which $p(\sigma) = \{M : M$ is a non-empty set of moves in $T$ at $\sigma\}$.

We say that $\sigma$ is **of type II** if $p(\sigma) = \{M : M$ is the set of all moves in $T$ at $\sigma\}$.

We say that $\sigma$ is **of type III** if there is a (unique) set $\Delta$ such that the set of all moves in $T$ at $\sigma$ is $\{(0, \tau) : \tau \in \Delta\} \cup \{(1, A) : A \subseteq \Delta\}$ and $p(\sigma) = \{M : M$ is a set of moves in $T$ at $\sigma$ such that $(1, \{\tau : (0, \tau) \in M\}) \in M\}$.

A position of type I or II can be thought of as a position at which player I or II, respectively, has to make a move. Note that each terminal node $\sigma$ of $T$ is either of type I (if $p(\sigma) = \emptyset$) or of type II (if $p(\sigma) = \{\emptyset\}$), but not of type III.

A position that is of type I and also of type II can be thought of as a position at which exactly one move can and must be made. Note that if this obligatory move is $(1, \emptyset)$, then this position is also of type III (with $\Delta = \emptyset$).

**5.3** Remark Each game $(T, P, X)$ can be considered as a generalized game of a special kind as follows: Let $(T, p, X)$ be the generalized game in which each position $\sigma$ is of type I if $\sigma \in P$ and of type II otherwise. Then the (winning) pseudostrategies for some player in the generalized game $(T, p, X)$ are precisely the (winning) pseudostrategies for that player in the game $(T, P, X)$.

Note that the (winning) *strategies* for player I in the game $(T, P, X)$ are precisely the (winning) pseudostrategies for player I in the generalized game $(T, s, X)$, where $s$ is defined by

$$s(\sigma) = \begin{cases} \{\{a\} : a \text{ is a move in } T \text{ at } \sigma\} & \text{if } \sigma \in P, \\ \{\{a : a \text{ is a move in } T \text{ at } \sigma\}\} & \text{otherwise.} \end{cases}$$

A simple example of a generalized game in which both players have a winning pseudostrategy is $(\{\langle\rangle, \langle 0\rangle\}, p, \emptyset)$, where $p(\langle\rangle) = p(\langle 0\rangle) = \{\emptyset\}$. In this generalized game, the tree $\{\langle\rangle\}$ is a winning pseudostrategy for player I (since $\emptyset \in p(\langle\rangle)$) and also for player II (since $\{0\} \notin p(\langle\rangle)$).

One also easily constructs a generalized game in which every position is of type III (with $\Delta = \{0\}$) and in which both players have a winning pseudostrategy (with exactly one infinite branch).

Suppose that $(T, p, X)$ is a generalized game such that for every $\sigma \in T$ and all sets $M$ and $M'$ of moves in $T$ at $\sigma$, if $M \subseteq M'$ and $M \in p(\sigma)$ then $M' \in p(\sigma)$. Then one easily verifies that if $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ are pseudostrategies for player I and player II in $(T, p, X)$, then $S_{\mathrm{I}} \cap S_{\mathrm{II}}$ is a tree without terminal nodes. Thus DC implies that at most one of the players has a winning pseudostrategy in $(T, p, X)$.

We now make some remarks about 'standard' (generalized) games.

For each game we can construct an 'equivalent' game $(T, P, X)$ in which the players alternately choose elements of some set $C$ and player I starts. In other words, $T = {}^{<\omega}C$ and $P = \{\sigma \in T : \mathrm{length}(\sigma) \text{ is even}\}$. Consider the generalized game $({}^{<\omega}(C \times C), c, Y)$, where $c$ is the function on ${}^{<\omega}(C \times C)$ with constant value $\{M \subseteq C \times C : \text{for some } a \in C, \text{ for all } b \in C, (a, b) \in M\}$, and $Y$ is the set of all infinite sequences of the form $\langle (a_0, a_1), (a_2, a_3), \ldots\rangle$ for some $\langle a_0, a_1, a_2, a_3, \ldots\rangle \in X$. Then one easily verifies that this generalized game is pseudodetermined if and only if the game $(T, P, X)$ is pseudodetermined.

For each generalized game $(T, p, X)$ we can construct an 'equivalent' generalized game in which each position is either of type III and of the form $\langle (0, a_0), \ldots, (0, a_{n-1})\rangle$ for some $\langle a_0, \ldots, a_{n-1}\rangle \in T$, or of the form $\langle (0, a_0), \ldots, (0, a_{n-1}), (1, A)\rangle$ for some $\langle a_0, \ldots, a_{n-1}\rangle \in T$ and some set $A$ of moves in $T$ at $\langle a_0, \ldots, a_{n-1}\rangle$; this terminal position is of type II if and only if $A \in p(\langle a_0, \ldots, a_{n-1}\rangle)$. If we add at these terminal positions an infinite sequence of obligatory moves $(1, \emptyset)$, leading to a win for player I if and only if $A \in p(\langle a_0, \ldots, a_{n-1}\rangle)$, then we find an 'equivalent' generalized game in which each position is of type III.

## 5B Pseudodeterminacy of basic open generalized games

In this section we prove, without using AC, that all trivial and all basic open generalized games are pseudodetermined.

**5.4** Theorem Every trivial generalized game is pseudodetermined.

Proof We will *define* for every trivial generalized game a winning pseudostrategy for one of the players. By symmetry, we only have to consider generalized games $G$ of the form $(T, p, \emptyset)$.

For every ordinal $\alpha$ we define, by transfinite induction, $A_\alpha$ as the set of all $\sigma \in T$ such that $\{a : \sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta\} \in p(\sigma)$. Put $B = \bigcup_\alpha A_\alpha$.

Case 1: $\langle\rangle \in B$.

Let $S$ be the unique tree such that for every $\sigma \in S$, $\sigma \in B$ and for all $a$, $\sigma^\frown\langle a\rangle \in S$ if and only if $\sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta$, where $\alpha$ is the least ordinal such that $\sigma \in A_\alpha$.

For every $\sigma \in S$, the set of all moves in $S$ at $\sigma$ is in $p(\sigma)$, so $S$ is a pseudostrategy for player I in $G$. $S$ is winning, in fact wellfounded.

Case 2: $\langle\rangle \notin B$.

Let $S$ be the unique tree such that for every $\sigma \in S$, $\sigma \in T \setminus B$ and for all $a$, $\sigma^\frown\langle a\rangle \in S$ if and only if $\sigma^\frown\langle a\rangle \in T \setminus B$.

Let $\sigma \in S$. Choose an ordinal $\alpha$ so large that for every move $a$ in $T$ at $\sigma$, if $\sigma^\frown\langle a\rangle \in B$ then $\sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta$. Then, since $\sigma \notin B$, $\sigma \notin A_\alpha$, so $\{a : \sigma^\frown\langle a\rangle \in T \setminus S\} = \{a : \sigma^\frown\langle a\rangle \in B\} = \{a : \sigma^\frown\langle a\rangle \in \bigcup_{\beta<\alpha} A_\beta\} \notin p(\sigma)$.

This proves that $S$ is a pseudostrategy for player II in $G$. Since the winning set for player II in $G$ is $[T]$, every pseudostrategy for player II in $G$ is winning.

So in both cases the trivial generalized game $G$ is pseudodetermined.

**5.5** Theorem Every basic open generalized game is pseudodetermined.

Proof We 'split' each basic open generalized game into some trivial generalized games. Let $(T, p, X)$ be a basic open generalized game. Choose $n \in \omega$ such that for all $x \in X$, for all $y \in [T]$, if $x|n = y|n$ then $y \in X$.

Let $\sigma \in T$ of length $n$. Define a generalized game $(T_\sigma, p_\sigma, X_\sigma)$ as follows: $T_\sigma$ is the tree $\{\tau : \sigma^\frown\tau \in T\}$, for all $\tau \in T_\sigma$, $p_\sigma(\tau) = p(\sigma^\frown\tau)$, and $X_\sigma = \{x \in [T_\sigma] : \sigma^\frown x \in X\}$. Then either $X_\sigma = \emptyset$ or $X_\sigma = [T_\sigma]$, so this generalized game is trivial. As in the proof of Theorem 5.4, we construct a winning pseudostrategy $S_\sigma$ for some player $W_\sigma$ in this generalized game.

Now consider the trivial generalized game $(T^{\leq n}, q, \emptyset)$, where $q$ is the function on $T^{\leq n}$ defined by

$$q(\sigma) = \begin{cases} p(\sigma) & \text{if length}(\sigma) < n, \\ \{\emptyset\} & \text{if length}(\sigma) = n \text{ and } W_\sigma = \text{I}, \\ \emptyset & \text{if length}(\sigma) = n \text{ and } W_\sigma = \text{II}. \end{cases}$$

By Theorem 5.4, we find a winning pseudostrategy $S$ for some player $W$ in this trivial generalized game.

Each $\sigma \in S$ of length $n$ is a terminal node of $T^{\leq n}$, so if $W = \text{I}$ then $\emptyset \in q(\sigma)$ and if $W = \text{II}$ then $\emptyset \notin q(\sigma)$. In both cases, $W_\sigma = W$.

Now one easily verifies that $S \cup \{\sigma^\frown\tau : \sigma \in S$ and length$(\sigma) = n$ and $\tau \in S_\sigma\}$ is a winning pseudostrategy for player $W$ in the basic open generalized game $(T, P, X)$.

Note that by Remark 5.3, Theorem 5.4 implies Theorem 2.4 and Theorem 5.5 implies Theorem 2.5.

## 5C Reducing coded Borel generalized games to basic open generalized games

Since we work with generalized games now and want to avoid the use of the axiom of choice, we have to adjust the definitions and lemmas of Section 1E, just like we did in Section 4D for preferential games.

The basic step, the reduction of open generalized games to basic open ones, will be described in Lemma 5.11. In contrast to the reduction of open preferential games to basic open ones (Section 4C), this basic step will be somewhat simpler than the reduction of open games to basic open ones (Section 1D).

The following definition is similar to Definition 1.15.

**5.6** DEFINITION Let $G = (T, p, X)$ be a generalized game, let $n \in \omega$ and let $J$ be a player. A **pseudostrategy for $J$ in $G$ up to positions of length $n$** is a pseudostrategy for that player in the generalized game $(T^{\leq n+1}, q, \emptyset)$, where $q$ is defined by

$$q(\sigma) = \begin{cases} p(\sigma) & \text{if length}(\sigma) \leq n, \\ \{\emptyset\} & \text{if length}(\sigma) = n+1 \text{ and } J = \text{I}, \\ \emptyset & \text{if length}(\sigma) = n+1 \text{ and } J = \text{II}. \end{cases}$$

One easily verifies that for every generalized game $G$ and tree $S$:
$S$ is a pseudostrategy for some player in $G$ if and only if for all $n \in \omega$, $S^{\leq n+1}$ is a pseudostrategy for that player in $G$ up to positions of length $n$.

The following definition is similar to Definition 1.16.

**5.7** DEFINITION Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_0 = (T_0, p_0, X_0)$ and $G_1 = (T_1, p_1, X_1)$ be generalized games such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Let $J$ be one of the players.

An **$f$-translator of pseudostrategies for player $J$ from $G_1$ to $G_0$** is a function $\phi$ that assigns to every pseudostrategy $S$ for player $J$ in $G_1$ a pseudostrategy $\phi(S)$ for player $J$ in $G_0$ such that for every infinite branch $x_0$ of $\phi(S)$, there is an infinite branch $x_1$ of $S$ such that $x_1 \circ f = x_0$.

We say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ such that:

**(i)** for every $n \in \omega$, domain$(\phi_n)$ is the set of all pseudostrategies for player $J$ in $G_1$ up to positions of length $f(n)$ and for every $S \in$ domain$(\phi_n)$, $\phi_n(S)$ is a pseudostrategy for player $J$ in $G_0$ up to positions of length $n$ and for every $m < n$, $\phi_n(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$;

**(ii)** for every pseudostrategy $S$ for $J$ in $G_1$ and every $m \in \omega$, $\phi(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$.

Note that if $\phi_0, \phi_1, \ldots$ are functions such that (i) holds, then there is a unique function $\phi$ from the set of all pseudostrategies for player $J$ in $G_1$ to the set of all pseudostrategies for player $J$ in $G_0$ such that (ii) holds.

We will construct, for every coded Borel generalized game, a basic open 'covering'. In order to do this without using the axiom of choice, we adjust Definition 1.17.

**5.8** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $G_0 = (T_0, p_0, X_0)$ be a generalized game.

An $f$**-covering of** $G_0$ is a triple $(G_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$, where:

**(i)** $G_1$ is a generalized game $(T_1, p_1, X_1)$ such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$, and $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$;

**(ii)** for each player $J$, $\Phi_J$ is of the form $(\phi, F, \langle \phi_0, \phi_1, \ldots \rangle)$, where $\phi$ is an $f$-translator of pseudostrategies for player $J$ from $G_1$ to $G_0$, $F$ is a function that assigns to every pseudostrategy $S$ for player $J$ in $G_1$ a function $F(S) : [\phi(S)] \longrightarrow [S]$ such that for all $x \in [\phi(S)]$, $F(S)(x) \circ f = x$, and $\phi_0, \phi_1, \ldots$ are functions that witness that $\phi$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then $T_1{}^{\leq n+1} = T_0{}^{\leq n+1}$, $p_1 | T_1{}^{\leq n} = p_0 | T_0{}^{\leq n}$, and for every $S \in$ domain$(\phi_n)$, $\phi_n(S) = S$.

This $f$-covering is called **basic open** if $X_1$ is a basic open subset of $[T_1]$.

**5.9** Lemma

**(i)** Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $(G_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ be an $f$-covering of some generalized game $G_0$. Suppose that $G_1$ is pseudodetermined. Then $G_0$ is also pseudodetermined.

**(ii)** Suppose that $((T_1, p_1, X_1), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ is an $f$-covering of some generalized game $(T_0, p_0, X_0)$. Let $Y_0 \subseteq [T_0]$ and define $Y_1 = \{x \in [T_1] : x \circ f \in Y_0\}$. Then $((T_1, p_1, Y_1), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ is an $f$-covering of $(T_0, p_0, Y_0)$.

**(iii)** Let $(G_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ be an $f$-covering of $G_0$ and let $(G_2, \Psi_{\mathrm{I}}, \Psi_{\mathrm{II}})$ be a $g$-covering of $G_1$. Then $(G_2, \Phi_{\mathrm{I}} \square \Psi_{\mathrm{I}}, \Phi_{\mathrm{II}} \square \Psi_{\mathrm{II}})$ is a $(g \circ f)$-covering of

$G_0$, where for every player $J$, if we write $\Phi_J = (\phi, F, \langle \phi_0, \phi_1, \ldots \rangle)$ and $\Psi_J = (\psi, G, \langle \psi_0, \psi_1, \ldots \rangle)$, and if we define, for every pseudostrategy $S$ for player $J$ in $G_2$, $H(S) = G(S) \circ F(\psi(S))$, then $\Phi_J \square \Psi_J = (\phi \circ \psi, H, \langle \phi_0 \circ \psi_{f(0)}, \phi_1 \circ \psi_{f(1)}, \ldots \rangle)$.

PROOF Just as in Lemma 1.18, these statements follow directly from the definition of 'covering'.

**5.10** LEMMA Let $G_0$ be a generalized game and, for every $n \in \omega$, let $(G_{n+1}, \Phi^n_{\mathrm{I}}, \Phi^n_{\mathrm{II}})$ be an $f_n$-covering of $G_n$.

Suppose that for every $n \in \omega$, $\lim_{m \to \omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ exists and is equal to $g_n$.

Then there is a (unique) generalized game $G$ and, for every $n \in \omega$, a (unique) $g_n$-covering $(G, \Psi^n_{\mathrm{I}}, \Psi^n_{\mathrm{II}})$ of of $G_n$ such that for each player $J$, $\Psi^n_J = \Phi^n_J \square \Psi^{n+1}_J$ (as defined in Lemma 5.9(iii)).

PROOF The proof is similar to that of Lemma 1.19:

If, for every $n \in \omega$, $G_n = (T_n, p_n, X_n)$, then $G = (T, p, X)$, where for every $i \in \omega$, for all large $m$, $T^{\leq i+1} = {T_m}^{\leq i+1}$ and $p|T^{\leq i} = p_m|{T_m}^{\leq i}$, and for every $n \in \omega$, $X = \{x \in [T] : x \circ g_n \in X_n\}$.

Let $J$ be a player. For each $n \in \omega$, write $\Phi^n_J = (\phi_n, F_n, \langle \phi^0_n, \phi^1_n, \ldots \rangle)$. Then, for every $n \in \omega$, $\Psi^n_J = (\psi_n, H_n, \langle \psi^0_n, \psi^1_n, \ldots \rangle)$, where for every $i \in \omega$, for all large $m$, $\psi^i_n = \phi^i_n \circ \phi^{f_n(i)}_{n+1} \circ \cdots \circ \phi^{f_{m-1} \circ \cdots \circ f_n(i)}_m$, and for every pseudostrategy $S$ for player $J$ in $G$ and every $x_n \in [\psi_n(S)]$, if we define inductively for every $m \geq n$, $x_{m+1} = F_m(\psi_{m+1}(S))(x_m)$, then $H(S)(x_n) = \lim_{m \to \omega} x_m$.

We now construct, as described in the beginning of this chapter, for every open or closed generalized game, an auxiliary basic open generalized game in which just one extra move is made.

**5.11** LEMMA Let $G_0$ be an open or closed generalized game. Let $k \in \omega$ and define $f$ as the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k\}$. (In other words, for all $n < k$, $f(n) = n$ and for all $n \geq k$, $f(n) = n + 1$.)

Then there is a basic open $f$-covering $(G_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ of $G_0$.

PROOF By Lemma 5.9(ii), we only have to consider the case that the generalized game $G_0 = (T_0, p_0, X_0)$ is open.

For every $\sigma \in T_0$ of length $k$, put $\Delta_\sigma = \{\tau \in T_0 : \sigma \subseteq \tau$ and for all $x \in [T_0]$, if $\tau \subseteq x$ then $x \in X_0\}$.

Define $T_1$ as the tree ${T_0}^{\leq k} \cup \{\sigma ^\frown \langle (1, A) \rangle ^\frown \rho : \sigma \in T_0$ of length $k$, $A \subseteq \Delta_\sigma$, and $\sigma ^\frown \rho \in T_0$ such that for all $n <$ length$(\rho)$, $\sigma ^\frown (\rho|n) \notin \Delta_\sigma\} \cup \{\sigma ^\frown \langle (0, \tau) \rangle ^\frown \rho : \sigma \in T_0$ of length $k$, $\tau \in \Delta_\sigma$, and $\sigma ^\frown \rho \in {T_0}^{\mathrm{via}\, \tau}\}$.

Note that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Put $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$. Then, since $X_0$ is open, $X_1 = \{x \in [T_1] : \text{for some } \tau,\ x(k) = (0, \tau)\}$. So $X_1$ is basic open.

Let $G_1$ be the generalized game $(T_1, p_1, X_1)$ in which:

- every $\sigma \in T_0$ of length $k$ is a position of type III (with $\Delta = \Delta_\sigma$);
- every position of the form $\sigma^\frown\langle(1, A)\rangle^\frown\rho$ such that $\sigma^\frown\rho \in \Delta_\sigma$, is a terminal position of type II if $\sigma^\frown\rho \in A$ and of type I otherwise;
- every position of the form $\sigma^\frown\langle(0, \tau)\rangle^\frown\rho$ such that $\sigma^\frown\rho \notin \Delta_\sigma$, is a position of type I and also of type II (with obligatory move $\tau(k + \text{length}(\rho))$);
- for all other positions $\sigma'$, the moves in $T_1$ at $\sigma'$ are precisely the moves in $T_0$ at $\sigma' \circ f$ and $p_1(\sigma') = p_0(\sigma' \circ f)$.

Let $J$ be some player and let $S$ be a pseudostrategy for player $J$ in $G_1$. For every $\sigma \in S$ of length $k$, put

$$A_\sigma = \begin{cases} \{\tau \in \Delta_\sigma : (0, \tau) \text{ is a move in } S \text{ at } \sigma\} & \text{if } J = \text{I}, \\ \{\tau \in \Delta_\sigma : (0, \tau) \text{ is not a move in } S \text{ at } \sigma\} & \text{if } J = \text{II}. \end{cases}$$

Since $S$ is a pseudostrategy for player $J$ in $G_1$ and $\sigma$ is a position of type III, $(1, A_\sigma)$ is a move in $S$ at $\sigma$. Furthermore, if $\tau = \sigma^\frown\rho \in \Delta_\sigma$ and $\sigma^\frown\langle(1, A_\sigma)\rangle^\frown\rho \in S$, then $\tau \in A_\sigma$ if $J = \text{I}$ and $\tau \notin A_\sigma$ if $J = \text{II}$. In both cases, $(0, \tau)$ is a move in $S$ at $\sigma$ and thus, since the moves $\rho(0), \rho(1), \ldots$ are obligatory, $\sigma^\frown\langle(0, \tau)\rangle^\frown\rho \in S$.

We now let $\phi_J(S)$ be the following pseudostrategy for player $J$ in $G_0$: $S^{\leq k} \cup \{\sigma^\frown\rho : \text{length}(\sigma) = k \text{ and } \sigma^\frown\langle(1, A_\sigma)\rangle^\frown\rho \in S\} \cup \{\sigma^\frown\rho : \text{length}(\sigma) = k \text{ and for some } \rho',\ \sigma^\frown\langle(1, A_\sigma)\rangle^\frown\rho' \in S \text{ and } \sigma^\frown\rho' \in \Delta_\sigma \text{ and } \sigma^\frown\langle(0, \sigma^\frown\rho')\rangle^\frown\rho \in S\}$.

Define a function $F_J(S) : [\phi_J(S)] \longrightarrow [S]$ as follows:
For every $x \in [\phi_J(S)]$, if $\sigma = x|k$ and $r$ is the infinite sequence for which $x = \sigma^\frown r$, then

$$F_J(S)(x) = \begin{cases} \sigma^\frown\langle(1, A_\sigma)\rangle^\frown r & \text{if there is no } \tau \in \Delta_\sigma \text{ such that } \tau \subseteq x, \\ \sigma^\frown\langle(0, \tau)\rangle^\frown r & \text{if } \tau \in \Delta_\sigma \text{ is minimal such that } \tau \subseteq x. \end{cases}$$

Then for all $x \in [\phi_J(S)]$, $F_J(S)(x) \circ f = x$. So $\phi_J$ is an $f$-translator of pseudostrategies for player $J$ from $G_1$ to $G_0$.

For every $n \in \omega$, we define a function $\phi_J^n$ just like $\phi_J$ but now from the set of all pseudostrategies for player $J$ in $G_1$ up to positions of length $f(n)$ to the set of all pseudostrategies for player $J$ in $G_0$ up to positions of length $n$. Then the functions $\phi_J^0, \phi_J^1, \ldots$ witness that $\phi_J$ is continuous. Put $\Phi_J = (\phi_J, F_J, \langle\phi_J^0, \phi_J^1, \ldots\rangle)$.

For each $n \in \omega$, if $f(n) = n$ then $n < k$, so $T_1{}^{\leq n+1} = T_0{}^{\leq n+1}$, $p_1|T_1{}^{\leq n} = p_0|T_0{}^{\leq n}$, and for every $S \in \text{domain}(\phi_J^n)$, $\phi_J^n(S) = S$.

Thus $(G_1, \Phi_{\text{I}}, \Phi_{\text{II}})$ is a basic open $f$-covering of the open generalized game $G_0$.

We now construct, for every coded Borel generalized game, a basic open covering.

**5.12** LEMMA Let $(T, p, X)$ be a coded Borel generalized game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that $\omega \setminus \text{range}(f)$ is infinite. Then $(T, p, X)$ has a basic open $f$-covering.

PROOF Choose a Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ for $X$. Define, for every $\gamma \in C$, a Borel subset $X_\gamma$ of $[T]$ as in Definition 2.13. So $X = X_{\langle\rangle}$.

We will not, as in the proof of Lemma 4.16, define a basic open $f$-covering of $(T, p, X)$ by transfinite induction to the rank of $C$. Instead, we define, for every $\gamma \in C$, $f_\gamma$, $g_\gamma$, $G_\gamma$, and $H_\gamma$ such that:

**(i)** $f_{\langle\rangle} = f$, $g_{\langle\rangle}$ is the identity on $\omega$, and $G_{\langle\rangle} = (T, p, X)$;

**(ii)** $f_\gamma$ and $g_\gamma$ are strictly increasing functions from $\omega$ to $\omega$ such that $\omega \setminus \text{range}(f_\gamma)$ is infinite and $G_\gamma$ is a generalized game $(T_\gamma, p_\gamma, Y_\gamma)$ such that for all $\sigma \in T_\gamma$, $\sigma \circ g_\gamma \in T$, and $Y_\gamma = \{x \in [T_\gamma] : x \circ g_\gamma \in X_\gamma\}$;

**(iii)** $H_\gamma$ is a basic open $f_\gamma$-covering $((U_\gamma, q_\gamma, Z_\gamma), \Phi^\gamma_{\mathrm{I}}, \Phi^\gamma_{\mathrm{II}})$ of $G_\gamma$.

For $\gamma = \langle\rangle$, $f_\gamma$, $g_\gamma$, and $G_\gamma$ are given by (i), and (ii) holds. Let $\gamma \in C$ and suppose that we have constructed $f_\gamma$, $g_\gamma$, and $G_\gamma$ such that (ii) holds. We construct some $H_\gamma$ such that (iii) holds and we construct, for every move $a$ in $C$ at $\gamma$, $f_{\gamma^\frown\langle a\rangle}$, $g_{\gamma^\frown\langle a\rangle}$ and $G_{\gamma^\frown\langle a\rangle}$ satisfying (ii) with $\gamma^\frown\langle a\rangle$ instead of $\gamma$, as follows.

First suppose that $c(\gamma) \in \{\bot, \top, \diamond\}$. Then $X_\gamma$ is a basic open subset of $[T]$, so $Y_\gamma$ is a basic open subset of $[T_\gamma]$. Now it is easy to construct a basic open $f_\gamma$-covering $H_\gamma$ of $G_\gamma$: The idea is to insert some trivial (obligatory) extra move at the right places, just as in the proof of Lemma 1.22. For every move $\tau$ in $C$ at $\gamma$, we put $f_{\gamma^\frown\langle\tau\rangle} = f_\gamma$, $g_{\gamma^\frown\langle\tau\rangle} = g_\gamma$, and $G_{\gamma^\frown\langle\tau\rangle} = (T_\gamma, p_\gamma, \{x \in [T_\gamma] : x \circ g_{\gamma^\frown\langle\tau\rangle} \in X_{\gamma^\frown\langle\tau\rangle}\})$.

Now suppose that $c(\gamma) \in \{\vee, \wedge\}$. Then the moves in $C$ at $\gamma$ are the natural numbers and $X_\gamma$ is either $\bigcup_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$ or $\bigcap_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$.

Enumerate the infinite set $\omega \setminus \text{range}(f_\gamma)$ in an increasing order $k_0, k_1, \ldots$. For every $n \in \omega$, let $h_n$ be the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k_{2^n} - 1, k_{2^n \cdot 2} - 1, k_{2^n \cdot 3} - 1, \ldots\}$. Since $\text{range}(h_n) \subseteq \text{range}(h_{n+1})$, we can define a strictly increasing $f_{\gamma^\frown\langle n\rangle} : \omega \longrightarrow \omega$ such that $h_{n+1} \circ f_{\gamma^\frown\langle n\rangle} = h_n$. Note that $\omega \setminus \text{range}(f_{\gamma^\frown\langle n\rangle})$ is infinite. Just as in the proof of Lemma 1.22, it is not difficult to verify that $h_n = \lim_{m\to\omega} f_{\gamma^\frown\langle m\rangle} \circ \cdots \circ f_{\gamma^\frown\langle n+1\rangle} \circ f_{\gamma^\frown\langle n\rangle}$.

