THE FINE STRUCTURE OF THE INTUITIONISTIC BOREL HIERARCHY

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Tous les géomètres seraient donc fins s’ils avaient la vue bonne

B. Pascal, Pensées, 21

Abstract. In intuitionistic analysis, a subset of a Polish space like $\mathbb{R}$ or $\mathcal{N}$ is called positively Borel if and only if it is an open subset of the space or a closed subset of the space or the result of forming either the countable union or the countable intersection of an infinite sequence of (earlier constructed) positively Borel subsets of the space. The operation of taking the complement is absent from this inductive definition, and, in fact, the complement of a positively Borel set is not always positively Borel itself (see Veldman, 2008a). The main result of Veldman (2008a) is that, assuming Brouwer’s Continuity Principle and an Axiom of Countable Choice, one may prove that the hierarchy formed by the positively Borel sets is genuinely growing: every level of the hierarchy contains sets that do not occur at any lower level. The purpose of the present paper is a different one: we want to explore the truly remarkable fine structure of the hierarchy. Brouwer’s Continuity Principle again is our main tool. A second axiom proposed by Brouwer, his Thesis on Bars is also used, but only incidentally.

§1. Introduction.

1.1. This is the second one in a series of papers on intuitionistic descriptive set theory. Our aim is to find out what becomes of the field of study opened up by É. Borel, H. Lebesgue, R. Baire, N. Lusin, A. Souslin, see Lusin (1930), Moschovakis (1980), and Kechris (1996), and others, if one tackles it from Brouwer’s intuitionistic point of view. As is explained in the introduction to Veldman (2008a), Brouwer was more radical than the French and Russian mathematicians who started classical descriptive set theory. They also had their doubts about some of Cantor’s and Zermelo’s assumptions, like Brouwer, but they never questioned classical logic. Brouwer however, in his search for a sensible treatment of the continuum, came to advocate a consistent constructive interpretation of the logical constants and the set-theoretical operations and he decided to reject the principle of the excluded middle as a valid principle of reasoning. In a mathematical context, where one considers infinite objects like subsets of the set of the natural numbers or infinite sequences of natural numbers, the principle of the excluded middle leads, as Brouwer explained, to absurd conclusions, that is, to statements that fail to be true when they are understood straightforwardly and constructively, and do not immediately make sense in a different way. Brouwer’s criticism of the logic of mathematical arguments went hand in hand with his suggestion to use some new axioms, in particular, his Continuity Principle and his
Thesis on bars. Once one comes to share Brouwer's view on how infinite mathematical objects and the continuum should be handled in thought and language, one may find these principles to be plausible starting points for our mathematical discourse.

In this series of papers, we follow Brouwer and we avoid the use of the principle of the excluded middle: the logic of our arguments will be intuitionistic logic. In addition, and unlike other constructivist mathematicians, we also use the axioms Brouwer suggested.

1.2. It is useful to remind the reader of three important theorems obtained in the earlier paper (Veldman, 2008a). We shall formulate them slightly differently than in Veldman (2008a) and first have to agree upon some notations and definitions.

We let \( \mathbb{N} \) denote the set of the natural numbers. \( \mathbb{N}^* \) is the set of all finite sequences of natural numbers. We let \( \langle \ldots \rangle \) be a fixed bijective mapping from \( \mathbb{N}^* \) onto \( \mathbb{N} \). Such a function is called a coding of the set of finite sequences of natural numbers: \( \langle a_0, a_1, \ldots, a_{k-1} \rangle \) is the code number of the finite sequence \( (a_0, a_1, \ldots, a_{k-1}) \). We assume that the empty sequence is coded by the number 0 and that for each finite sequence \( (a_0, a_1, \ldots, a_{k-1}) \), for every \( i < k \), the code number \( \langle a_0, a_1, \ldots, a_{k-1} \rangle \) is greater than \( a_i \). We let \( \text{length} \) be the function from \( \mathbb{N} \) to \( \mathbb{N} \) that associates to any natural number \( a \) the length of the finite sequence coded by \( a \).

We assume that there is a function \( a, i \mapsto a(i) \) from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \), such that, for every \( k \), for every \( a \), if \( \text{length}(a) = k \), then \( a = (a(0), a(1), \ldots, a(k-1)) \).

We let \( * \) denote concatenation: \( * \) is a function from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \) such that, for all \( m, n \), \( m * n \) is the code number of the finite sequence obtained by putting the sequence coded by \( n \) behind the sequence coded by \( m \).

For all \( m, n, m \) is an initial part of \( n \), notation: \( m \trianglelefteq n \), if and only if there exists \( p \) such that \( n = m * p \); and \( n \) is an immediate successor of \( m \) if and only if there exists \( p \) such that \( n = m * (p) \).

We define another function, called \( J \), from \( \mathbb{N} \times \mathbb{N} \) to \( \mathbb{N} \): for all \( m, n : J(m, n) := \langle m \rangle * n \). It is easy to see that \( J \) is a bijective mapping from \( \mathbb{N} \times \mathbb{N} \) onto \( \mathbb{N} \setminus \{0\} \).

We let \( K, L \) be the inverse functions of \( J \), that is, \( K \) and \( L \) are functions from \( \mathbb{N} \setminus \{0\} \) to \( \mathbb{N} \) and for each \( m, m \neq 0 : J(K(m), L(m)) = m \).

\( J \) is a nonsurjective pairing function on \( \mathbb{N} \).

We let Baire space \( \mathcal{N} \) be the set of all infinite sequences \( a = a(0), a(1), a(2), \ldots \) of natural numbers.

The intuitionistic mathematician sometimes calls this set the universal spread.

Let \( a, b \) belong to \( \mathcal{N} \). \( a \) is apart from \( b \), notation: \( a \nE b \), if and only if there exists \( n \) such that \( a(n) \neq b(n) \).

We define, for all \( a \), for all \( m, n, a^m(n) := a(J(m, n)) \). \( a^m \) is called the \( m \)-th subsequence of \( a \). We also define, for all \( a \), for all \( m, n, a^{m,n} := (a^m)^n \).

We define, given any \( a \) and any \( n, a(n) := \langle a(0), a(1), \ldots, a(n-1) \rangle \).

If confusion is unlikely to arise, we sometimes write \( \overline{a}n \) for \( \overline{a}(n) \).

We also define, given any \( a \) and any \( s, s \) is an initial part of \( a \), or: \( a \) passes through \( s \), or: \( s \) contains \( a \), if and only if, for some \( n, \overline{a}n = s \).

A subset \( X \) of \( \mathcal{N} \) is basic open if and only if either \( X \) is empty or there exists \( s \) such that \( X \) is the set of \( a \) in \( \mathcal{N} \) passing through \( s \). A subset \( X \) of \( \mathcal{N} \) is open if and only if \( X \) is a countable union of basic open sets. One may prove that a subset \( X \) of \( \mathcal{N} \) is open if and only if there exists \( b \) in \( \mathcal{N} \) such that, for every \( a \) in \( \mathcal{N} \), \( a \) belongs to \( X \) if and only if, for some \( n, b(\overline{a}n) \neq 0 \). A subset \( X \) of \( \mathcal{N} \) is closed if and only if there is an open subset \( Y \) of \( \mathcal{N} \) such that \( X \) is the set of all \( a \) in \( \mathcal{N} \) such that the assumption \( "a \) belongs to \( Y" \) leads to a contradiction. One may prove that a subset \( X \) of \( \mathcal{N} \) is closed if and only if there exists \( b \) in \( \mathcal{N} \) such that, for every \( a \) in \( \mathcal{N} \), \( a \) belongs to \( X \) if and only if, for all \( n, b(\overline{a}n) = 0 \).
Note that every closed subset of $\mathcal{N}$ is a countable intersection of open subsets of $\mathcal{N}$.

A subset of $\mathcal{N}$ will be called positively Borel if and only if it is obtained from open subsets of $\mathcal{N}$ by the repeated use of the operations of countable intersection and countable union.

An element $\gamma$ of $\mathcal{N}$ is said to be the code of a continuous function from $\mathcal{N}$ to $\mathcal{N}$ if and only if $\gamma(\langle \rangle) = 0$ and, for every $\alpha$ in $\mathcal{N}$, there exists $n$ such that $\gamma(\langle \alpha n \rangle) \neq 0$. For every $\gamma$ in $\mathcal{N}$, if $\gamma$ codes a continuous function from $\mathcal{N}$ to $\mathcal{N}$ and $\alpha$ belongs to $\mathcal{N}$, we let $\gamma | \alpha$ be the sequence $\beta$ in $\mathcal{N}$ such that, for each $n$, there exists $p$ with the property that $\gamma(\langle n \rangle * \alpha p) = \beta(n) + 1$, and, for each $q < p$, $\gamma(\langle n \rangle * \alpha q) = 0$.

Let $A, B$ be subsets of $\mathcal{N}$ and let $\gamma$ be an element of $\mathcal{N}$ coding a continuous function from $\mathcal{N}$ to $\mathcal{N}$. $\gamma$ is said to be the code of a function reducing $A$ to $B$ if and only if, for each $\alpha$, $\alpha$ belongs to $A$ if and only if $\gamma | \alpha$ belongs to $B$. Note that, if $\gamma$ reduces $A$ to $B$, then $\gamma$ translates every question about membership of $A$, ("does $\alpha$ belong to $A"?)", into an equivalent question about membership in $B$, ("does $\gamma | \alpha$ belong to $B"?"). $A$ is reducible to $B$, notation: $A \leq B$, if and only if there exists $\gamma$ in $\mathcal{N}$ coding a continuous function from $\mathcal{N}$ to $\mathcal{N}$ reducing $A$ to $B$.

This notion of reducibility resembles the notion of many-one reducibility in recursion theory and it plays a key role in the development of our subject. In classical descriptive set theory, it is often called Wadge-reducibility (see Kechris, 1996).

We let $E_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for some $n$, $\alpha(n) \neq 0$, and we let $A_1$ be the set of all $\alpha$ in $\mathcal{N}$ such that, for all $n$, $\alpha(n) = 0$. It is not difficult to see that, for all subsets $X$ of $\mathcal{N}$, $X$ is open if and only if $X$ reduces to $E_1$, and $X$ is closed if and only if $X$ reduces to $A_1$.

Every subset of $\mathcal{N}$ reducing to a positively Borel subset of $\mathcal{N}$ is positively Borel itself.

Let $(X_0, Y_0), (X_1, Y_1), (X_2, Y_2), \ldots$ be an infinite sequence of positively Borel sets. The sequence is called repetitive if and only if, for each $m$, there exists $n$ such that $n > m$ and $X_n = X_m$ and $Y_n = Y_m$.

We now introduce the class of the special complementary pairs of leading positively Borel sets, or, more briefly, special pairs:

(i) The pair $(E_1, A_1)$ is a special complementary pair of leading positively Borel sets.

