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The Nagata automorphism is shifted linearizable

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Abstract

A polynomial automorphism F is called *shifted linearizable* if there exists a linear map L such that LF is linearizable. We prove that the Nagata automorphism $N := (X - Y\Delta - Z\Delta^2, Y + Z\Delta, Z)$ where $\Delta = XZ + Y^2$ is shifted linearizable. More precisely, defining $L_{(a,b,c)}$ as the diagonal linear map having a, b, c on its diagonal, we prove that if $ac = b^2$, then $L_{(a,b,c)}N$ is linearizable if and only if $bc \neq 1$. We do this as part of a significantly larger theory: for example, any exponent of a homogeneous locally finite derivation is shifted linearizable. We pose the conjecture that the group generated by the linearizable automorphisms may generate the group of automorphisms, and explain why this is a natural question.

1 Preliminaries

1.1 Introduction

One of the main problems in affine algebraic geometry is to understand the polynomial automorphism group of affine spaces. In particular, it would be very useful to find some generators of these groups. The case of dimension one is easy : every automorphism of the affine line is indeed affine. (For a polynomial map, to be affine means to be of degree 1.)

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In dimension two, the situation is well known too. The Jung-van der Kulk-theorem asserts that the automorphism group of the affine plane is generated by affine and de Joncquière subgroups [14, 15]. Therefore, every automorphism of $\mathbb{A}^2$ is called *tame*.

The case of dimension 3 is still open. Recently, Umirbaev and Shestakov solved in [22, 23], the thirty years old *tame generators problem* by proving that some automorphism of $\mathbb{C}^3$ are not tame and in particular that the famous Nagata map is non tame.

Actually, there are several candidate generator sets for the automorphism group of $\mathbb{A}^n$ (see section 4).

Nevertheless, from a "geometric point of view", it is important to find generators which do not depend on choice of coordinates. Related, finding normal subgroups of the automorphism group, is important in itself (and almost the same question, actually). Notice that, since a non tame automorphism may be conjugate to a tame one (theorem 3.3 gives such an example), the notion of tame automorphism is not a relevant geometric notion.

Therefore, it seems natural to define *tamizable* automorphisms, i.e. automorphisms which are conjugate to a tame one. In particular, it leads us to the following questions :

1. Is the Nagata automorphism tamizable?

2. Are all automorphisms of $\mathbb{C}^3$ tamizable?

Note that if the answer to the first question is negative, then it will be very difficult to prove it. (The concept of degree is not invariant under conjugation, and so, the proof of Umirbaev-Shestakov does not give ideas for this.)

In this paper, we will investigate the second question and study what consequences a positive answer will give. It will lead us to consider the subgroup $\mathrm{GLIN}_n(\mathbb{C}) \subseteq \mathrm{GA}_n(\mathbb{C})$ generated by linearizable automorphisms. It turns out that this group contains all tame automorphisms, and, more surprising, that the Nagata automorphism belongs to $\mathrm{GLIN}_3(\mathbb{C})$.

More precisely, we will show that *"twice Nagata"* is even linearizable! "Twice Nagata" stands for the map $(2I) \circ N$, i.e. each component of the Nagata automorphism multiplied by 2. Then

$$N^{\frac{4}{3}}(2N)N^{\frac{-4}{3}} = 2I$$

as explained in theorem 3.3. In fact, we will prove that if D is a homogeneous locally finite derivation on $\mathbb{C}^{[n]}$, then there exists $s \in \mathbb{C}^*$ such that $s\exp(D) = (sI) \circ \exp(D)$ is linearizable. We say that $\exp(D)$ is *shifted linearizable*.

In the analytic realm, this is a known local fact, due to the Poincaré-Siegel theorem (see [2], chapter 5, or 8.3.1. of [6]). Roughly, this theorem states that for almost all $s \in \mathbb{C}^*$, and analytic map F satisfying $F(0) = 0$, sF is holomorphically linearizable locally around 0. This theorem was the starting point of a very interesting story[1] about the (negative) solution of the Markus-Yamabe conjecture and its link to the Jacobian conjecture, see [4, 7, 8]. One of the conjectures which was posed and killed "along the way" of this story was Meister's Linearization conjecture (see page 186 of [6] or [5]). However, the current article can be seen as a partial positive answer to a generalized Meister's conjecture – in fact, to such an extent that we revive a reformulate Meister's conjecture:

Meister's Linearization Problem: For which $F \in \mathrm{GA}_n(\mathbb{C})$ does there exist some $s \in \mathbb{C}^*$ such that sF is linearizable?