Let $h$ be the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k_0\}$. Then one easily verifies that $f_\gamma = h \circ h_0$.

Define inductively for every $n \in \omega$ a strictly increasing $g_{\gamma^\frown\langle n\rangle} : \omega \longrightarrow \omega$ by $g_{\gamma^\frown\langle 0\rangle} = g_\gamma$ and for all $n \in \omega$, $g_{\gamma^\frown\langle n+1\rangle} = f_{\gamma^\frown\langle n\rangle} \circ g_{\gamma^\frown\langle n\rangle}$.

Note that for all $\sigma \in T_\gamma$, $\sigma \circ g_{\gamma^\frown\langle 0\rangle} \in T$. Let $G_{\gamma^\frown\langle 0\rangle}$ be the generalized game $(T_\gamma, p_\gamma, \{x \in [T_\gamma] : x \circ g_{\gamma^\frown\langle 0\rangle} \in X_{\gamma^\frown\langle 0\rangle}\})$. By induction on $C$, we know how to construct the basic open $f_{\gamma^\frown\langle 0\rangle}$-covering $H_{\gamma^\frown\langle 0\rangle}$ of $G_{\gamma^\frown\langle 0\rangle}$.

Given $H_{\gamma^\frown\langle n\rangle}$ for some $n \in \omega$, note that for all $\sigma \in U_{\gamma^\frown\langle n\rangle}$, $\sigma \circ f_{\gamma^\frown\langle n\rangle} \circ g_{\gamma^\frown\langle n\rangle} \in T$ and $Z_{\gamma^\frown\langle n\rangle} = \{x \in [U_{\gamma^\frown\langle n\rangle}] : x \circ g_{\gamma^\frown\langle n+1\rangle} \in X_{\gamma^\frown\langle n\rangle}\}$. Let $G_{\gamma^\frown\langle n+1\rangle}$ be the generalized game $(U_{\gamma^\frown\langle n\rangle}, q_{\gamma^\frown\langle n\rangle}, \{x \in [U_{\gamma^\frown\langle n\rangle}] : x \circ g_{\gamma^\frown\langle n+1\rangle} \in X_{\gamma^\frown\langle n+1\rangle}\})$. By induction on $C$, we know how to construct $H_{\gamma^\frown\langle n+1\rangle}$.

For every $n \in \omega$, let $G^n$ be the generalized game $(T_{\gamma^\frown\langle n\rangle}, p_{\gamma^\frown\langle n\rangle}, \{x \in [T_{\gamma^\frown\langle n\rangle}] : x \circ g_{\gamma^\frown\langle n\rangle} \in X_\gamma\})$, Note that $G^0 = G_\gamma$. By Lemma 5.9(ii), $(G^{n+1}, \Phi_{\mathrm{I}}^{\gamma^\frown\langle n\rangle}, \Phi_{\mathrm{II}}^{\gamma^\frown\langle n\rangle})$ is an $f_{\gamma^\frown\langle n\rangle}$-covering of $G^n$.

Let $((T', p', X'), \Psi^0_{\mathrm{I}}, \Psi^0_{\mathrm{II}})$ be the $h_0$-covering of $G^0$ given by Lemma 5.10. Then $X' = \{x \in [T'] : x \circ h_0 \circ g_\gamma \in X_\gamma\}$. Thus $X'$ is either $\bigcup_{n\in\omega} X'_n$ or $\bigcap_{n\in\omega} X'_n$, where, for every $n \in \omega$, $X'_n = \{x \in [T'] : x \circ h_0 \circ g_\gamma \in X_{\gamma^\frown\langle n\rangle}\}$.

Since $h_0 \circ g_{\gamma^\frown\langle 0\rangle} = h_0 \circ g_\gamma$ and for every $n$, $h_{n+1} \circ g_{\gamma^\frown\langle n+1\rangle} = h_{n+1} \circ f_{\gamma^\frown\langle n\rangle} \circ g_{\gamma^\frown\langle n\rangle} = h_n \circ g_{\gamma^\frown\langle n\rangle}$, we have that for every $n$, $X'_n = \{x \in [T'] : x \circ h_{n+1} \in Z_{\gamma^\frown\langle n\rangle}\}$, so $X'_n$ is a basic open subset of $[T']$.

This implies that the generalized game $(T', p', X')$ is either open or closed. Let $(G', \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ be the basic open $h$-covering of $(T', p', X')$ given by Lemma 5.11. We let $H_\gamma$ be the $(h \circ h_0)$-covering $(G', \Psi^0_{\mathrm{I}} \square \Phi_{\mathrm{I}}, \Psi^0_{\mathrm{II}} \square \Phi_{\mathrm{II}})$ of $G^0$ given by Lemma 5.9(iii). So $H_\gamma$ is a basic open $f_\gamma$-covering of $G_\gamma$.

In the last part of the proof we described, for each $\gamma \in C$ such that $c(\gamma) \in \{\vee, \wedge\}$ and for each $n \in \omega$, how:

- $f_{\gamma^\frown\langle n\rangle}$ depends on $f_\gamma$;
- $g_{\gamma^\frown\langle 0\rangle}$ depends on $g_\gamma$;
- $g_{\gamma^\frown\langle n+1\rangle}$ depends on $g_{\gamma^\frown\langle n\rangle}$ and $f_{\gamma^\frown\langle n\rangle}$.

So these functions could have been defined by induction to the length of $\gamma$. For such $\gamma$ and $n$ we also described how:

- $G_{\gamma^\frown\langle 0\rangle}$ depends on $G_\gamma$ and $g_{\gamma^\frown\langle 0\rangle}$;
- $G_{\gamma^\frown\langle n+1\rangle}$ depends on $H_{\gamma^\frown\langle n\rangle}$ and $g_{\gamma^\frown\langle n+1\rangle}$;
- $H_\gamma$ depends on $f_{\gamma^\frown\langle m\rangle}$, $g_{\gamma^\frown\langle m\rangle}$, and $H_{\gamma^\frown\langle m\rangle}$ for all $m \in \omega$.

For the definition of these objects, it is essential that $C$ is wellfounded.

To be more precise, the definition of $f_\gamma$, $g_\gamma$, $G_\gamma$, and $H_\gamma$ for $\gamma \in C$ is justified by proving the following by induction on $C$: For every $\gamma \in C$, for all $f_\gamma, g_\gamma$, and $G_\gamma$ such that (ii) holds, there is a unique function $F$ on the set $\{\delta \in C : \gamma \subseteq \delta\}$ such that for some $H_\gamma$, (iii) holds and $F(\gamma) = (f_\gamma, g_\gamma, G_\gamma, H_\gamma)$, and such that for every $\delta \in \text{domain}(F)$, $F|(\{\delta\} \cup \{\delta^\frown\langle a\rangle : a$ is a move in $C$ at $\delta\})$ satisfies some condition. One can find this condition by inspecting the proof carefully.

**5.13** THEOREM Every coded Borel generalized game is pseudodetermined.

PROOF Let $G_0$ be a coded Borel generalized game. Define $f : \omega \longrightarrow \omega$ by $f(n) = 2n$. Then $\omega \setminus \text{range}(f)$ is infinite. By Lemma 5.12, there is an $f$-covering $(G_1, \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ of $G_0$ such that the generalized game $G_1$ is basic open. By Theorem 5.5, $G_1$ is pseudodetermined. So, by Lemma 5.9(i), the generalized game $G_0$ is pseudodetermined.

By Remark 5.3 and Theorem 5.13, every coded Borel game is pseudodetermined.

# 6 Quasi-Borel games

D.A. Martin [1990] introduced quasi-Borel sets and proved, using the axiom of choice, that these sets are the same as the so-called $\Delta^1_1$ sets. He also extended his proof of Borel determinacy to quasi-Borel games.

In this chapter, we give a definition of quasi-Borel sets that differs slightly from Martin's. We use the operation of 'mixing' instead of 'open-separated union'. We prove, without using the axiom of choice, that all coded quasi-Borel games are pseudodetermined. We also introduce *absolutely* $\Delta^1_1$ *sets* and prove that these sets coincide with the coded quasi-Borel sets.

## 6A Quasi-Borel sets

**6.1** Definition Let $T$ be a tree and let $n \in \omega$. By $T^{=n}$ we denote the set of all elements of $T$ of length $n$. Suppose that for each $\tau \in T^{=n}$, some subset $X_\tau$ of $[T]$ is given. Define $X = \{x \in [T] : x \in X_{x|n}\}$, so $X = \bigcup_{\tau \in T^{=n}} \{x \in X_\tau : \tau \subseteq x\}$. We call $X$ the **mix** of the sets $X_\tau$ ($\tau \in T^{=n}$).

Let $\mathcal{C}$ be a collection of subsets of $[T]$. We say that $\mathcal{C}$ is **closed under mixing with respect to** $T$ if for every $n \in \omega$ and $F : T^{=n} \longrightarrow \mathcal{C}$, the mix of the sets $F(\tau)$ ($\tau \in T^{=n}$) is an element of $\mathcal{C}$.

A **quasi-Borel** subset of $[T]$ is a set that belongs to every collection of subsets of $[T]$ that contains $\emptyset$ and $[T]$, and is closed under mixing with respect to $T$, countable union, and countable intersection.

Note that if $X$ is the mix of the sets $X_\tau$ ($\tau \in T^{=n}$), then $X$ is the union of the sets $X_\tau \cap [T^{\mathrm{via}\,\tau}]$ ($\tau \in T^{=n}$), and $[T] \setminus X$ is the mix of the sets $[T] \setminus X_\tau$ ($\tau \in T^{=n}$). Using this, one easily verifies that for every tree $T$ and every $X \subseteq [T]$:

- $X$ is quasi-Borel if and only if $[T] \setminus X$ is quasi-Borel;
- $X$ is basic open if and only if $X$ is a mix of trivial subsets of $[T]$;
- if $X$ is Borel, then $X$ is quasi-Borel;
- if the tree $T$ is countable and $X$ is quasi-Borel, then $X$ is Borel.

So for countable trees $T$, the quasi-Borel subsets of $[T]$ are the same as the Borel subsets of $[T]$.

**6.2** Remark CAC implies that not every quasi-Borel set is Borel:
Let $T$ be the tree $\{\langle\rangle\} \cup \{\langle B\rangle^\frown\sigma : B$ is a Borel subset of ${}^\omega 2$ and $\sigma \in {}^{<\omega}2\}$. Put $X = \{\langle B\rangle^\frown x : B$ is a Borel subset of ${}^\omega 2$ and $x \in B\}$.

Then $X$ is a quasi-Borel subset of $[T]$, since it is the mix of the sets $X_{\langle B\rangle}$ ($\langle B\rangle \in T$), where for each Borel subset $B$ of ${}^{\omega}2$, $X_{\langle B\rangle}$ is the Borel subset $\{\langle B\rangle^\frown x : x \in B\}$ of $[T]$.

Now suppose that $X$ is a Borel subset of $[T]$. Then, by CAC, the Borel rank $\alpha$ of $X$ is finite or countable. But this would imply that each Borel subset $B$ of ${}^{\omega}2$ has Borel rank at most $\alpha$, whereas it is well-known that CAC implies that for each finite or countable ordinal $\beta$, some Borel subset of ${}^{\omega}2$ of rank $\beta$ exists.

Let $f : \omega \longrightarrow \omega$ be strictly increasing and consider trees $T_0$ and $T_1$ such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. In Remark 1.14 we defined a (continuous) function $p : [T_1] \longrightarrow [T_0]$ by $p(x) = x \circ f$. Let $X \subseteq [T_0]$. Then one easily verifies that:

- if $X$ is a trivial subset of $[T_0]$, then $p^{-1}X$ is a trivial subset of $[T_1]$;
- if $X$ is the union or intersection of the subsets $X_n$ of $[T_0]$ ($n \in \omega$), then $p^{-1}X$ is the union or intersection of the subsets $p^{-1}X_n$ of $[T_1]$ ($n \in \omega$);
- if $X$ is the mix of the subsets $X_\tau$ of $[T_0]$ ($\tau \in {T_0}^{=n}$) and $m \in \omega$ such that $f^{-1}m = n$ (for example $m = f(n)$), then $p^{-1}X$ is the mix of the subsets $p^{-1}X_{\tau\circ f}$ of $[T_1]$ ($\tau \in {T_1}^{=m}$);
- if $X$ is a (quasi-)Borel subset of $[T_0]$, then $p^{-1}X$ is a (quasi-)Borel subset of $[T_1]$.

## 6B Reducing quasi-Borel games to basic open games

In Section 1E we proved, using AC, that every Borel game is determined.

The idea was to reduce each Borel game $(T, P, X)$ to a basic open game, by iterating the basic step of reducing open games to basic open games. In the induction step, $X$ is a countable union or intersection of 'simpler' Borel sets $X_n$ ($n \in \omega$). We used Lemma 1.19 to reduce $(T, P, X)$ to some open or closed game by treating $X_0, X_1, \ldots$ *successively.*

In order to extend this result to quasi-Borel games, we must also consider the case that the quasi-Borel set $X$ is a mix of 'simpler' quasi-Borel sets $X_\tau$ ($\tau \in T^{=n}$). We will use the following lemma to reduce $(T, P, X)$ to some open and closed game by treating the sets $X_\tau$ *simultaneously.*

**6.3** Lemma Let $(T, P, X)$ be a game. Let $n \in \omega$ and let $f : \omega \longrightarrow \omega$ be strictly increasing such that $f|n$ is the identity on $n$. In other words, for all $m < n$, $f(m) = m$. Suppose that for each $\tau \in T^{=n}$, some $f$-covering $(U_\tau, Q_\tau, Y_\tau)$ of the game $(T^{\mathrm{via}\,\tau}, P \cap T^{\mathrm{via}\,\tau}, X \cap [T^{\mathrm{via}\,\tau}])$ is given.

Then AC implies that there is a (unique) $f$-covering $(U, Q, Y)$ of $(T, P, X)$ such that for every $\tau \in T^{=n}$, $U_\tau = U^{\mathrm{via}\,\tau}$, $Q_\tau = Q \cap U_\tau$, and $Y_\tau = Y \cap [U_\tau]$.

Proof Let $U$ be the unique tree for which $U^{\leq n} = T^{\leq n}$ and for all $\tau \in U^{=n}$, $U^{\text{via}\,\tau} = U_\tau$. Define $Q = (P \cap T^{\leq n}) \cup \bigcup_{\tau \in T^{=n}} Q_\tau$ and put $Y = \bigcup_{\tau \in T^{=n}} Y_\tau$.

Then for all $\sigma \in U$, $\sigma \circ f \in T$, and $Y = \{y \in [U] : y \circ f \in X\}$. For all $m \in \omega$, if $f(m) = m$ then $U^{\leq m+1} = T^{\leq m+1}$ and $Q \cap U^{\leq m+1} = P \cap T^{\leq m+1}$.

Choose (using AC) for every $\tau \in T^{=n}$ an $f$-translator $\phi_\tau$ of strategies for player I from $(U_\tau, Q_\tau, Y_\tau)$ to $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, X \cap [T^{\text{via}\,\tau}])$ and functions $\phi_\tau^0, \phi_\tau^1, \ldots$ witnessing that $\phi_\tau$ is continuous.

For each strategy $S$ for player I in $(U, Q, Y)$, we describe a strategy $\phi(S)$ for I in $(T, P, X)$ as follows:

Player I follows $S$ until a position $\tau$ of length $n$ is reached. Then he follows $\phi_\tau(S^{\text{via}\,\tau})$. (Note that $S^{\text{via}\,\tau}$ is a strategy for I in $(U_\tau, Q_\tau, Y_\tau)$.)

In other words, $\phi(S) = S^{\leq n} \cup \bigcup_{\tau \in S^{=n}} \phi_\tau(S^{\text{via}\,\tau})$.

One easily verifies that $\phi$ is a continuous $f$-translator of strategies for player I from $(U, Q, Y)$ to $(T, P, X)$. In the same way, a continuous $f$-translator for player II can be found. Thus $(U, Q, Y)$ is an $f$-covering of $(T, P, X)$.

We now generalize Lemma 1.22.

**6.4** Lemma Let $G_0$ be a quasi-Borel game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that for infinitely many natural numbers $k$, neither $k$ nor $k+1$ is in the range of $f$.

Then AC implies that there is a basic open $f$-covering $G_1$ of $G_0$.

Proof Let $\Omega$, $T$ and $\mathcal{C}$ be as in the proof of Lemma 1.22. We have already proved that $\mathcal{C}$ contains every basic open subset of $[T]$ (so, in particular, $\mathcal{C}$ contains the trivial subsets of $[T]$) and that $\mathcal{C}$ is closed under countable union and countable intersection. Since we want to prove that $\mathcal{C}$ contains every quasi-Borel subset of $[T]$, it is enough to show that $\mathcal{C}$ is closed under mixing with respect to $T$.

So suppose that $X$ is the mix of the sets $X_\tau$ ($\tau \in T^{=n}$), where $n \in \omega$ and for all $\tau \in T^{=n}$, $X_\tau \in \mathcal{C}$. In order to prove that $X \in \mathcal{C}$, let $g : \omega \longrightarrow \omega$ be strictly increasing. Let $(T', P', X')$ be a game such that for all $\sigma \in T'$, $\sigma \circ g \in T$, and $X' = \{x \in [T'] : x \circ g \in X\}$. Let $f \in \Omega$. We must find a basic open $f$-covering of $(T', P', X')$.

Just as in the proof of Lemma 1.22, we find strictly increasing functions $g'$ and $h$ from $\omega$ to $\omega$ such that $f = h \circ g'$, $g' \in \Omega$, and for some $k \in \omega$, $\text{range}(h) = \omega \setminus \{k, k+1\}$.

Claim There are strictly increasing functions $l$ and $r$ from $\omega$ to $\omega$ such that $g' = l \circ r$, $r \in \Omega$ and for all $m < g(n)$, $r(m) = m$.

Proof of claim Put $k = g'(g(n)) - g(n)$. Define $l$ and $r$ by

$$l(m) = \begin{cases} g'(m) & \text{if } m < g(n), \\ m + k & \text{if } m \geq g(n), \end{cases} \quad r(m) = \begin{cases} m & \text{if } m < g(n), \\ g'(m) - k & \text{if } m \geq g(n). \end{cases}$$

Then it is clear that $g' = l \circ r$ and $r \in \Omega$.

Let $\tau \in T'$ of length $g(n)$, so $\tau \circ g \in T^{=n}$. For each $x \in [T'^{\text{via}\,\tau}]$, $\tau \circ g \subseteq x \circ g$, so $x \circ g \in X$ if and only if $x \circ g \in X_{\tau \circ g}$. So $X' \cap [T'^{\text{via}\,\tau}] = \{x \in [T'^{\text{via}\,\tau}] : x \circ g \in X_{\tau \circ g}\}$. Thus, since $X_{\tau \circ g} \in \mathcal{C}$ and $r \in \Omega$, the game $(T'^{\text{via}\,\tau}, P' \cap T'^{\text{via}\,\tau}, X' \cap [T'^{\text{via}\,\tau}])$ has some basic open $r$-covering.

Choose, using AC, for every $\tau \in T'^{=g(n)}$ some basic open $r$-covering $(U_\tau, Q_\tau, Y_\tau)$ of that game. By Lemma 6.3, there is a (unique) $r$-covering $(U, Q, Y)$ of $(T', P', X')$ such that for every $\tau \in T'^{=g(n)}$, $U_\tau = U^{\text{via}\,\tau}$, $Q_\tau = Q \cap U_\tau$, and $Y_\tau = Y \cap [U_\tau]$. Note that $Y$ is an open (and also closed) subset of $[U]$, since it is the union of the basic open sets $Y_\tau$ ($\tau \in T'^{=g(n)}$).

Just as in the proof of Lemma 1.22, we can easily construct an $l$-covering $G$ of the game $(U, Q, Y)$ by inserting some trivial extra move at the right places. Since $(U, Q, Y)$ is open, the game $G$ is also open. By Lemma 1.20 and AC, $G$ has some basic open $h$-covering $G'$. By Lemma 1.18(iii), $G'$ is an $(h \circ l \circ r)$-covering of $(T', P', X')$, so $G'$ is a basic open $f$-covering of $(T', P', X')$.

**6.5** Theorem AC implies that every quasi-Borel game is determined.

Proof Just like the proof of Theorem 1.24, with Lemma 6.4 instead of Lemma 1.22.

## 6C Quasi-Borel codes

In Chapter 3 we proved, using DC, that every Borel game is pseudodetermined. This result cannot be extended to quasi-Borel games. To be more precise, if ZF together with AC and the statement that there is a so-called strongly inaccessible cardinal, is consistent, then, in ZF, DC does not imply that every quasi-Borel game is pseudodetermined.

This follows from the next proposition and the following theorem of R.M. Solovay [1970]: If there is a model of ZF and AC in which there is a strongly inaccessible cardinal, then there is a model of ZF and DC in which every uncountable set of real numbers has a *perfect* subset, that is, a non-empty closed subset without isolated points. (Solovay observes that the converse of his theorem also holds.)

**6.6** Proposition If every uncountable set of real numbers has a perfect subset, then there is a quasi-Borel game that is not pseudodetermined.

Proof Suppose that every uncountable set of real numbers has a perfect subset.

Consider the game $G$ that is played as follows: Player I chooses a finite or countable subset $A$ of ${}^{\omega}2$; then player II chooses $a_0, a_1, \ldots \in 2$.

Player I wins the play $\langle A, a_0, a_1, \ldots \rangle$ if and only if $\langle a_0, a_1, \ldots \rangle \in A$. The winning set for player I in $G$ is a mix of finite or countable sets, so $G$ is a quasi-Borel game. Now suppose that $G$ is pseudodetermined. We will derive a contradiction.

Clearly, player I has no winning pseudostrategy in $G$: After a move $A$ of player I, player II can play moves $a_0, a_1, \ldots$ such that $\langle a_0, a_1, \ldots \rangle \in {}^{\omega}2 \setminus A$. So player II has a winning pseudostrategy in $G$. Since each of his moves is either 0 or 1, player II even has a winning strategy. Consequently, there is a function $F$ that assigns to each finite or countable subset $A$ of ${}^{\omega}2$, an element of ${}^{\omega}2 \setminus A$.

We define a function $f$ from $\omega_1$, the least uncountable ordinal, to ${}^{\omega}2$ by transfinite induction: For every finite or countable ordinal $\alpha$, $f(\alpha) = F(\{f(\beta) : \beta < \alpha\})$. Note that if $\beta < \alpha < \omega_1$ then $f(\alpha) \neq f(\beta)$. So $f$ is an injective function. Let $g$ be some bijection from ${}^{\omega}2$ to the set $\mathbb{R}$ of all real numbers and put $X = \mathrm{range}(g \circ f)$. Then $X$ is an uncountable set of real numbers. So, by our assumption, $X$ has some perfect subset $P$. Using the fact that $P$ is perfect, one easily finds an injective function from ${}^{\omega}2$ to $P$. So there is an injective function from $\mathbb{R}$ to $X$. Since $g \circ f$ is a bijection from $\omega_1$ to $X$, we see that $\mathbb{R}$ can be wellordered. It is well-known that this implies that there is a set of real numbers without a perfect subset: Using a wellordering of $\mathbb{R}$ and an injective function from the set of all perfect sets of real numbers to $\mathbb{R}$, one easily constructs, by transfinite induction, disjoint sets $A$ and $B$ of real numbers such that every perfect set of real numbers intersects both $A$ and $B$. So neither $\mathbb{R} \setminus A$ nor $\mathbb{R} \setminus B$ has a perfect subset. Since at least one of both sets is uncountable, this contradicts our assumption.

We now generalize the concept of a Borel code. A *quasi-Borel code* tells us how a quasi-Borel set is constructed from trivial sets by means of the operations of mixing, countable union and countable intersection.

The definition of quasi-Borel codes is very similar to the definition of Borel codes in Definition 2.13. In fact, we chose these definitions in order to ensure that each Borel code is a quasi-Borel code.

**6.7** DEFINITION Let $T$ be a tree. A **quasi-Borel code** with respect to $T$ is a function $c$ from a wellfounded tree $C$ to the set $\{\bot, \top, \vee, \wedge, \Diamond\}$ such that for every $\gamma \in C$:

- if $c(\gamma) \in \{\bot, \top\}$, then $\gamma$ is a terminal node of $C$;
- if $c(\gamma) \in \{\vee, \wedge\}$, then the moves in $C$ at $\gamma$ are the natural numbers;
- if $c(\gamma) = \Diamond$ then for some $n \in \omega$, the moves in $C$ at $\gamma$ are the elements of $T^{=n}$.

We define, by induction on the wellfounded tree $C$, for every $\gamma \in C$ a (quasi-Borel) subset $X_\gamma$ of $[T]$ as follows:

- if $c(\gamma)$ is $\bot$ or $\top$ then $X_\gamma$ is the trivial set $\emptyset$ or $[T]$, respectively;
- if $c(\gamma)$ is $\vee$ or $\wedge$ then $X_\gamma$ is $\bigcup_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$ or $\bigcap_{n\in\omega} X_{\gamma^\frown\langle n\rangle}$, respectively;
- if $c(\gamma) = \Diamond$ and for some $n \in \omega$, the moves in $C$ at $\gamma$ are the elements of $T^{=n}$, then $X_\gamma$ is the mix of the sets $X_{\gamma^\frown\langle\tau\rangle}$ ($\tau \in T^{=n}$); in other words, $X = \{x \in [T] : x \in X_{\gamma^\frown\langle x|n\rangle}\}$.

We say that $c$ **is a quasi-Borel code for** $X_{\langle\rangle}$.

A **coded quasi-Borel** subset of $[T]$ is a set $X$ such that there is a quasi-Borel code (with respect to $T$) for $X$.

Let $c$ be a quasi-Borel code for some $X \subseteq [T]$. Just as in Remark 2.14, we can define, for every $x \in [T]$, a game $G_x$ on a finite or countable, wellfounded tree such that $x \in X$ if and only if player I has a winning strategy in $G_x$.

**6.8** Definition Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $T_0$ and $T_1$ be trees such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Suppose that $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \Diamond\}$ is a quasi-Borel for some subset $X$ of $[T_0]$. Define a tree $C'$ and a function $\Gamma : C' \longrightarrow C$ as follows:

- $\Gamma(\langle\rangle) = \langle\rangle$;
- for every $\gamma \in C'$ such that $c(\Gamma(\gamma)) \in \{\bot, \top, \vee, \wedge\}$, the moves in $C'$ at $\gamma$ are the same as the moves in $C$ at $\Gamma(\gamma)$ and for each move $n$ in $C'$ at $\gamma$, $\Gamma(\gamma^\frown\langle n\rangle) = \Gamma(\gamma)^\frown\langle n\rangle$;
- for every $\gamma \in C'$ such that $c(\Gamma(\gamma)) = \Diamond$, the moves in $C'$ at $\gamma$ are the elements $\tau$ of $T_1$ of minimal length for which $\tau\circ f$ is a move in $C$ at $\Gamma(\gamma)$, and for each move $\tau$ in $C'$ at $\gamma$, $\Gamma(\gamma^\frown\langle\tau\rangle) = \Gamma(\gamma)^\frown\langle\tau \circ f\rangle$.

Then $c \circ \Gamma$ is quasi-Borel code for $\{x \in [T_1] : x \circ f \in X\}$. We call $c \circ \Gamma$ the **quasi-Borel code associated with** $T_1$, $f$, and $c$.

Note that the rank of the wellfounded tree $C'$ is less than or equal to the rank of the wellfounded tree $C$.