(ii) For every infinite and repetitive sequence $(X_0, Y_0), (X_1, Y_1), (X_2, Y_2), \ldots$ of special pairs, the pair of sets $\{(\alpha \in \mathcal{N} \text{ for some } n, \alpha^n \in Y_n), \{\alpha \in \mathcal{N} \text{ for all } n, \alpha^n \in X_n\}\}$ also is a special pair.

(iii) Every special complementary pair of leading positively Borel sets is obtained from the pair $(E_1, A_1)$ by the repeated application of (ii).

Let $(X, Y)$ be a special complementary pair of leading positively Borel sets. The set $X$ is called the additive member of the pair and the set $Y$ is called the multiplicative member of the pair.

It is not difficult to prove that, for every special pair $(X, Y)$, every element of $X$ is apart from every element of $Y$.

A subset $Z$ of $\mathcal{N}$ is positively Borel if and only if there exists a special complementary pair of leading positively Borel sets $(X, Y)$ such that $Z$ reduces to $X$.

Let $\mathcal{K}$ be a class of positively Borel subsets of $\mathcal{N}$. $\mathcal{K}$ is called a canonical class of positively Borel sets if and only if there exists a special complementary pair of leading positively Borel sets $X, Y$, such that either $\mathcal{K}$ is the class of all subsets of $\mathcal{N}$ reducing to $X$, $(\mathcal{K}$ then is called a additive class or a sum class), or $\mathcal{K}$ is the class of all subsets of $\mathcal{N}$ reducing to $Y$, $(\mathcal{K}$ then is called a multiplicative class or a product class).
One may prove the following result without using one of the new axioms of intuitionistic analysis (see Veldman, 2008a, theorem 5.2):

**Theorem 1.1.** (Classical Borel Hierarchy Theorem) Let \((X, Y)\) be a special complementary pair of leading positively Borel sets. If either \(X\) reduces to \(Y\) or \(Y\) reduces to \(X\), then there exists \(\alpha\) in \(\mathbb{N}\) belonging neither to \(X\) nor to \(Y\).

We cannot prove, without making further assumptions, that the statement that, for some special pair \((X, Y)\), there exists \(\alpha\) not belonging to either \(X\) or \(Y\) leads to a contradiction. In fact, if we should assume Church’s Thesis in the form: “Every function \(a\) from \(\mathbb{N}\) to \(\mathbb{N}\) is given by an algorithm in the sense of Church or Turing”, we would find that there exists a special pair \(X, Y\) such that \(X\) reduces to \(Y\) and thus discover some \(\alpha\) belonging neither to \(X\) nor to \(Y\) (see Veldman, 2008a, subsections 0.9 and 5.6).

Brouwer’s Continuity Principle, on the other hand, enables us to prove that no canonical class exhausts the collection of all positively Borel subsets of \(\mathbb{N}\) (see Veldman, 2008a, theorems 7.9 and 7.10):

**Theorem 1.2.** (Intuitionistic Borel Hierarchy Theorem) Let \((X, Y)\) be a special complementary pair of leading positively Borel sets.

1. The set \(X\) positively fails to reduce to \(Y\), that is, for every \(\gamma\) in \(\mathbb{N}\) coding a continuous function from \(\mathbb{N}\) to \(\mathbb{N}\), if, for each \(\alpha\) in \(X\), \(\gamma\) \(\mid \alpha\) belongs to \(Y\), then there exists also \(\alpha\) in \(Y\) such that \(\gamma\) \(\mid \alpha\) belongs to \(Y\).
2. The set \(Y\) positively fails to reduce to \(X\), that is, for every \(\gamma\) in \(\mathbb{N}\) coding a continuous function from \(\mathbb{N}\) to \(\mathbb{N}\), if, for each \(\alpha\) in \(Y\), \(\gamma\) \(\mid \alpha\) belongs to \(X\), then there exists also \(\alpha\) in \(X\) such that \(\gamma\) \(\mid \alpha\) belongs to \(X\).

Let \(X, Y\) be subsets of \(\mathbb{N}\). We let \(D(X, Y)\) be the set of all \(\alpha\) in \(\mathbb{N}\) such that either \(\alpha^0\) belongs to \(X\) or \(\alpha^1\) belongs to \(Y\). We call the set \(D(X, Y)\) the disjunction of the sets \(X, Y\). The following result is a consequence of Veldman (2008a, lemma 8.8) see also Veldman (2003b):

**Theorem 1.3.** (The persistent difficulty of disjunction) Let \((X, Y)\) be a special complementary pair of leading positively Borel sets.

The set \(D(A_1, Y)\) does not reduce to the set \(Y\).

(The statement in Veldman (2008a) is slightly stronger).

Theorem 1.3 may be considered as a first important statement on the fine structure of the intuitionistic Borel Hierarchy. The simplest case of this theorem is obtained for \(X = E_1, Y = A_1\). We may conclude that the set \(D(A_1, A_1)\) does not reduce to the set \(A_1\), that is: there exist a subset of \(\mathbb{N}\) that is the union of two closed sets and fails to be a closed set itself. Theorems 3.4 and 8.8 in Veldman (2008a) both enable one to conclude that this set does not coincide with a countable union of open sets.

1.3. Once we agree to accept and use or, at least, to try out Brouwer’s Continuity Principle as an axiom we enter a new world and discover many facts for which there does not exist a classical counterpart (see Veldman, 2001). The most famous consequence is Brouwer’s theorem that every function from \(\mathbb{R}\) to \(\mathbb{R}\) is continuous. The principle plays a crucial role in the proofs of the Theorems 1.2 and 1.3 mentioned in Subsection 1.2.

An early observation in connection with the subject of this paper is the following. The union of the two closed sets \([0, 1]\) and \([1, 2]\), as a subset of the set \(\mathbb{R}\) of the real numbers, behaves like the set \(D^2(A_1)\) we considered in the previous Subsection 1.2 as a subset of
\( N \colon [0, 1] \cup [1, 2] \) is not a closed subset of \( \mathbb{R} \); it is not even a countable intersection of open subsets of \( \mathbb{R} \). In order to see this, it suffices to consider a real number \( x \) that is floating around 1 in the sense that we cannot prove either one of the statements \( x \leq 1, x \geq 1 \). A number like \( x \) must belong to every open set containing \([0, 1] \cup [1, 2]\) and, therefore, also to every countable intersection of open subsets of \( \mathbb{R} \) containing \([0, 1] \cup [1, 2]\) but we have no argument proving \( x \) to belong to \([0, 1] \cup [1, 2]\) itself.

Brouwer's Continuity Principle implies that \([0, 1] \cup [1, 2]\) is not closed in an even stronger sense: not only are we unable to prove that \([0, 1] \cup [1, 2]\) is a closed subset of \( \mathbb{R} \), but the assumption that \([0, 1] \cup [1, 2]\) be a closed subset of \( \mathbb{R} \) leads to a contradiction. The statement that \([0, 1] \cup [1, 2]\) be a closed subset of \( \mathbb{R} \) is equivalent to the statement that the set \([0, 2]\) really coincide with the set \([0, 1] \cup [1, 2]\). For this reason, we want to call the set \([0, 1] \cup [1, 2]\) a proper subset of the set \([0, 2]\). Note, however, that we are unable to indicate an element \( x \) of \( \mathbb{R} \) that does belong to \([0,2]\) but not to \([0, 1] \cup [1, 2]\), as, for every \( x \) in \( \mathbb{R} \), \( \neg \neg (x \leq 1 \lor x \geq 1) \).

The general definition is as follows. Let \( X \) be a (real) subset of \( \mathbb{R} \) and \( Y \) a (real) subset of \( X \).

\( Y \) is a proper subset of \( X \), or: \( X \) is a proper extension of \( Y \), if and only if the assumption that every element of \( X \) really coincides with an element of \( Y \) leads to a contradiction.

As we saw above, it may occur that \( Y \) is a proper subset of \( X \), while, at the same time, there is no element of \( X \) that does not belong to \( Y \).

We introduce the same notion for subsets of \( N \). Let \( X \) be a subset of \( N \) and \( Y \) a subset of \( X \).

\( Y \) is a proper subset of \( X \), or: \( X \) is a proper extension of \( Y \), if and only if the assumption that every element of \( X \) is an element of \( Y \) leads to a contradiction.

One may prove, using Brouwer's Continuity Principle, that the set \( D^2(A_1) \) is a proper subset of the set \( D^2(A_1) \) although, for every \( a \) in \( D^2(A_1) \), \( \neg \neg (a^0 = \emptyset \lor a^1 = \emptyset) \) (see Theorem 5.4(i) and (ii)).

The result that there exist unions of two closed sets that fail to be closed may be extended: as we shall see, in Theorem 5.6(iv), there are, in \( N \), unions of three closed sets different from every union of two closed sets, and unions of four closed sets different from every union of three closed sets, and so on. Such sets exist in \( \mathbb{R} \) as well as in \( N \).

(We restrict ourselves, in this paper, to the exemplary space \( N \). It is not very difficult to extend our results to other Polish spaces).

The above observations offer a first glimpse of the astonishing fine structure of the intuitionistic (positive) Borel hierarchy we want to study in this paper.

Apart from this introductory section, the paper consists of five sections. The results of the paper are contained in Sections 3, 4, 5, and 6. At the beginning of each section there is a short introduction giving some information on its contents.

In Section 3, we mainly study members of the class \( F_\sigma \), also called \( \Sigma_2^0 \), that is, the class consisting of countable unions of closed subsets of \( N \). Note that finite unions of closed sets and also countable sets belong to this class.

In particular, we study countable sets whose closure coincides with their double complement. From a classical point of view such sets are just countable and closed. In classical set theory, the subsets of \( N \) that are both countable and closed are known to form a hierarchy, the so-called Cantor-Bendixson-hierarchy. This hierarchy is closely connected to the operation of taking the Cantor-Bendixson-derivative of a given subset of \( N \). We find that this hierarchy exists intuitionistically as well as classically and that it admits of a new intuitionistic characterization, by means of the notion "perhaps". The notion "perhaps" has been mentioned and discussed earlier in Veldman (1995, 1999, 2003a,
Using this notion, we define, given some subset $X$ of $\mathcal{N}$, a collection of so-called perhapsive extensions of $X$. Every perhapsive extension $Y$ of $X$ has the property: $X \subseteq Y \subseteq X^{-c}$. (For each subset $X$ of $\mathcal{N}$, we let $X^{-c}$, the complement of $X$, be the set of all $a$ in $\mathcal{N}$ such that the assumption: “$a$ belongs to $X$” leads to a contradiction.)

In Section 4, we introduce the notion “Almost” that is related to the notion “Perhaps” from Section 3. For every subset $X$ of $\mathcal{N}$, the set $\text{Almost}(X)$ is defined as the union of all perhapsive extensions of $X$.