This article is organized as follows. In section *1: Preliminaries* we define notations and mention well-known facts on derivations. In section *2: Shifted linearizability* we show how to shift-linearize homogeneous derivations. In section *3: When is Nagata shifted linearizable?* we use the previous section on Nagata's map as an example, and explain exactly for which shifts it is linearizable and when it isn't. (We will prove that sN is linearizable if and only if $s \neq 1, -1$.) In the last section 4 we will discuss how the results of this article influence the current conjectures on generators of $\mathrm{GA}_n(\mathbb{C})$.

1.2 Notations and definitions

Let R be a commutative ring with one. (In this article, R will be $\mathbb{C}$ almost exclusively.) $R^{[n]}$ will denote the polynomial ring in n variables over R. $\mathrm{GA}_n(R)$ will denote the group of polynomial automorphisms on $R^{[n]}$. We will denote I for the identity map. ∂_X $(\partial_Y, \partial_Z, \ldots)$ will denote the derivative to the variable X $(Y, Z, \ldots)$.

An R-derivation (or simply derivation if no confusion is possible) on an R-algebra A is an R-linear map $D : A \longrightarrow A$ that satisfies the Leibniz rule $D(ab) = aD(b) + bD(a)$ for each $a, b \in A$. The set of R-derivations (or derivations) on A is denoted by $\mathrm{DER}_R(A)$ (or $\mathrm{DER}(A)$). The set of R-derivations on $R^{[n]}$ is denoted by $\mathrm{DER}_n(R)$. $\mathrm{DER}(A)$ forms a Lie algebra, as any two derivations D, E the map $[D, E] := DE - ED$ is again a derivation, as can be easily checked. A *locally nilpotent derivation* is a derivation D for which each $a \in A$ one finds an $m \in \mathbb{N}$ such that $D^m(a) = 0$. For example: $D = \partial_X$ on $\mathbb{C}[X]$. If $R = k$, a field, we define a *locally finite derivation* as a derivation D for which each $a \in A$ the k-span of $a, D(a), D^2(a), \ldots$ is finite dimensional. For example: $D = (X + 1)\partial_X$ on $\mathbb{C}[X]$. We use $\mathrm{LND}_n(k), \mathrm{LFD}_n(k)$ for the sets of locally nilpotent resp. locally finite derivations on $k^{[n]}$.

If D is a derivation on a ring A containing $\mathbb{Q}$, then one can define the map $\exp(TD) : A[[T]] \longrightarrow A[[T]]$ as the map sending f to $\sum_{i=0}^{\infty} \frac{T^i}{i!} D^i(f)$. It is an automorphism of $A[[T]]$, and its inverse is $\exp(-TD)$. In case D is locally nilpotent, the map $\exp(D) : A \longrightarrow A$ is well-defined and again an automorphism (with inverse $\exp(-D)$). In case D is locally finite, one cannot always define the exponential

[1] To save space we have to refer to [6] page 185 and beyond, or the review [1]

map. For one, the field k must satisfy "$a \in k$ then $\sum_{i=0}^{\infty} \frac{a}{i!} \in k$". We will only take exponents of locally finite derivations in case $k = \mathbb{C}$.

We define the derivation δ on $\mathbb{C}[X, Y, Z]$ and the polynomial $\Delta \in \mathbb{C}[X, Y, Z]$ by $\delta := -2Y\partial_X + Z\partial_Y$, and $\Delta := XZ + Y^2$. $\Delta\delta$ will be the Nagata derivation, and N will denote the Nagata automorphism:

$$N = \exp(\Delta\delta) = (X - 2Y\Delta - Z\Delta^2, Y + Z\Delta, Z).$$

If $\lambda \in \mathbb{C}$, we denote N^λ the following automorphism of $\mathbb{C}[X, Y, Z]$:

$$N^\lambda := \exp(\lambda\Delta\delta) = (X - \lambda 2Y\Delta - \frac{1}{2}\lambda^2 Z\Delta^2, Y + \lambda Z\Delta, Z).$$

.

Note that one can also use this formula to define N^λ as an automorphism of $k[X, Y, Z]$ for any field of characteristic $\mathrm{char}(k) \neq 2$ and any $\lambda \in k$.

