A note on $k[z]$-automorphisms in two variables

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Abstract

We prove that for a polynomial $f \in k[x,y,z]$ equivalent are: (1)f is a $k[z]$-coordinate of $k[z][x,y]$, and (2) $k[x,y,z]/(f) \cong k^{[2]}$ and $f(x,y,a)$ is a coordinate in $k[x,y]$ for some $a \in k$. This solves a special case of the Abhyankar-Sathaye conjecture. As a consequence we see that a coordinate $f \in k[x,y,z]$ which is also a $k(z)$-coordinate, is a $k[z]$-coordinate. We discuss a method for constructing automorphisms of $k[x,y,z]$, and observe that the Nagata automorphism occurs naturally as the first non-trivial automorphism obtained by this method - essentially linking Nagata with a non-tame R-automorphism of $R[x]$, where $R = k[z]/(z^2)$.

Introduction

Let k be a field of characteristic zero. The most famous $k[z]$-automorphism of $k[x,y,z]$ is undoubtedly Nagata's automorphism $\sigma : k^3 \to k^3$ given by

$$\sigma = (x - 2sy - s^2 z, y + sz, z), \text{ where } s = xz + y^2$$

In a landmark paper [6] Shestakov and Umirbaev solved the long standing Nagata Conjecture, asserting that σ is not tame. In January 2007 the authors were rewarded with the Moore prize for the best research paper in the last six years. Nagata's automorphism can be constructed in several ways: for example it was constructed by

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Nagata in [5] as the composition ${\sigma_1}^{-1}\sigma_2\sigma_1$, where $\sigma_1 = (x + z^{-1}y^2, y)$ and $\sigma_2 = (x, y + z^2x)$.

Another construction uses locally nilpotent derivations, namely one easily verifies that $\sigma = exp(sD)$, where D is the locally nilpotent derivation given by $D = -2y\partial_x + z\partial_y$. Yet another construction can be found in [2] and [1]. It goes roughly as follows (for a detailed description we refer to section 1): start with an element p in $k[z]$, which is no constant, and form the quotient ring $R = k[z]/(p)$. Consider the polynomial ring in one variable x over R and let $a(x) \in Aut_R R[x]$. To this R-automorphism one constructs a $k[z]$-automorphism (f_1, f_2) of $k[z][x, y]$, which in turn gives a k-automorphism of $k[x, y, z]$. Now the Nagata automorphism can be found by taking the simplest non-trivial case in the above construction, namely $p = z^2$ and $a(x) = x + \overline{z}x^2$, (see Remark 2.3).

As a consequence of the main result of this paper, Proposition 2.1, one obtains that in general the k-automorphism (f_1, f_2, z) is tame if and only if the $k[z]$-automorphism (f_1, f_2) is tame if and only if $a(x)$ is tame in $Aut_R R[x]$, which just means that $a(x)$ has degree one in x. Consequently Nagata's example is non-tame. The proof of this result obviously uses one of the main results of [6] which asserts that a $k[z]$-automorphism of $k[x, y, z]$ is tame if and only if it is tame as a k-automorphism.

In the last section we give a result on $k[z]$-coordinates which to our knowledge is new. It asserts that a polynomial in $k[x, y, z]$ is a $k[z]$-coordinate if and only if $k[x, y, z]/(f)$ is k-isomorphic to a polynomial ring in two variables over k and $f(x, y, a)$ is a coordinate in $k[x, y]$ for some a in k. Hence this result proves a special case of the Abhyankar-Sathaye Conjecture and furthermore it shows that if f is a coordinate in $k[x, y, z]$ which is also a $k(z)$-coordinate, then it is a $k[z]$-coordinate.

1 Constructing R-automorphisms of $R[x, y]$

In this section we recall a construction of R-coordinates (R-automorphisms) which already can be found in [2] and [1].