A subset $X$ of Baire space $\mathcal{N}$ is called located if one may decide, for every finite sequence $s = s(0), s(1), \ldots, s(n-1)$ of natural numbers, if there exists $a$ in $X$ such that, for ever $i < n$, $a(i) = s(i)$, or not. Subsets of Baire space $\mathcal{N}$ that are both closed and located are traditionally called spreads (see Subsection 2.3.2). Not every closed subset of $\mathcal{N}$ is a spread (see Veldman, 2006a, theorem 9.5). (An example of a closed subset of $\mathcal{N}$ that one cannot prove to be a spread is the set of all $a$ in $\mathcal{N}$ such that, for each $n$, there is no uninterrupted sequence of 99 nines in the first $n$ digits of the decimal expansion of $n$.)

In Section 4, we also extend some of our observations on countable sets in Section 3 to countable unions of spreads.

In Section 5, we study finite unions of closed sets and how they behave under the operation of intersection. We provide an algorithm by which one may decide, for any two members $X, Y$ of a large class of such sets, if $X$ reduces to $Y$ or not.

In Section 6, we prove that, given a sequence $X_0, X_1, \ldots$ of subsets of $\mathcal{N}$ strictly increasing in complexity, that is, for each $n$, $X_n$ reduces to $X_{n+1}$ but not conversely, there are various ways of finding a productive upper bound, that is, a subset $Y$ of $\mathcal{N}$ with the property that, for each $n$, $X_n$ reduces to $Y$ and such that $Y$ itself is the first element of a new sequence of subsets of $\mathcal{N}$ that strictly increases in complexity. The upper bounds that we consider have the additional property that, if each of the sets $X_n$ belongs to the class $\Sigma^0_2$, then so does the upper bound $Y$ and also every element of the new sequence of subsets of $\mathcal{N}$ we construct with $Y$ as a first element. We prove the existence of two more hierarchies within the class $\Sigma^0_2$, the so-called disjunctive and conjunctive Cantor-Bendixson-hierarchies. In order to do so, we have to go into the intuitionistic way of handling transfinite induction.

At the end of the section, we note that the results we obtained for the class $\Sigma^0_2$ extend to other classes of the intuitionistic Borel hierarchy.

In Section 2, we briefly describe the axioms of intuitionistic analysis. The reader who is already familiar with intuitionistic mathematics may skip this section.

The titles of the remaining sections are as follows.

1. The axioms of intuitionistic analysis.
2. Rediscovering, perhaps, the Cantor-Bendixson hierarchy.
3. Perhaps and Almost.
4. Finite unions of closed sets.
5. Forming limits and finding more hierarchies.

This paper formed part of a larger report, ultimately finding its origin in Veldman (1981), see also Veldman (1990), that has been refereed and revised a number of times. I thank the referees involved from the bottom of my heart for their kind and generous efforts. Many improvements have been made upon their suggestion. One of them also helped me to improve my rendering into English of the sentence “$a$ belongs to Perhaps($X, Y$)” in Subsection 3.16. I am now following his proposal. The referee of the latest version of the paper did not think my choice of the term “perhaps” a felicitous one, but I decided to
maintain it. He studied the paper very seriously and made many useful and encouraging comments. I am deeply grateful that he, like his predecessors, unselfishly spent so much time and energy on my work.

§2. The axioms of intuitionistic analysis.

2.1. We are contributing to intuitionistic analysis. In our statements and arguments, the logical constants have their constructive meaning and we follow the rules of intuitionistic logic. In particular, a disjunctive statement $A \lor B$ is considered proven only if either $A$ or $B$ is proven and a proof of an existential statement $\exists x \in V[A(x)]$ has to provide one with a particular element $x_0$ from the set $V$ and a proof of the corresponding statement $A(x_0)$.

In addition we are going to use a number of axioms, some of which do not hold upon a classical reading of the connectives and the quantifiers. In Veldman (2008a) we have made an attempt to explain why one might decide to accept these axioms. We now feel entitled to simply list them, with almost no comment.

2.2. We first mention four axioms of countable choice.

We use $m, n, \ldots$ as variables over $\mathbb{N}$, and $\alpha, \beta, \ldots$ as variables over $\mathcal{N}$. $\mathcal{N}$ is sometimes called Baire space. Cantor space $\mathcal{C}$ is the set of all $\alpha$ in $\mathcal{N}$ that assume no other values than 0,1.

2.2.1. First Axiom of Countable Choice: For every binary relation $R$ on $\mathbb{N}$, if for every $m$ there exists $n$ such that $mRn$, then there exists $\alpha$ such that, for every $m$, $mR(\alpha(m))$.

Observe that we cannot, like nonintuitionistic mathematicians, define $\alpha$ by saying: let $\alpha(m)$ be the least $n$ such that $mRn$. One may be unable to find the least such $n$, for instance, if one knows $0R1$ but cannot decide if $0R0$ or not.

2.2.2. Second Axiom of Countable Choice: For every binary relation $R \subseteq \mathbb{N} \times \mathcal{N}$, if for every $m$ there exists $\alpha$ such that $mRa$, then there exists $\alpha$ such that, for every $m$, $mR(\alpha^m)$.

2.3. The following axiom is classically false. It makes that intuitionistic analysis is not a subsystem of classical analysis.

2.3.1. Brouwer’s Continuity Principle: For every binary relation $R \subseteq \mathcal{N} \times \mathbb{N}$, if for every $\alpha$ there exists $m$ such that $\alpha Rm$, then for every $\alpha$ there exist $m, n$ such that for every $\beta$, if $\alpha \beta n = \beta n$, then $\beta Rm$.

Brouwer’s Continuity Principle, making its first appearance in Brouwer (1918), is a crucial assumption for many results in this paper.

2.3.2. Let $X$ be a subset of $\mathcal{N}$. We let the sequential closure, or, more simply, the closure of $X$, notation $\overline{X}$, be the set of all $\alpha$ in $\mathcal{N}$ such that for each $n$ there exists $\beta$ in $X$ passing through $\overline{\alpha}n$. Note that, by the Second Axiom of Countable Choice, for all $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $\overline{X}$ if and only if there exists $\beta$ in $\mathcal{N}$ such that, for each $n$, $\beta^m$ belongs to $X$ and $\overline{\alpha}n = \beta^m n$.

$X$ is sequentially closed if and only if $X$ coincides with its closure $\overline{X}$. If a subset $X$ of $\mathcal{N}$ is closed in the sense of Subsection 1.2, $X$ is a countable intersection of basic open sets and there exists $\beta$ in $\mathcal{C}$ such that, for every $\alpha$ in $\mathcal{N}$, $\alpha$ belongs to $X$ if and only if, for each $n$, $\beta(\overline{\alpha}n) = 1$. Every subset of $\mathcal{N}$ that is closed in the sense of Subsection 1.2 is sequentially closed, but the converse is not generally true.

$X$ is a spread if and only if $X$ is sequentially closed and, in addition, $X$ is a located subset of $\mathcal{N}$, that is, there exists $\gamma$ such that, for every natural number $s$, $s$ contains an element of $X$ if and only if $\gamma(s) = 1$. 
Every spread is subset of \( \mathcal{N} \) that is closed in the sense of Subsection 1.2, that is, a countable intersection of basic open sets, but the converse is not generally true.

Deviating from Brouwer’s usage, we also want to call the empty set a spread.

2.3.3. We let \( \text{Fun} \) be the set of all \( \gamma \) coding a continuous function from \( \mathcal{N} \) to \( \mathbb{N} \), that is such that, for every \( a \), there exists \( n \) such that \( \gamma (\overline{an}) \neq 0 \). For every \( \gamma \) in \( \text{Fun} \), every \( a \), we let \( \gamma (a) \) be the natural number \( p \) such that there exists \( n \) such that \( \gamma (\overline{an}) = p + 1 \) and, for every \( m < n \), \( \gamma (\overline{am}) = 0 \).

Observe that, if \( \gamma \) belongs to \( \text{Fun} \) and \( \gamma (0) = 0 \), then for each \( n \), \( \gamma^n \) belongs to \( \text{Fun} \), and \( \gamma \) codes a continuous function from \( \mathcal{N} \) to \( \mathcal{N} \). Recall that, in Subsection 1.2, we defined, for every \( \gamma \) in \( \text{Fun} \) such that \( \gamma (0) = 0 \), for every infinite sequence \( \alpha \), an infinite sequence \( \gamma |\alpha \) as follows: for each \( n \), \( (\gamma |\alpha) (n) := \gamma^n (\alpha) \).

2.3.4. In Veldman (2008a), the following theorem is proved.

2.3.5. **Theorem:** (Extension of Brouwer’s Continuity Principle to spreads)

Let \( X \) be a nonempty spread and \( R \) a subset of \( X \times \mathbb{N} \).

If for every \( a \) in \( X \) there exists \( m \) such that \( a Rm \), then for every \( a \) in \( X \) there exist \( m, n \) such that for every \( \beta \) in \( X \), if \( \overline{an} = \overline{bn} \), then \( \beta Rm \).

2.3.6. Let \( X \) be a spread. We let \( \text{Fun}_X^0 \) be the set of all \( \gamma \) such that, for every \( a \) in \( X \), there exists \( n \) such that \( \gamma (\overline{an}) \neq 0 \). For every \( \gamma \) in \( \text{Fun}_X^0 \), every \( a \) in \( X \), we let \( \gamma (a) \) be the natural number \( p \) such that there exists \( n \) such that \( \gamma (\overline{an}) = p + 1 \), and for every \( m < n \), \( \gamma (\overline{am}) = 0 \).

We let \( \text{Fun}_X^1 \) be the set of all \( \gamma \) such that \( \gamma (0) = 0 \) and, for each \( n \), \( \gamma^n \) belongs to \( \text{Fun}_X^0 \).

For every \( \gamma \) in \( \text{Fun}_X^1 \), for every \( a \) in \( X \), we define the element \( \gamma |\alpha \) of \( \mathcal{N} \) as follows: for each \( n \), \( (\gamma |\alpha) (n) := \gamma^n (\alpha) \).

If \( \gamma \) belongs to \( \text{Fun}_X^0 \) we say that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathbb{N} \).

If \( \gamma \) belongs to \( \text{Fun}_X^1 \) we say that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \).

In particular, an element of \( \text{Fun} \) will be called a function from \( \mathcal{N} \) to \( \mathbb{N} \), and an element \( \gamma \) of \( \text{Fun} \) such that \( \gamma (0) = 0 \) will be called a function from \( \mathcal{N} \) to \( \mathcal{N} \).