**6.9** Proposition AC implies that for every tree $T$, every quasi-Borel subset of $[T]$ is coded quasi-Borel.

Proof The unique function from $\{\langle\rangle\}$ to $\{\bot\}$ is a quasi-Borel code for $\emptyset$ and the unique function from $\{\langle\rangle\}$ to $\{\top\}$ is a quasi-Borel code for $[T]$.

Let $\langle X_0, X_1, \ldots\rangle$ be an infinite sequence of coded quasi-Borel subsets of $[T]$. By the axiom of choice we can choose, for each $n \in \omega$, some quasi-Borel code $c_n$ for $X_n$. Using these, we define quasi-Borel codes for $\bigcup_{n\in\omega} X_n$ and $\bigcap_{n\in\omega} X_n$, just as in the proof of Proposition 2.15.

In a similar way, we find, using AC, a quasi-Borel code for each mix of coded quasi-Borel subsets of $[T]$.

**6.10** Remark We will prove, in ZF, that every coded quasi-Borel game is pseudodetermined. We already know, by Proposition 6.6, that DC does not imply that every quasi-Borel game is pseudodetermined. Therefore, DC does not imply that every quasi-Borel set has a code. On the other hand, by Proposition 2.15, CAC implies that every Borel set has a code.

In Section 6B we proved, using AC, the determinacy of quasi-Borel games by extending the result of Chapter 1. We now prove the pseudodeterminacy of coded quasi-Borel (generalized) games by extending the result of Chapter 5. (One may also use preferential games, but then one should first define quasi-Borel coded preferential games. This definition will be similar to Definition 4.8.)

**6.11** Lemma Let $(T, p, X)$ be a generalized game. Let $n \in \omega$ and let $f : \omega \longrightarrow \omega$ be strictly increasing such that $f|n$ is the identity on $n$. Suppose that for every $\tau \in T^{=n}$, some $f$-covering $((U_\tau, q_\tau, Y_\tau), \Phi_{\mathrm{I}}^\tau, \Phi_{\mathrm{II}}^\tau)$ of the generalized game $(T^{\mathrm{via}\,\tau}, p_\tau, X \cap [T^{\mathrm{via}\,\tau}])$ is given, where $p_\tau$ is the function on $T^{\mathrm{via}\,\tau}$ defined by:
$$p_\tau(\sigma) = \begin{cases} \{\{\tau(m)\}\} & \text{if } m < n \text{ and } \sigma = \tau|m, \\ p(\sigma) & \text{if } \tau \subseteq \sigma \text{ and } \sigma \in T. \end{cases}$$

Then there is a (unique) $f$-covering $((U, q, Y), \Phi_{\mathrm{I}}, \Phi_{\mathrm{II}})$ of $(T, p, X)$ such that for every $\tau \in T^{=n}$, $U_\tau = U^{\mathrm{via}\,\tau}$, for all $\sigma \in U$, if $\tau \subseteq \sigma$ then $q(\sigma) = q_\tau(\sigma)$, $Y_\tau = Y \cap [U_\tau]$, and for each player $J$, if we write $\Phi_J^\tau = (\phi^\tau, F^\tau, \langle \phi_0^\tau, \phi_1^\tau, \ldots\rangle)$ and $\Phi_J = (\phi, F, \langle \phi_0, \phi_1, \ldots\rangle)$, then for every pseudostrategy $S$ for player $J$ in $(U, q, Y)$ such that $\tau \in S$, $F(S)|[\phi(S)^{\mathrm{via}\,\tau}] = F^\tau(S^{\mathrm{via}\,\tau})$, and for every $m \in \omega$ and every pseudostrategy $S$ for player $J$ in $(U, q, Y)$ up to positions of length $f(m)$ such that $\tau \in S$, $\phi_m(S)^{\mathrm{via}\,\tau} = \phi_m^\tau(S^{\mathrm{via}\,\tau})$.

Proof Similar to the proof of Lemma 6.3.

We now generalize Lemma 5.12.

**6.12** Lemma Let $(T, p, X)$ be a coded quasi-Borel generalized game. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that $\omega \setminus \mathrm{range}(f)$ is infinite. Then $(T, p, X)$ has a basic open $f$-covering.

Proof Choose a quasi-Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ for $X$. Define, for every $\gamma \in C$, a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7.

We will define a tree $D$ and a function $\pi : D \longrightarrow C$ such that for all $\gamma, \gamma' \in D$, $\text{length}(\gamma) = \text{length}(\pi(\gamma))$ and if $\gamma \subseteq \gamma'$ then $\pi(\gamma) \subseteq \pi(\gamma')$. (Since the tree $C$ is wellfounded, $D$ will also be wellfounded.) At the same time we will define, for every $\gamma \in D$, $f_\gamma$, $g_\gamma$, $G_\gamma$, and $H_\gamma$ such that (i), (ii), and (iii) in the proof of Lemma 5.12 hold (after replacing $X_\gamma$ in (ii) by $X_{\pi(\gamma)}$).

We put $\pi(\langle\rangle) = \langle\rangle$, $f_{\langle\rangle} = f$, $G_{\langle\rangle} = (T, p, X)$ and we let $g_{\langle\rangle}$ be the identity on $\omega$. Let $\gamma \in D$ and suppose that we have defined $\pi(\gamma)$, $f_\gamma$, $g_\gamma$ and $G_\gamma$. We define $H_\gamma$ and the set of all moves in $D$ at $\gamma$, and for every move $a$ in $D$ at $\gamma$, we define $\pi(\gamma^\frown\langle a\rangle)$, $f_{\gamma^\frown\langle a\rangle}$, $g_{\gamma^\frown\langle a\rangle}$ and $G_{\gamma^\frown\langle a\rangle}$, as follows.

If $c(\pi(\gamma)) \in \{\bot, \top, \vee, \wedge\}$, then the moves in $D$ at $\gamma$ are the same as the moves in $C$ at $\pi(\gamma)$ and for each move $n$ in $D$ at $\gamma$, $\pi(\gamma^\frown\langle n\rangle) = \pi(\gamma)^\frown\langle n\rangle$. The other objects are defined as in the proof of Lemma 5.12.

Now suppose that $c(\pi(\gamma)) = \Diamond$. The construction of the basic open $f_\gamma$-covering $H_\gamma$ of $G_\gamma = (T_\gamma, p_\gamma, Y_\gamma)$ is similar to the construction in the proof of Lemma 6.4. It is easy to find some strictly increasing functions $h$, $l$ and $r$ from $\omega$ to $\omega$ such that $f_\gamma = h \circ l \circ r$, for some $k \in \omega$, $\text{range}(h) = \omega \setminus \{k\}$, $\omega \setminus \text{range}(r)$ is infinite, and for all $m < g_\gamma(n)$, $r(m) = m$.

Let $n \in \omega$ such that the moves in $C$ at $\pi(\gamma)$ are the elements of $T^{=n}$. So $X_{\pi(\gamma)}$ is the mix of the sets $X_{\pi(\gamma)^\frown\langle\tau\rangle}$ ($\tau \in T^{=n}$). We let the moves in $D$ at $\gamma$ be the elements of $T_\gamma$ of length $g_\gamma(n)$.

Let $\tau$ be a move in $D$ at $\gamma$. Then $\tau \circ g_\gamma$ is a move in $C$ at $\pi(\gamma)$. We put $\pi(\gamma^\frown\langle\tau\rangle) = \pi(\gamma)^\frown\langle\tau \circ g_\gamma\rangle$, $f_{\gamma^\frown\langle\tau\rangle} = r$, $g_{\gamma^\frown\langle\tau\rangle} = g_\gamma$, and $T_{\gamma^\frown\langle\tau\rangle} = T_\gamma^{\text{via}\,\tau}$. We define a function $p_{\gamma^\frown\langle\tau\rangle}$ on $T_{\gamma^\frown\langle\tau\rangle}$ as follows:

$$p_{\gamma^\frown\langle\tau\rangle}(\sigma) = \begin{cases} \{\{\tau(m)\}\} & \text{if } m < g_\gamma(n) \text{ and } \sigma = \tau|m, \\ p_\gamma(\sigma) & \text{if } \tau \subseteq \sigma \text{ and } \sigma \in T_\gamma. \end{cases}$$

We put $Y_{\gamma^\frown\langle\tau\rangle} = Y_\gamma \cap [T_{\gamma^\frown\langle\tau\rangle}]$ and $G_{\gamma^\frown\langle\tau\rangle} = (T_{\gamma^\frown\langle\tau\rangle}, p_{\gamma^\frown\langle\tau\rangle}, Y_{\gamma^\frown\langle\tau\rangle})$. For each $y \in [T_{\gamma^\frown\langle\tau\rangle}]$, since $\tau \circ g_\gamma \subseteq y \circ g_\gamma$, we have that $y \circ g_\gamma \in X_{\pi(\gamma)}$ if and only if $y \circ g_\gamma \in X_{\pi(\gamma^\frown\langle\tau\rangle)}$. So $Y_{\gamma^\frown\langle\tau\rangle} = \{y \in [T_{\gamma^\frown\langle\tau\rangle}] : y \circ g_{\gamma^\frown\langle\tau\rangle} \in X_{\pi(\gamma^\frown\langle\tau\rangle)}\}$.

By induction, we know how to construct, for each move $\tau$ in $D$ at $\gamma$, a basic open $f_{\gamma^\frown\langle\tau\rangle}$-covering $H_{\gamma^\frown\langle\tau\rangle}$ of $G_{\gamma^\frown\langle\tau\rangle}$. By Lemma 6.11, these $r$-coverings induce some $r$-covering $((U, q, Y), \Phi_{\text{I}}, \Phi_{\text{II}})$ of $G_\gamma$. Just as in the proof of Lemma 6.4, we see that $Y$ is an open (and also closed) subset of $[U]$.

Just as in the proof of Lemma 5.12, we easily construct an $l$-covering $((U', q', Y'), \Phi'_{\text{I}}, \Phi'_{\text{II}})$ of the generalized game $(U, q, Y)$ by inserting some trivial extra move at the right places. Since $(U, q, Y)$ is open, $(U', q', Y')$ is also open. By Lemma 5.11, $(U', q', Y')$ has some basic open $h$-covering $((U'', q'', Y''), \Phi''_{\text{I}}, \Phi''_{\text{II}})$. By Lemma 5.9(iii),

$((U'', q'', Y''), \Phi_{\mathrm{I}} \square \Phi'_{\mathrm{I}} \square \Phi''_{\mathrm{I}}, \Phi_{\mathrm{II}} \square \Phi'_{\mathrm{II}} \square \Phi''_{\mathrm{II}})$ is an $(h \circ l \circ r)$-covering of $G_\gamma$. We let $H_\gamma$ be this basic open $f$-covering of $G_\gamma$.

**6.13** THEOREM Every coded quasi-Borel generalized game is pseudodetermined.

PROOF Just like the proof of Theorem 5.13, with Lemma 6.12 instead of Lemma 5.12.

## 6D Suslin codes and $\Delta^1_1$ sets

In this section, we will see that, for each tree $T$, the quasi-Borel subsets of $[T]$ are related to the $\omega$-*Suslin* subsets of $[T]$ (also called *analytic* or $\Sigma^1_1$ sets). We introduce, for each set $E$, $E$-*Suslin* sets and we give some properties. Then we consider the special case $E = \omega$.

**6.14** DEFINITION Let $T$ be a tree. A **Suslin code** with respect to $T$ is a pair $(s, S)$ such that $s : \omega \longrightarrow \omega$ is strictly increasing, $S$ is a tree and for all $\sigma \in S$, $\sigma \circ s \in T$. The **extra moves** of $(s, S)$ are the sets of the form $\sigma(n)$ for some $\sigma \in S$ and some $n < \text{length}(\sigma)$ such that $n \notin \text{range}(s)$. Let $X$ be the subset $\{x \circ s : x \in [S]\}$ of $[T]$. We say that $(s, S)$ **is a Suslin code for** $X$.

Let $E$ be a set. A subset of $[T]$ is called $E$-**Suslin** if it has a Suslin code whose extra moves are elements of $E$.

A subset $X$ of $[T]$ is called $\Delta^1_1$ if both $X$ and $[T] \setminus X$ are $\omega$-Suslin.

One easily verifies that every subset of $[T]$ is $[T]$-Suslin and that for every finite set $E$, the $E$-Suslin subsets of $[T]$ are the same as the closed subsets of $[T]$.

**6.15** PROPOSITION Let $f : \omega \longrightarrow \omega$ be strictly increasing. Suppose that $T_0$ and $T_1$ are trees such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$. Define $p : [T_1] \longrightarrow [T_0]$ by $p(x) = x \circ f$. Let $E$ be a set and let $X$ be an $E$-Suslin subset of $[T_0]$. Then $p^{-1}X$ is an $E$-Suslin subset of $[T_1]$.

PROOF Let $(s_0, S_0)$ be a Suslin code for $X$ whose extra moves are in $E$. Just as in the proof of Proposition 3.3, we easily find strictly increasing functions $g$ and $s_1$ from $\omega$ to $\omega$ such that $g \circ s_0 = s_1 \circ f$, $\text{range}(g) \cap \text{range}(s_1) = \text{range}(g \circ s_0)$, and $\text{range}(g) \cup \text{range}(s_1) = \omega$.

Define $S_1$ as the set of all finite sequences $\sigma$ such that $\sigma \circ g \in S_0$ and $\sigma \circ s_1 \in T_1$. Then $S_1$ is a tree and $(s_1, S_1)$ is a Suslin code for $p^{-1}X$, since for each $y \in [T_1]$, the following three statements are equivalent: $p(y) \in X$; for some $x \in [S_0]$, $y \circ f = x \circ s_0$; for some $z \in [S_1]$, $y = z \circ s_1$.

In order to see that $p^{-1}X$ is $E$-Suslin, let $\sigma \in S_1$ and let $n < \text{length}(\sigma)$ such that $n \notin \text{range}(s_1)$. Then $n = g(m)$ for some $m \in \omega \setminus \text{range}(s_0)$, so $\sigma(n) = \sigma \circ g(m) \in E$.

**6.16** Proposition AC implies that for each tree $T$ and set $E$, the collection of $E$-Suslin subsets of $[T]$ is closed under countable intersection, union over $E$, and mixing with respect to $T$.

Proof Assume that AC holds. Let $T$ be a tree and let $E$ be a set. Since the $\emptyset$-Suslin subsets of $[T]$ are the closed subsets of $[T]$, we may assume that $E \neq \emptyset$.

Let $X$ be an $E$-Suslin subset of $[T]$. Then $X$ has a Suslin code $(s, S)$ with extra moves in $E$ such that $\omega \setminus \text{range}(s)$ is infinite, since we can insert an extra move at infinitely many places. We have some freedom to rearrange the extra moves of $(s, S)$: Let $f$ be a bijection from $\omega$ to $\omega$ such that $f \circ s$ is strictly increasing. Define $S'$ as the set of all finite sequences $\tau$ such that for some $\sigma \in S$, $\tau \circ f \subseteq \sigma$. Then one easily verifies that $(f \circ s, S')$ is also a Suslin code for $X$ with extra moves in $E$. Given $s$ and some $n \in \omega$, we can choose $f$ such that $f \circ s(n) = n$. Moreover, since $\omega \setminus \text{range}(s)$ is infinite, we can choose $f$ such that for all $n \in \omega$, $f \circ s(n) = 2n$.

For each $n \in \omega$, let $X_n$ be some $E$-Suslin subset of $[T]$. Choose some Suslin code $(s_0, S_0)$ for $X_0$ whose extra moves are in $E$. By Proposition 6.15, $\{x \in [S_0] : x \circ s_0 \in X_1\}$ is an $E$-Suslin subset of $[S_0]$. So we can choose a Suslin code $(s_1, S_1)$ for this set whose extra moves are in $E$, such that $s_1(s_0(0)) = s_0(0)$. Using AC, we can repeat this argument and we find, for each $n \in \omega$, a Suslin code $(s_{n+1}, S_{n+1})$ for the $E$-Suslin set $\{x \in [S_n] : x \circ s_n \circ \cdots \circ s_0 \in X_{n+1}\}$ such that $s_{n+1}(s_n \circ \cdots \circ s_0(n)) = s_n \circ \cdots \circ s_0(n)$.

Put $s = \lim_{n \to \omega} s_n \circ \cdots \circ s_0$ and let $S$ be the set of all finite sequences $\sigma$ such that for all large $n$, $\sigma \in S_n$. Then one easily verifies, using AC, that $(s, S)$ is a Suslin code for $\bigcap_{n \in \omega} X_n$ whose extra moves are in $E$. This proves that each countable intersection of $E$-Suslin sets is $E$-Suslin.

To see that each union over $E$ of $E$-Suslin sets is $E$-Suslin, suppose that for each $e \in E$, some $E$-Suslin subset $X_e$ of $[T]$ is given. Define functions $s$ and $t$ on $\omega$ by $s(n) = 2n$ and $t(n) = 2n + 1$ for all $n \in \omega$. Choose, using AC, for each $e \in E$, a Suslin code $(s, S_e)$ for $X_e$ with extra moves in $E$. Let $S$ be the tree $\{\langle\rangle\} \cup \{\langle e \rangle^\frown \sigma : e \in E \text{ and } \sigma \in S_e\}$. Then one easily verifies that $(t, S)$ is a Suslin code for $\bigcup_{e \in E} X_e$ whose extra moves are in $E$.

Finally, let $n \in \omega$ and assume that for each $\tau \in T^{=n}$, some $E$-Suslin subset $X_\tau$ of $[T]$ is given. Choose a strictly increasing $s : \omega \longrightarrow \omega$ such that $\omega \setminus \text{range}(s)$ is infinite and $s(n) = n$. Choose, using AC, for each $\tau \in T^{=n}$, a Suslin code $(s, S_\tau)$ for $X_\tau$ with extra moves in $E$. Let $S$ be the tree $T^{\leq n} \cup \bigcup_{\tau \in T^{=n}} \{\sigma \in S_\tau : \tau \subseteq \sigma\}$. Then $(s, S)$ is a Suslin code for the mix of the sets $X_\tau$ $(\tau \in T^{=n})$ whose extra moves are in $E$.

By taking $E = \omega$ in this proposition, we see that the axiom of choice implies that each quasi-Borel set is $\omega$-Suslin.

Complements of $\omega$-Suslin sets of real numbers need not be $\omega$-Suslin, but they are $\omega_1$-Suslin (see Moschovakis [1980], pages 43 and 84). This result can be generalized to uncountable trees, without using the axiom of choice:

**6.17** PROPOSITION Let $T$ be a tree, let $X$ be an $\omega$-Suslin subset of $[T]$ and let $\alpha$ be an uncountable ordinal. Then $[T] \setminus X$ is $\alpha$-Suslin.

PROOF Let $(s, S)$ be a Suslin code for $X$ whose extra moves are in $\omega$. For each $x \in [T]$, let $S_x$ be the finite or countable tree $\{\sigma \in S : \sigma \circ s \subseteq x\}$. Then, by Propositions 2.8(i) and 2.9 and since $\alpha$ is uncountable, the following statements are equivalent:

- $x \notin X$;
- $[S_x] = \emptyset$;
- the tree $S_x$ is wellfounded;
- there is a function $\rho : S_x \longrightarrow \alpha$ such that for each $\sigma \in S_x$ and each move $a$ in $S_x$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$.

Define $f : \omega \longrightarrow \omega$ by $f(n) = 2n + 1$. We will define a Suslin code $(f, U_\alpha)$ for $[T] \setminus X$ such that for each $u \in [U_\alpha]$, the extra moves $u(0), u(2), \ldots$ code a function $\rho : S_{u\circ f} \longrightarrow \alpha$ as above.

We can define a function $H : T \longrightarrow S$ such that for all $\tau \in T$, $H(\tau) \circ s \subseteq \tau$ and for all $\sigma \in S$, for some $n \geq \text{length}(\sigma \circ s)$, for all $\tau \in T^{=n}$, if $\sigma \circ s \subseteq \tau$ then $H(\tau) = \sigma$. For example, choose a surjection $h$ from $\omega$ to the countable tree ${}^{<\omega}\omega$ and define, for each $\tau \in T$, say of length $n$, $H(\tau)$ as follows: If there is a $\sigma \in S$ such that $\sigma \circ s \subseteq \tau$, $\text{length}(\sigma) = \text{length}(h(n))$, and for all $i < \text{length}(h(n))$, $i \in \text{range}(s)$ or $\sigma(i) = h(n)(i)$, then $H(\tau)$ is this unique $\sigma$; otherwise, $H(\tau) = \langle\rangle$.

Let $U_\alpha$ be the set of all finite sequences $\tau$ such that $\tau \circ f \in T$ and for all even $m, n < \text{length}(\tau)$, $\tau(m) \in \alpha$ and if for some $\sigma$, for some $a$, $\sigma = H((\tau|m) \circ f)$ and $\sigma^\frown\langle a\rangle = H((\tau|n) \circ f)$ then $\tau(m) > \tau(n)$. Then $(f, U_\alpha)$ is a Suslin code for $[T] \setminus X$, since for each $x \in [T]$, $S_x = \{H(x|k) : k \in \omega\}$. So $[T] \setminus X$ is $\alpha$-Suslin.

Note that, by Theorem 2.16, for each countable tree $T$, the coded Borel subsets of $[T]$ are the same as the $\Delta^1_1$ subsets of $[T]$. We now prove, using the principle of dependent choices, that for each tree, the coded quasi-Borel sets coincide with the $\Delta^1_1$ sets.

**6.18** PROPOSITION Let $T$ be a tree and let $X \subseteq [T]$. Then DC implies that $X$ is coded quasi-Borel if and only if $X$ is $\Delta^1_1$.

PROOF Suppose that $X$ is coded quasi-Borel. Since $[T] \setminus X$ is also coded quasi-Borel, it is enough to show that $X$ is $\omega$-Suslin. Choose a quasi-Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \Diamond\}$ for $X$ and define, for every $\gamma \in C$, a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7. Define $f : \omega \longrightarrow \omega$ by $f(n) = 2n+1$ for every $n \in \omega$. Then we can *define*, by induction on the wellfounded tree $C$, for every $\gamma \in C$ a Suslin code $(f, A_\gamma)$ for $X_\gamma$ whose extra moves are in $\omega$, exactly as in the first part of the proof of Theorem 2.16. Note that we get similar Suslin codes by following the proof of Proposition 6.16 for $E = \omega$. In this proof, the axiom of choice can be avoided by using the quasi-Borel code $c$.

Now suppose that DC holds and that $X$ is $\Delta^1_1$. Choose Suslin codes $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ for $X$ and $[T] \setminus X$, respectively, whose extra moves are in $\omega$. Just as in the proof of Proposition 3.3, we easily find strictly increasing functions $f_{\mathrm{I}}$, $f_{\mathrm{II}}$, and $f$ from $\omega$ to $\omega$ such that $f_{\mathrm{I}} \circ s_{\mathrm{I}} = f_{\mathrm{II}} \circ s_{\mathrm{II}} = f$, $\mathrm{range}(f_{\mathrm{I}}) \cap \mathrm{range}(f_{\mathrm{II}}) = \mathrm{range}(f)$, and $\mathrm{range}(f_{\mathrm{I}}) \cup \mathrm{range}(f_{\mathrm{II}}) = \omega$. Let $D$ be the set of all finite sequences $\sigma$ such that $\sigma \circ f_{\mathrm{I}} \in S_{\mathrm{I}}$ and $\sigma \circ f_{\mathrm{II}} \in S_{\mathrm{II}}$. Since $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are Suslin codes for disjoint sets, the tree $D$ has no infinite branches. So, by DC and Proposition 2.8(ii), the tree $D$ is wellfounded.

We will use $D$ to construct a quasi-Borel code for $X$. We first explain the idea in terms of games. For each $x \in [T]$, define a finite or countable wellfounded tree $D_x = \{\sigma \in D : \sigma \circ f \subseteq x\}$ and a game $G_x = (D_x, \{\sigma \in D_x : \mathrm{length}(\sigma) \in \mathrm{range}(f_{\mathrm{I}})\}, \emptyset)$. So in $G_x$, at a position $\sigma$ of length $n$, the following happens: If $n \in \mathrm{range}(f_{\mathrm{I}}) \setminus \mathrm{range}(f_{\mathrm{II}})$, then player I chooses a natural number that is a move in $S_{\mathrm{I}}$ at $\sigma \circ f_{\mathrm{I}}$; if $n \in \mathrm{range}(f_{\mathrm{II}}) \setminus \mathrm{range}(f_{\mathrm{I}})$, then player II chooses a natural number that is a move in $S_{\mathrm{II}}$ at $\sigma \circ f_{\mathrm{II}}$, and finally, if $n \in \mathrm{range}(f)$, say $n = f(m)$, then player I has to play the move $x(m)$. The game ends as soon as a player cannot make a move. Now one easily verifies that $x \in X$ if and only if player I has a winning strategy in the game $G_x$. In fact, since there is either some $y \in [S_{\mathrm{I}}]$ such that $y \circ s_{\mathrm{I}} = x$, or some $y \in [S_{\mathrm{II}}]$ such that $y \circ s_{\mathrm{II}} = x$, one of the players can win in such a way that his moves do not depend on the moves of his opponent.

We define a tree $C$ and functions $c$ and $d$ on $C$. We put $d(\langle\rangle) = \langle\rangle$ and for each $\gamma \in C$:

- if $d(\gamma) \notin D$ (in other words, if $d(\gamma) \circ f_{\mathrm{I}} \notin S_{\mathrm{I}}$ or $d(\gamma) \circ f_{\mathrm{II}} \notin S_{\mathrm{II}}$) then $c(\gamma) = \begin{cases} \bot & \text{if } d(\gamma) \circ f_{\mathrm{I}} \notin S_{\mathrm{I}}, \\ \top & \text{otherwise,} \end{cases}$ and $\gamma$ is a terminal node of $C$;
- if $d(\gamma) \in D$ and $d(\gamma)$ has length $f(n)$ for some $n$, then $c(\gamma) = \Diamond$, the moves in $C$ at $\gamma$ are the elements of $T^{=n+1}$, and for each move $\tau$ in $C$ at $\gamma$, $d(\gamma^\frown\langle\tau\rangle) = d(\gamma)^\frown\langle\tau(n)\rangle$;

- if $d(\gamma) \in D$ and length$(d(\gamma)) \notin$ range$(f)$, then
  $c(\gamma) = \begin{cases} \vee & \text{if length}(d(\gamma)) \in \text{range}(f_{\mathrm{I}}), \\ \wedge & \text{if length}(d(\gamma)) \in \text{range}(f_{\mathrm{II}}), \end{cases}$
  and the moves in $C$ at $\gamma$ are the natural numbers, and for each $n \in \omega$, $d(\gamma^\frown\langle n\rangle) = d(\gamma)^\frown\langle n\rangle$.

Now $C$ is a wellfounded tree and $c$ is a quasi-Borel code with respect to $T$. For every $\gamma \in C$, we define a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7. We want to show that $X = X_{\langle\rangle}$.

Let $x \in X$. Choose $y \in [S_{\mathrm{I}}]$ such that $y \circ s_{\mathrm{I}} = x$. Then one easily proves, by induction on $C$, that for each $\gamma \in C$ such that $d(\gamma) \circ f_{\mathrm{I}} \subseteq y$, $x \in X_\gamma$. Thus $x \in X_{\langle\rangle}$. In a similar way we see that if $x \in [T] \setminus X$, then $x \notin X_{\langle\rangle}$.

By modifying the second part of this proof slightly, we see that if DC holds and $X_{\mathrm{I}}$ and $X_{\mathrm{II}}$ are disjoint $\omega$-Suslin subsets of $[T]$, then there is a coded quasi-Borel subset $X$ of $[T]$ that *separates* $X_{\mathrm{I}}$ from $X_{\mathrm{II}}$, that is, such that $X_{\mathrm{I}} \subseteq X$ and $X_{\mathrm{II}} \subseteq [T] \setminus X$. If the tree $T$ is countable, then DC is not needed in the proof, so then disjoint $\omega$-Suslin subsets of $[T]$ can be separated by a Borel subset of $[T]$. This is the *Separation Theorem* of N. Lusin [1927] (see Moschovakis [1980], pages 86 and 114).