1.3 A basic result

Lemma 1.1. *Let $D \in \mathrm{LND}(\mathbb{C}^{[n]})$, and $p \in \mathbb{C}^{[n]}$, $p \neq 0$. If $\exp(D)(p) = \lambda p$, then $\lambda = 1$, and $D(p) = 0$.*

Proof. Let $q \in \mathbb{N}$ such that $D^q(p) \neq 0, D^{q+1}(p) = 0$. Then $D^q(p) = D^q(\exp(D))(p) = D^q(\lambda p) = \lambda D^q(p)$ hence $\lambda = 1$. Assume $q \geq 1$. Now $0 = D^{q-1}(0) = D^{q-1}(\exp(D)(p) - p) = D^{q-1}(\sum_{i=1}^{q}(i!)^{-1}D^i(p)) = D^q(p)$. Contradiction, hence $q = 0$. $\square$

2 Shifted linearizability

2.1 Definition

We will define $F \in \mathrm{GA}_n(\mathbb{C})$ to be *shifted linearizable* if there exists a linear map $L \in \mathrm{GL}_n(\mathbb{C})$ such that LF is linearizable, i.e. exist $G \in \mathrm{GA}_n(\mathbb{C})$ and $L' \in \mathrm{GL}_n(\mathbb{C})$ such that $G^{-1}LFG = L'$.

A special case is if sF is linearizable, where $s \in \mathbb{C}^*$. In this case $L = sI$.

2.2 Noncommuting derivations forming a Lie algebra

Well-known is that any two-dimensional Lie algebra over $\mathbb{C}$ which is non-commutative is essentially the Lie algebra $\mathbb{C}X + \mathbb{C}Y$ where $[X, Y] = X$. This Lie algebra turns up in this section as the sub Lie algebra of $\mathrm{DER}_n(\mathbb{C})$ generated by two derivations D, E satisfying $[E, D] = D$.

Lemma 2.1. *Let D, E be derivations, $E \in \mathrm{LFD}_n(\mathbb{C})$, such that $[E, D] = \alpha D$ where $\alpha \in \mathbb{C}$. Then*

$$\exp(\beta E)D = e^{\alpha\beta}D\exp(\beta E)$$

for any $\beta \in \mathbb{C}$.

The assumption $E \in \mathrm{LFD}_n(\mathbb{C})$ is only here to make sure that $\exp(\beta E)$ is well-defined. However, if one interprets β as a variable in the ring $\mathbb{C}^{[n]}[[\beta]]$, this assumption is not necessary.

Proof. One can compute this directly, but easier is to use the well-known formulae

$$\exp(A)B\exp(-A) = \exp([A,-]) \circ B$$

where A, B are elements of a Lie algebra. In this case, conjugating D by $\exp(\beta E)$ yields

$$(\exp[\beta E,-])\circ D = I+\beta[E,D]+\frac{\beta^2}{2!}[E,[E,D]]+\ldots = D+\beta\alpha D+\frac{(\beta\alpha)^2}{2!}D+\ldots = e^{\beta\alpha}D.$$

This concludes the proof. □

Corollary 2.2. *Let $D, E \in \mathrm{LFD}_n(\mathbb{C})$ and suppose $[D,E] = \alpha D$ where $\alpha \in \mathbb{C}$. Then for any $\beta, \lambda \in \mathbb{C}$ we have*

$$\exp(\beta E)\exp(\lambda D) = \exp(e^{\alpha\beta}\lambda D)\exp(\beta E).$$

In particular, if $\alpha\beta \in 2\pi i\mathbb{Z}$ then $\exp(\beta E)$ and $\exp(\lambda D)$ commute for each $\lambda \in \mathbb{C}$.

Proof. Follows from lemma 2.1, which one can use to show that

$$\exp(\beta E)D^i = (e^{\alpha\beta})^i D^i \exp(\beta E).$$

□

Corollary 2.3. *Let $D, E \in \mathrm{LFD}_n(\mathbb{C})$ and suppose $[D,E] = \alpha D$ where $\alpha \in \mathbb{C}$. Then for any $\beta, \lambda \in \mathbb{C}$, $\exp(\beta E)\exp(\lambda D)$ is conjugate to $\exp(\beta E)$ as long as $\alpha\beta \notin 2\pi i\mathbb{Z}$. In particular,*

$$\exp(-\mu D)(\exp(\beta E)\exp(\lambda D))\exp(\mu D) = \exp(\beta E)$$

where $\mu = \lambda(e^{-\alpha\beta} - 1)^{-1}$.