Suppose that \( Z \) is a subset of \( \mathcal{N} \) and \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \) in \( \mathcal{N} \), \( \gamma |\alpha \) belongs to \( Z \). We then say that \( \gamma \) is a function from \( \mathcal{N} \) to \( Z \).

2.4. The following axiom is a little bit stronger than Brouwer’s Continuity Principle.

2.4.1. **First Axiom of Continuous Choice.** For every binary relation \( R \subseteq \mathcal{N} \times \mathbb{N} \), if, for every \( a \), there exists \( m \) such that \( a Rm \), then there exists \( \gamma \) in \( \text{Fun} \) such that, for every \( a \), \( a R (\gamma (a)) \).

2.4.2. **Second Axiom of Continuous Choice:** For every binary relation \( R \subseteq \mathcal{N} \times \mathcal{N} \), if, for every \( \alpha \), there exists \( \beta \) such that \( a R \beta \), then there exists \( \gamma \) in \( \text{Fun} \) such that \( \gamma (0) = 0 \) and, for every \( a \), \( a R (\gamma |\alpha) \).

This axiom implies the two Axioms of Countable Choice and the First Axiom of Continuous Choice.

The Second Axiom of Continuous Choice causes the collapse of the projective hierarchy (see Veldman, 2006a, theorem 9.16).

In Veldman (2008a), the following theorem is proved.
2.4.3. **Theorem:** (Extension of the Axioms of Continuous Choice to spreads)
Let $X$ be a spread.

(i) For every binary relation $R \subseteq X \times \mathbb{N}$, if, for every $a$ in $X$, there exists $m$ such that $a R m$, then there exists a function $\gamma$ from $X$ to $\mathbb{N}$ such that, for every $a$ in $X$, $a R (\gamma(a))$.

(ii) For every binary relation $R \subseteq X \times \mathbb{N}$, if, for every $a$ in $X$ there exists $\beta$ such that $a R \beta$, then there exists a function from $X$ to $\mathbb{N}$ such that, for every $a$ in $X$, $a R (\gamma \mid a)$.

2.5. We need something like countable ordinals and introduce stumps. We have taken the word “stump” from Brouwer (1954) but are giving it a slightly different meaning. For each $n$, we let $n$ be the element of $\mathbb{N}$ with the constant value $n$.

2.5.1. The set $\text{Stp}$ of stumps is a subset of Baire space $\mathbb{N}$ and is defined as follows.

(i) $0$ is a stump. We sometimes call $0$ the empty stump.

(ii) For all $\beta$ in $\mathbb{N}$, if, for each $n$, $\beta^n$ is a stump, and $\beta(0) = 1$, then $\beta$ itself is a stump. We call the stumps $\beta^0, \beta^1, \ldots$ the immediate substumps of the stump $\beta$.

(iii) Clauses (i) and (ii) produce all stumps.

Note that for every stump $\beta$, if $\beta(0) = 0$, then $\beta = 0$, and, if $\beta(0) = 1$, then $\beta \neq 0$, so we may decide if $\beta$ is the empty stump or not.

For every stump $\beta$, we define the successor of $\beta$, notation: $\beta^+$ or $S(\beta)$, by: $(S(\beta))(0) = 1$ and for every $n$, $(S(\beta))^n = \beta$.

We define a sequence $0^*, 1^*, \ldots$ of stumps by induction, as follows. $0^* := 0$ and, for each $n$, $(n + 1)^* := S(n^*)$. Thus we obtain a natural embedding of the set $\mathbb{N}$ into the set $\text{Stp}$.

(We slightly changed the definition of the set $\text{Stp}$ used in Veldman (2008a), interchanging the roles of 0 and 1. We now follow the definition used in Veldman (2006b)).

2.5.2. First Principle of Induction on the set $\text{Stp}$ of stumps:

For every subset $P$ of the set $\text{Stp}$ of stumps, if the empty stump $0$ belongs to $P$, and every nonempty stump $\beta$ belongs to $P$ as soon as each one of its immediate substumps $\beta^0, \beta^1, \ldots$ belongs to $P$, then $P$ coincides with $\text{Stp}$.

2.5.3. For every $\beta$, for every $n$, we say that $n$ belongs to $\beta$ if and only if $\beta(n) = 1$.

Let $\beta$ be a stump. The set of all finite sequences of natural numbers whose code number belongs to $\beta$ is more like a “stump” in the sense given to this word by Brouwer. We mention four important properties of this set.

(i) We may decide, for every finite sequence of natural numbers, if its code number belongs to $\beta$ or not.

(ii) Every initial part of a number belonging to $\beta$ belongs to $\beta$.

(iii) For every $\gamma$ in $\mathbb{N}$, we may calculate $n$ such that $\gamma n$ does not belong to $\beta$.

(iv) For every $\delta$ in $\mathbb{N}$, if (1) every initial part of a number belonging to $\delta$ belongs to $\delta$, and (2) every number belonging to $\delta$ belongs to $\beta$ and (3) for all $n$, if $\delta(n) \neq 0$, then $\delta(n) = 1$, then $\delta$ itself is a stump.

These properties may be verified by induction on the set $\text{Stp}$ of stumps.
Observe that there is no finite sequence whose code number belongs to \( 0 \). This explains why \( 0 \) is sometimes called the *empty stump*.

As we observed in Subsection 2.5.1, we may decide, for every stump \( \beta \), if \( \beta = 0 \) or not.

2.5.4. From now on we use \( \sigma, \tau, \ldots \) as variables on the set \( \text{Stp} \). We define binary relations \( <, \leq \) on the set \( \text{Stp} \) of stumps as follows:

(i) for every stump \( \sigma \), \( 0 < \sigma \) and for no stump \( \sigma \), \( \sigma < 0 \), and

(ii) for all stumps \( \sigma, \tau \) such that \( \tau \neq 0 \), \( \tau \leq \sigma \) if and only if, for each \( n \), \( \tau^n < \sigma \), and \( \sigma < \tau \) if and only if, for some \( n \), \( \sigma \leq \tau^n \).

One may prove, by straightforward (transfinite) induction on the set of stumps that the relations \( <, \leq \) are transitive and that, for all stumps \( \sigma, \tau \), if \( \sigma < \tau \), then \( \sigma \leq \tau \). Another useful fact is that, for all stumps \( \sigma, \tau, \rho \), if \( \sigma \leq \tau \) and \( \tau < \rho \), then \( \sigma < \rho \).

In general, it is impossible, given stumps \( \sigma, \tau \) to decide if \( \sigma < \tau \) or not. An example making this clear is given in Veldman (2008a). The relation \( \leq \) also fails to be decidable on \( \text{Stp} \).

A subset \( P \) of \( \text{Stp} \) is called *hereditary* if and only if for every stump \( \sigma \), \( \sigma \) belongs to \( P \) if every \( t < \sigma \) belongs to \( P \).

2.5.5. *Second Principle of Induction on the set \( \text{Stp} \) of stumps:* Every hereditary subset of \( \text{Stp} \) coincides with \( \text{Stp} \).

The proof is straightforward.

Observe that this principle does not imply that every inhabited set \( P \) of stumps contains an element \( \sigma \) that, for all \( \tau \) in \( P \), \( \sigma \leq \tau \). Actually, it is not even true that every inhabited subset of \( \{0^*, 1^*\} \) has a least element.

Theorem 6.7 enunciates an important Principle of Induction on the set \( \text{Stp}^* \) of nonempty finite sequences of stumps. The proof of this theorem forms a rather long intermezzo in Section 6, that starts in Subsection 6.8 and is concluded in Subsection 6.18.

In Veldman (2004) some other principles of induction on stumps are explained and used.

2.6. We now consider the assumption that underlies the famous Bar Theorem. It will play a role in this paper when we come to discuss the notion *Almost* in Section 4 and also figures in various results about (strictly) analytic and co-analytic sets in Veldman (2006a, section 9).

2.6.1. A subset \( P \) of \( \mathbb{N} \) will be called a *bar in \( \mathcal{N} \)* if and only if for each \( a \) there exists \( n \) such that \( \overline{an} \) belongs to \( P \).

2.6.2. *Brouwer’s Thesis on Bars:*

For every subset \( P \) of \( \mathbb{N} \), if \( P \) is a bar in \( \mathcal{N} \), then there exists a stump \( \beta \) such that the set of all elements of \( P \) belonging to \( \beta \) is a bar in \( \mathcal{N} \).

Brouwer thought that his thesis could be seen to be true by reflection on the possible structure of a (canonical) proof of the fact “for every \( a \) there exists \( n \) such that \( P(\overline{an}) \)”. We shall not discuss his argument at this place.

The above formulation of Brouwer’s thesis does not literally occur in Brouwer’s writings. As was discovered by S. C. Kleene (see Kleene & Vesley, 1965), Brouwer used the fundamental assumption underlying his famous bar theorem incorrectly, and we believe the above formulation of his “Thesis”, a term we introduced because of its analogy to Church’s Thesis, comes close to what he really intended (see Veldman, 2006b).
2.6.3. The Fan Theorem is both the most famous and a rather weak consequence of Brouwer’s Thesis (see Veldman, 2008b).
A fan or finitary spread is a subset $F$ of Baire space $N$ such that there exists $\beta$ with the following two properties:

(i) for every $\alpha$, $\alpha$ belongs to $F$ if and only if, for each $n$, $\beta(\bar{\alpha}n) = 0$, and

(ii) for each $n$ such that $\beta(n) = 0$ there exists $m$ such that for all $k$, if $\beta(n \ast (k)) = 0$, then $k < m$.

Let $X$ be a subset of $N$ and let $P$ be a subset of $\mathbb{N}$. $P$ is called a bar in $X$ if for every $\alpha$ in $X$ there exists $n$ such that $\bar{\alpha}n$ belongs to $P$.

2.6.4. Unrestricted Fan Theorem:

Let $F \subseteq N$ be a fan. For every subset $P$ of $\mathbb{N}$, if $P$ is a bar in $F$, then some finite subset of $P$ is a bar in $F$.

Brouwer used the Fan Theorem for proving that every real function defined on $[0,1]$ is uniformly continuous on $[0,1]$ (see Brouwer, 1927).

Note that, in the formulation of the Unrestricted Fan Theorem, we do not require $P$ to be a decidable subset of $\mathbb{N}$, as one does in the usual (Restricted) Fan theorem.

More information on various formulations of the Fan Theorem may be found in Veldman (2005a).

The most important example of a fan is Cantor space $C$, the set of all $\alpha$ in $N$ that assume no other value than 0, 1. Using a weak form of the First Axiom of Countable Choice one may derive the (Unrestricted) Fan Theorem for arbitrary fans from the (Unrestricted) Fan Theorem for $C$ only (see Veldman, 2005a).

In this paper, the Fan Theorem is not used.