Let R be a commutative ring and let $p \in R$ be neither a unit nor a zero-divisor in R. Put $\overline{R} = R/Rp$. Let $a(x), b(x)$ in $R[x]$ be such that

$$\overline{a}(\overline{b}(x)) = x = \overline{b}(\overline{a}(x)) \text{ in } \overline{R}[x].$$

Equivalently

$$a(b(x)) = x = b(a(x))(modp) \text{ in } R[x] \quad (1)$$

To such an element $\overline{a}(x) \in Aut_{\overline{R}}\overline{R}[x]$ one constructs an element of $Aut_R R[x,y]$ as follows. Put

$$f_1 = py + a(x) \qquad (2)$$

Since $f_1 = a(x)(\text{mod}p)$ it follows from (1) that $b(f_1) - x = 0(\text{mod}p)$. Consequently, since p is a non-zero divisor in R, there exists a unique element f_2 in $R[x,y]$ such that

$$b(f_1) - x = pf_2 \quad (3)$$

Lemma 1.1 $F = (f_1, f_2) \in Aut_R R[x,y]$.

Proof. Let $g_1 = b(x) - py$. Then $a(g_1) = a(b(x)) = x(\text{mod})p$. So

$$x - a(g_1) = pg_2 \quad (4)$$

for some $g_2 \in R[x,y]$.

Now we will show that $G = (g_1, g_2)$ is the inverse of F: namely by (3)

$$g_1(f_1, f_2) = b(f_1) - pf_2 = x \quad (5)$$

Furthermore, using (4), (5) and (2) we obtain

$$pg_2(f_1, f_2) = f_1 - a(g_1(f_1, f_2)) = f_1 - a(x) = py.$$

So, since p is no zero divisor in R we get $g_2(f_1, f_2) = y$.

2 Tame R-automorphisms of $R[x,y]$

Let again R be a commutative ring and n a positive integer. An R-automorphism of $R^{[n]}$ is called *tame* if it is a finite product of automorphisms of the form

$$(x_1, \ldots, x_{i-1}, ux_i + v(x), x_{i+1}, \ldots, x_n)$$

where $u \in R^*$ and $v(x) \in R^{[n]}$ does not contain x_i. The group of tame automorphisms of $R^{[n]}$ is denoted by $T(n, R)$. So $T(1, R)$ consists of the elements $ux_1 + v$ with $u \in R^*$ and $v \in R$ arbitrary.

From now on we assume that R is a *domain*.

Keeping the notations from the previous section, the main result of this section, Proposition 2.1, asserts that if one starts with an element $\overline{a}(x) \in Aut_{\overline{R}}\overline{R}[x]$ and constructs the corresponding element $F = (f_1, f_2)$ in $Aut_R R[x, y]$, then F is tame if and only if $a(x)$ is tame, in other words if and only if $a(x)$ is of degree one in x. More precisely

Proposition 2.1 *Let $\overline{a}(x) \in Aut_{\overline{R}}\overline{R}[x]$ with inverse $\overline{b}(x)$, $f_1 = py + a(x)$ and $f_2 = (b(f_1) - x)/p$. Then there is equivalence between*
i) $\overline{a}(x) \in T(1, \overline{R})$.
ii) $(f_1, f_2) \in T(2, R)$.
Furthermore, in this situation $a(x) = a_0 + a_1x + px^2\tilde{a}(x)$, for some a_0, a_1 in R and $\tilde{a}(x) \in R[x]$ and there exist c, d in R such that $da_1 - cp = 1$ and $f_2 = cx + d(y + x^2\tilde{a}(x)) + \tilde{b}(f_1)$ for some $\tilde{b}(x) \in R[x]$.

Proof. Write $a(x) = \sum_{i=0}^{d} a_i(x)x^i$ and $b(x) = \sum_{i=0}^{e} b_i x^i$. Assume i). Then $\overline{a_1} \in (\overline{R})^*$ and $\overline{a_i} = 0$ for all $i \geq 2$. Since $\overline{a_1} \in (\overline{R})^*$ there exist $c, d \in R$ such that $da_1 - cp = 1$. Since $\overline{a_i} = 0$ for $i \geq 2$ it follows that p divides each such a_i in R, so

$$a(x) = a_0 + a_1x + px^2\tilde{a}(x) \text{ for some } \tilde{a}(x) \in R[x].$$

Since $da_1 = 1(\text{mod } p)$ it follows that the inverse of $\overline{a_0} + \overline{a_1}x$ is equal to $\overline{d}(x - \overline{a_0})$, whence $\overline{b}(x) = \overline{d}(x - \overline{a_0})$, so $b(x) = d(x - a_0) + p\tilde{b}(x)$, for some $\tilde{b}(x) \in R[x]$. Consequently

$$f_2 = (\tilde{b}(f_1) - x)/p = (d(py + a_1x + px^2\tilde{a}(x)) + p\tilde{b}(f_1) - x)/p.$$