Lusin [1917] stated that a set of real numbers is Borel if and only if it is a continuous, injective image of a closed set (see Moschovakis [1980], page 114). Using the principle of dependent choices, we find a similar characterization of coded quasi-Borel sets.

**6.19** Proposition Let $T$ be a tree and let $X \subseteq [T]$. Then DC implies that $X$ is coded quasi-Borel if and only if $X$ has a Suslin code $(s, S)$ whose extra moves are in $\omega$ and such that for all $y, z \in [S]$, if $y \circ s = z \circ s$ then $y = z$.

Proof Suppose that $X$ has some quasi-Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$. Define, for every $\gamma \in C$, a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7. For every $x \in [T]$, let $C_x$ be the wellfounded tree $\{\gamma \in C : \text{for all } i < \text{length}(\gamma), \text{ if } c(\gamma|i) = \diamond \text{ then } \gamma(i) \subseteq x\}$. Define two functions $a_x$ and $b_x$ on $C_x$ as follows: For every $\gamma \in C_x$, $a_x(\gamma) = \begin{cases} 1 & \text{if } x \in X_\gamma, \\ 0 & \text{otherwise,} \end{cases}$
and $b_x(\gamma) = \begin{cases} \min\{k \in \omega : x \in X_{\gamma^\frown\langle k\rangle}\} & \text{if } c(\gamma) = \vee \text{ and } x \in X_\gamma, \\ \min\{k \in \omega : x \notin X_{\gamma^\frown\langle k\rangle}\} & \text{if } c(\gamma) = \wedge \text{ and } x \notin X_\gamma, \\ 0 & \text{otherwise.} \end{cases}$
We will define a Suslin code $(s, S)$ for $X$ such that for each $y \in [S]$, the extra moves code the functions $a_{y\circ s}$ and $b_{y\circ s}$.

We can define a function $H : T \longrightarrow C$ such that for each $x \in [T]$, $C_x = \{H(x|k) : k \in \omega\}$. For example, choose a surjection $h$ from $\omega$

to the countable tree ${}^{<\omega}\omega$ and define, for each $\tau \in T$, say of length $n$, $H(\tau)$ as follows: If there is a $\gamma \in C$ such that $\text{length}(\gamma) = \text{length}(h(n))$ and for all $i < \text{length}(h(n))$, $(c(\gamma|i) = \diamond$ and $\gamma(i) \subseteq \tau)$ or $(c(\gamma|i) \neq \diamond$ and $\gamma(i) = h(n)(i))$, then $H(\tau)$ is this unique $\gamma$; otherwise, $H(\tau) = \langle\rangle$.

Define $s : \omega \longrightarrow \omega$ by $s(n) = 3n + 2$. Let $S$ be the set of all finite sequences $\sigma$ such that $\sigma \circ s \in T$, for all $i < \text{length}(\sigma)$, if $i \notin \text{range}(s)$ then $\sigma(i) \in \omega$ and such that for all $m$ such that $3m + 1 < \text{length}(\sigma)$, if we put $\gamma = H(\sigma \circ s|m)$, then we have the following:

- if $\gamma = \langle\rangle$ then $\sigma(3m) = 1$;
- if $c(\gamma) = \top$ then $\sigma(3m) = 1$ and $\sigma(3m + 1) = 0$;
- if $c(\gamma) = \bot$ then $\sigma(3m) = 0$ and $\sigma(3m + 1) = 0$;
- if $c(\gamma) = \diamond$ or $(c(\gamma) = \vee$ and $\sigma(3m) = 0)$ or $(c(\gamma) = \wedge$ and $\sigma(3m) = 1)$ then $\sigma(3m+1) = 0$ and for all $n, a$, if $3n < \text{length}(\sigma)$ and $H(\sigma \circ s|n) = \gamma^\frown\langle a\rangle$ then $\sigma(3m) = \sigma(3n)$;
- if $(c(\gamma) = \vee$ and $\sigma(3m) = 1)$ or $(c(\gamma) = \wedge$ and $\sigma(3m) = 0)$ then for all $n$ and $a$ such that $3n < \text{length}(\sigma)$ and $H(\sigma \circ s|n) = \gamma^\frown\langle a\rangle$, if $a < \sigma(3m + 1)$ then $\sigma(3m) \neq \sigma(3n)$ and if $a = \sigma(3m + 1)$ then $\sigma(3m) = \sigma(3n)$.

Let $x \in X$. Define an infinite sequence $y$ as follows: For all $n \in \omega$, $y(3n) = a_x(H(x|n))$, $y(3n + 1) = b_x(H(x|n))$, and $y(3n + 2) = x(n)$. Then $y \in [S]$ and $y \circ s = x$.

Now let $y \in [S]$ and put $x = y \circ s$. Then one easily proves, by induction on the wellfounded tree $C_x$, that for all $\gamma \in C_x$, for all $n$, if $H(x|n) = \gamma$ then $y(3n) = a_x(\gamma)$ and $y(3n + 1) = b_x(\gamma)$. This implies that $x \in X_{\langle\rangle} = X$ and that $y$ is the *unique* $z \in [S]$ for which $x = z \circ s$.

Before proving the other direction, let us mention another proof. First suppose that for each $\gamma \in C$ such that $c(\gamma) = \vee$, the sets $X_{\gamma^\frown\langle 0\rangle}, X_{\gamma^\frown\langle 1\rangle}, \dots$ are disjoint. (Let us call such a quasi-Borel code $c$ *disjointed*.) Define, using the quasi-Borel code $c$, a Suslin code $(f, A_{\langle\rangle})$ for $X$ exactly as in the first part of the proof of Theorem 2.16. Let $x \in [T]$. Define a game $G_x$ on $C_x$ as in Remark 2.14. Then there is a bijection from $\{y \in [A_{\langle\rangle}] : y \circ f = x\}$ to the set of all strategies for player I in $G_x$. Since $c$ is disjointed and each strategy for player I in $G_x$ is a subset of $\{\gamma \in C_x : x \in X_\gamma\}$, player I has *at most one* strategy in $G_x$. This implies that for all $y, z \in [A_{\langle\rangle}]$, if $y \circ f = z \circ f$ then $y = z$.

If $c$ is not disjointed, then we can find a disjointed quasi-Borel code for $X$, using the fact that for each infinite sequence $Y_0, Y_1, \dots$ of subsets of $[T]$, $\bigcup_{n\in\omega} Y_n = \bigcup_{n\in\omega}(Y_n \setminus \bigcup_{m<n} Y_m)$. To be more precise, we can define, by induction on $C$, for each $\gamma \in C$, disjointed quasi-Borel codes for $X_\gamma$ and $[T] \setminus X_\gamma$.

Proof (continued) Suppose that DC holds and that $X$ has a Suslin code $(s, S)$ whose extra moves are in $\omega$ and such that for all $y, z \in [S]$, if $y \circ s = z \circ s$ then $y = z$. Let $D$ be the tree $\{\langle\rangle\} \cup \{\langle n\rangle : n \in \omega\} \cup \{\langle n, \sigma\rangle : n \in \omega$ and $\sigma \in S^{=n}\} \cup \{\langle n, \sigma, \tau, a_n, b_n, a_{n+1}, b_{n+1}, \ldots\rangle : n = \text{length}(\sigma) = \text{length}(\tau)$ and $\sigma \neq \tau$ and $\sigma^\frown\langle a_n, a_{n+1}, \ldots\rangle \in S$ and $\tau^\frown\langle b_n, b_{n+1}, \ldots\rangle \in S$ and $\tau^\frown\langle b_n, b_{n+1}, \ldots\rangle \circ s \subseteq \sigma^\frown\langle a_n, a_{n+1}, \ldots\rangle \circ s\}$. Then $D$ has no infinite branch, since for each $d \in [D]$, if we put $y = d(1)^\frown\langle d(3), d(5), \ldots\rangle$ and $z = d(2)^\frown\langle d(4), d(6), \ldots\rangle$, then $y, z \in [S]$ and $y \circ s = z \circ s$ but $y \neq z$. So, by DC and Proposition 2.8(ii), the tree $D$ is wellfounded.

For each $x \in [T]$, define a finite or countable wellfounded tree $D_x = \{\delta \in D : \delta(1)^\frown\langle\delta(3), \delta(5), \ldots\rangle \circ s \subseteq x\}$ and a game $G_x = (D_x, \{\delta \in D_x : \text{length}(\delta) \text{ is odd}\}, \emptyset)$. So player II starts this game by choosing a natural number $n$. Then player I chooses $\langle a_0, \ldots, a_{n-1}\rangle$, player II chooses $\langle b_0, \ldots, b_{n-1}\rangle \neq \langle a_0, \ldots, a_{n-1}\rangle$, player I chooses $a_n$, player II chooses $b_n$, I chooses $a_{n+1}$, and so on, such that for each $m$, $\langle a_0, \ldots, a_{m-1}\rangle$ and $\langle b_0, \ldots, b_{m-1}\rangle$ are elements of $\{\sigma \in S : \sigma \circ s \subseteq x\}$.

If $x \in X$, then for some $y \in [S]$, $y \circ s = x$, so player I can win $G_x$ easily by choosing $a_m = y(m)$ for each $m$ (until a position is reached at which player II cannot make a move). Now suppose that player I has a winning strategy $U$ in $G_x$. For each $n \in \omega$, let $\sigma_n$ be the unique move in $U$ at $\langle n\rangle$.

Claim For each $n \in \omega$, $\sigma_n \subseteq \sigma_{n+1}$.

Proof of claim Suppose that for some $n$, $\sigma_n \neq \sigma_{n+1}|n$. Write $\sigma_n = \langle a_0, \ldots, a_{n-1}\rangle$ and $\sigma_{n+1} = \langle b_0, \ldots, b_n\rangle$. If player I follows strategy $U$ and player II's first move is $n$, then player I will play $\langle a_0, \ldots, a_{n-1}\rangle$, so player II may answer by playing the move $\langle b_0, \ldots, b_{n-1}\rangle$. Then player I will play some move $a_n$ and player II may play the move $b_n$. On the other hand, if player II's first move is $n + 1$, then player I plays $\langle b_0, \ldots, b_n\rangle$ and player II may play $\langle a_0, \ldots, a_n\rangle$. Continuing in this way, we see that there are unique infinite sequences $y = \langle a_0, a_1, \ldots\rangle$ and $z = \langle b_0, b_1, \ldots\rangle$ such that for each $m > n$, both $\langle n, \langle a_0, \ldots, a_{n-1}\rangle, \langle b_0, \ldots, b_{n-1}\rangle, a_n, b_n, \ldots, a_m\rangle$ and $\langle n + 1, \langle b_0, \ldots, b_n\rangle, \langle a_0, \ldots, a_n\rangle, b_{n+1}, a_{n+1}, \ldots, b_m\rangle$ are in $U$. Now $y, z \in [S]$ and $y \circ s = x = z \circ s$, but $y \neq z$. This contradicts one of our assumptions.

Let $y$ be the infinite sequence $\bigcup_{n\in\omega} \sigma_n$. Then $y \in [S]$ and $y \circ s = x$. So $x \in X$.

Thus, for each $x \in [T]$, $x \in X$ if and only if player I has a winning strategy in the game $G_x$. Using this and the wellfoundedness of $D$, it is not difficult to construct a quasi-Borel code for $X$.

The role of DC in Propositions 6.18 and 6.19 can be made clear by weakening the concept of being coded quasi-Borel. Let $T$ be a tree and let $X \subseteq [T]$. Let us say that $X$ is *weakly coded quasi-Borel* if there is a tree $C$ without infinite branches (but not necessarily wellfounded) and a function $c : C \longrightarrow \{\perp, \top, \vee, \wedge, \Diamond\}$ as in Definition 6.7, such that, for each $x \in [T]$, $x \in X$ if and only if player I has a winning strategy in $G_x$, where the game $G_x$ is defined just as in Remark 2.14. Then we can prove, without using DC, that the following statements are equivalent:

- $X$ is weakly coded quasi-Borel;
- $X$ is $\Delta^1_1$;
- $X$ has a Suslin code $(s, S)$ whose extra moves are in $\omega$ and such that for all $y, z \in [S]$, if $y \circ s = z \circ s$ then $y = z$.

We can also avoid the use of DC in Proposition 6.18 by strengthening the concept of being $\Delta^1_1$. We do this in the next section.

## 6E Incompatible Suslin codes and absolutely $\Delta^1_1$ sets

Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes with respect to some tree $T$. Let $M$ be a transitive class model of ZF such that $(s_{\mathrm{I}}, S_{\mathrm{I}})$, $(s_{\mathrm{II}}, S_{\mathrm{II}})$, and $T$ are in $M$. Suppose that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are Suslin codes for disjoint sets. Then there are no infinite branches $y, z \in M$ of $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$, respectively, such that $y \circ s_{\mathrm{I}} = z \circ s_{\mathrm{II}}$. Thus $M \models ((s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are Suslin codes for disjoint sets). We will see that if $M \models$DC, then the converse also holds. On the other hand, if there is a tree $D \in M$ without terminal nodes such that $D$ has an infinite branch, but no infinite branch $x \in M$, and if $d$ is the identity on $\omega$, then $M \models ((d, D)$ and $(d, D)$ are Suslin codes for disjoint sets), whereas $(d, D)$ is a Suslin code for a non-empty set.

**6.20** Definition A **quasi-branch** of a tree $T$ is a pair $(d, D)$ such that $d : \omega \longrightarrow \omega$ is strictly increasing, $D$ is a tree without terminal nodes, and for all $\sigma \in D$, $\sigma \circ d \in T$.

Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes with respect to some tree $T$. We say that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ is **compatible** with $(s_{\mathrm{II}}, S_{\mathrm{II}})$ if for some tree $D$ without terminal nodes, there are strictly increasing functions $d_{\mathrm{I}}$ and $d_{\mathrm{II}}$ from $\omega$ to $\omega$, such that $d_{\mathrm{I}} \circ s_{\mathrm{I}} = d_{\mathrm{II}} \circ s_{\mathrm{II}}$ and for all $\sigma \in D$, $\sigma \circ d_{\mathrm{I}} \in S_{\mathrm{I}}$ and $\sigma \circ d_{\mathrm{II}} \in S_{\mathrm{II}}$. If not, then $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are said to be **incompatible**.

Let $T$ be a tree. A quasi-branch of $T$ could also have been defined as a pair $(d, D)$ such that $D$ is a $d$-tactic for player I in the game $(T, T, \emptyset)$, or as a Suslin code $(d, D)$ with respect to $T$ such that $D$ has no terminal nodes.

It is clear that $T$ is wellfounded if and only if $T$ has no quasi-branch. Note that for each infinite branch $x$ of $T$, $(d, \{x|n : n \in \omega\})$ is a quasi-branch of $T$, where $d$ is the identity on $\omega$.

Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes with respect to $T$. Observe that if $D$, $d_{\mathrm{I}}$, and $d_{\mathrm{II}}$ witness that these Suslin codes are compatible, then $(d_{\mathrm{I}}, D)$ is a quasi-branch of $S_{\mathrm{I}}$ and $(d_{\mathrm{II}}, D)$ is a quasi-branch of $S_{\mathrm{II}}$.

Now let $f_{\mathrm{I}}$ and $f_{\mathrm{II}}$ be strictly increasing such that $f_{\mathrm{I}} \circ s_{\mathrm{I}} = f_{\mathrm{II}} \circ s_{\mathrm{II}}$, $\mathrm{range}(f_{\mathrm{I}}) \cap \mathrm{range}(f_{\mathrm{II}}) = \mathrm{range}(f_{\mathrm{I}} \circ s_{\mathrm{I}})$, and $\mathrm{range}(f_{\mathrm{I}}) \cup \mathrm{range}(f_{\mathrm{II}}) = \omega$. Define a tree $C = \{\sigma : \sigma \circ f_{\mathrm{I}} \in S_{\mathrm{I}} \text{ and } \sigma \circ f_{\mathrm{II}} \in S_{\mathrm{II}}\}$. Then one easily verifies that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible if and only if $C$ is wellfounded.

Thus, by Lemma 2.17, for each transitive class model $M$ of ZF such that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are in $M$, $M \models ((s_{\mathrm{I}}, S_{\mathrm{I}})$ is compatible with $(s_{\mathrm{II}}, S_{\mathrm{II}}))$ if and only if $(s_{\mathrm{I}}, S_{\mathrm{I}})$ is compatible with $(s_{\mathrm{II}}, S_{\mathrm{II}})$.

Also note that $[C] = \emptyset$ if and only if $\{x \circ s_{\mathrm{I}} : x \in [S_{\mathrm{I}}]\} \cap \{x \circ s_{\mathrm{II}} : x \in [S_{\mathrm{II}}]\} = \emptyset$. Suppose that the trees $S_{\mathrm{I}}$ and $S_{\mathrm{II}}$ are countable or that DC holds. Then, by Proposition 2.8, $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible if and only if they are Suslin codes for disjoint subsets of $[T]$.

**6.21** Proposition Let $T$ be a tree and let $X \subseteq [T]$. Then $X$ is coded quasi-Borel if and only if there are Suslin codes $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ for $X$ and $[T] \setminus X$, respectively, whose extra moves are in $\omega$ and such that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible.

Proof Suppose that $X$ has a quasi-Borel code $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$. Define, for every $\gamma \in C$, a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7. Define $f : \omega \longrightarrow \omega$ by $f(n) = 2n + 1$ for every $n \in \omega$. Define, exactly as in the first part of the proof of Theorem 2.16, by induction on the wellfounded tree $C$, for every $\gamma \in C$ a Suslin code $(f, A_\gamma)$ for $X_\gamma$ whose extra moves are in $\omega$, and similarly (by using the quasi-Borel code $\{(\bot, \top), (\top, \bot), (\vee, \wedge), (\wedge, \vee), (\diamond, \diamond)\} \circ c$ for $[T] \setminus X$) a Suslin code $(f, B_\gamma)$ for $[T] \setminus X_\gamma$. We now prove, by induction on the wellfounded tree $C$, that for each $\gamma \in C$, $(f, A_\gamma)$ and $(f, B_\gamma)$ are incompatible.

If $c(\gamma) \in \{\bot, \top\}$, then this is clear, since nothing is compatible with $(f, \{\langle\rangle\})$.

If $c(\gamma) = \vee$, then this follows from the induction assumption and the following simple properties: Everything that is compatible with $(f, A_\gamma)$, is, for some $\langle n \rangle \in A_\gamma$, compatible with $(f, {A_\gamma}^{\mathrm{via}\,\langle n \rangle})$, and thus also compatible with $(f, A_{\gamma^\frown\langle n \rangle})$. Everything that is compatible with $(f, B_\gamma)$, is compatible with $(f, B_{\gamma^\frown\langle n \rangle})$ for each $n \in \omega$. For $c(\gamma) = \wedge$, the same holds, with the roles of $A_\gamma$ and $B_\gamma$ exchanged.

Now suppose that $c(\gamma) = \diamond$ and that $D$, $d_{\mathrm{I}}$, and $d_{\mathrm{II}}$ witness that $(f, A_\gamma)$ is compatible with $(f, B_\gamma)$. Since $(d_{\mathrm{I}} \circ f, D)$ is a quasi-branch of $T$, we can choose some $\tau \in D$ such that $\tau \circ d_{\mathrm{I}} \circ f$ is a move in $C$

at $\gamma$. Put $a = \tau \circ d_{\mathrm{I}} \circ f = \tau \circ d_{\mathrm{II}} \circ f$. Then $D^{\text{via}\,\tau}$, $d_{\mathrm{I}}$, and $d_{\mathrm{II}}$ witness that $(f, A_{\gamma^\frown\langle a\rangle})$ is compatible with $(f, B_{\gamma^\frown\langle a\rangle})$. This contradicts the induction assumption.

To prove the other direction, suppose that $X$ and $[T] \setminus X$ have incompatible Suslin codes $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$, respectively, whose extra moves are in $\omega$. Now the proof continues as the second part of the proof of Proposition 6.18, but instead of using DC to see that the tree $D$ is wellfounded, we use the assumption that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible.

Suppose that $M$ is a transitive class model of ZF and let $(s_{\mathrm{I}}, S_{\mathrm{I}}), (s_{\mathrm{II}}, S_{\mathrm{II}}), T \in M$ such that $M \models$ ($T$ is a tree and for some $X \subseteq [T]$, $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible Suslin codes for $X$ and $[T] \setminus X$, respectively, whose extra moves are in $\omega$). Then $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible Suslin codes for some subsets $X_{\mathrm{I}}$ and $X_{\mathrm{II}}$ of $[T]$, respectively, so $X_{\mathrm{I}} \cap X_{\mathrm{II}} = \emptyset$. But not necessarily $X_{\mathrm{I}} \cup X_{\mathrm{II}} = [T]$ (for example if $S_{\mathrm{I}} = S_{\mathrm{II}} = \{\langle\rangle\}$ and $T$ has an infinite branch, but not one in $M$).

Therefore we introduce a characterization of coded quasi-Borel sets in terms of Suslin codes, which is more 'absolute' than the one given in Proposition 6.21.

**6.22** DEFINITION Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes with respect to some tree $T$. We say that $(S_{\mathrm{I}}, s_{\mathrm{I}})$ and $(S_{\mathrm{II}}, s_{\mathrm{II}})$ are **complementary** with respect to $T$ if these Suslin codes are incompatible and each quasi-branch of $T$ is compatible with $(s_{\mathrm{I}}, S_{\mathrm{I}})$ or with $(s_{\mathrm{II}}, S_{\mathrm{II}})$.

We call a subset $X$ of $[T]$ **absolutely** $\Delta^1_1$ if there are complementary Suslin codes $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ with respect to $T$, whose extra moves are natural numbers, and such that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ is a Suslin code for $X$.

Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes for subsets $X_{\mathrm{I}}$ and $X_{\mathrm{II}}$ of $[T]$, respectively.

One easily verifies that if DC holds, then $(S_{\mathrm{I}}, s_{\mathrm{I}})$ and $(S_{\mathrm{II}}, s_{\mathrm{II}})$ are complementary if and only if $X_{\mathrm{II}} = [T] \setminus X_{\mathrm{I}}$. So DC implies that the absolutely $\Delta^1_1$ subsets of $[T]$ are the same as the $\Delta^1_1$ subsets of $[T]$. We will see in the proof of the following theorem that we do not need DC in order to prove that each absolutely $\Delta^1_1$ subset of $[T]$ is $\Delta^1_1$.

Now suppose that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are complementary with respect to $T$. Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $T'$ be a tree such that for all $\sigma \in T'$, $\sigma \circ f \in T$. Define $p : [T'] \longrightarrow [T]$ by $p(x) = x \circ f$. Define Suslin codes $(s'_{\mathrm{I}}, S'_{\mathrm{I}})$ and $(s'_{\mathrm{II}}, S'_{\mathrm{II}})$ for the subsets $p^{-1}X_{\mathrm{I}}$ and $p^{-1}X_{\mathrm{II}}$ of $[T']$, respectively, just as in the proof of Proposition 6.15. Then one easily verifies that $(s'_{\mathrm{I}}, S'_{\mathrm{I}})$ and $(s'_{\mathrm{II}}, S'_{\mathrm{II}})$ are complementary with respect to $T'$.

**6.23** THEOREM Let $T$ be a tree and let $X \subseteq [T]$. Then $X$ is absolutely $\Delta^1_1$ if and only if $X$ is coded quasi-Borel.

PROOF Suppose that $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ witness that $X$ is absolutely $\Delta^1_1$. Then $(s_{\mathrm{I}}, S_{\mathrm{I}})$ is a Suslin code for $X$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ is a Suslin code for some subset $Y$ of $[T]$. Let $x \in [T]$. Let $d$ be the identity on $\omega$ and consider the tree $D = \{x|n : n \in \omega\}$. Then, for $J = \mathrm{I}$ or $J = \mathrm{II}$, $(d, D)$ is compatible with $(s_J, S_J)$. Since $s_J \circ d = s_J = d \circ s_J$, $\mathrm{range}(s_J) \cap \mathrm{range}(d) = \mathrm{range}(s_J)$, and $\mathrm{range}(s_J) \cup \mathrm{range}(d) = \omega$, this implies that the tree $C := \{\sigma \in S_J : \sigma \circ s_J \in D\}$ is not wellfounded. Therefore, since $C$ is countable, $C$ has an infinite branch and thus $x \in X$ if $J = \mathrm{I}$ and $x \in Y$ if $J = \mathrm{II}$. So $X \cup Y = [T]$. Since $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible, $X \cap Y = \emptyset$. So $Y = [T] \setminus X$. By Proposition 6.21, this implies that $X$ is coded quasi-Borel.

Now assume that $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ is a quasi-Borel code for $X$. We will prove, by induction on the wellfounded tree $C$, that for each $\gamma \in C$, the Suslin codes $(f, A_\gamma)$ and $(f, B_\gamma)$ that we defined in the proof of Proposition 6.21, are complementary. Since $(f, A_{\langle\rangle})$ is a Suslin code for $X$, this shows that $X$ is absolutely $\Delta^1_1$.

So let $\gamma \in C$. We have already seen that $(f, A_\gamma)$ and $(f, B_\gamma)$ are incompatible. Let $(d, D)$ be a quasi-branch of $T$. We must prove that $(d, D)$ is compatible with $(f, A_\gamma)$ or with $(f, B_\gamma)$.

If $c(\gamma) \in \{\bot, \top\}$, then this follows from the fact that $(d, D)$ is compatible with $(f, \{\langle 0, x_0, 0, x_1, \ldots\rangle : \langle x_0, x_1, \ldots\rangle \in T\})$.

If $c(\gamma) = \diamond$, then, since $D$ has no terminal nodes, there is a move $a$ in $C$ at $\gamma$ and some $\tau \in D$ such that $a = \tau \circ d$. The tree $D^{\mathrm{via}\,\tau}$ has no terminal nodes. By the induction hypothesis, $(d, D^{\mathrm{via}\,\tau})$ is compatible with $(f, A_{\gamma^\frown\langle a\rangle})$ or with $(f, B_{\gamma^\frown\langle a\rangle})$. For each $\sigma \in D^{\mathrm{via}\,\tau}$, $\sigma \circ d \subseteq a$ or $a \subseteq \sigma \circ d$. Therefore $(d, D^{\mathrm{via}\,\tau})$ is also compatible with $(f, A_\gamma)$ or with $(f, B_\gamma)$. Thus $(d, D)$ is compatible with $(f, A_\gamma)$ or with $(f, B_\gamma)$.

Now suppose that $c(\gamma) = \vee$. Recall that $B_\gamma = \{\langle a_0, x_0, a_1, x_1, \ldots\rangle :$ for all $n \in \omega$, $\langle a_{\pi(n,0)}, x_0, a_{\pi(n,1)}, x_1, a_{\pi(n,2)}, \ldots\rangle \in B_{\gamma^\frown\langle n\rangle}\}$ for some bijection $\pi : \omega \times \omega \longrightarrow \omega$. For each $n \in \omega$, we let $(b_n, B^n)$ be the Suslin code that is obtained from $(f, B_{\gamma^\frown\langle n\rangle})$ by rearranging the moves so that they are ordered like the corresponding moves in $B_\gamma$. To be more precise, we let $B^n$ be the set of all finite sequences that are of the form $\langle x_0, \ldots, x_{\pi(n,0)-1}, c_0, x_{\pi(n,0)}, \ldots, x_{\pi(n,1)-1}, c_1, x_{\pi(n,1)}, \ldots\rangle$ for some $\langle c_0, x_0, c_1, x_1, \ldots\rangle \in B_{\gamma^\frown\langle n\rangle}$, and we define $b_n : \omega \longrightarrow \omega$ by $b_n(k) = k + \min\{m \in \omega : \pi(n,m) > k\}$ for all $k \in \omega$.