§3. Rediscovering, perhaps, the Cantor-Bendixson hierarchy. We introduce a class of enumerable subsets of $N$ with the property that their closure coincides with their double complement. The classical mathematician would call these sets countable and closed, but intuitionistically, only the finite sets in our class are closed subsets of $N$. We show how these sets, in four different ways, form a hierarchy, and we also prove that the resulting hierarchies more or less coincide. Each of the hierarchies is connected with a partial ordering of the class of sets we study. Let us call these partial orderings $\preceq_0$, $\preceq_1$, $\preceq_2$, and $\preceq_3$.

The first partial ordering, $\preceq_0$, is defined as follows: $X \preceq_0 Y$ if and only if $X$ embeds into $Y$, that is, there exists a continuous function from the sequential closure $X$ of $X$ to the sequential closure $Y$ of $Y$ that embeds $X$ one-to-one into $Y$.

The second partial ordering, $\preceq_1$, is the important relation $\preceq$ of reducibility, introduced in Subsection 1.2: $X \preceq_1 Y$ if and only if $X$ reduces to $Y$, that is, there exists a continuous function $\gamma$ from $N$ to $N$ such that, for every $\alpha$, $\alpha$ belongs to $X$ if and only if $\gamma(\alpha)$ belongs to $Y$.

In order to define the third and fourth partial ordering we need some auxiliary notions.

First, we consider Cantor’s operation of taking the derivative of a given subset of $N$. Iterating it, even into the transfinite, we obtain, for any given subset $X$ of $N$, the collection of the Cantor-Bendixson-derivatives of the set $X$.

Secondly, we introduce a binary operation, called Perhaps, on the class of subsets of $N$, and, use it to define, again by a transfinite iteration process, for any subset $X$ of $N$, the collection of the perhapsive extensions of $X$.

Both the Cantor-Bendixson-derivatives and the perhapsive extensions of a given set $X$ will be “numbered” by the intuitionistic equivalent of countable ordinals, that is, by stumps.
It now turns out that, for every set \( X \) from our class, (i) the empty set is a Cantor-Bendixson-derivative of \( X \), and (ii) the closure \( \overline{X} \) of \( X \) is a perhapsive extension of \( X \).

The third partial ordering, \( \leq_2 \), may be described, roughly, as follows: \( X \leq_2 Y \) if and only if the “first” stump such that the Cantor-Bendixson-derivative of \( X \) connected with this stump is the empty set, (one might call this stump: the Cantor-Bendixson-rank of \( X \)), comes earlier than the “first” stump such that the Cantor-Bendixson-derivative of \( Y \) connected with this stump is the empty set.

The fourth partial ordering, \( \leq_3 \), may be described, roughly, as follows: \( X \leq_3 Y \) if and only if the “first” stump such that the perhapsive extension of \( X \) connected with this stump coincides with \( X \), (one might call this stump: the perhapsive rank of \( X \)), comes earlier than the “first” stump such that the perhapsive extension of \( Y \) connected with this stump coincides with \( Y \).

We are using the words “first” and “earlier than” here in a loose sense. In the subsection itself we will be more precise.

In Veldman (2003a), results similar to the results obtained in this section are obtained for certain subsets of \( \mathbb{R} \).

### 3.1

For every \( s \) in \( \mathbb{N} \), every \( \alpha \), we let \( s \ast \alpha \) be the element of \( \mathcal{N} \) that we obtain by putting the infinite sequence \( \alpha \) behind the finite sequence (coded by) \( s \).

For every \( s \) in \( \mathbb{N} \), for every subset \( X \) of \( \mathbb{N} \) we let \( s \ast X \) be the set of all infinite sequences \( s \ast \alpha \), where \( \alpha \) belongs to \( X \). We now define, for every stump \( \sigma \), a subset \( CB_\sigma \) of \( \mathcal{N} \) by means of the following inductive definition:

1. \( CB_0 := \emptyset \)
2. For every nonempty stump \( \sigma \), \( CB_\sigma := \{0\} \cup \bigcup_{n \in \mathbb{N}} \overline{0n} \ast \{1\} \ast CB_\sigma^n \).

The letters \( CB \) have been chosen in honour of G. Cantor and I. Bendixson.

A subset \( X \) of \( \mathcal{N} \) will be called a Cantor-Bendixson-set if, for some stump \( \sigma \), \( X \) coincides with \( CB_\sigma \).

Observe that \( CB_0 = CB_0^* = \emptyset \), \( CB_1 = CB_1^* = \{0\} \) and \( CB_2 = CB_2^* = \{0\} \cup \{\overline{0n} \ast \{1\} \ast 0 | n \in \mathbb{N}\} \).

### 3.2

Let \( X \) be a subset of \( \mathcal{N} \). Recall, from Subsection 2.3.2, that the (sequential) closure of \( X \), notation \( \overline{X} \), is the set of all \( \alpha \) in \( \mathcal{N} \) such that for every \( n \) there exists an element of \( X \) passing through \( \overline{\alpha n} \). If we may decide, for every \( s \), if \( s \) contains an element of \( X \) or not, then \( \overline{X} \) is a closed subset of \( \mathcal{N} \) and a spread.

For all natural numbers \( s, t \) we define: \( s \) is incompatible with \( t \), or: \( s, t \) are disjoint, notation: \( s \perp t \), if and only if (we are considering \( s, t \) as codes of finite sequence of natural numbers), \( s \) is not an initial part of \( t \) and \( t \) is not an initial part of \( s \).

Cantor space \( C \) is the set of all \( \alpha \) in \( \mathcal{N} \) that assume no other values than 0, 1. \( CB_2^* \) is the set of all \( \alpha \) in \( C \) that assume the value 1 either not at all or exactly one time, that is, either, for each \( n \), \( \alpha(n) = 0 \) or, for some \( m \), \( \alpha(m) = 1 \); while, for each \( n \neq m \), \( \alpha(n) = 0 \). The sequential closure \( CB_2^* \) of \( CB_2 \) is the set of all \( \alpha \) in \( C \) that assume the value 1 at most one time, that is, for all \( m, n \), if \( \alpha(m) = \alpha(n) = 1 \), then \( m = n \).

One of the consequences of the next theorem is that the sets \( \overline{CB_2} \) and \( CB_2 \) do not coincide. It follows that the set \( CB_2 \) is not sequentially closed in the sense of Subsection 2.3.2 and also not closed in the sense of Subsection 1.2.

The reader should compare the second item of the next theorem with a result that we are to prove in Section 5, Theorem 5.4(i): the union of two closed sets, in general, is not a closed set itself. The second item of the next theorem points out that the union of two closed and, in a sense, clearly disjoint sets is closed.
Recall that, according to the definition in Subsection 2.6.3, a subset $F$ of Baire space $\mathcal{N}$ is a finitary spread or a fan if and only if there exists $\beta$ with the following two properties:

(i) for every $\alpha$, $\alpha$ belongs to $F$ if and only if, for each $n$, $\beta(\alpha n) = 0$, and
(ii) for each $n$ such that $\beta(n) = 0$ there exists $m$ such that, for all $k$, if $\beta(n \ast k) = 0$, then $k < m$.

Note that every spread that is a subset of Cantor space $\mathcal{C}$ is a finitary spread.

For each subset $X$ of $\mathcal{N}$, for each $s$ in $\mathbb{N}$, we let $X \cap s$ be the set of all $\alpha$ in $X$ passing through $s$.

3.3. Theorem:

(i) For every nonempty stump $\sigma$, for every $\alpha$ in $C B_\sigma$, one may decide: $\alpha = 0$ or $\alpha \neq 0$.
(ii) For all $s, t$, for all subsets $A, B$ of $\mathcal{N}$, if $s \vdash t$, then $(s \ast A) \cup (t \ast B)$ is a closed subset of $\mathcal{N}$, (respectively, a sequentially closed subset of $\mathcal{N}$) if and only if both $A$ and $B$ are closed subsets of $\mathcal{N}$, (respectively, sequentially closed subsets of $\mathcal{N}$).
(iii) For every stump $\sigma$, $C B_\sigma$ is a finitary spread.
(iv) For every stump $\sigma$, if $C B_\sigma$ is sequentially closed, then $C B_\sigma$ is a finite set, and, therefore, if $C B_\sigma$ is infinite, then $C B_\sigma$ is not sequentially closed.

Proof. We leave the proofs of (i) and (ii) to the reader.

(iii) We have to prove, that we may decide, for every stump $\sigma$, for every $s$, if $s$, (decoded into a finite sequence of natural numbers), contains an element of $C B_\sigma$ or not. The proof is straightforward, by induction on the set of stumps, and left to the reader.

(iv) We use induction on the set of stumps. There is nothing to prove if $\sigma$ is the empty stump. Assume that $\sigma$ is a nonempty stump and suppose that $C B_\sigma$ is sequentially closed, that is, $C B_\sigma$ coincides with $C B_\sigma$. By (i), we may decide, for every $\alpha$ in $C B_\sigma$, either $\alpha = 0$ or $\alpha \neq 0$. According to (iii), $C B_\sigma$ is a spread containing $0$. Applying Brouwer’s Continuity Principle we find $n$ such that either every $\alpha$ in $C B_\sigma$ passing through $\alpha n$ coincides with $0$, or every $\alpha$ in $C B_\sigma$ passing through $\alpha n$ is apart from $0$. As the latter alternative is absurd, we are left with the conclusion that every $\alpha$ in $C B_\sigma$ passing through $\alpha n$ coincides with $0$, and that the set $C B_\sigma$ coincides with $\{0\} \cup \bigcup_{i < n} \overline{\alpha i} \ast (1) \ast C B_{\sigma i}$. Using (ii) and the induction hypothesis, we conclude that, for each $i$ such that $i < n$, the set $C B_{\sigma i}$ is closed and finite, and thus that $C B_\sigma$ is a finite set. □

3.4. Let $X$ be a subset of $\mathcal{N}$. $X$ will be called enumerable if and only if there exists an enumeration of $X$, that is, an element $\alpha$ of $\mathcal{N}$ such that $X$ coincides with the set $\{\alpha^0, \alpha^1, \ldots\}$. According to this somewhat narrow definition, the empty set is not enumerable. Note that every enumerable subset of $\mathcal{N}$ belongs to the class $\Sigma^0_2$.

Let $X$ be a subset of $\mathcal{N}$. The complement of $X$, notation $X^\bot$, is the set of all $\alpha$ such that $\alpha$ does not belong to $X$, that is, the assumption that $\alpha$ does belong to $X$ leads to a contradiction.

By a well-known rule of intuitionistic logic, every subset $X$ of $\mathcal{N}$ is a subset of its double complement $X^{\bot\bot}$ but the converse is not generally true. One easily proves, however, that, for every subset $X$ of $\mathcal{N}$, $X^{\bot\bot}$ coincides with $X^\bot$.