Proof. ¿From corollary 2.2, we replace λ by $-e^{-\alpha\beta}\mu$ to get

$$\exp(\beta E)\exp(-e^{-\alpha\beta}\mu D) = \exp(-\mu D)\exp(\beta E).$$

This means that

$$\exp(-\mu D)(\exp(\beta E)\exp(\lambda D))\exp(\mu D) = \exp(\beta E)\exp((-e^{-\alpha\beta}\mu + \lambda + \mu)D).$$

Setting $-e^{-\alpha\beta}\mu + \lambda + \mu = 0$ yields $\mu = \lambda(e^{-\alpha\beta} - 1)^{-1}$. □

2.3 Linearizing exponents of monomial homogeneous derivations

As an application of the previous section we will show how to shift-linearize exponents of monomial homogeneous derivations.

A grading deg on $\mathbb{C}^{[n]}$ is called *monomial* if each monomial (or equivalently, each variable X_i) is homogeneous. It is the typical grading one puts on $\mathbb{C}^{[n]}$: one assigns weights to the variables X_i. In fact, let us state

$$w_i := \deg(X_i)$$

for this article. A homogeneous derivation is a derivation that sends homogeneous elements to homogeneous elements – in this article, homogeneous w.r.t. some *monomial* grading. It is not too difficult to check that there exists a unique k such that a homogeneous element of degree d is sent to a homogeneous element of degree $d+k$ or to the zero element. We say that D is *homogeneous of degree* k.

(Above, we did not specify in which set w_i, d, k are. Typical is to have them in $\mathbb{N}, \mathbb{Z}$, or even $\mathbb{R}$, and that is what we think of in this article. It is however possible to choose a grading which takes values in a group, i.e. a group grading. The above explanation makes sense for this.)

For this section, define the derivation associated to deg as $E := E_{\deg} := \sum_{i=1}^{n} w_i X_i \partial_{X_i}$. ($E$ stands for Euler derivation.) The goal of this section is to prove the following theorem:

Theorem 2.4. *If $D \in \mathrm{LFD}_n(\mathbb{C})$ is homogeneous of degree $k \neq 0$ w.r.t. a monomial grading, then* $\exp(D)$ *is shifted linearizable.*

Proof. Follows immediately from corollary 2.3 and lemma 2.5 below, and the observation that $\exp(E)$ is a linear map: the diagonal map $(e^{w_1}X_1, \ldots, e^{w_n}X_n)$. ☐

Lemma 2.5. *Let D be a homogeneous derivation of degree k with respect to a monomial grading* deg*. Then $[E_{\deg}, D] = kD$. In particular, if $k = 0$, then D and $E_{\deg}$ commute.*

Proof. Let $M := X_1^{v_1} \cdots X_n^{v_n}$ ($v_i \in \mathbb{N}$) be an arbitrary monomial of degree d. Then $\deg(D(M)) = d + k$, and $E(M) = \sum_{i=1}^{n} v_i w_i M = dM$. Similarly $E(DM) = (d + k)D(M)$. Now one can see that

$$\begin{aligned} [E, D](M) = E(D(M)) - D(E(M)) = & \ (d+k)DM - D(dM) \\ = & \ kD(M). \end{aligned}$$

Thus, $[E, D] = kD$. ☐

3 When is Nagata shifted linearizable?

3.1 Using Nagata's homogeneousness

For the rest of this section, $D := \Delta\delta$ will be Nagata's derivation. The Nagata derivation D is homogeneous to several monomial gradings. The set of monomial gradings form a vector space (for if $\deg_1, \deg_2$ are the associated degree functions, then $\deg_1 + \deg_2$ and $c \deg_1$ where $c \in \mathbb{C}$ are degree functions associated to a grading too).

Let us explain how we find all homogeneous derivations for the Nagata derivation. More details on such procedure one can find in [18] and pages 228-234 of [6], where it is explained how to do this to prove that Robert's derivation is a counterexample to Hilbert's 14th problem. First, notice that the variables X, Y, Z are homogeneous, lets say of degree s, t, u respectively. These values determine the degree function deg completely. Now we need to satisfy the following two requirements:
(1) $D(X), D(Y), D(Z)$ all are homogeneous,
(2) $\deg(D(X)) - \deg(X) = \deg(D(Y)) - \deg(Y)$. (This condition comes from the fact that there should be a constant d (which is the degree of D) for which we have: any homogeneous H is homogeneous of degree n, then $D(H)$ is of degree $n + d$ or $D(H) = 0$.)

¿From $D(X), D(Y)$ homogeneous we derive that $\Delta = XZ + Y^2$ is homogeneous, and thus $s+u = 2t$. Now (1) is satisfied. From (2) we get that $s-(t+2t) = t-(u+2t)$ which yields the exact same equation $s + u = 2t$. Thus $[\deg(X), \deg(Y), \deg(Z)] = [s, t, 2t - s]$ and the derivation is of degree $3t - s$. The degree function is associated with the (semisimple) derivation $E := sX\partial_X + tY\partial_Y + (2t - s)Z\partial_Z$ and the diagonal linear map $\exp(E) := (e^s X, e^t Y, e^{2t-s} Z)$.