Put $d_0 = d$. We can find, inductively, for each $n \in \omega$, strictly increasing functions $d_{n+1}$, $d^n$, and $b^n$ from $\omega$ to $\omega$ such that $d_{n+1} = d^n \circ d_n = b^n \circ b_n$, $\mathrm{range}(d^n) \cap \mathrm{range}(b^n) = \mathrm{range}(d_{n+1})$, and $\mathrm{range}(d^n) \cup \mathrm{range}(b^n) = \omega$.

Suppose that $(d, D)$ and $(f, A_\gamma)$ are incompatible. Put $D_0 = D$. We define, inductively, for each $n \in \omega$, a tree $D_{n+1}$ without terminal nodes such that for each $\sigma \in D_{n+1}$, $\sigma \circ b^n \in B^n$ (and thus $\sigma \circ d_{n+1} \in T$), and such that $\{\sigma \circ d^n : \sigma \in D_{n+1}\} = D_n$, as follows: $D_{n+1}$ is the union of all trees $E$ without terminal nodes, such that for all $\sigma \in E$, $\sigma \circ b^n \in B^n$ and $\sigma \circ d^n \in D_n$. In order to see that $D_{n+1}$ has the properties that we mentioned, let $\tau \in D_n$. It in enough to find a $\sigma \in D_{n+1}$ such that $\sigma \circ d^n = \tau$. The tree ${D_n}^{\text{via}\,\tau}$ has no terminal node. Since $(d, D)$ is incompatible with $(f, A_\gamma)$, the Suslin code $(d_n, {D_n}^{\text{via}\,\tau})$ is also incompatible with $(f, A_\gamma)$, and thus it is incompatible with $(f, A_{\gamma^\frown\langle n\rangle})$. By the induction hypothesis, $(d_n, {D_n}^{\text{via}\,\tau})$ must be compatible with $(f, B_{\gamma^\frown\langle n\rangle})$, and thus also with $(b_n, B^n)$. This easily implies that there is some tree $E$ without terminal nodes such that for all $\sigma \in E$, $\sigma \circ b^n \in B^n$ and $\sigma \circ d^n \in {D_n}^{\text{via}\,\tau}$. Choose $\sigma \in E \subseteq D_{n+1}$ such that $\sigma \circ d^n$ and $\tau$ have the same length. Then $\sigma \circ d^n = \tau$.

Define $e = \lim_{n\to\omega} d^n \circ \cdots \circ d^0$, so $e \circ d = \lim_{n\to\omega} d_n$. Let $E$ be the set of all finite sequences $\sigma$ such that for all large $m$, $\sigma \in D_m$. Note that for each $n \in \omega$, if $m$ is so large that $d_m(n) = e \circ d(n)$, then $E^{\leq e\circ d(n)+1} = {D_m}^{\leq e\circ d(n)+1}$. This implies that $E$ is a tree without terminal nodes. Let $b : \omega \longrightarrow \omega$ be strictly increasing such that $b \circ f = e \circ d$, $\text{range}(b) \cap \text{range}(e) = \text{range}(e \circ d)$, and $\text{range}(b) \cup \text{range}(e) = \omega$. Then $E$, $e$, and $b$ witness that $(d, D)$ is compatible with $(f, B_\gamma)$.

For $c(\gamma) = \wedge$, the same holds, with the roles of $A_\gamma$ and $B_\gamma$ exchanged.

We will see that this characterization of coded quasi-Borel sets is 'absolute' in a certain sense. We need the following lemma.

**6.24** Lemma Let $(s, S)$ be a Suslin code with respect to some tree $T$, whose extra moves are in $\omega$. Then we can define, for each ordinal $\alpha$, a Suslin code $(f, U_\alpha)$ with extra moves in $\alpha$ that is incompatible with $(s, S)$, such that for each quasi-branch $(d, D)$ of $T$, $(d, D)$ is compatible with $(s, S)$ or, for some ordinal $\alpha$, $(d, D)$ is compatible with $(f, U_\alpha)$.

Proof Define $f : \omega \longrightarrow \omega$ by $f(n) = 2n + 1$. We define a function $H : T \longrightarrow S$ and, for each ordinal $\alpha$, a tree $U_\alpha$, just like we did (for uncountable $\alpha$) in the proof of Proposition 6.17.

Let $\alpha$ be an ordinal. Suppose that $D$, $d_{\mathrm{I}}$, and $d_{\mathrm{II}}$ witness that $(s, S)$ is compatible with $(f, U_\alpha)$. Let $B$ be the set of all ordinals $\beta < \alpha$ such that for some $\sigma \in S$, $\delta \in D$, $\tau \in U_\alpha$, and some even $n < \text{length}(\tau)$, the following holds: $\sigma \subseteq \delta \circ d_{\mathrm{I}}$, $\tau = \delta \circ d_{\mathrm{II}}$, $\sigma = H((\tau|n) \circ f)$, and $\beta = \tau(n)$. Using the definition of $H$ and the fact that $D$ has no terminal nodes, it is not difficult to see that $B$ is non-empty and has no least element. This contradiction shows that $(s, S)$ and $(f, U_\alpha)$ are incompatible.

Now let $(d, D)$ be a quasi-branch of $T$ and suppose that $(d, D)$ and $(s, S)$ are incompatible. Choose strictly increasing functions $c_{\mathrm{I}}$ and $c_{\mathrm{II}}$ such that $c_{\mathrm{I}} \circ d = c_{\mathrm{II}} \circ s$, $\operatorname{range}(c_{\mathrm{I}}) \cap \operatorname{range}(c_{\mathrm{II}}) = \operatorname{range}(c_{\mathrm{I}} \circ d)$, and $\operatorname{range}(c_{\mathrm{I}}) \cup \operatorname{range}(c_{\mathrm{II}}) = \omega$. Let $C$ be the tree $\{\gamma : \gamma \circ c_{\mathrm{I}} \in D$ and $\gamma \circ c_{\mathrm{II}} \in S\}$. Then, since $(d, D)$ and $(s, S)$ are incompatible, $C$ is wellfounded. By Proposition 2.9, there is some ordinal $\alpha$ and some function $\rho : C \longrightarrow \alpha$ such that for each $\gamma \in C$ and each move $a$ in $C$ at $\gamma$, $\rho(\gamma) > \rho(\gamma^\frown\langle a\rangle)$. Define a function $\rho' : \{\delta \in D : \operatorname{length}(\delta) \in \operatorname{range}(d)\} \longrightarrow \alpha$ as follows: For each $n \in \omega$ and $\delta \in D^{=d(n)}$, if we put $\tau = \delta \circ d \in T^{=n}$ and $\sigma = H(\tau) \in S$, then $\sigma \circ s \subseteq \tau$, so there is a unique $\gamma \in C$ of length $c_{\mathrm{II}}(\operatorname{length}(\sigma))$ such that $\gamma \circ c_{\mathrm{I}} \subseteq \delta$ and $\gamma \circ c_{\mathrm{II}} = \sigma$; we let $\rho'(\delta) = \rho(\gamma)$.

Define $e_{\mathrm{I}}$ as the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{n + d(n) : n \in \omega\}$. Let $E$ be the set of all finite sequences $\sigma$ such that $\sigma \circ e_{\mathrm{I}} \in D$ and for each $n$, if $n + d(n) < \operatorname{length}(\sigma)$ then $\sigma(n + d(n)) = \rho'(\delta)$, where $\delta = (\sigma|(n + d(n))) \circ e_{\mathrm{I}} \in D^{=d(n)}$. Then $E$ is a tree without terminal nodes. Define $e_{\mathrm{II}} : \omega \longrightarrow \omega$ as follows: For all $n \in \omega$, $e_{\mathrm{II}}(2n) = n + d(n)$ and $e_{\mathrm{II}}(2n + 1) = n + d(n) + 1$. Then one easily verifies that $e_{\mathrm{II}} \circ f = e_{\mathrm{I}} \circ d$ and for all $\sigma \in E$, $\sigma \circ e_{\mathrm{II}} \in U_\alpha$. So $E$, $e_{\mathrm{I}}$, and $e_{\mathrm{II}}$ witness that $(d, D)$ is compatible with $(f, U_\alpha)$.

Note that DC implies that for each uncountable ordinal $\alpha$, $(s, S)$ and $(f, U_\alpha)$ are complementary. But it is consistent with ZF that, for certain countable trees $S$ and $T$ and uncountable $\alpha$, $(s, S)$ and $(f, U_\alpha)$ are not complementary. To see this, let us call an ordinal $\alpha$ *low* if $\alpha = 0$ or for some infinite sequence $g$ of ordinals, $\beta$ is the least ordinal that is greater than each of the ordinals $g(n)$ $(n \in \omega)$. Each finite or countable ordinal is low. It is consistent with ZF that $\omega_1$, the least uncountable ordinal, is low (see Jech [1973], page 142). M. Gitik [1980] proved, assuming the consistency of some large cardinal hypothesis, that it is consistent with ZF that each ordinal is low. Now let $T$ be the tree ${}^{<\omega}2$. Choose a bijection $h : \omega \longrightarrow {}^{<\omega}\omega$ such that for all $m, n$, if $h(m) \subseteq h(n)$ then $m \leq n$ (so $h(0) = \langle\rangle$). Put $s = f$ and define $S$ as the tree $\{\sigma \in {}^{<\omega}\omega : \sigma \circ s \in T$ and for all $n$, if $2n + 1 < \operatorname{length}(\sigma)$ and $h(n) \subseteq \sigma \circ s$ then $\sigma(2n + 1) = 1\}$. Let $\alpha$ be an ordinal such that for all $\beta \leq \alpha$, $\beta$ is low. We will prove that $(s, S)$ and $(f, U_\alpha)$ are not complementary, by defining a quasi-branch $(d, D)$ of $T$ that is neither compatible with $(s, S)$ nor with $(f, U_\alpha)$.

Define $d : \omega \longrightarrow \omega$ by $d(n) = 3n + 1$. Let $D$ be the unique tree without terminal nodes such that for each $k$, $D^{=3k}$ is the set of all finite sequences of the form $\langle \beta_0, b_0, g_0, \ldots, \beta_{k-1}, b_{k-1}, g_{k-1}\rangle$ such that for each $n < k$:

- $\beta_n$ is a low ordinal, if $\beta_n = 0$ then $b_n = 0$ and $g_n : \omega \longrightarrow \{0\}$, and if $\beta_n > 0$ then $b_n = 1$ and $g_n$ witnesses that $\beta_n$ is low;
- if $n = 0$ then $\beta_n = \alpha$, and if $n > 0$ and $m, j$ are the unique natural numbers for which $h(n) = h(m)^\frown\langle j\rangle$, then $\beta_n = g_m(j)$.

Then for all $\delta \in D$, $\delta \circ d \in T$.

Suppose that $E$, $e_{\mathrm{I}}$, and $e_{\mathrm{II}}$ witness that $(d, D)$ is compatible with $(s, S)$. Consider the set $B$ of all ordinals $\beta$ such that for some $m, n \in \omega$ and for some $\sigma \in E$, if we write $\sigma \circ e_{\mathrm{I}} = \langle \beta_0, b_0, g_0, \beta_1, b_1, g_1, \ldots\rangle \in D$ and $\sigma \circ e_{\mathrm{II}} = \langle a_0, b_0, a_1, b_1, \ldots\rangle \in S$, then $h(n) = \langle a_0, \ldots, a_{m-1}\rangle$ and $\beta = \beta_n$. Note that for each such $\beta$, if $\sigma$ is long enough, then $b_n = 1$ since $\langle a_0, b_0, \ldots, a_n, b_n\rangle \in S$, and for some $n'$, $h(n') = \langle a_0, \ldots, a_m\rangle$, so $\beta_{n'} = g_n(a_m) < \beta$. Therefore the non-empty set $B$ has no least element. This contradiction shows that $(d, D)$ and $(s, S)$ are incompatible.

Now suppose that $E$, $e_{\mathrm{I}}$, and $e_{\mathrm{II}}$ witness that $(d, D)$ is compatible with $(f, U_\alpha)$. Let $B$ be the set of all ordinals $\beta$ such that for some $k, n \in \omega$ and some $\sigma \in E$, if we write $h(n) = \langle a_0, \ldots, a_{m-1}\rangle$, $\sigma \circ e_{\mathrm{II}} = \langle \alpha_0, b_0, \alpha_1, b_1, \ldots\rangle \in U_\alpha$, and $\sigma \circ e_{\mathrm{I}} = \langle \beta_0, b_0, g_0, \beta_1, b_1, g_1, \ldots\rangle \in D$, then $H(\langle b_0, b_1, \ldots, b_{k-1}\rangle) = \langle a_0, b_0, \ldots, a_{m-1}, b_{m-1}\rangle$ and $\beta = \beta_n > \alpha_k$. Note that for each such $\beta$, if $\sigma$ is long enough, then, since $\beta_n$ is the least ordinal that is greater than each of the ordinals $g_n(0), g_n(1), \ldots$, there is some $a_m \in \omega$ such that $g_n(a_m) \geq \alpha_k$. Let $n'$ be such that $h(n') = \langle a_0, \ldots, a_m\rangle$. Then $\beta_{n'} \geq \alpha_k$. Now $\langle a_0, b_0, \ldots, a_{m-1}, b_{m-1}, a_m\rangle \in S$, so for some $k'$, $H(\langle b_0, b_1, \ldots, b_{k'-1}\rangle) = \langle a_0, b_0, \ldots, a_{m-1}, b_{m-1}, a_m\rangle$. So $\alpha_k > \alpha_{k'}$ and thus $\beta_{n'} > \alpha_{k'}$, so $b_{n'} = 1$. Therefore $\langle a_0, b_0, \ldots, a_m, b_m\rangle \in S$ and for some $k''$, $H(\langle b_0, b_1, \ldots, b_{k''-1}\rangle) = \langle a_0, b_0, \ldots, a_m, b_m\rangle$. Since $\beta_{n'} > \alpha_{k''}$, we conclude that $\beta_{n'} \in B$. This shows that $B$ has no least element. But $\alpha \in B$ (take $n = 0$, $k \in \omega$ such that for all $\tau \in T^{=k}$, $H(\tau) = \langle\rangle$, and $\sigma \in E$ long enough). This contradiction shows that $(d, D)$ and $(f, U_\alpha)$ are incompatible.

**6.25** PROPOSITION Let $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ be Suslin codes with respect to some tree $T$, whose extra moves are natural numbers. Suppose that $M$ is a transitive class model of ZF such that $(s_{\mathrm{I}}, S_{\mathrm{I}}), (s_{\mathrm{II}}, S_{\mathrm{II}}), T \in M$. Suppose that each ordinal is an element of $M$. Then $M \models ((s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are complementary with respect to $T)$ if and only if $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are complementary with respect to $T$.

PROOF Define, for each ordinal $\alpha$, Suslin codes $(f, U^{\mathrm{I}}_\alpha)$ and $(f, U^{\mathrm{II}}_\alpha)$ as in Lemma 6.24. Now the following statements are equivalent:

**(i)** $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are complementary with respect to $T$;

**(ii)** $(s_{\mathrm{I}}, S_{\mathrm{I}})$ and $(s_{\mathrm{II}}, S_{\mathrm{II}})$ are incompatible and for all ordinals $\alpha$ and $\beta$, $(f, U^{\mathrm{I}}_\alpha)$ and $(f, U^{\mathrm{II}}_\beta)$ are incompatible.

To see this, first suppose that (i) holds. Suppose that for some ordinals $\alpha$ and $\beta$, $D$, $d_{\mathrm{I}}$, and $d_{\mathrm{II}}$ witness that $(f, U^{\mathrm{I}}_\alpha)$ is compatible with $(f, U^{\mathrm{II}}_\beta)$. Put $d = d_{\mathrm{I}} \circ f = d_{\mathrm{II}} \circ f$. By (i), we know that $(d, D)$ is compatible with $(s_J, S_J)$ for $J = \mathrm{I}$ or $J = \mathrm{II}$. Let us say that $J = \mathrm{I}$ and let $E$, $e$, and $e'$ witness that $(d, D)$ is compatible with $(s_{\mathrm{I}}, S_{\mathrm{I}})$. But then $E$, $e \circ d_{\mathrm{I}}$, and $e'$ witness that $(f, U^{\mathrm{I}}_\alpha)$ is compatible with $(s_{\mathrm{I}}, S_{\mathrm{I}})$. This contradiction with Lemma 6.24 shows that (ii) holds.

Now suppose that (ii) holds. Let $(d, D)$ be a quasi-branch of $T$ and suppose that $(d, D)$ is not compatible with $(s_{\mathrm{I}}, S_{\mathrm{I}})$. Then, by Lemma 6.24, for some ordinal $\alpha$, there are $E$, $e$, and $e'$ that witness that $(d, D)$ is compatible with $(f, U^{\mathrm{I}}_\alpha)$. If $(e \circ d, E)$ is compatible with $(s_{\mathrm{II}}, S_{\mathrm{II}})$, then $(d, D)$ is also compatible with $(s_{\mathrm{II}}, S_{\mathrm{II}})$ and we are done. If not, then, by Lemma 6.24, for some ordinal $\beta$, $(e' \circ f, E)$ is compatible with $(f, U^{\mathrm{II}}_\beta)$. But this would imply that $(f, U^{\mathrm{I}}_\alpha)$ is also compatible with $(f, U^{\mathrm{II}}_\beta)$, which contradicts (ii).

Note that for each ordinal $\alpha$, since $\alpha \in M$, the Suslin codes $(f, U^{\mathrm{I}}_\alpha)$ and $(f, U^{\mathrm{I}}_\alpha)$ are also in $M$. We have already seen that for all Suslin codes $(d, D), (e, E) \in M$, $M \models ((d, D)$ and $(e, E)$ are incompatible) if and only if $(d, D)$ and $(e, E)$ are incompatible. To complete the proof, we observe that, since $M$ is a model of ZF, $M \models$ (the statements (i) and (ii) are equivalent).

The assumption that $M$ contains all ordinals is relevant, even if we assume that AC holds in the universe and in $M$. The idea is that if there is a transitive model of ZF that does not contain all ordinals, then there is a countable transitive set model $M$ of ZF of minimal set theoretical rank. Now $M \models$ (there is no countable transitive set model of ZF). Furthermore, some $(s, S)$ witnesses that $\{x \in {}^\omega 2 : x$ does not 'code' a wellfounded countable transitive set model of ZF$\}$ is $\omega$-Suslin. The statement that $(s, S)$ and $(s, \{\langle\rangle\})$ are complementary with respect to ${}^{<\omega}2$, holds in $M$ but not in the universe.

# 7 Strongly winning pseudostrategies

In this chapter we prove that in every coded quasi-Borel game, one of the players has a 'strongly winning' pseudostrategy. Intuitively, this means that such a pseudostrategy is winning in every extension of the universe. We will see that this concept can be formulated in some different ways. The principle of dependent choices implies that the concepts 'strongly winning' and 'winning' are equivalent.

## 7A Strongly winning tactics

In Chapter 3 we proved, using the principle of dependent choices, that in every Borel game $G$, player I has a winning tactic or player II has a winning pseudostrategy. If $G$ is coded Borel (or coded quasi-Borel), then we can avoid the use of DC by *defining* a basic open tactical covering of $G$ (just like we did in Chapter 4 for coded Borel preferential games and in Chapter 5 for coded Borel generalized games). But, in order to deduce from this result that every coded (quasi-)Borel game is pseudodetermined, we still need DC to rule out the possibility that both players have a winning tactic in $G$.

In this section we introduce the concept of *strongly winning* tactics in coded quasi-Borel games. DC implies that 'strongly winning' is equivalent to 'winning', but we can prove, without using DC, that there is no coded quasi-Borel game in which both players have a strongly winning tactic. In the next section, we will use this to prove that in every coded quasi-Borel game exactly one of the players has a strongly winning pseudostrategy. In fact, we will reduce each coded quasi-Borel game to some auxiliary game that is strongly basic open in the sense of the following definition.

**7.1** Definition Let $T$ be a tree and let $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ be a quasi-Borel code for some subset $X$ of $[T]$. Define, by induction on $C$, a relation between elements of $T$ and $C$ as follows: For each $\sigma \in T$ and $\gamma \in C$, we say that $\sigma$ **forces** $\gamma$ with respect to $T$ and $c$ if for some $k \geq \text{length}(\sigma)$, for each $\tau \in T^{=k}$ such that $\sigma \subseteq \tau$, the following holds:

- $c(\gamma) \neq \bot$;
- if $c(\gamma) = \vee$ then for some $n \in \omega$, $\tau$ forces $\gamma^\frown\langle n\rangle$;
- if $c(\gamma) = \wedge$ then for all $n \in \omega$, $\tau$ forces $\gamma^\frown\langle n\rangle$;
- if $c(\gamma) = \diamond$ then for some $n \leq k$, the moves in $C$ at $\gamma$ are the elements of $T^{=n}$ and $\tau$ forces $\gamma^\frown\langle \tau|n\rangle$.

We say that $\sigma$ **immediately forces** $\gamma$ if we can take $k = \text{length}(\sigma)$, in other words if the four statements above hold for $\tau = \sigma$.

We say that $\sigma$ (**immediately**) **avoids** $\gamma$ with respect to $T$ and $c$ if $\sigma$ (immediately) forces $\gamma$ with respect to $T$ and the quasi-Borel code $\{(\bot, \top), (\top, \bot), (\vee, \wedge), (\wedge, \vee), (\diamond, \diamond)\} \circ c$ for $[T] \setminus X$.

We say that $X$ is **strongly basic open** with respect to $T$ and $c$ if for some $n \in \omega$, for each $\sigma \in T^{=n}$, $\sigma$ immediately forces or immediately avoids $\langle\rangle$ with respect to $T$ and $c$.

Define, for every $\gamma \in C$, a quasi-Borel subset $X_\gamma$ of $[T]$ as in Definition 6.7. Then one easily verifies, by induction on $C$, that for every $\gamma \in C$ and $\sigma \in T$:

- if $\sigma$ forces $\gamma$, then for each $\tau \in T$ such that $\sigma \subseteq \tau$, $\tau$ forces $\gamma$, and for each $x \in [T]$ such that $\sigma \subseteq x$, $x \in X_\gamma$;
- if $\sigma$ avoids $\gamma$, then for each $\tau \in T$ such that $\sigma \subseteq \tau$, $\tau$ avoids $\gamma$, and for each $x \in [T]$ such that $\sigma \subseteq x$, $x \notin X_\gamma$;
- if $T$ has no terminal node, then $\sigma$ does not both force and avoid $\gamma$.

This implies that if $X$ is strongly basic open with respect to $T$ and $c$, then $X$ is a basic open subset of $[T]$.

One also easily proves that if for some infinite sequence $x$, $T$ is the tree $\{x|n : n \in \omega\}$, then $\langle\rangle$ forces $\langle\rangle$ if and only if $x \in X$, and $\langle\rangle$ avoids $\langle\rangle$ if and only if $x \notin X$.

Let $f : \omega \longrightarrow \omega$ be strictly increasing and let $T'$ be a tree such that for all $\sigma \in T'$, $\sigma \circ f \in T$. Define $X' = \{x \in [T'] : x \circ f \in X\}$ and let $c' : C' \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ be the quasi-Borel code for $X'$ associated with $T'$, $f$, and $c$, so $c' = c \circ \Gamma$ for a certain $\Gamma : C' \longrightarrow C$. Then one easily verifies, by induction on $C'$, that for every $\gamma \in C'$ and $\sigma \in T'$, if $\sigma \circ f$ (immediately) forces $\Gamma(\gamma)$ with respect to $T$ and $c$, then $\sigma$ (immediately) forces $\gamma$ with respect to $T'$ and $c'$. Thus, if $X$ is strongly basic open with respect to $T$ and $c$, then $X'$ is strongly basic open with respect to $T'$ and $c'$.

We will use the 'forcing' relation to define the concept of a strongly winning tactic.

**7.2** Definition Let $(T, P, X)$ be a game, let $c$ be a quasi-Borel code for $X$, and let $s : \omega \longrightarrow \omega$ be strictly increasing. We call an $s$-tactic $S$ for player I in $(T, P, X)$ **strongly winning** with respect to $c$ if for every quasi-branch $(d, D)$ of $S$, there is a quasi-branch $(e, E)$ of $D$ such that $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and the quasi-Borel code associated with $E$, $e \circ d \circ s$, and $c$.

An $s$-tactic for player II in $(T, P, X)$ is called **strongly winning** with respect to $c$ if it is a strongly winning $s$-tactic for player I in the game $(T, T \setminus P, [T] \setminus X)$ with respect to the quasi-Borel code $\{(\bot, \top), (\top, \bot), (\vee, \wedge), (\wedge, \vee), (\diamond, \diamond)\} \circ c$ for $[T] \setminus X$.

A pseudostrategy for some player in $(T, P, X)$ is called **strongly winning** with respect to $c$ if it is a strongly winning $t$-tactic with respect to $c$, where $t$ is the identity on $\omega$.

**7.3** Example Consider a game of the form $(T, P, \emptyset)$. The unique function $c : \{\langle\rangle\} \longrightarrow \{\bot\}$ is a quasi-Borel code for the winning set for player I in this game. Now one easily verifies that each tactic for player II in this game is strongly winning (with respect to $c$) and that a tactic $S$ for player I in this game is strongly winning if and only if $S$ has no quasi-branch if and only if $S$ is wellfounded (whereas $S$ is winning if and only if $[S] = \emptyset$).

We call a game $(T, P, X)$ *strongly basic open* with respect to $c$ if $c$ is a quasi-Borel code for $X$ and $X$ is strongly basic open with respect to $T$ and $c$.

**7.4** Theorem Let $(T, P, X)$ be a strongly basic open game with respect to $c$. Then one of the players has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

Proof Choose $n \in \omega$ such that for each $\sigma \in T^{=n}$, $\sigma$ (immediately) forces or avoids $\langle\rangle$ with respect to $T$ and $c$.

Let $\sigma \in T^{=n}$. Consider the game $G_\sigma = (T^{\text{via}\,\sigma}, P \cap T^{\text{via}\,\sigma}, X \cap [T^{\text{via}\,\sigma}])$. If $\sigma$ avoids $\langle\rangle$ then the winning set for player I in $G_\sigma$ is empty, so, by Remark 2.10, player I has a wellfounded pseudostrategy or player II has a pseudostrategy in this trivial game. If $\sigma$ forces $\langle\rangle$ then the same holds with I and II interchanged. Furthermore, if $\sigma$ both forces and avoids $\langle\rangle$ then $T^{\text{via}\,\sigma}$ is wellfounded. So we can find a player $W_\sigma$ and a pseudostrategy $S_\sigma$ for $W_\sigma$ in $G_\sigma$ such that if either $\sigma$ forces $\langle\rangle$ and $W_\sigma = \text{II}$ or $\sigma$ avoids $\langle\rangle$ and $W_\sigma = \text{I}$, then $S_\sigma$ is wellfounded.

Continuing as in the proof of Theorem 2.5, we find a pseudostrategy $S$ for some player $W$ in $(T, P, X)$ such that for all $\sigma \in S^{=n}$, $W_\sigma = W$ and $S^{\text{via}\,\sigma} = S_\sigma$. In order to see that $S$ is strongly winning with respect to $c$, let $(d, D)$ be a quasi-branch of $S$. Choose some $\sigma' \in D^{=d(n)}$. Put $\sigma = \sigma' \circ d$ and $E = D^{\text{via}\,\sigma'}$. Let $e$ be the identity on $\omega$. Then $(e, E)$ is a quasi-branch of $D$ and $(d, E)$ is a quasi-branch of $S_\sigma$, so $S_\sigma$ is not wellfounded. If $W = \text{I}$, then we conclude that $\sigma$ does not avoid $\langle\rangle$, so $\sigma$ forces $\langle\rangle$ with respect to $T$ and $c$ and therefore $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and the quasi-Borel $c'$ code associated with $E$, $d$, and $c$. Similarly, if $W = \text{II}$ then $\langle\rangle$ avoids $\langle\rangle$ with respect to $E$ and $c'$.