Also note that, for all propositions $P, Q$, if $P \rightarrow Q$, then $\neg Q \rightarrow \neg P$ and $(\neg \neg P) \rightarrow (\neg \neg Q)$.
3.5. **Theorem:**

(i) For all \( s \), for every subset \( A \) of \( \mathcal{N} \), \( (s \ast A) \) coincides with \( s \ast (A \) coincides with \( A \).

(ii) For every nonempty stump \( \sigma \), \( C B_\sigma \) is an enumerable subset of \( \mathcal{N} \) and belongs to \( \Sigma^0_2 \).

(iii) For every \( \sigma \), the set \( \overline{C B_\sigma} \) coincides with the set \( (C B_\sigma) \).

**Proof.** We leave the proofs of (i) and (ii) to the reader.

As to (iii), note that, for every stump \( \sigma \), \((C B_\sigma) \) is a subset of \( \overline{C B_\sigma} \), as for every \( \alpha \) in \((C B_\sigma) \), for every \( n \), it is not true that \( \overline{\alpha n} \) does not contain a member of \( C B_\sigma \), and, therefore, by Theorem 3.3.(iii), \( \overline{\alpha n} \) contains a member of \( C B_\sigma \).

In order to prove that, for every stump \( \sigma \), the set \( \overline{C B_\sigma} \) is a subset of the set \( (C B_\sigma) \) we use induction on the set of stumps.

There is nothing to prove if \( \sigma \) is the empty stump.

Assume that \( \sigma \) is a nonempty stump and that, for each \( n \), the set \( \overline{C B_\sigma n} \) is a subset of the set \( (C B_\sigma n) \). Let \( \alpha \) belong to \( \overline{C B_\sigma} \). We distinguish two cases.

**First case:** \( \alpha = 0 \). Then \( \alpha \) belongs to \( C B_\sigma \).

**Second case:** \( \alpha \neq 0 \). Calculate \( n, \beta \) such that \( \alpha = \overline{\alpha n} \ast (1) \ast \beta \).

Observe that \( \beta \) belongs to \( \overline{C B_\sigma n} \) and therefore also to \( (C B_\sigma n) \). Using (i), we conclude that \( \alpha \) belongs to \( (C B_\sigma) \).

We thus see that if either \( \alpha = 0 \) or \( \alpha \neq 0 \), then \( \alpha \) belongs to \( (C B_\sigma) \). We recall that \( (C B_\sigma) \) is a subset of \((C B_\sigma) \). Therefore, \( (C B_\sigma) \) is a subset of \( (C B_\sigma) \). □

3.6. Note that, in classical, nonintuitionistic mathematics, every Cantor-Bendixson set is a closed subset of \( \mathcal{N} \).

Let \( X \) be a subset of \( \mathcal{N} \) and \( P \) a subset of \( \mathcal{N} \). As we agreed in Subsection 2.6.1, \( P \) is a bar in \( X \) if and only if every element of \( X \) has an initial part in \( P \).

The first item of the next theorem establishes that the closure of a Cantor-Bendixson set satisfies the conclusion of the Fan Theorem. Note that we prove this in an elementary way, by straightforward induction, without invoking either Brouwer’s Thesis itself, or its famous consequence, the Fan Theorem.

\( \Sigma^0_1 \) is the class of all open subsets of \( \mathcal{N} \), \( \Pi^0_1 \) is the class of all closed subsets of \( \mathcal{N} \), \( \Sigma^0_2 \) is the class of all subsets of \( \mathcal{N} \) such that there is an infinite sequence \( X_0, X_1, \ldots \) of elements of \( \Pi^0_1 \) with the property \( X = \bigcup_{n \in \mathbb{N}} X_n \) and \( \Pi^0_2 \) is the class of all subsets of \( \mathcal{N} \) such that there is an infinite sequence \( X_0, X_1, \ldots \) of elements of \( \Sigma^0_1 \) with the property \( X = \bigcap_{n \in \mathbb{N}} X_n \).

3.7. **Theorem:**

(i) For every stump \( \sigma \), for every subset \( P \) of \( \mathbb{N} \), if \( P \) is a bar in the countable set \( C B_\sigma \), then some finite subset of \( P \) is a bar in \( \overline{C B_\sigma} \).

(ii) For every stump \( \sigma \), for every open subset \( Y \) of \( \mathcal{N} \), if the set \( C B_\sigma \) is a subset of \( Y \), then its closure \( \overline{C B_\sigma} \) is a subset of \( Y \).

(iii) For every stump \( \sigma \), if the set \( C B_\sigma \) belongs to the class \( \Pi^0_2 \), then \( C B_\sigma \) is a finite set, and, therefore, if \( C B_\sigma \) is infinite, then \( C B_\sigma \) does not belong to \( \Pi^0_2 \).

**Proof.**

(i) We use induction on the set of stumps. The statement is trivially true if \( \sigma \) is the empty stump. Assume that \( \sigma \) is a nonempty stump and that the statement holds true for every set \( C B_\sigma n \). Also assume that \( P \) is a bar in \( C B_\sigma \). Calculate \( m \) such that \( \overline{\alpha m} \)
belongs to $P$. Note that, for each $j < m$, the set $Q_j$ consisting of all $t$ such that either $\overline{\sum_{j}} (1) * t$ belongs to $P$ or $t = t$ and, for some $i \leq j$, $\overline{\sum_{i}}$ belongs to $P$ is a bar in $CB_{\sigma}$. Using the induction hypothesis, find, for each $j < m$, a finite subset $R_j$ of $Q_j$ that is a bar in $CB_{\sigma}$. Note that the finite set $R := \overline{\sum_{m}} \cup \bigcup_{j < m} \overline{\sum_{j}} (1) * R_j$ is a bar in $CB_{\sigma}$ and that every element of $R$ has an initial part in $P$. For every $t$ in $R$, let $s_t$ be an element of $P$ that is an initial part of $t$. The set $T$ consisting of all $s_t$, where $t$ belongs to $R$, is a bar in $CB_{\sigma}$ and a finite subset of $P$.

(ii) Let $Y$ be an open subset of $\mathcal{N}$ and let $\sigma$ be a stump such that $CB_{\sigma}$ is a subset of $Y$. Let $P$ be the set of all $s$ in $\mathbb{N}$ such that every $\alpha$ passing through $s$ belongs to $Y$. Note that $P$ is a bar in $CB_{\sigma}$. Using (i), find a finite subset $Q$ of $P$ such that $Q$ is a bar in $CB_{\sigma}$. Clearly, $CB_{\sigma}$ is a subset of $Y$.

(iii) Assume that $\sigma$ is a stump such that $CB_{\sigma}$ belongs to $\Pi_2^0$. Determine a sequence $G_0, G_1, \ldots$ of open subsets of $\mathcal{N}$ such that $CB_{\sigma} = \bigcap_{n \in \mathbb{N}} G_n$. According to (ii), the closure $CB_{\sigma}$ of $CB_{\sigma}$ is a subset of $\bigcap_{n \in \mathbb{N}} G_n$, and, therefore, $CB_{\sigma} = CB_{\sigma}$ and $CB_{\sigma}$ is a closed subset of $\mathcal{N}$. Using Theorem 3.3(iv), we conclude that $CB_{\sigma}$ is a finite set.

3.8. Let $X$, $Y$ be spreads and let $\gamma$ be a function from $X$ to $Y$. $\gamma$ embeds $X$ into $Y$, or: $\gamma$ is an embedding of $X$ into $Y$ if and only if for all $\alpha, \beta$ in $X$, if $\alpha$ is apart from $\beta$, then $\gamma(\alpha)$ is apart from $\gamma(\beta)$. $X$ embeds into $Y$ if and only if some $\gamma$ embeds $X$ into $Y$.

Let $\beta$ belong to $\mathcal{N}$. $\beta$ is repetitive if and only if for each $m$ there exists $n$ such that $n > m$ and $\beta^n \beta^m$.

We now introduce the set $Hrs$ of the hereditarily repetitive stumps. $Hrs$ is a subset of $Stp$ and is given by the following inductive definition:

(i) $\emptyset$ is a hereditarily repetitive stump.
(ii) For all $\beta$, if, for each $n$, $\beta^n$ is a hereditarily repetitive stump, and $\beta$ is repetitive, and $\beta(0) = 1$, then $\beta$ itself is a hereditarily repetitive stump.
(iii) Clauses (i) and (ii) produce all hereditarily repetitive stumps.

Observe that, for each $n, n^*$ is a hereditarily repetitive stump.

One may prove that, for every stump $\sigma$, there exists a hereditarily repetitive stump $\tau$ such that both $\sigma \leq \tau$ and $\tau \leq \sigma$. In this sense, the restriction to hereditarily repetitive stumps does no harm. The reason we are introducing them is that we were able to develop some of our arguments for hereditarily repetitive stumps whereas we did not always see how to do so for stumps in general. We also used hereditarily repetitive stumps in our treatment of the Borel hierarchy theorem in Veldman (2008a).

3.9. Theorem:

(i) For all hereditarily repetitive stumps $\sigma$, for all $n$, $CB_{\sigma}$ embeds into $CB_{\sigma} \cap \overline{\sum_{n}}$.
(ii) For all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $CB_{\sigma}$ embeds into $CB_{\tau}$.
(iii) For all hereditarily repetitive stumps $\sigma, \tau$, if $CB_{\sigma}$ embeds into $CB_{\tau}$, then $\sigma \leq \tau$.
(iv) For all hereditarily repetitive stumps $\sigma$, for all $n$, $CB_{\sigma}$ reduces to $CB_{\sigma} \cap \overline{\sum_{n}}$.
(v) For all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $CB_{\sigma}$ reduces to $CB_{\tau}$.
(vi) For all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $CB_{\tau}$ does not reduce to $CB_{\sigma}$.
Proof.

(i) There is nothing to prove if \( \sigma = \emptyset \), as \( CB_\emptyset = \emptyset \) and \( \emptyset \) embeds into every subset of \( \mathcal{N} \).

Now let \( \sigma \) be a nonempty hereditarily repetitive stump and let \( n \) be a natural number. Using the Second Axiom of Countable Choice, find a strictly increasing \( \gamma \) such that \( \gamma (0) > n \) and, for each \( i \), \( \sigma^i = \sigma \gamma (i) \). Let \( \delta \) be a function from \( CB_\sigma \) to \( CB_\sigma \) such that \( \delta (\emptyset) = \emptyset \) and, for each \( i \), for each \( \alpha \) in \( CB_\sigma \), \( \delta (\emptyset i \ast \langle 1 \rangle \ast \alpha) = \emptyset (\gamma (i)) \ast \langle 1 \rangle \ast \alpha \). Clearly, \( \delta \) embeds \( CB_\sigma \) into \( CB_\sigma \cap \emptyset n \).