Thus, for the Nagata derivation, the set of gradings for which it is homogeneous, is two dimensional. A possible basis is $\{\deg_1, \deg_2\}$ where

$$\deg_1((X, Y, Z)) = (1, 0, -1),$$
$$\deg_2((X, Y, Z)) = (0, 1, 2).$$

The degree function $\deg_1$ corresponds to the (semisimple) derivation $E_1 := X\partial_X - Z\partial_Z$, where $\deg_2$ corresponds to $E_2 := Y\partial_Y + 2Z\partial_Z$. Any degree function $\deg = s\deg_1 + t\deg_2$ ($s, t \in \mathbb{C}$) which is a linear combination of $\deg_1, \deg_2$ corresponds to $E := sE_1 + tE_2$. The linear map corresponding to the linear combination $sE_1 + tE_2$ is $L_{s,t} := (e^s X, e^t Y, e^{2t-s} Z)$. The set of these maps is exactly the set

$$\mathcal{L} := \{(aX, bY, cZ) \mid ac = b^2, abc \neq 0\}.$$

Thus we have proven the following lemma:

Lemma 3.1. *D is of degree 0 with respect to* $\deg = s\deg_1 + t\deg_2$ *if and only if* $s = 3t$.

Definition 3.2. Let us define $L_b := (b^3X, bY, b^{-1}Z)$, and $\mathcal{L}_0$ as the set $\{L_b \mid b \neq 0\}$. Note that
$$\mathcal{L}_0 := \{(aX, bY, cZ) \mid ac = b^2, bc = 1\}.$$

3.2 Explicit formulae for shifted linearizableness of the Nagata map

One can use corollary 2.3, theorem 2.4, and results of the previous section, to immediately get formulas for many linear maps $L \in \mathcal{L}$ which satisfy LN is linearizable. However, let us give the following formulas, which are slightly more elegant, and can be easily checked directly. Moreover, they work for any field k of characteristic $\mathrm{char}(k) \neq 2$ (see subsection 1.2). To be clear, for this section, we are working over a field k satisfying $\mathrm{char}(k) \neq 2$. Write $L_{(a,b,c)} := (aX, bY, cZ)$ where $ac = b^2$. The following formulas can be easily checked:

- $L_{(a,b,c)}^{-1} D L_{(a,b,c)} = (bc)^{-1}D$, which implies
- $L_{(a,b,c)}^{-1} \exp(\lambda D) L_{(a,b,c)} = \exp(b^{-1}c^{-1}\lambda D)$, which implies
- $N^{\lambda} L_{(a,b,c)} = L_{(a,b,c)} N^{b^{-1}c^{-1}\lambda}$.

Using the latter equation, the following is easy:

Theorem 3.3. *Let $a, b, c \in k^*$, $ac = b^2$, and $bc \neq 1$. Then $L_{(a,b,c)}N^{\lambda}$ is conjugate to $L_{(a,b,c)}$. In particular, choosing $\mu = bc\lambda(1-bc)^{-1}$, we have*
$$N^{-\mu}(L_{(a,b,c)}N^{\lambda})N^{\mu} = L_{(a,b,c)}.$$

The particular case that L is a multiple of the identity, gives the formulae for $s \in k^*$:
$$N^{-\frac{s^2\lambda}{1-s^2}}(sN^{\lambda})N^{\frac{s^2\lambda}{1-s^2}} = sI.$$
This gives the formula for $s = 2, \lambda = 1$ from the introduction. In the same introduction it was announced that we can linearize for any $s \neq 1, -1$, which indeed follows from this.

Remark 3.4. Maps $L_{(a,b,c)}N$ as in theorem 3.3, are non-tame (provided $\mathrm{char}(k) = 0$) but linearizable (and in particular, tamizable).

3.3 The non-linearizable case

We will now consider what happens if the grading of the previous section is such that D is homogeneous of degree 0. By lemma 2.5 this means that E commutes with D, and hence also $\exp(E)$ commutes with $\exp(D)$. By lemma 3.1 and definition 3.2, we can say $\exp(E) \in \mathcal{L}_0$, i.e. $\exp(E) = L_b = (b^3X, bY, b^{-1}Z)$ for some $b \in \mathbb{C}^*$. Now there are several ways of showing that L_bN^{λ} is not linearizable, we will use invariants.