**7.5** Proposition Let $S$ be a tactic for some player in a game $(T, P, X)$ and let $c$ be a quasi-Borel code for $X$. Then:

**(i)** if $S$ is strongly winning with respect to $c$, then $S$ is winning;

**(ii)** if DC holds and $S$ is winning, then $S$ is strongly winning with respect to $c$.

Proof By symmetry, we may assume that $S$ is an $s$-tactic for player I.

Suppose that $S$ is strongly winning. Let $x$ be an infinite branch of $S$. Define $D$ as the tree $\{x|n : n \in \omega\}$ and $d$ as the identity on $\omega$. Since $S$ is strongly winning, $D$ has a quasi-branch $(e, E)$ such that $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and the quasi-Borel code $c'$ associated with $E$, $e \circ s$, and $c$. Now suppose that $x \circ s \notin X$. Then $\langle\rangle$ avoids $\langle\rangle$ with respect to $D$ and the quasi-Borel code associated with $D$, $s$, and $c$, and thus $\langle\rangle$ both forces and avoids $\langle\rangle$ with respect to $E$ and $c'$. Since this is impossible, we conclude that $x \circ s \in X$. This proves that the $s$-tactic $S$ for player I in $(T, P, X)$ is winning.

Now suppose that DC holds and that $S$ is winning. Let $(d, D)$ be a quasi-branch of $S$. By DC and Proposition 2.6, $D$ has some infinite branch $x$. Put $E = \{x|n : n \in \omega\}$ and let $e$ be the identity on $\omega$. Let $c'$ be the quasi-Borel code associated with $E$, $d \circ s$, and $c$. Then, since $x \circ d \in [S]$, $x \circ d \circ s \in X$. So $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and $c'$. This proves that $S$ is strongly winning with respect to $c$.

**7.6** Proposition There is no game $(T, P, X)$ and quasi-Borel code $c$ for $X$, such that both players have a strongly winning tactic in $(T, P, X)$ with respect to $c$.

Proof Suppose that player I has a strongly winning $s_{\mathrm{I}}$-tactic $S_{\mathrm{I}}$ and that player II has a strongly winning $s_{\mathrm{II}}$-tactic $S_{\mathrm{II}}$. Just as in the proof of Proposition 3.3, we can define strictly increasing functions $f_{\mathrm{I}}$, $f_{\mathrm{II}}$, and $f$ from $\omega$ to $\omega$ such that $f_{\mathrm{II}} \circ s_{\mathrm{II}} = f_{\mathrm{I}} \circ s_{\mathrm{I}} = f$, and a tree $D$ without terminal nodes such that for all $\sigma \in D$, $\sigma \circ f_{\mathrm{I}} \in S_{\mathrm{I}}$ and $\sigma \circ f_{\mathrm{II}} \in S_{\mathrm{II}}$.

Since $S_{\mathrm{I}}$ is strongly winning, $D$ has some quasi-branch $(e, E)$ such that $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and the quasi-Borel code associated with $E$, $e \circ f$, and $c$. Since $S_{\mathrm{II}}$ is strongly winning, $E$ has some quasi-branch $(e', E')$ such that $\langle\rangle$ avoids $\langle\rangle$ with respect to $E'$ and the quasi-Borel code $c'$ associated with $E'$, $e' \circ e \circ f$, and $c$. But then $\langle\rangle$ both forces and avoids $\langle\rangle$ with respect to $E'$ and $c'$. We have seen that this is impossible.

## 7B Reducing coded quasi-Borel games to strongly basic open games

In this section, we prove that in every coded quasi-Borel game exactly one of the players has a strongly winning pseudostrategy, by reducing each coded quasi-Borel game to some auxiliary strongly basic open game.

In Definition 1.16, we introduced translators of strategies from one game to another in such a way that winning strategies are translated into winning strategies. We want to do something similar for strongly winning pseudostrategies.

**7.7** DEFINITION Let $f : \omega \longrightarrow \omega$ be strictly increasing and consider games $G_0 = (T_0, P_0, X_0)$ and $G_1 = (T_1, P_1, X_1)$ such that for all $\sigma \in T_1$, $\sigma \circ f \in T_0$ and such that $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$.

A **strong $f$-translator of pseudostrategies for player II from $G_1$ to $G_0$** is a function $\phi$ that assigns to every pseudostrategy $S$ for player II in $G_1$ a pseudostrategy $\phi(S)$ for player II in $G_0$ such that every quasi-branch of $\phi(S)$ is compatible with $(f, S)$.

We say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ such that:

**(i)** for every $n \in \omega$, domain$(\phi_n)$ is the set of pseudostrategies for player II in $G_1$ up to positions of length $f(n)$ and for every $S \in$ domain$(\phi_n)$, $\phi_n(S)$ is a pseudostrategy for player II in $G_0$ up to positions of length $n$ and for every $m < n$, $\phi_n(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$;

**(ii)** for every pseudostrategy $S$ for II in $G_1$ and every $m \in \omega$, $\phi(S)^{\leq m+1} = \phi_m(S^{\leq f(m)+1})$;

**(iii)** for every $n \in \omega$, if $f(n) = n$, then ${T_1}^{\leq n+1} = {T_0}^{\leq n+1}$, $P_1 \cap {T_1}^{\leq n} = P_0 \cap {T_0}^{\leq n}$, and for every $S \in$ domain$(\phi_n)$, $\phi_n(S) = S$.

**7.8** LEMMA Let $f$, $G_0$, and $G_1$ be as in Definition 7.7. Suppose that $X_0$ has some quasi-Borel code $c_0$ and define $c_1$ as the quasi-Borel code associated with $T_1$, $f$, and $c_0$. Let $\phi$ be a strong $f$-translator of pseudostrategies for player II from $G_1$ to $G_0$. Suppose that II has a strongly winning pseudostrategy $S$ in $G_1$ with respect to $c_1$. Then $\phi(S)$ is a strongly winning pseudostrategy for II in $G_0$ with respect to $c_0$.

PROOF Let $(d, D)$ be a quasi-branch of $\phi(S)$. Let $E$, $e$, and $d'$ witness that $(d, D)$ is compatible with $(f, S)$. Since $(d', E)$ is a quasi-branch of the strongly winning pseudostrategy $S$, $E$ has a quasi-branch $(e', E')$ such that $\langle\rangle$ avoids $\langle\rangle$ with respect to $E'$ and the quasi-Borel code $c'$ associated with $E'$, $e' \circ d'$, and $c_1$. Since $(e' \circ e, E')$ is a quasi-branch of $D$ and $c'$ is the quasi-Borel code associated with $E'$, $e' \circ e \circ d$, and $c_0$, this proves that $\phi(S)$ is strongly winning with respect to $c_0$.

**7.9** LEMMA Suppose that for each $n \in \omega$, $G_n$ is a game, $f_n : \omega \longrightarrow \omega$ is strictly increasing, $\phi_n$ is a strong $f_n$-translator of pseudostrategies for player II from $G_{n+1}$ to $G_n$, and $\phi_n^0, \phi_n^1, \ldots$ witness that $\phi_n$ is continuous.

Assume that for each $n \in \omega$, $g_n = \lim_{m\to\omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ exists. Then there is a (unique) game $G$ and, for each $n \in \omega$, a continuous, strong $g_n$-translator $\psi_n$ of pseudostrategies for player II from $G$ to $G_n$ such that $\phi_n \circ \psi_{n+1} = \psi_n$.

Proof Define a game $G$ and, for all $n, i \in \omega$, functions $\psi_n^i$ and $\psi_n$, just as in the proof of Lemma 1.19 (with 'pseudostrategy for II' instead of 'strategy for I'). Let $n \in \omega$. In order to see that $\psi_n$ is a continuous, strong $g_n$-translator of pseudostrategies for player II from $G$ to $G_n$, let $S$ be a pseudostrategy for II in $G$ and let $(d_n, D_n)$ be a quasi-branch of the pseudostrategy $\psi_n(S)$ for II in $G_n$. We must show that $(d_n, D_n)$ is compatible with $(g_n, S)$.

Just as in the proof of Proposition 3.3, we easily find strictly increasing functions $e_n$ and $d_{n+1}$ from $\omega$ to $\omega$ such that $e_n \circ d_n = d_{n+1} \circ f_n$, $\text{range}(e_n) \cap \text{range}(d_{n+1}) = \text{range}(e_n \circ d_n)$, and $\text{range}(e_n) \cup \text{range}(d_{n+1}) = \omega$. Note that for all $i$, if $f_n(i) = i$ then $d_{n+1}(i) = d_n(i)$. Define $D_{n+1}$ as the union of all trees $E$ such that $E$, $e_n$, and $d_{n+1}$ witness that $(d_n, D_n)$ is compatible with $(f_n, \psi_{n+1}(S))$. For each $\tau \in D_n$, the quasi-branch $(d_n, {D_n}^{\text{via}\,\tau})$ of $\psi_n(S)$ is compatible with $(f_n, \psi_{n+1}(S))$, since $\psi_n(S) = \phi_n(\psi_{n+1}(S))$ and $\phi_n$ is a strong $f_n$-translator. This easily implies that $D_{n+1}$ is a tree without terminal nodes and $\{\sigma \circ e_n : \sigma \in D_{n+1}\} = D_n$.

Repeating this argument, we find, for each $m \geq n$, some $D_{m+1}$, $e_m$, and $d_{m+1}$ witnessing that $(d_m, D_m)$ is compatible with $(f_m, \psi_{m+1}(S))$ and such that for all $i$, if $e_m(i) = i$ then ${D_m}^{\leq i} = {D_{m+1}}^{\leq i}$. Using the fact that for each $i \in \omega$, for all large $m$, $f_m(i) = i$, $d_{m+1}(i) = d_m(i)$, and $e_m(i) = i$, we find a tree $D$ without terminal nodes such that for each $i \in \omega$, for all large $m$, $D^{\leq i} = {D_m}^{\leq i}$. We also see that both $d = \lim_{m\to\omega} d_m$ and $h_n = \lim_{m\to\omega} e_m \circ \cdots \circ e_{n+1} \circ e_n$ exist, and that $h_n \circ d_n = d \circ g_n$. For each $i$, for all large $m$, $\psi_m(S)^{\leq i} = S^{\leq i}$, so $(d, D)$ is a quasi-branch of $S$. Thus $D$, $h_n$, and $d$ witness that $(d_n, D_n)$ is compatible with $(g_n, S)$.

Note that we used a similar argument in the last part of the proof of Theorem 6.23.

**7.10** Definition Let $f$, $G_0$, and $G_1$ be as in Definition 7.7 and let $E$ be some set.

A **strong $f$-translator of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$** is a function $\phi$ that assigns to every pair $(s_1, S_1)$ for which $s_1 : \omega \longrightarrow \omega$ is strictly increasing and $S_1$ is an $s_1$-tactic for player I in $G_1$ whose extra moves are in $E$, a pair $(s_0, S_0)$ such that $s_0 : \omega \longrightarrow \omega$ is strictly increasing, $S_0$ is an $s_0$-tactic for player I in $G_0$ whose extra moves are in $E$, and such that for every quasi-branch $(d, D)$ of $S_0$, $(d \circ s_0, D)$ is compatible with $(s_1 \circ f, S_1)$.

We say that $\phi$ is **continuous** if there are functions $\phi_0, \phi_1, \ldots$ that satisfy the conditions (i) and (ii) in Definition 3.6, such that for all $n \in \omega$, if $f(n) = n$ then for all $(\sigma, S) \in \text{domain}(\phi_n)$, $\phi_n((\sigma, S)) = (\sigma, S)$.

It is not difficult to formulate and prove lemmas similar to the Lemmas 7.8 and 7.9, with 'strong $f$-translators of tactics for player I with extra moves in $E$' instead of 'strong $f$-translators of pseudostrategies for player II'. The result is similar to Lemma 3.9, but now we have *strong* translators of pseudostrategies for II and tactics for I. Another difference is that we do not use DC.

In Chapter 3 we only considered Borel games, but now we consider (coded) quasi-Borel games. Therefore we need a lemma similar to Lemma 6.3.

**7.11** LEMMA Let $(T, P, X)$ be a game. Let $E$ be a set such that ${}^{<\omega}E \subseteq E$. Let $n \in \omega$ and let $f : \omega \longrightarrow \omega$ be strictly increasing such that $f|n$ is the identity on $n$. For each $\tau \in T^{=n}$, let $\phi_\tau$ and $\phi^E_\tau$ be continuous, strong $f$-translators of pseudostrategies for II and tactics for I with extra moves in $E$, respectively, from a game $(U_\tau, Q_\tau, Y_\tau)$ to the game $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, X \cap [T^{\text{via}\,\tau}])$.

Then there is a game $(U, Q, Y)$ such that for every $\tau \in T^{=n}$, $U_\tau = U^{\text{via}\,\tau}$, $Q_\tau = Q \cap U_\tau$, and $Y_\tau = Y \cap [U_\tau]$, and there are continuous, strong $f$-translators $\phi$ and $\phi^E$ of pseudostrategies for II and tactics for I with extra moves in $E$, respectively, from $(U, Q, Y)$ to $(T, P, X)$.

PROOF Define $(U, Q, Y)$ just as in the proof of Lemma 6.3. Note that $Y = \{y \in [U] : y \circ f \in X\}$.

For each pseudostrategy $S$ for player II in $(U, Q, Y)$, $\phi(S)$ is the pseudostrategy $S^{\leq n} \cup \bigcup_{\tau \in S^{=n}} \phi_\tau(S^{\text{via}\,\tau})$ for II in $(T, P, X)$. Let $(d, D)$ be a quasi-branch of $\phi(S)$. Choose $\delta \in D$ such that $\tau = \delta \circ d \in T^{=n}$. Then $(d, D^{\text{via}\,\delta})$ is a quasi-branch of $\phi_\tau(S^{\text{via}\,\tau})$ and thus compatible with $(f, S^{\text{via}\,\tau})$. So $(d, D)$ is compatible with $(f, S)$. Now one easily verifies that $\phi$ is a continuous, strong $f$-translator.

The definition of $\phi^E$ is more complicated, although the idea is the same. Let $S_1$ be an $s_1$-tactic for player I in $(U, Q, Y)$ whose extra moves are in $E$. We let $\phi^E((s_1, S_1)) = (s_0, S_0)$, where $s_0 : \omega \longrightarrow \omega$ is defined inductively by $s_0(m) = \begin{cases} s_1(m) & \text{if } f(m) = m, \\ 1 & \text{if } f(m) > m = 0, \\ s_0(m-1) + 2 & \text{if } f(m) > m > 0, \end{cases}$ and $S_0$ is the $s_0$-tactic for player I in $(T, P, X)$ that is described as follows:

Player I follows $S_1$ until a position $\tau = \langle a_0, \ldots, a_{n-1} \rangle$ in $(T, P, X)$ is reached, corresponding to some $\tau' \in S_1$ such that $\tau' \circ s_1 = \tau$. Let $(s_\tau, S_\tau) = \phi^E_\tau((s_1, {S_1}^{\text{via}\,\tau'}))$. So $S_\tau$ is an $s_\tau$-tactic for I in the game

$(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, X \cap [T^{\text{via}\,\tau}])$ with extra moves in $E$. If player I followed this tactic, then he would make a certain finite sequence $\sigma_m$ of extra moves immediately before move $a_m$ is played, for each $m \geq n$. The length of this finite sequence may depend on $m$, but, if $f(m) > m$, also on $\tau$. Therefore player I follows $S_\tau$ with the following adjustment: For each $m \geq n$ such that $f(m) > m$, instead of making the *finite sequence* $\sigma_m$ of extra moves, player I makes the *single* move $\sigma_m$. Since ${}^{<\omega}E \subseteq E$, this extra move is also in $E$.

One easily verifies that $\phi^E$ is a continuous, strong $f$-translator of tactics for I with extra moves in $E$.

In Section 1D, we constructed, for each open game $G_0$ and $k \in \omega$, some basic open game $G_1$ and translators of strategies for I and II from $G_1$ to $G_0$. Lemma 3.10 shows that we can also translate tactics for I and pseudostrategies for II from $G_1$ to $G_0$. The following lemma shows that these translators are in fact strong translators.

**7.12** LEMMA Let $G_0 = (T_0, P_0, X_0)$ be an open game and let $\Delta \subseteq T_0$ such that $X_0 = \{x \in [T_0] : \text{for some } \tau \in \Delta,\ \tau \subseteq x\}$ and for all $\sigma \in \Delta$, for all $\tau \in T_0$, if $\sigma \subseteq \tau$ then $\tau \in \Delta$. Let $k \in \omega$ and let $f$ be the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{k, k+1\}$. Then there is a basic open game $G_1 = (T_1, P_1, X_1)$ such that $T_0 = \{\sigma \circ f : \sigma \in T_1\}$, $X_1 = \{x \in [T_1] : x \circ f \in X_0\}$, and for each $\sigma \in {T_1}^{=k+2}$, either for all $\tau \in T_1$, if $\sigma \subseteq \tau$ and $\tau \circ f \in \Delta$ then $\tau$ is a terminal node of $T_1$, or for some $\tau \in T_1$, $\sigma \subseteq \tau$ and $\tau \circ f \in \Delta$ and ${T_1}^{\text{via}\,\sigma} = {T_1}^{\text{via}\,\tau}$. Furthermore, there is a continuous, strong $f$-translator of pseudostrategies for player II from $G_1$ to $G_0$ and for every set $E$ such that $T_1 \subseteq {}^{<\omega}E$, there is a continuous, strong $f$-translator of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$.

PROOF For each $\sigma \in {T_0}^{=k}$, let $\Delta_\sigma = \{\tau \in \Delta : \sigma \subseteq \tau\}$. Apart from this, we define $G_1$ exactly as in Section 1D. Then each $\sigma \in {T_1}^{=k+2}$ satisfies one of the two conditions above: the first one if $\sigma(k+1) = 1$ and the second one otherwise.

Define a function $F$ and a continuous $f$-translator $\phi_{\text{II}}$ of pseudostrategies for player II from $G_1$ to $G_0$, just as in the proof of Lemma 3.10. In order to see that $\phi_{\text{II}}$ is a continuous, *strong* $f$-translator of pseudostrategies for II from $G_1$ to $G_0$, let $S$ be a pseudostrategy for player II in $G_1$ and let $(d, D)$ be a quasi-branch of $\phi(S)$. We must show that $(d, D)$ is compatible with $(f, S)$. Let $g$ be the unique strictly increasing function from $\omega$ to $\omega$ whose range is $\omega \setminus \{d(k), d(k)+1\}$. Define
$e : \omega \longrightarrow \omega$ by $e(n) = \begin{cases} d(n) & \text{if } n \leq k, \\ d(k)+1 & \text{if } n = k+1, \\ d(n-2)+2 & \text{if } n \geq k+2. \end{cases}$ Then $g \circ d = e \circ f$.

Choose some $\delta \in D^{=d(k)}$ and put $\sigma = \delta \circ d$.

CASE 1: For all $\delta' \in D$, if $\delta \subseteq \delta'$ then $\delta' \circ d \notin \Delta$. For some $A$, $F(S, \sigma, 1) = \sigma^\frown\langle A, 1\rangle$. Let $E$ be the set of all finite sequences $\rho$ such that for some $\rho'$, $\rho \subseteq \delta^\frown\langle A, 1\rangle^\frown\rho'$ and $\delta^\frown\rho' \in D$.

CASE 2: For some $\delta' \in D$, $\delta \subseteq \delta'$ and $\delta' \circ d \in \Delta$. Choose such $\delta'$ of minimal length and put $\tau = \delta' \circ d$. For some $A$, $F(S, \sigma, \tau) = \sigma^\frown\langle A, \tau\rangle$. Now let $E$ be the set of all finite sequences $\rho$ such that for some $\rho'$, $\rho \subseteq \delta^\frown\langle A, \tau\rangle^\frown\rho'$ and $\delta' \subseteq \delta^\frown\rho' \in D$.

In both cases, for each $\rho \in E$, $\rho \circ e \in S$, since $\rho \circ g \circ d \in \phi(S)$. Thus $E$, $g$, and $e$ witness that $(d, D)$ is compatible with $(f, S)$.

Now let $E$ be a set such that $T_1 \subseteq {}^{<\omega}E$. Define a continuous $f$-translator $\phi_{\mathrm{I}}$ of tactics for player I with extra moves in $E$, from $G_1$ to $G_0$, just as in the proof of Lemma 3.10. Using a similar argument as above, we see that $\phi_{\mathrm{I}}$ is a strong $f$-translator.

In Lemma 3.11 we proved, using the principle of dependent choices, that each Borel game has a basic open tactical covering. In a similar way we now construct, without using DC, for each coded quasi-Borel game some 'strong tactical covering'.

**7.13** LEMMA Let $G = (T, P, X)$ be a game and suppose that $X$ has some quasi-Borel code $c$. Let $f : \omega \longrightarrow \omega$ be strictly increasing such that for infinitely many natural numbers $k$, neither $k$ nor $k+1$ is in the range of $f$. Then there is some game $G' = (T', P', X')$ such that:

**(i)** $T = \{\sigma \circ f : \sigma \in T'\}$;

**(ii)** $G'$ is a strongly basic open game with respect to the quasi-Borel code $c'$ associated with $T'$, $f$, and $c$ (so $X' = \{x \in [T'] : x \circ f \in X\}$);

**(iii)** there is a strong $f$-translator $\phi$ of pseudostrategies for player II from $G'$ to $G$ and there are functions $\phi_0, \phi_1, \ldots$ witnessing that $\phi$ is continuous;

**(iv)** for every set $E$ such that $T' \subseteq {}^{<\omega}E \subseteq E$, there is a strong $f$-translator $\phi^E$ of tactics for player I with extra moves in $E$, from $G'$ to $G$ and there are functions $\phi^E_0, \phi^E_1, \ldots$ witnessing that $\phi^E$ is continuous, such that for every $n \in \omega$, if $f(n) = n$, then for every $(\sigma, S) \in \mathrm{domain}(\phi^E_n)$, $\phi^E_n((\sigma, S)) = (\sigma, S)$.

PROOF Put $C = \mathrm{domain}(c)$. We will *define* $G'$ and functions $\phi$, $\phi_n$, $\phi^E$, and $\phi^E_n$ as above, by transfinite induction to the rank of the wellfounded tree $C$. We will call $G'$ the *strong tactical $f$-covering* of $G$ with respect to $c$. For each move $a$ in $C$ at $\langle\rangle$, we let $C_a$ be the wellfounded tree

$\{\gamma : \langle a\rangle^\frown\gamma \in C\}$ and we define a function $c_a$ on $C_a$ by $c_a(\gamma) = c(\langle a\rangle^\frown\gamma)$. So $c_a$ is a quasi-Borel code of some $X_a \subseteq [T]$.

First suppose that $c(\langle\rangle) \in \{\bot, \top\}$. Then $X$ is strongly basic open with respect to $T$ and $c$. We define $G'$ just as in the proof of Lemma 1.22, by inserting some trivial extra move at the right places. So $X'$ is strongly basic open with respect to $T'$ and the quasi-Borel code associated with $T'$, $f$, and $c$. We define continuous, strong $f$-translators from $G'$ to $G$ just as in the proof of Lemma 3.11.

Now suppose that $c(\langle\rangle) = \vee$. Then $X = \bigcup_{n\in\omega} X_n$. Just as in the proof of Lemma 1.22, we find strictly increasing functions $g_0$ and $h$ from $\omega$ to $\omega$ and some $k \in \omega$ such that $f = h\circ g_0$ and $\mathrm{range}(h) = \omega\setminus\{k, k+1\}$. We also find strictly increasing functions $f_0, g_1, f_1, g_2, f_2, \ldots$ such that for all $n \in \omega$, $g_n = \lim_{m\to\omega} f_m \circ \cdots \circ f_{n+1} \circ f_n$ and such that for each $n$, for infinitely many natural numbers $m$, neither $m$ nor $m + 1$ is in the range of $f_n$.

We define, inductively, for each $n \in \omega$, a game $(T_n, P_n, Y_n)$ as follows: We let $T_0 = T$ and $P_0 = P$. For each $n \in \omega$, $Y_n$ is the quasi-Borel set coded by the quasi-Borel code $d_n$ associated with $T_n$, $f_{n-1} \circ \cdots \circ f_0$, and $c_n$, and, for some $Z_n$, $(T_{n+1}, P_{n+1}, Z_n)$ is the strong tactical $f_n$-covering of $(T_n, P_n, Y_n)$ with respect to $d_n$.

For every $n \in \omega$, let $G_n$ be the game $(T_n, P_n, \{x \in [T_n] : x\circ f_{n-1}\circ\cdots\circ f_0 \in X\})$, so the translators that we associate with the strong tactical $f_n$-covering $(T_{n+1}, P_{n+1}, Z_n)$ are also continuous, strong $f_n$-translators from $G_{n+1}$ to $G_n$. Now let $G'' = (T'', P'', X'')$ be the 'limit' of the infinite sequence $G_0, G_1, \ldots$, given by Lemma 7.9. Define $c''$ as the quasi-Borel code for $X''$ associated with $T''$, $g_0$, and $c$. Then for each $n \in \omega$, there is some $m$ such that for each $\sigma \in T''^{=m}$, $\sigma$ immediately forces or immediately avoids $\langle n\rangle$ with respect to $T''$ and $c''$. This follows from the fact that $Z_n$ is strongly basic open with respect to $T_{n+1}$ and the quasi-Borel code associated with $T_{n+1}$, $f_n$, and $d_n$.

Thus, if we put $\Delta = \{\tau \in T'' : \text{for some } n \in \omega,\ \tau \text{ immediately forces } \langle n\rangle\}$, then $X''$ is the open set $\{x \in [T''] : \text{for some } \tau \in \Delta,\ \tau \subseteq x\}$. By Lemma 7.12, we find a basic open game $G' = (T', P', X')$ and certain continuous, strong $h$-translators from $G'$ to $G''$. Composing these with the $g_0$-translators from $G''$ to $G$ that are given by Lemma 7.9 and the remark following Definition 7.10, we get continuous, strong $f$-translators from $G'$ to $G$. In order to see that $X'$ is strongly basic open with respect to $T'$ and the quasi-Borel code $c'$ associated with $T'$, $f$, and $c$, consider some $\sigma \in T'^{=k+2}$. By Lemma 7.12, we can distinguish two cases.

Case 1: For all $\tau \in T'$, if $\sigma \subseteq \tau$ and $\tau \circ h \in \Delta$ then $\tau$ is a terminal node of $T'$.

Let $n \in \omega$ and choose $m \geq k$ such that for each $\rho \in T''^{=m}$, $\rho$ immediately forces or immediately avoids $\langle n \rangle$ with respect to $T''$ and $c''$. Then for each $\tau \in T'^{=m+3}$ such that $\sigma \subseteq \tau$, $\tau|(m+2)$ is not a terminal node of $T'$, so $\tau \circ h|m \notin \Delta$ and therefore $\tau$ immediately avoids $\langle n \rangle$ with respect to $T'$ and $c'$. This shows that $\sigma$ avoids $\langle n \rangle$ with respect to $T'$ and $c'$. Since this holds for each $n \in \omega$, we conclude that $\sigma$ immediately avoids $\langle\rangle$ with respect to $T'$ and $c'$.