(ii) We use induction on the set of hereditarily repetitive stumps. Observe that, for every Cantor-Bendixson-set \( X \), the identity function embeds \( \emptyset = CB_0 \) into \( X \).

Now assume that \( \sigma \) is a nonempty hereditarily repetitive stump and that \( \tau \) is a hereditarily repetitive stump such that \( \sigma \leq \tau \). Using the Second Axiom of Countable Choice, find \( \delta \) such that, for each \( i \), \( \delta^i \) is an embedding from \( CB_{\sigma^i} \) into \( CB_{\tau^i} \). Let \( \epsilon \) be a function from \( CB_{\sigma} \) to \( CB_{\tau} \) with the property that, for each \( i \), for each \( \alpha \) in \( CB_{\sigma^i} \), \( \epsilon (\emptyset i \ast \langle 1 \rangle \ast \alpha) = \emptyset (\gamma (i)) \ast \langle 1 \rangle \ast (\delta^i \alpha) \). Clearly, \( \epsilon \) embeds \( CB_{\sigma} \) into \( CB_{\tau} \).

(iii) We use induction on the set of hereditarily repetitive stumps.

Note that, if we take \( \sigma = \emptyset \), the statement holds, as, for every hereditarily repetitive stump \( \tau \), \( \emptyset \leq \tau \).

Now assume that \( \sigma \) is a nonempty hereditarily repetitive stump and that the statement holds for each immediate substump \( \sigma^n \) of \( \sigma \).

Assume that \( \tau \) is a hereditarily repetitive stump and \( \gamma \) embeds \( CB_\tau \) into \( CB_\tau \). Note that \( CB_\tau \neq \emptyset \), and, therefore, \( \tau \) is a nonempty stump. We want to prove: \( \sigma \leq \tau \), that is, for each \( m \), there exists \( q \) such that \( \sigma^m \leq \tau^q \).

Let \( m \) belong to \( \mathbb{N} \). Note that, if \( \sigma^m = \emptyset \), then, for all \( q \), \( \sigma^m \leq \tau^q \). Let us assume, therefore, that \( \sigma^m \neq \emptyset \). Calculate \( p \) such that \( p > m \) and \( \sigma^m = \sigma^p \). Observe that either \( \gamma (\emptyset p \ast \langle 1 \rangle \ast \emptyset) \) is apart from \( \emptyset \) or \( \gamma (\emptyset p \ast \langle 1 \rangle \ast \emptyset) \) is apart from \( \emptyset \). Assume the latter and calculate \( q \) such that \( \gamma (\emptyset p \ast \langle 1 \rangle \ast \emptyset) \) passes through \( \emptyset q \ast \langle 1 \rangle \). Calculate \( r \) such that for every \( \alpha \) in \( CB_\tau \), if \( \alpha \) passes through \( \emptyset p \ast \langle 1 \rangle \ast \emptyset r \), then \( \gamma \alpha \) passes through \( \emptyset q \ast \langle 1 \rangle \). Let \( \delta \) be a function from \( CB_\tau \) to \( \mathcal{N} \) such that, for every \( \alpha \) in \( CB_\tau \), \( \delta \alpha \) maps the sequence \( \emptyset p \ast \langle 1 \rangle \ast \alpha \) onto \( \emptyset q \ast \langle 1 \rangle \ast \delta \alpha \). Observe that \( \delta \) embeds \( CB_\tau \cap \emptyset r \) into \( CB_\tau \). Using (i), we conclude that \( CB_\tau \) itself embeds into \( CB_\tau \), and, therefore, by the induction hypothesis, \( \sigma^m = \sigma^p \leq \tau^q \).

The case that \( \gamma (\emptyset m \ast \langle 1 \rangle \ast \emptyset) \) is apart from \( \emptyset \) is treated similarly.

Clearly, for all \( m \), there exists \( q \) such that \( \sigma^m \leq \tau^q \), and, therefore, \( \sigma \leq \tau \).

(iv) The statement is true if \( \sigma = \emptyset \), as \( CB_\emptyset = \emptyset \), and, for each \( n \), \( CB_1 \cap \emptyset n = \emptyset \).

Now let \( \sigma \) be a nonempty hereditarily repetitive stump and let \( n \) be a natural number. Using the Second Axiom of Countable Choice, find a strictly increasing \( \gamma \) such that \( \gamma (0) > n \) and, for each \( i \), \( \sigma^i = \sigma \gamma (i) \). Let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that \( \delta (\emptyset) = \emptyset \) and, for each \( i \), for each \( \alpha \), \( \delta (\emptyset i \ast \langle 1 \rangle \ast \alpha) = \emptyset (\gamma (i)) \ast \langle 1 \rangle \ast \alpha \), and, for each \( j > 0 \), if, for some \( i \), \( \alpha \) passes through \( \emptyset i \ast \langle j \rangle \), then, for some \( i \), \( \delta \alpha \) passes through \( \emptyset i \ast \langle j \rangle \).

Clearly, \( \delta \) reduces \( CB_\sigma \) to \( CB_\sigma \cap \emptyset n \).

(v) We use induction on the set of stumps.
The statement is true if $\sigma = \emptyset$, as $CB_\emptyset = \emptyset$, and $\emptyset$ reduces to every subset $X$ of $N$ such that there exists $\alpha$ not belonging to $X$.

Now assume that $\sigma$ is a nonempty hereditarily repetitive stump and that $\tau$ is a hereditarily repetitive stump such that $\sigma \leq \tau$. Using the Second Axiom of Countable Choice, find a strictly increasing $\gamma$ such that, for every $i$, $\sigma^i \leq \tau^{\gamma(i)}$. Using the induction hypothesis and again the Second Axiom of Countable Choice, find $\delta$ such that, for each $i$, $\delta^i$ is a function from $N$ to $N$ reducing $CB_{\sigma^i}$ into $CB_{\tau^{\gamma(i)}}$.

Let $\varepsilon$ be a function from $N$ to $N$ with the property that, for each $i$, $\varepsilon(\sigma^i) \leq \tau^{\gamma(i)}$. Using the Second Axiom of Countable Choice, find $\sigma^m$ such that, for each $i$, $\varepsilon(\sigma^i) \leq \tau^{\gamma(i)}$. Using the induction hypothesis and again the Second Axiom of Countable Choice, find $\delta$ such that, for each $i$, $\varepsilon(\sigma^i) \leq \tau^{\gamma(i)}$.

Clearly, $\varepsilon$ reduces $CB_\sigma$ to $CB_\tau$.

(vi) We again use induction on the set of hereditarily repetitive stumps. In order to see that the statement holds if $\sigma = \emptyset$, note that every set reducing to $CB_\emptyset$ is empty, and that, for every $\tau$, if $CB_\tau$ is empty, then $\tau = \emptyset$. It follows that, if $\emptyset < \tau$, then $CB_\tau$ does not reduce to $CB_\emptyset$.

Now assume that $\sigma$ is a nonempty hereditarily repetitive stump and that the statement holds for each immediate substump $\sigma^n$ of $\sigma$.

Also assume that $\tau$ is a hereditarily repetitive stump and $\sigma < \tau$ and $\gamma$ is a function from $N$ to $N$ reducing $CB_\tau$ to $CB_\sigma$. We have to derive a contradiction.

Calculate $q$ such that $\sigma < \tau^q$. Note that, like $\sigma$, $\tau^q$ must be nonempty. We claim that $\gamma$ maps $\sigma^q \leq \tau^q$ onto $\emptyset$, and prove this claim as follows.

Suppose $\gamma | (\sigma^q \leq \tau^q)$ is apart from $\emptyset$. Find $m$ such that $\gamma | (\sigma^q \leq \tau^q)$ passes through $\sigma^m \leq \tau^q$. Calculate $t$ such that, for every $\beta$, if $\beta$ passes through $\sigma^q \leq \tau^q$, then $\gamma | \beta$ passes through $\sigma^m \leq \tau^q$.

Construct a function $\delta$ from $N$ to $N$ such that, for every $\sigma$ passing through $\sigma^q \leq \tau^q$, $\gamma | (\sigma^q \leq \tau^q)$ equals $\sigma^m \leq \tau^q$, and observe that $\delta$ reduces $CB_{\tau^q} \cap \sigma^q$ to $CB_{\sigma^m}$. As, according to (iv), $CB_{\tau^q}$ reduces to $CB_{\tau^q} \cap \sigma^q$, $CB_{\tau^q}$ itself also reduces to $CB_{\sigma^m}$. We now have a contradiction, as $\sigma^m < \sigma \leq \tau^q$, and, therefore (see Subsection 2.5.5), $\sigma^m < \tau^q$, and, by the induction hypothesis, $CB_{\tau^q}$ does not reduce to $CB_{\sigma^m}$.

As $\tau$ is hereditarily repetitive, there exists a strictly increasing sequence $q = q_0 < q_1 < \cdots$ such that for each $n$, $\tau^{q_n} = \tau^q$, and, therefore, $\gamma$ maps $\sigma^q_n \leq \tau^q$ onto $\emptyset$.

Consider the closure $X$ of the set $\{\sigma^q_n \leq \tau^q | n \in N\}$. Observe that $X$ is a spread containing $\emptyset$ and that $\gamma$ maps every element of $X$ onto $\emptyset$, and thus into $CB_{\sigma}$. As $\gamma$ reduces $CB_{\tau}$ to $CB_{\sigma}$, $X$ is a subset of $CB_\tau$, and, by Theorem 4.3(i), every $\alpha$ in $X$ either coincides with $\emptyset$ or is apart from $\emptyset$. Applying Brouwer's Continuity Principle we find $m$ such that either every $\alpha$ in $X$ passing through $\sigma^m$ coincides with $\emptyset$ or every $\alpha$ in $X$ passing through $\sigma^m$ is apart from $\emptyset$, an obvious contradiction.

We thus have shown that $CB_{\tau}$ does not reduce to $CB_{\sigma}$.

3.10. Note that the formulation of Theorem 3.9(vi) differs from the formulation of Theorem 3.9(iii). We were unable to prove the stronger statement:

For all hereditarily repetitive stumps $\alpha, \beta$, if $CB_\alpha$ reduces to $CB_\beta$, then $\alpha \leq \beta$.

Some of the results we obtained thus far contrast starkly with some theorems in classical descriptive set theory: there, by a result of W. Wadge (see Kechris, 1996, p. 169) every set
that belongs to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \) is \( \Sigma^0_2 \)-complete, that is, every set belonging to \( \Sigma^0_2 \) reduces to \( X \). As a consequence, all sets that belong to \( \Sigma^0_2 \) but not to \( \Pi^0_2 \) are of the same reducibility degree. Here, we find large hierarchies formed by sets from \( \Sigma^0_2 \setminus \Pi^0_2 \).