Using $(da_1 - 1)x = cpx$ it follows that

$$f_2 = cx + d(y + x^2\tilde{a}(x)) + \tilde{b}(f_1).$$

So

$$(f_1, f_2) = (a_1x + p(y + x^2\tilde{a}(x)) + a_0, cx + d(y + x^2\tilde{a}(x)) + \tilde{b}(f_1)).$$

Now one easily verifies that $(f_1, f_2) \in T(2, R)$, since

$$(x - a_0, y) \circ (x, y - \tilde{b}(x)) \circ (f_1, f_2) \circ (x, y - x^2\tilde{a}(x)) = (a_1x + py, cx + dy)$$

and $ad - pc = 1$. So i) implies ii).
Conversely, assume ii). If $\overline{a}(x) \notin T(1, \overline{R})$, then $d_1 :=\deg_x\overline{a}(x) \geq 2$ and we can write

$$a(x) = \sum_{i=0}^{d_1} a_i x^i + px^{d_1+1}\tilde{a}(x)$$

for some $\tilde{a}(x)$ in $R[x]$ and p does not divide a_{d_1}. Similarly, let $e_1 := \deg_x \overline{b}(x)$. Since $\overline{b}(x)$ is the inverse of $\overline{a}(x)$ and $d_1 \geq 2$, it follows that $e_1 \geq 2$. So we can write $b(x) = \sum_{i=1}^{e_1} b_i x^i + p\tilde{b}(x)$, for some b_i in R and $\tilde{b}(x)$ in $R[x]$, where p does not divide b_{e_1}. Then

$$f_1 = p(y + x^{d_1+1}\tilde{a}(x)) + \sum_{i=0}^{d_1} a_i x^i$$

and

$$f_2 = \frac{1}{p}[\sum_{j=0}^{e_1} b_j(p(y + x^{d_1+1}\tilde{a}(x)) + \sum_{i=0}^{d_1} a_i x^i)^j - x] + f^{e_1+1}\tilde{b}(f_1).$$

Since $(f_1, f_2) \in T(2, R)$ it follows that $(f_1, f_2 - f_1^{e_1+1}\tilde{b}(f_1)) \in T(2, R)$ and hence, replacing y by $y - x^{d_1+1}\tilde{a}(x)$, that

$$(py + \sum_{i=0}^{d_1} a_i x^i, \frac{1}{p}[\sum_{j=0}^{e_1} b_j(py + \sum_{i=0}^{d_1} a_i x^i)^j - x] \in T(2, R).$$

Since $d_1 \geq 2$ the highest degree xy-term of f_1(resp.f_2) equals $a_{d_1}x^{d_1}$(resp. $\frac{1}{p}b_{e_1}(a_{d_1}x^{d_1})^{e_1}$). So by Corollary 5.1.6 of [3], using that $e_1 \geq 2$, it follows that there exists c in R such that

$$\frac{1}{p}b_{e_1}a_{d_1}^{e_1}x^{d_1e_1} = ca_{d_1}^{e_1}x^{d_1e_1}$$

whence $b_{e_1} = pc$, so p divides b_{e_1}, a contradiction. So $\overline{a}(x) \in T(1, \overline{R})$ as desired.

Corollary 2.2 *Let k be a field of characteristic zero and $p \in k[z]$, but not in k. Put $R = k[z]/(p)$. Let $a(z, x)$ and $b(z, x)$ in $k[z, x]$ be such that $a(\overline{z}, x) \in Aut_R R[x]$ with inverse $b(\overline{z}, x)$. Put $f_1 = py + a(z, x)$ and $f_2 = (b(z, f_1) - x)/p$. Then $(f_1, f_2, z) \in T(3, k)$ if and only if $(f_1, f_2) \in T(2, k[z])$ if and only if $a(\overline{z}, x) \in T(1, R)$ if and only if $deg_x a(\overline{z}, x) = 1$.*

Proof. Proposition 2.1 gives the equivalence of the first two statements and a result of [6] gives the equivalence of the second and third statement. The last equivalence is obvious.