CASE 2: For some $\tau \in T'$, $\sigma \subseteq \tau$ and $\tau \circ h \in \Delta$ and $T'^{\text{via}\,\sigma} = T'^{\text{via}\,\tau}$. Choose such a $\tau$. Then there is some $n \in \omega$ such that $\tau \circ h$ immediately forces $\langle n \rangle$ with respect to $T''$ and $c''$. So $\tau$ immediately forces $\langle n \rangle$ with respect to $T'$ and $c'$. Since for each $\rho \in T'$ of the same length as $\tau$, if $\sigma \subseteq \rho$ then $\rho = \tau$, this shows that $\sigma$ forces $\langle n \rangle$ with respect to $T'$ and $c'$. Thus $\sigma$ immediately forces $\langle\rangle$ with respect to $T'$ and $c'$.

If $c(\langle\rangle) = \wedge$, then $X = \bigcap_{n\in\omega} X_n$. We construct $G''$ as above, but now this game is closed. We continue with 'forces' and 'avoids' exchanged everywhere, and with the following adjustment: Applying Lemma 7.12 to the open game $(T'', P'', [T''] \setminus X'')$, we find a basic open game $(T', P', [T'] \setminus X')$. The continuous, strong $h$-translators that we find, are also translators from $G' = (T', P', X')$ to $G''$.

Finally suppose that $c(\langle\rangle) = \diamond$. So, for some $n \in \omega$, $X$ is the mix of the sets $X_\tau$ ($\tau \in T^{=n}$). Just as in the proof of Lemma 6.4, we easily find strictly increasing functions $h$, $l$, and $r$ from $\omega$ to $\omega$ and some $k \in \omega$ such that $f = h \circ l \circ r$, range$(h) = \omega \setminus \{k, k+1\}$, for infinitely many natural numbers $m$, neither $m$ nor $m+1$ is in the range of $r$, and for all $m < n$, $r(m) = m$. We can do this in such a way that $k + 2 \geq f(n)$.

Let $\tau \in T^{=n}$. Let $d_\tau$ be the quasi-Borel code associated with $T^{\text{via}\,\tau}$, the identity on $\omega$, and $c_\tau$. Then $d_\tau$ is a quasi-Borel code for $X \cap [T^{\text{via}\,\tau}]$. Let $(U_\tau, Q_\tau, Y_\tau)$ be the strong tactical $r$-covering of $(T^{\text{via}\,\tau}, P \cap T^{\text{via}\,\tau}, X \cap [T^{\text{via}\,\tau}])$ with respect to $d_\tau$.

By Lemma 7.11, we find a game $(U, Q, Y)$ such that for every $\tau \in T^{=n}$, $U_\tau = U^{\text{via}\,\tau}$, $Q_\tau = Q \cap U_\tau$, and $Y_\tau = Y \cap [U_\tau]$, and continuous, strong $r$-translators from $(U, Q, Y)$ to $(T, P, X)$. We easily construct a game $G'' = (U', Q', Y')$ and continuous, strong $l$-translators from $G''$ to $(U, Q, Y)$ by inserting some trivial extra move at the right places.

Let $c''$ be the quasi-Borel code for $Y'$, associated with $U'$, $l \circ r$, and $c$. Then for some $n' \leq k$, the moves in $c''$ at $\langle\rangle$ are precisely the elements $\tau'$ of $U'^{=n'}$. For each such $\tau'$, there is some $m \geq n'$ such that for each $\sigma \in U'^{=m}$, if $\tau' \subseteq \sigma$, then $\sigma$ immediately forces or immediately avoids $\langle \tau' \rangle$ with respect to $U'$ and $c''$. This follows from the fact that, for $\tau = \tau' \circ l \circ r$, $Y_\tau$ is strongly basic open with respect to $U_\tau$ and the quasi-Borel code associated with $U_\tau$, $r$, and $d_\tau$.

Thus, if we put $\Delta = \{\sigma \in U' : \text{length}(\sigma) \geq n'$ and $\sigma$ immediately forces $\langle \sigma | n' \rangle\}$, then $Y'$ is the open set $\{x \in [U'] :$ for some $\tau \in \Delta$, $\tau \subseteq x\}$. By Lemma 7.12, we find a basic open game $G' = (T', P', X')$ and certain continuous, strong $h$-translators from $G'$ to $G''$. Composing these with the $(l \circ r)$-translators from $G''$ to $G$ that we already have, we get continuous, strong $f$-translators from $G'$ to $G$. In order to see that $X'$ is strongly basic open with respect to $T'$ and the quasi-Borel code $c'$ associated with $T'$, $f$, and $c$, consider some $\sigma \in T'^{=k+2}$. Put $\tau' = \sigma | n' = \sigma \circ h | n'$. By Lemma 7.12, we can distinguish two cases.

Case 1: For all $\tau \in T'$, if $\sigma \subseteq \tau$ and $\tau \circ h \in \Delta$ then $\tau$ is a terminal node of $T'$.
Choose $m \geq k$ such that for each $\rho \in U'^{=m}$, if $\tau' \subseteq \rho$, then $\rho$ immediately forces or immediately avoids $\langle \tau' \rangle$ with respect to $U'$ and $c''$. Then for each $\tau \in T'^{=m+3}$ such that $\sigma \subseteq \tau$, $\tau | (m+2)$ is not a terminal node of $T'$, so $\tau \circ h | m \notin \Delta$ and therefore $\tau$ immediately avoids $\langle \tau' \rangle$ with respect to $T'$ and $c'$. This shows that $\sigma$ avoids $\langle \tau' \rangle$ with respect to $T'$ and $c'$. Since $\tau' \subseteq \sigma$, we conclude that $\sigma$ immediately avoids $\langle\rangle$ with respect to $T'$ and $c'$.

Case 2: For some $\tau \in T'$, $\sigma \subseteq \tau$ and $\tau \circ h \in \Delta$ and $T'^{\text{via}\,\sigma} = T'^{\text{via}\,\tau}$.
Choose such a $\tau$. Then $\tau \circ h$ immediately forces $\langle \tau' \rangle$ with respect to $U'$ and $c''$. So $\tau$ immediately forces $\langle \tau' \rangle$ with respect to $T'$ and $c'$. Since for each $\rho \in T'$ of the same length as $\tau$, if $\sigma \subseteq \rho$ then $\rho = \tau$, this shows that $\sigma$ forces $\langle \tau' \rangle$ with respect to $T'$ and $c'$. Thus $\sigma$ immediately forces $\langle\rangle$ with respect to $T'$ and $c'$.

We now give a proof of the main result of this chapter, similar to the proof of Theorem 3.12.

**7.14** Theorem Let $(T, P, X)$ be a game and suppose that $X$ has a quasi-Borel code $c$. Then exactly one of the players has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

Proof By Proposition 7.6, at most one of the players has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

Define $f : \omega \longrightarrow \omega$ by $f(n) = 3n$, and put $G = (T, P, X)$. Let $G'$ be as in Lemma 7.13. So $G'$ is strongly basic open with respect to the quasi-Borel code $c'$ associated with $T'$, $f$, and $c$. By Theorem 7.4, some player $J$ has a strongly winning pseudostrategy in $G'$ with respect to $c'$. If $J = \text{II}$, then, by the Lemmas 7.8 and 7.13, player II has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$. In a similar way, using the fact that each pseudostrategy is a tactic without extra moves and for some set $E$, $T' \subseteq {}^{<\omega}E \subseteq E$, we see that if $J = \mathrm{I}$, then player I has a strongly winning tactic in $(T, P, X)$ with respect to $c$.

The game $(T, T \setminus P, [T] \setminus X)$ is also coded quasi-Borel. Therefore, in $(T, P, X)$, either player I has a strongly winning pseudostrategy or player II has a strongly winning tactic with respect to $c$.

Thus, in $(T, P, X)$, either both players have a strongly winning tactic or at least one of the players has a strongly winning pseudostrategy. The first possibility is ruled out by Proposition 7.6.

## 7C Winning in each extension of the universe

In this section, we show that the concept 'strongly winning' is related to some concepts that we introduced in Chapter 6. We will see that it is also related to (generic) extensions of the universe. We need the following lemma.

**7.15** Lemma Let $c : C \longrightarrow \{\bot, \top, \vee, \wedge, \diamond\}$ be a quasi-Borel code with respect to some tree $T$. For each $\gamma \in C$, we define a Suslin code $(f, A_\gamma)$ as in the proof of Theorem 2.16. Let $(d, D)$ be a Suslin code with respect to $T$. Let $c'$ be the quasi-Borel code associated with $D$, $d$, and $c$. Put $C' = \mathrm{domain}(c')$ and recall that $c' = c \circ \Gamma$ for a certain $\Gamma : C' \longrightarrow C$. Let $\tau \in D$ and $\gamma' \in C'$ and put $\gamma = \Gamma(\gamma')$. Suppose that there is a strictly increasing $g : \omega \longrightarrow \omega$ and an $F : \{\sigma \in D : \tau \subseteq \sigma\} \longrightarrow A_\gamma$ such that for all $\sigma, \sigma' \in \mathrm{domain}(F)$ and $n \in \omega$, $F(\sigma) \circ f \subseteq \sigma \circ d$ and if $\sigma \subseteq \sigma'$ then $F(\sigma) \subseteq F(\sigma')$ and if $\mathrm{length}(\sigma) \geq g(n)$ then $\mathrm{length}(F(\sigma)) \geq n$. Then $\tau$ forces $\gamma'$ with respect to $D$ and $c'$.

In particular, for $(d, D) = (f, A_{\langle\rangle})$, $\langle\rangle$ forces $\langle\rangle$ with respect to $A_{\langle\rangle}$ and $c'$.

Proof We prove this by induction on $C$ (or on $C'$). Note that $c'(\gamma') = c(\gamma)$.

For $c(\gamma) = \top$, this is trivial, and for $c(\gamma) = \bot$, ${A_\gamma}^{=1} = \emptyset$ and thus for each $k \geq g(1)$, there is no $\sigma \in D^{=k}$ such that $\tau \subseteq \sigma$.

If $c(\gamma) = \diamond$, then there is some $k \geq \mathrm{length}(\tau)$ such that for each $\sigma \in \mathrm{domain}(F)$ of length $k$, there is a unique move $a'$ in $C'$ at $\gamma'$ such that $a' \subseteq \sigma$, and a unique move $a$ in $C$ at $\gamma$ such that $a \subseteq F(\sigma) \circ f$. For such $\sigma$, $a$ and $a'$ we have that $a = a' \circ d$ and $\Gamma(\gamma'^\frown\langle a'\rangle) = \gamma^\frown\langle a\rangle$. Since $\{F(\sigma') : \sigma' \in D \text{ and } \sigma \subseteq \sigma'\} \subseteq \{\rho \in A_\gamma : a \subseteq \rho \circ f\} \subseteq A_{\gamma^\frown\langle a\rangle}$, we know by the induction hypothesis that $\sigma$ forces $\gamma'^\frown\langle a'\rangle$. This proves that $\tau$ forces $\gamma'$.

If $c(\gamma) = \vee$, choose $k \geq \mathrm{length}(\tau)$ such that $k \geq g(1)$. Let $\sigma \in \mathrm{domain}(F)$ of length $k$. Let $n$ be the natural number $F(\sigma)(0)$. Then $\Gamma(\gamma'^\frown\langle n\rangle) = \gamma^\frown\langle n\rangle$ and for each $\sigma' \in D$ such that $\sigma \subseteq \sigma'$, if we write $F(\sigma') = \langle n, x_0, a_0, x_1, a_1, \ldots\rangle$, then $\langle a_0, x_0, a_1, x_1, \ldots\rangle \in A_{\gamma^\frown\langle n\rangle}$. Thus, by the induction hypothesis, $\sigma$ forces $\gamma'^\frown\langle n\rangle$. This proves that $\tau$ forces $\gamma'$.

Finally, suppose that $c(\gamma) = \wedge$. Let $n \in \omega$. Then $\Gamma(\gamma'^\frown\langle n\rangle) = \gamma^\frown\langle n\rangle$ and, for some bijection $\pi : \omega\times\omega \longrightarrow \omega$, for each $\sigma' \in \text{domain}(F)$, if $F(\sigma') = \langle a_0, x_0, a_1, x_1, \ldots\rangle$, then $\langle a_{\pi(n,0)}, x_0, a_{\pi(n,1)}, x_1, a_{\pi(n,2)}, \ldots\rangle \in A_{\gamma^\frown\langle n\rangle}$. Thus, by the induction hypothesis, $\tau$ forces $\gamma'^\frown\langle n\rangle$. Since this holds for each $n \in \omega$, we have that $\tau$ forces $\gamma'$. In fact, $\tau$ immediately forces $\gamma'$.

In the following theorem, we give some equivalent formulations of the concept of a strongly winning tactic.

**7.16** THEOREM Let $(T, P, X)$ be a game and suppose that $X$ has a quasi-Borel code $c$. Define Suslin codes $(f, A_{\langle\rangle})$ and $(f, B_{\langle\rangle})$ for $X$ and $[T]\setminus X$, respectively, as in the proof of Proposition 6.21. Let $S$ be an $s$-tactic for player I in $(T, P, X)$. Then the following statements are equivalent:

- **(i)** $S$ is strongly winning with respect to $c$;
- **(ii)** $(s, S)$ and $(f, B_{\langle\rangle})$ are incompatible;
- **(iii)** for every quasi-branch $(d, D)$ of $S$, $(d \circ s, D)$ is compatible with $(f, A_{\langle\rangle})$;
- **(iv)** for every quasi-branch $(d, D)$ of $S$, there exists a tree $D' \subseteq D$ without terminal nodes such that $\langle\rangle$ forces $\langle\rangle$ with respect to $D'$ and the quasi-Borel code associated with $D'$, $d \circ s$, and $c$.

Furthermore, if $X$ is strongly basic open with respect to $c$ then for some $n$, for all $\tau \in T^{=n}$, at least one of the trees $\{\sigma \in A_{\langle\rangle} : \sigma \circ f \in T^{\text{via}\,\tau}\}$ and $\{\sigma \in B_{\langle\rangle} : \sigma \circ f \in T^{\text{via}\,\tau}\}$ is wellfounded.

PROOF In order to prove that (i) implies (ii), suppose that $S$ is strongly winning and $D$, $d$, and $d'$ witness that $(s, S)$ is compatible with $(f, B_{\langle\rangle})$. We will derive a contradiction. Since $(d, D)$ is a quasi-branch of $S$, $D$ has a quasi-branch $(e, E)$ such that $\langle\rangle$ forces $\langle\rangle$ with respect to $E$ and the quasi-Borel code $c'$ associated with $E$, $e \circ d \circ s$, and $c$. On the other hand, by Lemma 7.15, $\langle\rangle$ avoids $\langle\rangle$ with respect to $B_{\langle\rangle}$ and the quasi-Borel code associated with $B_{\langle\rangle}$, $f$, and $c$. Since $(e \circ d', E)$ is a quasi-branch of $B_{\langle\rangle}$, we would have that $\langle\rangle$ both forces and avoids $\langle\rangle$ with respect to $E$ and $c'$.

By (the proof of) Theorem 6.23, the Suslin codes $(f, A_{\langle\rangle})$ and $(f, B_{\langle\rangle})$ are complementary, so (ii) implies (iii).

Now suppose that (iii) holds. Let $(d, D)$ be a quasi-branch of $S$. Then there are $E$, $e$, and $e'$ witnessing that $(d \circ s, D)$ is compatible with $(f, A_{\langle\rangle})$. We may assume, without loss of generality, that $\text{range}(e) \cup \text{range}(e') = \omega$ and thus the extra moves of $(e, E)$ are natural numbers.

Therefore we can find a subtree $E'$ of $E$ without terminal nodes such that for each $\sigma \in E'$ whose length is not in range($e$), there is exactly one move in $E'$ at $\sigma$. Define a subtree $D'$ of $D$ without terminal nodes by $D' = \{\sigma \circ e : \sigma \in E'\}$. Define $F : D' \longrightarrow A_{\langle\rangle}$ as follows: For each $\tau \in D'$, $F(\tau) = \sigma \circ e'$, where $\sigma$ is the unique element of $E'$ for which $\tau = \sigma \circ e$ and length($\sigma$) = $e$(length($\tau$)). Then, by Lemma 7.15, $\langle\rangle$ forces $\langle\rangle$ with respect to $D'$ and the quasi-Borel code associated with $D'$, $d \circ s$, and $c$. This proves that (iii) implies (iv).

It is trivial that (iv) implies (i).

Now suppose that $X$ is strongly basic open with respect to $c$. Choose $n \in \omega$ such that for each $\tau \in T^{=n}$, $\tau$ either forces or avoids $\langle\rangle$ with respect to $T$ and $c$. Let $\tau \in T^{=n}$ such that $\tau$ avoids $\langle\rangle$. Suppose that the tree $\{\sigma \in A_{\langle\rangle} : \sigma \circ f \in T^{\text{via}\,\tau}\}$ has some subtree $D$ without terminal nodes. Then $\langle\rangle$ avoids $\langle\rangle$ with respect to $D$ and the quasi-Borel code $c'$ associated with $D$, $f$, and $c$. But, by Lemma 7.15, since $D \subseteq A_{\langle\rangle}$, $\langle\rangle$ also forces $\langle\rangle$ with respect to $D$ and $c'$, which is impossible. In the same way we see that for each $\tau \in T^{=n}$ such that $\tau$ forces $\langle\rangle$, the tree $\{\sigma \in B_{\langle\rangle} : \sigma \circ f \in T^{\text{via}\,\tau}\}$ is wellfounded.

If $S$ is countable, then, since the extra moves of $(f, B_{\langle\rangle})$ are natural numbers, statement (ii) in this theorem holds if and only if the Suslin codes $(s, S)$ and $(f, B_{\langle\rangle})$ code disjoint sets. Thus, a countable $s$-tactic $S$ for player I in $(T, P, X)$ is strongly winning with respect to $c$ if and only if $S$ is winning.

Also note that if $s$ is the identity on $\omega$ (and thus $S$ is a pseudostrategy for player I in $(T, P, X)$) then statement (ii) in this theorem holds if and only if the tree $\{\sigma \in B_{\langle\rangle} : \sigma \circ f \in S\}$ is wellfounded.

In the rest of this chapter, we will use the equivalence of the statements (i) and (ii) above instead of Definition 7.2. So one may think of statement (ii) as a *definition* of the concept of a strongly winning tactic for player I. Similarly, an $s$-tactic $S$ for player II is strongly winning if and only if $(s, S)$ and $(f, A_{\langle\rangle})$ are incompatible. With this new definition, Proposition 7.5 follows easily from the observation that an $s$-tactic $S$ for player I is winning if and only if the Suslin codes $(s, S)$ and $(f, B_{\langle\rangle})$ code disjoint subsets of $[T]$. Proposition 7.6 follows from the observation that if $S_{\text{I}}$ is an $s_{\text{I}}$-tactic for player I and $S_{\text{II}}$ is an $s_{\text{II}}$-tactic for player II in the same game, then $(s_{\text{I}}, S_{\text{I}})$ is compatible with $(s_{\text{II}}, S_{\text{II}})$. Since the Suslin codes $(f, A_{\langle\rangle})$ and $(f, B_{\langle\rangle})$ are complementary, this implies that $(s_{\text{I}}, S_{\text{I}})$ is compatible with $(f, B_{\langle\rangle})$ or $(s_{\text{II}}, S_{\text{II}})$ is compatible with $(f, A_{\langle\rangle})$, so at least one of the tactics is not strongly winning.

Statement (ii) in Theorem 7.16 is 'absolute' for transitive class models of ZF. Thus, for each transitive class model $M$ of ZF such that $T$, $P$, $c$, $s$, and $S$ are in $M$, $M \models$ ($S$ is strongly winning with respect to $c$) if and only if $S$ is strongly winning with respect to $c$. Now let $X' \in M$ such that

$M \models$ ($X'$ is the quasi-Borel subset of $[T]$ coded by $c$). Then one easily shows that $X' = X \cap M$. Thus, if $S$ is a winning $s$-tactic for player I in $(T, P, X)$, then $M \models$ ($S$ is a winning $s$-tactic for player I in $(T, P, X')$). Note that if DC holds in $M$ and $M \models$ ($S$ is a winning $s$-tactic for player I in $(T, P, X')$), then $S$ is a winning $s$-tactic for player I in $(T, P, X)$, by Proposition 7.5(i) and since Proposition 7.5(ii) holds in $M$.

The *universe* V is the class of all sets. Until now, we have only considered 'subuniverses' $M$ of V. In the proof of Theorem 2.18, we essentially used the following: For each transitive countable set $A$, there is a transitive class model $M$ of ZF such that $A \in M$ and $M \models$AC.

We also want to consider V as a transitive 'subuniverse' of certain models $N$ of ZF. The method of 'forcing' makes this possible (even if AC is not true in V, see for example Jech [1978], page 171). By this method, we can find, for each partial ordering $\leq$ of a non-empty set $P$, some 'generic extension' V$[G]$ of V. This is a model $N$ of ZF such that V is a transitive subuniverse of $N$ that contains all ordinals of $N$, and such that for some $G \in N$, $G$ is V-*generic*. This means that $G$ is a subset of $P$ such that for all $g \in G$, for all $p \in P$, if $g \leq p$ then $p \in G$, and for all $g, g' \in G$, for some $p \in P$, $p \leq g$ and $p \leq g'$, and such that $G$ intersects each $D \in$ V that is a *dense* subset of $P$, i.e., such that for all $p \in P$, for some $d \in D$, $d \leq p$. Note that if V itself has a V-generic element, then we can take $N =$ V.

**7.17** Lemma For each set $A$, there is a generic extension $N$ of V such that in $N$, $A$ is finite or countable.

Proof Put $P = {}^{<\omega}A$ and define a partial ordering $\leq$ of $P$ by $p \leq q$ if and only if $q \subseteq p$. Let $N$ be a generic extension of V with respect to this partial ordering and let $G \in N$ be $V$-generic. Then for all $g, g' \in G$, $g \subseteq g'$ or $g' \subseteq g$, so there is a finite or infinite sequence $F \in N$ such that for all $g \in G$, $g \subseteq F$. For each $a \in A$, the set $\{d \in P : a \in \text{range}(d)\}$ is a dense subset of $P$ and an element of $V$, so it intersects $G$ and thus $a \in \text{range}(F)$. So $A \subseteq \text{range}(F)$ and therefore $A$ is finite or countable in $N$.

This lemma is, in the following sense, just as strong as the method of forcing itself: Let $\leq$ be a partial ordering of some non-empty set $P$. By applying Lemma 7.17 to the set $A = P \cup \{D \subseteq P : D \text{ is dense}\}$, we find a generic extension $N$ of V, such that in $N$, both $P$ and $\{D \in \text{V} : D \text{ is a dense subset of } P\}$ are finite or countable. Using this, one easily finds a V-generic $G \in N$.

**7.18** Proposition Let $(T, P, X)$ be a game and suppose that $X$ has a quasi-Borel code $c$. Let $S$ be an $s$-tactic for player I in $(T, P, X)$. Then $S$ is strongly winning with respect to $c$ if and only if in each generic extension $N$ of V, $S$ is a winning $s$-tactic for I in $(T, P, X')$, where $X'$ is the subset of $[T]$ coded by $c$ (in $N$).

PROOF If $S$ is strongly winning with respect to $c$, then in each generic extension $N$ of V, $S$ is also strongly winning since 'strongly winning' is absolute for transitive class models of ZF. Thus, since Proposition 7.5(i) holds in $N$, $S$ is winning in the corresponding game in $N$.

Now suppose that in each generic extension of V, $S$ is winning in the corresponding game. Define a Suslin code $(f, B_{\langle\rangle})$ for $[T] \setminus X$ as in Theorem 7.16. By Lemma 7.17, there is some generic extension $N$ of V in which $S$ is finite or countable. In $N$, $S$ is a finite or countable winning $s$-tactic, so $S$ is strongly winning with respect to $c$ in $N$. By absoluteness, $S$ is also strongly winning with respect to $c$ in the transitive subuniverse V.

## 7D Strongly winning pseudostrategies in coded quasi-Borel games

D.A. Martin proved, using the axiom of choice, that each Borel game is determined (Theorem 1.24). For coded Borel games on countable trees, the use of AC can be avoided by considering some class model of ZF in which AC holds (Theorem 2.18). Since each coded quasi-Borel game on a countable tree is coded Borel, this implies that each coded quasi-Borel game on a *countable* tree is pseudodetermined.

This section is an elaboration of an idea of J.R. Steel. We prove that each coded quasi-Borel game on an infinite tree $T$ is pseudodetermined, by considering some generic extension $N$ of V in which $T$ is countable.

**7.19** LEMMA Let $G$ be a game of the form $(T, P, \emptyset)$ and let $M$ be a transitive class model of ZF such that $G \in M$. Then $M \models$ (II has a pseudostrategy in $G$) if and only if II has a pseudostrategy in $G$.

PROOF First suppose that for some $S \in M$, $M \models$ ($S$ is a pseudostrategy for II in $G$). Then $S$ is a pseudostrategy for II in $G$. Now suppose that there is no $S \in M$ such that $M \models$ ($S$ is a pseudostrategy for II in $G$). Since Remark 2.10 holds in $M$, there is some $S \in M$ such that $M \models$ ($S$ is a wellfounded pseudostrategy for I in $G$), and thus, by Lemma 2.17, $S$ is a wellfounded pseudostrategy for I in $G$. This implies that there is no pseudostrategy for player II in $G$.

**7.20** LEMMA Let $(T, P, X)$ be a game and suppose that $X$ has a quasi-Borel code $c$. Then we can define, for each ordinal $\alpha$, a game $(U_\alpha, Q_\alpha, \emptyset)$, such that player II has a strongly winning pseudostrategy in $(T, P, X)$

with respect to $c$ if and only if for some $\alpha$, II has a pseudostrategy in $(U_\alpha, Q_\alpha, \emptyset)$.

PROOF Define a Suslin code $(f, A_{\langle\rangle})$ for $X$ with extra moves in $\omega$, as in the proof of Proposition 6.18. Let $\alpha$ be an ordinal. Define a Suslin code $(f, U_\alpha)$ with extra moves in $\alpha$, using some function $H : T \longrightarrow A_{\langle\rangle}$, as in the proof of Proposition 6.17. Put $Q_\alpha = \{\sigma \in U_\alpha : \text{length}(\sigma)$ is odd and $\sigma \circ f \in P\}$. So the game $(U_\alpha, Q_\alpha, \emptyset)$ is played as follows: At a position $\langle \beta_0, a_0, \ldots, \beta_{k-1}, a_{k-1}\rangle$ of even length, player II chooses some $\beta_k < \alpha$ such that for all $m, n \leq k$, if for some $a$, $H(\langle a_0, \ldots, a_{m-1}\rangle)^\frown\langle a\rangle = H(\langle a_0, \ldots, a_{n-1}\rangle)$ then $\beta_m > \beta_n$. At a position $\langle \beta_0, a_0, \ldots, \beta_{k-1}, a_{k-1}, \beta_k\rangle$ of odd length, the player whose turn it is at position $\langle a_0, \ldots, a_{k-1}\rangle$ in $(T, P, X)$, chooses a move $a_k$ in $T$ at $\langle a_0, \ldots, a_{k-1}\rangle$. If an infinite sequence $\langle \beta_0, a_0, \beta_1, \ldots\rangle$ of moves is played, then player II wins. (Note that in that case player II also wins play $\langle a_0, a_1, \ldots\rangle$ in $(T, P, X)$, since $(f, A_{\langle\rangle})$ and $(f, U_\alpha)$ code disjoint subsets of $[T]$.)