Definition 3.5. Let $\varphi \in \mathrm{GA}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$. Then $\mathcal{E}_\mu(\varphi) := \{p \in \mathbb{C}[X_1, \ldots, X_n] \mid \varphi(p) = \mu p\}$ is defined as the eigenspace of φ with respect to μ.

If $L_b N$ is linearizable, it will be linearizable to L_b (as the linear part is equal to L_b). We will show that $\mathcal{E}_1(L_b N^\lambda)$ and $\mathcal{E}_1(L_b)$ are so different that they contradict the following property:

Lemma 3.6. *If $\varphi, \tilde{\varphi} \in \mathrm{GA}_n(\mathbb{C})$ are conjugate (i.e. there exists $\sigma \in \mathrm{GA}_n(\mathbb{C})$ such that $\tilde{\varphi} = \sigma^{-1}\varphi\sigma$) then $\mathcal{E}_\mu(\varphi)$ and $\mathcal{E}_\mu(\tilde{\varphi})$ are isomorphic (in fact, $\mathcal{E}_\mu(\tilde{\varphi}) = \sigma^{-1}(\mathcal{E}_\mu(\varphi))$).*

Proof.

$$\begin{aligned} p \in \mathcal{E}_\mu(\varphi) &\iff \\ \varphi(p) = \mu p &\iff \\ \varphi\sigma\sigma^{-1}(p) = \mu p &\iff \\ \sigma^{-1}\varphi\sigma\sigma^{-1}(p) = \mu\sigma^{-1}(p) &\iff \\ \sigma^{-1}(p) \in \mathcal{E}_\mu(\sigma^{-1}\varphi\sigma). & \end{aligned}$$

□

Lemma 3.7. *Let $b \in \mathbb{C}^*$ be no root of unity, $\lambda \in \mathbb{C}^*$. Then $L_b N^\lambda(p) = p$ for some $p \in \mathbb{C}[X, Y, Z]$ if and only if $p \in \mathbb{C}[Z^2\Delta]$.*

Corollary 3.8. *$L_b N^\lambda$ is not linearizable for any $b, \lambda \in \mathbb{C}^*$.*

Proof of lemma 3.7. Give weights $w(X) = 3, w(Y) = 1, w(Z) = -1$ making $A := \mathbb{C}[X, Y, Z]$ into a graded algebra $\oplus_{n\in\mathbb{Z}} A_n$. D and L_b are homogeneous: $L_b(A_n) = A_n$ and $D(A_n) \subseteq A_n$. Because of the latter, N^λ is homogeneous too: $N^\lambda(A_n) \subseteq A_n$ (actually "$=$" since it is an automorphism). Hence $L_b N^\lambda(A_n) = A_n$. For L_b we have $L_b(p) = b^n p$ if $p \in A_n$.

We want to find all p such that $L_b N^\lambda(p) = p$. It suffices to classify all such p which are homogeneous. Let $n = \deg(p)$. It now must hold that $N^\lambda(p) = L_b^{-1}(p) = b^{-n}p$. Because of lemma 1.1, we have $b^{-n} = 1$ and $p \in \ker \Delta\delta$. Hence, since b is no root of unity we get $n = 0$, and so $p \in \ker \Delta\delta \cap A_0 = \mathbb{C}[\Delta, Z] \cap A_0$. Since $\Delta \in A_2$ and $Z \in A_{-1}$, we get that $p \in \mathbb{C}[\Delta Z^2]$. It is easy to check that such p indeed satisfy $L_b N^\lambda(p) = p$. □

Proof of corollary 3.8. Assume $L_b N^\lambda$ is linearizable. We split the proof in two cases:
Let $b^m = 1$ for some $m \in \mathbb{N}^*$: Thus there exists some $\varphi \in \mathrm{GA}_3(\mathbb{C})$ such that $\varphi^{-1} L_b N^\lambda \varphi = L_b$. Thus $I = (L_b)^m = (\varphi^{-1} L_b N^\lambda \varphi)^m = \varphi^{-1} N^{m\lambda} \varphi$. Thus $N^{m\lambda}$ must be the identity, which implies that $m = 0$, contradiction.
b is no root of unity: By lemma 3.7 $\mathcal{E}_1(L_b N^\lambda)$ is isomorphic to $\mathbb{C}[\Delta Z^2]$. By lemma 3.6, we must have that $E_1(L_b N^\lambda)$ is isomorphic to $E_1(L_b)$. However, their transcendence degrees differ. □

4 Generators of $\mathrm{GA}_n(\mathbb{C})$ and conjectures

4.1 Tamizable automorphisms

The following definition and the problems 1 and 2 were given to us by A. Dubouloz.