Remark 2.3 If $p = z^2$ and $a(z, x) = x + zx^2$ one obtains the simplest non-tame automorphism, namely $x + \overline{z}x^2$. The corresponding (non-tame) k-automorphism (f_1, f_2, z) is, apart from a permutation of the variables x and y, the Nagata automorphism σ. More precisely

$$\sigma = (y, x, z) \circ (f_1, f_2, z) \circ (y, x, z).$$

So Nagata's automorphism is the simplest non-tame automorphism "coming from" a one dimensional example.

3 A remark on $k[z]$-coordinates

Throughout this section k is a field of characteristic zero. If $n \geq 1$ and f is a k-coordinate in the polynomial ring $k^{[n]}$ then $k^{[n]}/(f)$ is k-isomorphic to a polynomial ring in $n-1$ variables over k. The Abhyankar-Sathaye Conjecture asserts that the converse is true. This conjecture is still open for all $n \geq 3$.

In this section we prove a special case of this conjecture in case $n = 3$. More precisely we show

Proposition 3.1 *Let $f \in k[x, y, z]$ be such that $A = k[x, y, z]/(f)$ is k-isomorphic to $k^{[2]}$ and that for some $a \in k$ the polynomial $f(x, y, a)$ is a coordinate in $k[x, y]$. Then f is a $k[z]$-coordinate in $k[x, y, z]$.*

To prove this proposition we need the following result from [4]:

Theorem 3.2 *Let R be a $\mathbb{Q}$-algebra and $f \in R[x, y]$. Let D be the derivation $f_y \partial_x - f_x \partial_y$ on $R[x, y]$. Then there is equivalence between*
i) f is a coordinate in $R[x, y]$.
ii) D is locally nilpotent and $1 \in R[x, y]f_x + R[x, y]f_y$.

Proof of Proposition 3.1 i) Let $\overline{k}$ be an algebraic closure of k and view f in $\overline{k}[x, y, z]$. The hypothesis implies that $\overline{k}[x, y, z]/(f) \simeq \overline{k}^{[2]}$ and that $f(x, y, a)$ is a coordinate in $\overline{k}[x, y]$. We will deduce in ii) below that f is a $\overline{k}[z]$-coordinate in $\overline{k}[x, y, z]$. It then follows from Theorem 3.2 that D is locally nilpotent on $\overline{k}[x, y, z]$, and hence on $k[x, y, z]$. Also we obtain from Theorem 3.2 that $1 \in \overline{k}[x, y, z]f_x + \overline{k}[x, y, z]f_y$, which implies that $1 \in k[x, y, z]f_x + k[x, y, z]f_y$, since f has coefficients in k. Then again applying Theorem 3.2 gives that f is a $k[z]$-coordinate of $k[x, y, z]$. So we may assume that $k = \overline{k}$.
ii) Now assume $k = \overline{k}$. The hypothesis implies that

$$A/(z - a) \simeq k[x, y, z]/(f, z - a) \simeq k[x, y]/(f(x, y, a)) \simeq k^{[1]}.$$

Since $A \simeq k^{[2]}$ the Abhyankar-Moh theorem implies that $z-a$ is a coordinate in A and hence so is $z-b$ for all $b \in k$. So $k[x,y]/(f(x,y,b)) \simeq A/(z-b) \simeq k^{[2]}$. Hence by the Abhyankar-Moh theorem $f(x,y,b)$ is a coordinate in $k[x,y]$ for all $b \in k$. In particular for all b in k the element 1 is in the ideal generated by $f_x(x,y,b)$ and $f_y(x,y,b)$ in $k[x,y]$. Consequently $f_x(x,y,z)$ and $f_y(x,y,z)$ have no common zero in k^3. So by the Nullstellensatz we have

(6) 1 is in the ideal generated by f_x and f_y in $k[z][x,y]$.

Now let $D = f_y\partial_x - f_x\partial_y$ and let d be the maximum of the x and y degrees of f_x and f_y. Since for each b in k the polynomial $f(x,y,b)$ is a coordinate in $k[x,y]$, the derivation D evaluated at $z=b$ is locally nilpotent and hence it follows from [3], Theorem 1.3.52 that $D^{d+2}(x)(z=b) = 0$ and $D^{d+2}(y)(z=b) = 0$ for all b in k. This implies that $D^{d+2}(x) = D^{d+2}(y) = 0$. So

(7) D is locally nilpotent on $k[z][x,y]$.

Then it follows from (6), (7) and Theorem 3.2 that f is a $k[z]$-coordinate of $k[x,y,z]$, as desired.

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