Suppose that player II has a strongly winning pseudostrategy $S$ in $(T, P, X)$ with respect to $c$. Then the tree $C = \{\sigma \in A_{\langle\rangle} : \sigma \circ f \in S\}$ is wellfounded, so there is some ordinal $\alpha$ and some $\rho : C \longrightarrow \alpha$ such that for each $\sigma \in C$ and each move $a$ in $C$ at $\sigma$, $\rho(\sigma) > \rho(\sigma^\frown\langle a\rangle)$. Now one easily verifies that $\{\sigma \in U_\alpha : \sigma \circ f \in S$ and for each even $n < \text{length}(\sigma)$, $\sigma(n) = \rho(H((\sigma|n) \circ f))\}$ is a pseudostrategy for player II in $(U_\alpha, Q_\alpha, \emptyset)$.

Now suppose that, for some ordinal $\alpha$, II has a pseudostrategy $S$ in $(U_\alpha, Q_\alpha, \emptyset)$. Since at each position $\sigma$ in $(U_\alpha, Q_\alpha, \emptyset)$ of even length, player II has to choose some ordinal, we may assume that for each $\sigma \in S$ of even length, there is exactly one move in $S$ at $\sigma$. Now put $S' = \{\sigma \circ f : \sigma \in S\}$. Then $S'$ is a pseudostrategy for player II in $(T, P, X)$. We have seen in Lemma 6.24 that the Suslin codes $(f, U_\alpha)$ and $(f, A_{\langle\rangle})$ are incompatible. In a similar way, we now show that $S'$ is strongly winning with respect to $c$. In other words, we prove that the tree $C = \{\sigma \in A_{\langle\rangle} : \sigma \circ f \in S'\}$ is wellfounded. Suppose that $C$ has some subtree $D$ without terminal nodes. Let $B$ be the set of all ordinals $\beta < \alpha$ such that for some $\langle \beta_0, a_0, \ldots, \beta_{k-1}, a_{k-1}, \beta_k\rangle \in S$, $\beta = \beta_k$ and for some $\delta \in D$, $\delta \circ f = \langle a_0, \ldots, a_{k-1}\rangle$ and $H(\delta \circ f) \subseteq \delta$. Using the fact that $D$ has no terminal nodes, one easily verifies that $B$ is non-empty and has no least element. This contradiction shows that $C$ is wellfounded.

**7.21** LEMMA Let $(T, P, X)$ be a game and suppose that $X$ has a quasi-Borel code $c$. Let $M$ be a transitive class model of ZF containing $T$,

$P$, $c$, and all ordinals. Let $X' \in M$ such that $M \models$ ($X'$ is the quasi-Borel subset of $[T]$ coded by $c$). Then $M \models$ (II has a strongly winning pseudostrategy in $(T, P, X')$ with respect to $c$) if and only if II has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

PROOF For each ordinal $\alpha$, define $G_\alpha$ as the game $(U_\alpha, Q_\alpha, \emptyset)$ in Lemma 7.20, so, since $\alpha \in M$, $G_\alpha \in M$. By Lemma 7.19, for each $\alpha$, $M \models$ (II has a pseudostrategy in $G_\alpha$) if and only if II has a pseudostrategy in $G_\alpha$. Since Lemma 7.20 holds both in the real universe and in $M$, we conclude that $M \models$ (II has a strongly winning pseudostrategy in $(T, P, X')$ with respect to $c$) if and only if II has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

This lemma, which we will use to give another proof of Theorem 7.14, still holds if we drop the assumption that $M$ contains all ordinals. We can prove this just like Lemma 7.19, with Theorem 7.14 instead of Remark 2.10, and using the fact that if for some player $J$ and some $S \in M$, $M \models$ ($S$ is a strongly winning pseudostrategy for $J$ in $(T, P, X')$ with respect to $c$), then $S$ is a strongly winning pseudostrategy for $J$ in $(T, P, X)$ with respect to $c$.

We now give another proof of the theorem that in each coded quasi-Borel game $(T, P, X)$, exactly one of the players has a strongly winning pseudostrategy with respect to a given quasi-Borel $c$.

NEW PROOF OF THEOREM 7.14 By Lemma 7.17, there is a generic extension $N$ of V such that in $N$, $T$ is finite or countable. Let $X' \in N$ such that, in $N$, $c$ is a quasi-Borel code for $X'$. Then $X'$ also has some Borel code $c'$, since $T$ is at most countable in $N$. Since Theorem 2.18 holds in $N$, the coded Borel game $(T, P, X')$ in $N$ in determined. Thus, in $N$, exactly one of the players has a winning strategy in $(T, P, X')$. In $N$, $T$ is finite or countable and thus the following statements are equivalent:
player II has a winning strategy in $(T, P, X')$;
player II has a finite or countable winning pseudostrategy in $(T, P, X')$;
player II has a strongly winning pseudostrategy in $(T, P, X')$ with respect to $c$.

By Lemma 7.21, the last statement is equivalent to: V $\models$ (II has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$). The same holds for the other player: In $N$, player I has a winning strategy in $(T, P, X')$ if and only if in V, player I has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$. Thus, in V, exactly one of the players has a strongly winning pseudostrategy in $(T, P, X)$ with respect to $c$.

Note that this also gives another proof of Proposition 7.6.

# References

**A. Church**

[1927] *Alternatives to Zermelo's assumption*, Transactions of the American Mathematical Society **29**, 178–208.

**M. Davis**

[1964] *Infinite games of perfect information*, in: M. Dresher, L.S. Shapley, A.W. Tucker (editors), *Advances in game theory*, Annals of Mathematics Studies **52**, Princeton University Press, Princeton, N.J., 85–101.

**J.E. Fenstad**

[1971] *The axiom of determinateness*, in: J.E. Fenstad (editor), *Proceedings of the Second Scandinavian Logic Symposium*, Studies in Logic and the Foundations of Mathematics **63**, North-Holland Publishing Company, Amsterdam–London, 41–61.

**H. Friedman**

[1971] *Higher set theory and mathematical practice*, Annals of Mathematical Logic **2**, 325–357.

[1981] *On the necessary use of abstract set theory*, Advances in Mathematics **41**, 209–280.

**D. Gale** and **F.M. Stewart**

[1953] *Infinite games with perfect information*, in: H.W. Kuhn and A.W. Tucker (editors), *Contributions to the theory of games, Volume II*, Annals of Mathematics Studies **28**, Princeton University Press, Princeton, N.J., 245–266.

**M. Gitik**

[1980] *All uncountable cardinals can be singular*, Israel Journal of Mathematics **35**, 61–88.

**J.D. Halpern** and **A. Lévy**

[1971] *The Boolean prime ideal theorem does not imply the axiom of choice*, in: D. Scott (editor), *Axiomatic set theory*, Proceedings of Symposia in Pure Mathematics **13**, Part 1, American Mathematical Society, Providence, R.I., 83–134.

**S.H. Hechler**

[1974] *On a game for which both players have a winning strategy*, Bulletin de l'Académie Polonaise des Sciences. Série des sciences mathématiques, astronomiques et physiques **22**, 751–753.

**T.J. Jech**

[1973] *The axiom of choice*, Studies in Logic and the Foundations of Mathematics **75**, North-Holland Publishing Company, Amsterdam–London–New York.

[1978] *Set theory*, Academic Press, New York–San Francisco–London.

**A.S. Kechris**

[1984] *The axiom of determinacy implies dependent choices in* L(ℝ), The Journal of Symbolic Logic **49**, 161–173.

**N. Lusin**

[1917] *Sur la classification de M. Baire*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris **164**, 91–94.

[1927] *Sur les ensembles analytiques*, Fundamenta Mathematicae **10**, 1–95.

**N. Lusin** and **W. Sierpiński**

[1918] *Sur quelques propriétés des ensembles (A)*, Bulletin International de l'Académie des Sciences de Cracovie. Série A: sciences mathématiques **1918**, 35–48.

**D.A. Martin**

[1970] *Measurable cardinals and analytic games*, Fundamenta Mathematicae **66**, 287–291.

[1975] *Borel determinacy*, Annals of Mathematics **102**, 363–371.

[1985] *A purely inductive proof of Borel determinacy*, in: A. Nerode and R.A. Shore (editors), *Recursion theory*, Proceedings of Symposia in Pure Mathematics **42**, American Mathematical Society, Providence, R.I., 303–308.

[1990] *An extension of Borel determinacy*, Annals of Pure and Applied Logic **49**, 279–293.

**D.A. Martin** and **J.R. Steel**

[1988] *Projective determinacy*, Proceedings of the National Academy of Sciences of the United States of America **85**, 6582–6586.

[1989] *A proof of projective determinacy*, Journal of the American Mathematical Society **2**, 71–125.

**R.D. Mauldin** (editor)

[1981] *The Scottish Book: Mathematics from the Scottish Café*, Birkhäuser, Boston–Basel–Stuttgart.

**G.H. Moore**

[1982] *Zermelo's axiom of choice. Its origins, development, and influence*, Studies in the History of Mathematics and Physical Sciences **8**, Springer-Verlag, New York–Heidelberg–Berlin.

**Y.N. Moschovakis**

[1980] *Descriptive set theory*, Studies in Logic and the Foundations of Mathematics **100**, North-Holland Publishing Company, Amsterdam–New York–Oxford.

**J. Mycielski**

[1964] *On the axiom of determinateness*, Fundamenta Mathematicae **53**, 205–224.

**J. Mycielski** and **H. Steinhaus**
[1962] *A mathematical axiom contradicting the axiom of choice*, Bulletin de l'Académie Polonaise des Sciences. Série des sciences mathématiques, astronomiques et physiques **10**, 1–3.
**J. Mycielski** and **S. Świerczkowski**
[1964] *On the Lebesgue measurability and the axiom of determinateness*, Fundamenta Mathematicae **54**, 67–71.
**J. Mycielski, S. Świerczkowski,** and **A. Zięba**
[1956] *On infinite positional games*, Bulletin de l'Académie Polonaise des Sciences. Classe 3 **4**, 485–488.
**J. Mycielski** and **A. Zięba**
[1955] *On infinite games*, Bulletin de l'Académie Polonaise des Sciences. Classe 3 **3**, 133–136.
**J.B. Paris**
[1972] *ZF* $\vdash \Sigma^0_4$ *determinateness*, The Journal of Symbolic Logic **37**, 661–667.
**R.M. Solovay**
[1970] *A model of set-theory in which every set of reals is Lebesgue measurable*, Annals of Mathematics. Second series **92**, 1–56.
**H. Steinhaus**
[1965] *Games, an informal talk*, American Mathematical Monthly **72**, 457–468.
**M. Suslin**
[1917] *Sur une définition des ensembles mesurables B sans nombres transfinis*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences, Paris **164**, 88–91.
**R. Telgárski**
[1987] *Topological games: On the 50$^{th}$ anniversary of the Banach–Mazur game*, The Rocky Mountain Journal of Mathematics **17**, 227–276.
**W.H. Woodin**
[1988] *Supercompact cardinals, sets of reals, and weakly homogeneous trees*, Proceedings of the National Academy of Sciences of the United States of America **85**, 6587–6591.
**P. Wolfe**
[1955] *The strict determinateness of certain infinite games*, Pacific Journal of Mathematics **5**, 841–847.
**E. Zermelo**
[1913] *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels*, in: E.W. Hobson and A.E.H. Love (editors), *Proceedings of the Fifth International Congress of Mathematicians, Vol. II*, Cambridge University Press, Cambridge, 501–504.

# Index

# Samenvatting

## Borel-gedetermineerdheid zonder keuze-axioma

Dit proefschrift gaat over oneindige spelen.

Een oneindig spel wordt gespeeld door twee spelers. Op elk moment kennen zij het eindige rijtje van de zetten die al gedaan zijn en doet één van beiden een nieuwe zet. Na oneindig veel zetten heeft één speler gewonnen.

De *posities* zijn de toegestane eindige rijtjes van zetten en vormen een *boom* $T$. De mogelijke oneindige rijen van zetten zijn de oneindige takken van $T$ en vormen een verzameling $[T]$. Deze wordt van de gebruikelijke topologie voorzien. Het spel ligt dus vast door de boom $T$, de verzameling $P$ van alle posities waarin speler I aan de beurt is om een zet te doen en de verzameling $X$ van alle oneindige rijen van zetten waarbij speler I uiteindelijk wint.

Het spel heet *gedetermineerd* als een van de spelers een winnnende *strategie* heeft, d.w.z. een precies voorgeschreven speelwijze waarbij hij altijd wint. Als $X$ een open deel van $[T]$ is, heet het spel *open*; als $X$ een Borel-verzameling is, spreken we van een Borel-spel. In hoofdstuk 1 geven we precieze definities van deze begrippen. We laten *eindposities* toe: posities waarin geen zet mogelijk is. In zulke posities verliest de speler die aan de beurt is.

In tegenstelling tot spelen die na *eindig* veel zetten afgelopen zijn, zijn sommige oneindige spelen niet gedetermineerd. Met behulp van het *keuze-axioma* AC construeerden D. Gale en F.M. Stewart [1953] zelfs een ongedetermineerd spel op een *aftelbare* boom. Zij toonden aan dat elk open spel $(T, P, X)$ gedetermineerd is. Deze uitspraak is equivalent met AC, maar voor het speciale geval dat de boom $T$ aftelbaar is, is AC niet nodig.

D.A. Martin [1975, 1985] bewees, met behulp van AC, dat elk Borel-spel $(T, P, X)$ gedetermineerd is. Het idee is om met inductie naar de complexiteit van de Borel-verzameling $X$ een open spel te construeren (op een veel grotere boom) zo dat elke (winnende) strategie in dat open hulpspel vertaald kan worden in een (winnende) strategie in het oorspronkelijke spel. In hoofdstuk 1 vereenvoudigen we dit bewijs door gebruik te maken van eindposities.

In hoofdstuk 2 bekijken we de rol van het keuze-axioma in gedetermineerdheidsbewijzen. Zonder AC kunnen we niet bewijzen dat elk open spel gedetermineerd is, maar wel dat zo'n spel *pseudogedetermineerd* is: Speler I of speler II heeft een winnende *pseudostrategie*, een winnende speelwijze waarbij de zetten van de speler niet precies voorgeschreven hoeven te zijn. Als we AC aannemen of ons beperken tot spelen op aftelbare bomen, dan zijn gedetermineerdheid en pseudogedetermineerdheid gelijkwaardig. Martin merkte op dat de gedetermineerdheid van Borel-spelen op *aftelbare* bomen al volgt uit het *aftelbare keuze-axioma* CAC, een zwakke versie van AC. Het idee hierachter is dat in bepaalde deelklassen van het universum het keuze-axioma waar is en dus elk Borel-spel gedetermineerd is. We werken dit uit in sectie 2E. CAC wordt alleen gebruikt om bij een Borel-verzameling $X$ een *Borel-code* te vinden die laat zien hoe $X$ ontstaat uit open en gesloten verzamelingen door herhaald aftelbare verenigingen of doorsnedes te nemen.

In dit proefschrift onderzoeken we Borel-spelen op willekeurig grote bomen. We tonen aan dat zulke spelen pseudogedetermineerd zijn, waarbij we slechts een zwakke vorm van het keuze-axioma aannemen. We kunnen Martins bewijs in grote lijnen blijven volgen, maar dit kost wel wat moeite: Het is niet voldoende om simpelweg overal 'strategie' te vervangen door 'pseudostrategie'. We geven drie verschillende manieren om het bewijs aan te passen. Telkens introduceren we een nieuw begrip:

- In hoofdstuk 3 introduceren we *taktieken*, speelwijzen waarbij soms extra zetten gedaan worden als 'geheugensteuntje'. Een pseudostrategie is een taktiek zonder extra zetten.

- In hoofdstuk 4 bekijken we spelen voorzien van een relatie $R$ tussen posities die uitdrukt dat sommige posities voor speler I 'gemakkelijker' en voor II 'moeilijker' zijn dan andere. We introduceren vervolgens *$R$-pseudostrategieën*, pseudostrategieën die voldoen aan bepaalde 'redelijke' eisen.

- In hoofdstuk 5 generalizeren we het begrip 'spel'. In *gegeneralizeerde* spelen zijn er behalve de posities waarin speler I of speler II een zet moet doen, ook nog posities waarin beide spelers in zekere zin *samen* een zet doen.

Martin [1990] breidde zijn bewijs van Borel-gedetermineerdheid uit tot *quasi-Borel-spelen*. Quasi-Borel-verzamelingen vallen samen met de zogenaamde $\Delta^1_1$-verzamelingen. Als de onderliggende boom aftelbaar is, dan vallen ze ook samen met de Borel-verzamelingen. In hoofdstuk 6 onderzoeken we de rol van AC hierbij. We laten zien, zonder enige vorm van AC te gebruiken, dat elk gecodeerd quasi-Borel-spel pseudogedetermineerd is.

In hoofdstuk 7 voeren we, voor gecodeerde quasi-Borel-spelen, *sterk winnende* pseudostrategieën in, pseudostrategieën die winnend zijn in elke 'generieke' uitbreiding van het universum. In elk gecodeerd quasi-Borel-spel $(T, P, X)$ heeft precies een speler een sterk winnende pseudostrategie. We geven hiervan twee bewijzen: een met behulp van taktieken en een door middel van 'forceren'. Dit laaste bewijs is een uitwerking van een idee van J.R. Steel, namelijk dat er een generieke uitbreiding is waarin $T$ hoogstens aftelbaar is en dus het corresponderende spel gedetermineerd is.

# Curriculum Vitae

Ik ben op 29 april 1964 in Boxmeer geboren en woon in Haps. In 1976 ging ik naar het Merletcollege in Cuijk. Op 9 juni 1982 haalde ik het diploma Atheneum B met als achtste vak Latijn. Ik was een van de Nederlandse deelnemers aan de Internationale Wiskunde Olympiade in juli 1981 (Washington D.C.) en in juli 1982 (Boedapest). Daarna ben ik wiskunde gaan studeren aan de Katholieke Universiteit Nijmegen. Het propedeutisch examen behaalde ik op 29 augustus 1983 en het doctoraal examen op 5 november 1987, beide met lof. Bovendien kreeg ik op 5 november 1987 onderwijsbevoegdheid in het vak wiskunde. Ik studeerde af in de grondslagen van de wiskunde bij dr. W.H.M. Veldman. Ik was studentassistent bij de vakken introductie operations research (voorjaar 1985), wiskunde I voor chemici (najaar 1985) en logica/introductie logica (najaar 1987).

Op 15 januari 1988 werd ik assistent in opleiding in dienst van de Stichting Katholieke Universiteit en ben ik verder gegaan met mijn afstudeeronderwerp, oneindige spelen in de verzamelingsleer zonder keuze-axioma. Van 1 september 1988 tot 1 september 1992 heb ik dat onderzoek voortgezet als onderzoeker in opleiding in dienst van NWO (Nederlandse Organisatie voor Wetenschappelijk Onderzoek). Dit project werd begeleid door W.H.M. Veldman en gesubsidieerd door SMC (Stichting Mathematisch Centrum). Het heeft geleid tot dit proefschrift. Bovendien heb ik geassisteerd bij practica logica en voortgezette logica voor wiskunde- en informaticastudenten van de KUN. Ik heb bijeenkomsten van de ASL (Association for Symbolic Logic) bezocht in augustus 1988 (Padua), juli/augustus 1989 (Berlijn) en augustus 1992 (Veszprém). Van 15 september tot 15 december 1990 bezocht ik op uitnodiging van Prof. Y.N. Moschovakis de Universiteit van Calfornië in Los Angeles (UCLA), waar ik ook met D.A. Martin en J.R. Steel mijn onderzoek besproken heb.

Sinds 16 november 1992 vervul ik mijn vervangende dienst bij de vakgroep wiskunde van de KUN. Een deel van mijn taak bestaat uit onderzoek in de toegepaste logica aan de Universiteit Utrecht en assistentie bij het logica-onderwijs aan filosofiestudenten van de KUN.

# Stellingen

behorende bij het proefschrift

**Borel determinacy without the axiom of choice**

van A.J.C. Hurkens

# I

Een *binaire stroomgraph* is een gerichte graph, eventueel met lussen en meervoudige pijlen, zó dat er twee verschillende punten $a$ en $z$ zijn met:

- uit $z$ vertrekt geen pijl en uit elk ander punt vertrekken er precies twee;
- voor elk punt $p$ is er een weg van $a$ naar $p$ en een weg van $p$ naar $z$.

Een binaire stroomgraph heet *irreducibel* als geen van zijn echte deelgraphen een binaire stroomgraph is.

De eindige irreducibele binaire stroomgraphen zijn precies de graphen die ontstaan uit een binaire stroomgraph met slechts twee punten door een aantal keren het volgende uit te voeren:

- Vervang een bestaande pijl, zeg van $u$ naar $v$, door drie nieuwe pijlen: een pijl van $u$ naar een nieuw punt $x$, een pijl van $x$ naar $v$ en een pijl van $x$ naar een bestaand punt $y \neq v$.

A.J.C. Hurkens, C.A.J. Hurkens, R.W. Whitty, *On generation of a class of flowgraphs*, in: M. Fiedler and J. Nešetřil (editors), *Combinatorics, graphs, complexity: proceedings of the Fourth Czechoslovak Symposium on Combinatorics: Prachatice – June 1990*, Society of Czechoslovakian Mathematicians and Physicists, Prague, 1991, 107–111.

# II

Noem een rijtje $x_1, x_2, \ldots, x_N$ reële getallen *gelijkmatig* als voor $n = 1, \ldots, N$ elk van de intervallen $[0, \frac{1}{n})$, $[\frac{1}{n}, \frac{2}{n})$, …, $[\frac{n-1}{n}, 1)$ precies één van de getallen $x_1, x_2, \ldots, x_n$ bevat.

Het volgende rijtje van 17 getallen is gelijkmatig:
$0, \frac{16}{17}, \frac{5}{11}, \frac{8}{11}, \frac{2}{7}, \frac{4}{7}, \frac{1}{7}, \frac{6}{7}, \frac{3}{8}, \frac{9}{14}, \frac{3}{14}, \frac{11}{14}, \frac{1}{2}, \frac{1}{14}, \frac{15}{17}, \frac{5}{16}, \frac{11}{17}.$

Er is geen gelijkmatig rijtje van 18 getallen. Dit kan bewezen worden door voor $n = 1, 2, \ldots, 18$ en voor $p, q \in \{\frac{i}{j} : j = 1, \ldots, 18, i = 0, 1, \ldots, j\}$ met $p < q$, de uitspraak 'minstens één van de getallen $x_1, x_2, \ldots, x_n$ zit in het interval $[p, q)$' in het vlak weer te geven met het halfopen lijnstuk $\{n\} \times [p, q)$ en dit op te vatten als een propositieletter. Vervolgens kan men in de propositielogica een tegenspraak afleiden uit formules van de vorm

- $\{n\} \times [\frac{t}{n}, \frac{t+1}{n})$;
- $\{n\} \times [p, q) \;\rightarrow\; \{n'\} \times [p', q')$, met $n \leq n'$, $p' \leq p$ en $q \leq q'$;
- $\{n\} \times [p, q) \;\rightarrow\; (\{n\} \times [p, r) \;\vee\; \{n\} \times [r, q))$;

- $\neg(\{n\} \times [\frac{t}{n}, r) \ \wedge \ \{n\} \times [r, \frac{t+1}{n}))$.

## III

De strukturen $\langle (0,1) \cup (1,2), < \rangle$ en $\langle (0,2), < \rangle$ zijn intuïtionistisch elementair equivalent. De eerste is namelijk een intuïtionistisch elementaire deelstruktuur van de tweede.

> Wim Veldman and Michaël Janssen, *Some observations on intuitionistically elementary properties of linear orderings*, Archive for Mathematical Logic **29** (1990), 171–185.

## IV

Zij $H(x, Y_0, \ldots, Y_d) \in \mathbb{C}[x, Y_0, \ldots, Y_d]$ zó dat er voor elke $p \in \mathbb{C}[x]$ een $\lambda \in \mathbb{C}$ en een $m \in \mathbb{N}$ is met $H(x, p, p', p'', \ldots, p^{(d)}) = \lambda x^m$. Dan zijn er $\lambda \in \mathbb{C}$ en $m \in \mathbb{N}$ met $H = \lambda x^m$.

> Kossivi Adjamagbo and Arno van den Essen, *Eulerian systems of partial differential equations and the Jacobian conjecture*, Journal of Pure and Applied Algebra **74** (1991), 1–15.

## V

Elke funktie van $\mathbb{C}$ naar $\mathbb{C}$ die afstanden 1 behoudt, is een isometrie.

> A. van Rooij, *Probleem*, Mededelingen van het Mathematisch Instituut, Nijmegen, nummer 83 (7 oktober 1988).

## VI

Als men 20 pionnen opstelt in het rooster $\mathbb{Z} \times \mathbb{Z}$ op de met o aangegeven posities, dan kan men in 19 zetten een pion in de oorsprong (aangegeven met x) krijgen. Bij een zet springt een pion in horizontale of verticale richting over een naburige pion (die weggenomen wordt) naar een leeg vakje.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | x | | | | |
| | | | | | | | | |
| | | | | | | | | |
| | | | | | | | | |
| o | o | o | o | o | o | o | | |
| | o | o | o | o | o | o | | |
| | | o | o | o | o | o | | |
| | | o | | o | | | | |

Het lukt niet om de oorsprong te bereiken als men begint met minder dan 20 pionnen in het halfvlak $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \leq -4\}$.

Met alleen pionnen in het halfvlak $\{(x, y) \in \mathbb{Z} \times \mathbb{Z} : y \leq -5\}$ kan de oorsprong niet in eindig veel zetten bereikt worden. De vraag of het in oneindig veel zetten mogelijk is, kan op verschillende manieren worden opgevat en beantwoord.

## VII

Zij $S$ een verzameling van eindige, niet-lege verzamelingen. Een verzameling $T$ die met elk element van $S$ een niet-lege doorsnede heeft, noemen we een *transversaal* van $S$. Als $T$ bovendien geen echte deelverzameling heeft die een transversaal van $S$ is, dan heet $T$ een *minimale* transversaal van $S$.

De uitspraak dat elke verzameling van eindige, niet-lege verzamelingen een minimale transversaal heeft, is equivalent met het keuze-axioma.

> Problem 10245, proposed by M.A. Bezem and A.J.C. Hurkens, The American Mathematical Monthly **99** (1992), 675.

## VIII

Noem een verzameling $b$ een *bouwsteen* van een verzameling $a$ als elke deelverzameling van $b$ een element is van $a$. In het bijzonder is dan $b$ zelf een element van $a$, zodat wegens het funderingsaxiomaschema het volgende *bouwsteen-inductie-principe* geldt:

- Wanneer we willen bewijzen dat elke verzameling $a$ een bepaalde eigenschap heeft, mogen we zonder de algemeenheid te schaden aannemen dat elke bouwsteen van $a$ die eigenschap heeft.

Dit inductie-principe volgt echter, zelfs met intuïtionistische logica, al uit het deelverzamelingsaxiomaschema:

- Voor elke eigenschap en elke verzameling $a$, is er een verzameling $b$ waarvan de elementen precies die elementen van $a$ zijn die de genoemde eigenschap hebben.

Met bouwsteen-inductie kan de cumulatieve hiërarchie zonder omwegen worden gedefinieerd. Dit is de essentie van Scotts alternatieve axiomatizering van de verzamelingsleer.

> D. Scott, *Axiomatizing set theory*, in: T.J. Jech (editor), *Axiomatic set theory*, Proceedings of Symposia in Pure Mathematics **13**, Part 2, American Mathematical Society, Providence, R.I., 1974, 207–214.

## IX

De formule $\varphi := \exists x \forall y\ y \notin x$ drukt uit dat er een lege verzameling is.

Met behulp van *klassieke* logica is $\varphi$ afleidbaar uit de volgende instantie van het funderingsaxiomaschema: $\forall v(\forall w(w \in v \to \varphi) \to \varphi) \to \forall v \varphi$.

Met *intuïtionistische* logica is $\varphi$ hier niet uit afleidbaar, maar wel indien ook het vervangingsaxiomaschema, verenigingsaxioma en extensionaliteits-axioma aangenomen worden.