Definition 4.1. A polynomial automorphism φ is called *tamizable* if there exists a polynomial automorphism σ such that $\sigma^{-1}\varphi\sigma$ is tame (in analog to linearizable and triangularizable).

Now let us repeat the conjectures from the introduction:

Problem 1. *Is N tamizable? (Is every automorphism of $\mathbb{C}^{[3]}$ tamizable?)*

Problem 2. *Is N tamizable by conjugation of an element of $\mathrm{GA}_2(\mathbb{C}[Z])$?*

Connected to this, we also mention the following problem, which we took from [9, p.120]:

Problem 3. *Every tame $\mathcal{G}_a$-action on $\mathbb{C}^3$ is conjugate to a triangular action.*

Note that the problems 1 and 3 cannot both be true.

4.2 Known conjectures

Since the "tame generators conjecture" (which hardly anyone believed because of the automorphism N) was disproved by Umirbaev-Shestakov in [22, 23] (and also before this feat was accomplished), several new conjectures have been made of "understandable" sets which could generate all of $\mathrm{GA}_n(\mathbb{C})$ for any n. We will mention several of them.

Conjecture 1. *Let k be a field of characteristic zero. Then $\mathrm{GA}_n(k) = \mathrm{GLND}_n(k)$, which is defined as $< e^{\mathrm{LND}_n(k)}, \mathrm{GL}_n(k) >$.*

Conjecture 2. $\mathrm{GA}_n(\mathbb{C}) = \mathrm{GLFD}_n(\mathbb{C})$*, which is defined as $< e^{\mathrm{LFD}_n(\mathbb{C})} >$.*

For $k = \mathbb{C}$, conjecture 2 is different than conjecture 1, as $\mathrm{GLND}_n(\mathbb{C}) \subseteq \mathrm{GLFD}_n(\mathbb{C})$ but it is not clear if all exponents of for example semisimple derivations are in the previous set. In fact, in our opinion, conjecture 2 is more natural, as it is obvious that $\mathrm{GLFD}_n(\mathbb{C})$ is a normal subgroup, but we do not know if the subgroup $\mathrm{GLND}_n(\mathbb{C})$ is normal.

Another one is the following, from [11] (where it is stated only for $k = \mathbb{C}$):

Conjecture 3. *Let k be a field. $\mathrm{GA}_n(k) = \mathrm{GLF}_n(k)$, where $\mathrm{GLF}_n(k)$ is the group generated by all locally finite polynomial automorphisms (which are polynomial automorphisms F for which the sequence $\{\deg(F^n)\}_{n\in\mathbb{N}}$ is bounded).*

The subgroup $\mathrm{GLF}_n(k)$ is normal for any field k: If $F \in \mathrm{GLF}_n(\mathbb{C})$, then the sequence $\{\deg(\varphi^{-1}F^m\varphi)\}_{m\in\mathbb{N}}$ is bounded by the bounded sequence $\{\deg(\varphi^{-1})\deg(F^m)\deg(\varphi)\}_{m\in\mathbb{N}}$.

Then there is the following conjecture, which to our knowledge originates from Shpilrain in [13, problem 2, p. 16] (there stated for $k = \mathbb{C}$):

Conjecture 4. $\mathrm{GA}_n(k) = \mathrm{GSHP}_n(k)$, *where* $\mathrm{GSHP}_n(k) =< \mathrm{GA}_{n-1}(k[X_n]), \mathrm{Aff}_n(k) >$, *interpreting* $\mathrm{GA}_{n-1}(k[X_n])$ *as the automorphisms in* $\mathrm{GA}_n(k)$ *which fix the last variable.*

He suggests immediately that this conjecture may have counterexamples in dimension 3 of the form $\exp(D)$ where $D \in \mathrm{LND}_3(\mathbb{C})$ which does not have coordinates in its kernel, as constructed by G. Freudenburg in [10]. Also, it is not clear if $\mathrm{GSHP}_n(k)$ is a normal subgroup of $\mathrm{GA}_3(k)$.

4.3 The group $\mathrm{GLIN}_n(k)$

Let us denote by $\mathrm{Lin}_n(k)$ the set of linearizable polynomial automorphisms. We define $\mathrm{GLIN}_n(k) :=< \mathrm{Lin}_n(k) >$ as the group generated by the linearizable automorphisms. This is by construction the smallest normal subgroup of $\mathrm{GA}_n(k)$ containing $\mathrm{GL}_n(k)$.

Lemma 4.2. *If* $\mathrm{char}(k) \neq 2$, *then* $\mathrm{GLIN}_n(k)$ *contains* $\mathrm{T}_n(k)$.

Proof. It suffices to show the lemma for an elementary map $E_f := (X_1+f, X_2, \ldots, X_n)$ where $f \in k[X_2, \ldots, X_n]$. Define $L := (2X_1, X_2, \ldots, X_n)$ which is in $\mathrm{GL}_n(k)$ as $\mathrm{char}(k) \neq 2$. The result follows since $E_f = L^{-1}(E_{-2f}LE_{2f})$. □

Remark 4.3. The first author will show in a future preprint that $(X + Y^3, Y) \in \mathrm{T}_2(\mathbb{F}_2)$ is not in $\mathrm{GLIN}_2(\mathbb{F}_2)$.

Corollary 4.4. *If* $\mathrm{char}(k) \neq 2$, *then* $\mathrm{GLIN}_n(k)$ *is the smallest normal subgroup of* $\mathrm{GA}_n(k)$ *containing* $\mathrm{T}_n(k)$.

In light of this lemma, *and* the result of theorem 3.3 (being $N \in \mathrm{GLIN}_n(\mathbb{C})$), it is natural to pose the following (as far as we know, new) conjecture:

Conjecture 5. $\mathrm{GLIN}_n(k) = \mathrm{GA}_n(k)$ *(if* $\mathrm{char}(k) \neq 2$*).*

For $\mathrm{char}(k) = 2$, one might replace $\mathrm{GLIN}_n(k)$ by the smallest normal subgroup of $\mathrm{GA}_n(k)$ containing $T_n(k)$. We remark that for $k = \mathbb{C}$ we have the following chain of inclusions:

$$\begin{array}{ccccc} & & \mathrm{GLIN}_n(\mathbb{C}) & & \\ & \subsetneq & & \subseteq & \\ \mathrm{TA}_n(\mathbb{C}) & & & & \mathrm{GLFD}_n(\mathbb{C}) \subseteq \mathrm{GLF}_n(\mathbb{C}) \subseteq \mathrm{GA}_n(\mathbb{C}) \ . \\ & \subsetneq & & \subseteq & \\ & & \mathrm{GLND}_n(\mathbb{C}) & & \end{array}$$

Any inequality or equality in this chain would be very interesting. (The set $\mathrm{GSHP}_n(\mathbb{C})$ is sort of separate.) Remark that $\mathrm{GLFD}_n, \mathrm{GLF}_n$ and GLIN_n are all normal, only the latter two can be defined over any field.

Let us recall the following conjecture from [17, 19]:

Conjecture 6. *Let $F \in \mathrm{GA}_n(\mathbb{F}_q)$. If q is even and $q \neq 2$, then only half (the even ones) of the bijections of $(\mathbb{F}_q)^n \longrightarrow (\mathbb{F}_q)^n$ are given by maps in $\mathrm{GA}_n(\mathbb{F}_q)$.*

Here, we say that a bijection of $(\mathbb{F}_q)^n$is even, if it is even if seen as an element of the permutation group on q^n elements. In [17], theorem 2.3, it is concluded that the tame automorphisms over $\mathbb{F}_q$ give all bijections in case q is odd or $q = 2$, and only the even bijections in case $q = 4, 8, 16, \ldots$.

Remark 4.5. If the conjecture 6 would *not* be true for some $q = 2^m, m \geq 2$, this would give a *ridiculously simple counterexample* to the the (already rejected) "tame generators problem" for $\mathbb{F}_q$. Also, it will imply that conjecture 4 is not true, and the smallest normal subgroup of $\mathrm{GA}_n(k)$ containing $T_n(k)$ does not equals $\mathrm{GA}_n(k)$ (and en passant conjecture 5 is not true for $k = \mathbb{F}_q$).

The remark follows from the fact that any conjugate of an even bijection is again even, and from the fact that any $F = (F_1(X,Y,Z), F_2(X,Y,Z), Z) \in GA_2(\mathbb{F}_{2^m}[Z])$, $m \geq 2$, is even: fix $Z = a \in \mathbb{F}_{2^m}$, and the map $F_a := (F_1(X,Y,a), F_2(X,Y,a), a)$ is a tame map on $\mathbb{F}_{2^m}^2 \times \{a\}$ by Jung-van der Kulk-theorem (and hence even because of theorem 2.3 in [17]).

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