Let \( X \) be a subset of \( \mathcal{N} \). We let the Cantor-Bendixson-derivative of \( X \), notation: \( X' \), be the set of all \( \alpha \) in \( \mathcal{N} \) such that, for each \( n \), there exists an element of \( X \) passing through \( \alpha n \), but apart from \( \alpha \).

Iterating the operation of taking the derivative, we define, by induction on the set of stumps, for every subset \( X \) of \( \mathcal{N} \) and every stump \( \sigma \), another subset of \( \mathcal{N} \), called the \( \sigma \)-th Cantor-Bendixson-derivative of \( X \), notation: \( \text{Der}(\sigma, X) \), as follows:

(i) \( \text{Der}(\emptyset, X) := X \)

(ii) For every nonempty stump \( \sigma \), \( \text{Der}(\sigma, X) := (\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, X))' \).

### 3.11. Lemma:

(i) For every subset \( X \) of \( \mathcal{N} \), the set \( X' \) is a subset of the closure \( \overline{X} \) of \( X \), and \( \overline{X} \) coincides with \( X' \) and with \( (\overline{X})' \).

(ii) For every subset \( X \) of \( \mathcal{N} \), for every nonempty stump \( \sigma \), \( \text{Der}(\sigma, X) \) coincides with \( \text{Der}(\sigma, \overline{X}) \).

(iii) For all subsets \( X, Y \) of \( \mathcal{N} \), if \( X \) is a subset of \( Y \), then \( X' \) is a subset of \( Y' \).

(iv) For all sequentially closed subsets \( X \) of \( \mathcal{N} \), for all stumps \( \sigma, \tau \), if \( \sigma \leq \tau \), then \( \text{Der}(\tau, X) \) is a subset of \( \text{Der}(\sigma, X) \).

(v) For every subset \( X \) of \( \mathcal{N} \), for each \( s \), \( (s * X)' \) coincides with \( s * (X') \).

(vi) For every stump \( \sigma \), for every subset \( X \) of \( \mathcal{N} \), for each \( s \), \( \text{Der}(\sigma, s * X) \) coincides with \( s * \text{Der}(\sigma, X) \).

**Proof.** We leave the proof of (i), (ii) and (iii) to the reader. □

### 3.12. This subsection forms, together with Theorem 3.13 and Subsection 3.14, a rather long intermezzo that interrupts the main line of the paper. The reader may skip this intermezzo and go on with Subsection 3.15.

In Section 1.2 we defined: a subset \( x \) of \( \mathcal{N} \) is a closed subset of \( \mathcal{N} \) if and only if there exists \( p \) in \( \mathcal{N} \) such that, for every \( a, a \) belongs to \( X \) if and only if, for all \( n \), \( p(\alpha n) = 0 \).

Now assume that \( \sigma \) is a nonempty stump and that the statement holds for every immediate substump \( \sigma^n \) of \( \sigma \). Assume that \( \tau \) is a stump and \( \sigma \leq \tau \). Observe that for every \( m \) there exists \( n \) such that \( \sigma^m \leq \tau^n \), and thus, by the induction hypothesis, for every sequentially closed subset \( X \) of \( \mathcal{N} \), \( \text{Der}(\tau^n, X) \) is a subset of \( \text{Der}(\sigma^n, X) \). It follows that, for every sequentially closed subset \( X \) of \( \mathcal{N} \), \( \text{Der}(\tau^n, X) \) is a subset of \( \bigcap_{m \in \mathbb{N}} \text{Der}(\sigma^m, X) \), and thus, by (iii), \( \text{Der}(\tau, X) = (\bigcap_{n \in \mathbb{N}} \text{Der}(\tau^n, X))' \) is a subset of \( \text{Der}(\sigma, X) = (\bigcap_{m \in \mathbb{N}} \text{Der}(\sigma^m, X))' \).

The proofs of (v) and (vi) are left to the reader. □
Let \( X \) be a subset of the set \( \mathbb{N} \) of the natural numbers. \( X \) will be called a **positively Borel subset of \( \mathbb{N} \)** if and only if the set \([\alpha | \alpha(0) \in X]\) is a positively Borel subset of Baire space \( \mathcal{N} \). The class of the positively Borel subsets of \( \mathbb{N} \) is the least class of subsets of \( \mathbb{N} \) containing the empty set and every singleton \( \{n\} \) that is closed under the operations of countable union and countable intersection, \( X \) will be called \( \Sigma_0^1 \) (or \( \Pi_1^0, \Sigma_2^0, \Pi_2^0 \), respectively) if and only if the set \([\{n\} * \alpha | \alpha \in \mathcal{N}\] is \( \Sigma_1^1 \) (or \( \Pi_1^0, \Sigma_2^0, \Pi_2^0 \), respectively).

A subset \( X \) of \( \mathbb{N} \) belongs to the class \( \Sigma_0^1 \) if and only if there exists a sequence \( X_0, X_1, \ldots \) of subsets of \( \mathbb{N} \) such that, for every \( n \), \( X_n \) either is the empty set or a singleton and \( X = \bigcup_{n \in \mathbb{N}} X_n \), that is (see Subsection 3.1) if and only if there exists \( \gamma \) in \( \mathcal{N} \) such that \( X \) coincides with the set \( E_\gamma \) consisting of all \( m \) such that, for some \( n \), \( \gamma(n) = m + 1 \). The \( \Sigma_1^1 \)-subsets of \( \mathbb{N} \) thus are the **enumerable** subsets of \( \mathbb{N} \).

A subset \( X \) of \( \mathbb{N} \) belongs to the class \( \Pi_1^0 \) if and only if \( X \) is co-enumerable, that is, if and only if there exists a \( \Sigma_1^0 \)-subset \( Y \) of \( \mathbb{N} \) such that \( X \) is the set of all \( m \) that do not belong to \( Y \). Note that \( X \) belongs to the class \( \Pi_1^0 \) if and only if, for some \( \gamma \), \( X \) is the set of all \( m \) in \( \mathbb{N} \) such that, for all \( n \), \( \gamma(n) \neq m + 1 \).

Corresponding characterizations may be given of the \( \Sigma_2^0 \)-subsets of \( \mathbb{N} \) and the \( \Pi_2^0 \)-subsets of \( \mathbb{N} \).

These distinctions and the notations we use are reminiscent of the recursion-theoretic arithmetical hierarchy. Note however that our context differs from the recursion-theoretic one: we do not ask that every function from \( \mathbb{N} \) to \( \mathbb{N} \) be given by a finite algorithm. \( \mathcal{N} \) is the set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \). Every function from \( \mathbb{N} \) to \( \mathbb{N} \) that is given by a finite algorithm belongs to \( \mathcal{N} \), but, conversely, given some \( \gamma \) in \( \mathcal{N} \), we do not want to assume that we are able to produce a finite algorithm calculating \( \gamma \). We do demand, however, that, given some \( \gamma \) in \( \mathcal{N} \) and some \( n \) in \( \mathbb{N} \), we are able to bring to light the value \( \gamma(n) \). In classical mathematics, one often introduces "functions" that do not satisfy the latter requirement. This explains why it is not true in intuitionistic mathematics, as it is in classical mathematics, that every subset of \( \mathbb{N} \) is enumerable by some function in \( \mathcal{N} \).

The following observations are important.

(i) The statement that not every subset of \( \mathbb{N} \) is enumerable does not mean that we are able to produce a set that is not enumerable. The famous **Brouwer-Kripke axiom** excludes this possibility. According to this axiom one may, given some well-defined proposition \( P \), not involving objects whose construction is not yet complete, determine \( \alpha \) in \( \mathcal{N} \) such that the proposition \( P \) is equivalent to the statement \( \exists n [\alpha(n) = 1] \).

Given a well-defined subset \( X \) of \( \mathbb{N} \), we may apply the axiom to every proposition of the form "\( m \in X \)" and then, using also the Second Axiom of Countable Choice, find \( \alpha \) such that, for every \( m, m \) belongs to \( X \) if and only if, for some \( n \), \( \alpha^m(n) = 1 \). Defining \( \gamma \) such that, for all \( m, n \), if \( \alpha^m(n) = 1 \), then \( \gamma((m, n)) = m + 1 \), and, if \( \alpha^m(n) \neq 1 \), then \( \gamma((m, n)) = 0 \), and also, for every \( p \), if there are no \( m, n \) such that \( p = (m, n) \), then \( \gamma(p) = 0 \), we find: \( X \) coincides with \( E_\gamma \), and \( X \) is enumerable. We do not want to use the Brouwer-Kripke axiom in this paper, but it warns us not to try to find an example of a subset \( X \) of \( \mathbb{N} \) that is not enumerable. Some further information on the axiom and its background is given in Veldman (2006a, subsections 9.12.3 and 9.14).

(ii) It is true, however, by the Second Axiom of Continuous Choice, (this axiom is the strongest version of Brouwer’s Continuity Principle (see Subsection 2.4.2)), that the assumption that every subset of \( \mathbb{N} \) is enumerable leads to contradiction. Even the
assumption that every subset of the singleton \{0\} is enumerable, is contradictory, as we will show in a moment (see Theorem 3.13(i)).

Let \(X\) be a sequentially closed subset of \(\mathcal{N}\). The frame of \(X\) is the set of all \(s\) in \(\mathbb{N}\) containing an element of \(X\). In general, the frame of a closed subset \(X\) of \(\mathcal{N}\) is not a decidable subset of \(\mathbb{N}\), that is, it may happen that we are unable to find \(\delta\) in \(\mathcal{N}\) such that, for every \(s\), \(\delta(s) = 1\) if and only if \(s\) belongs to the frame of \(X\). If the frame of \(X\) is a decidable subset of \(\mathbb{N}\), then \(X\) itself is a spread, and, in particular, a closed subset of \(\mathcal{N}\) in the sense of Subsection 1.2.

A subset \(X\) of the set \(\mathbb{N}\) of the natural numbers will be called an analytic subset of \(\mathbb{N}\) if and only if the set \(\{(m) \ast a | m \in X\}\) is an analytic subset of Baire space \(\mathcal{N}\). Analytic subsets of Baire space are considered in Veldman (2006a, section 9). It is proven there that every positively Borel subset of \(\mathcal{N}\) is analytic, and that there exist analytic subsets of \(\mathcal{N}\) that positively fail to be positively Borel (see Veldman, 2006a, theorem 9.12.1).

### 3.13. Theorem:

(i) Not every co-enumerable subset of \(\{0\}\) is an enumerable subset of \(\{0\}\).

(ii) For every subset \(X\) of \(\mathcal{N}\), if \(X\) is a countable union of spreads, then \(X'\) is a sequentially closed \(\Pi^0_2\)-subset of \(\mathcal{N}\).

(iii) For every closed subset \(X\) of \(\mathcal{N}\), and, more generally, for every sequentially closed positively Borel subset \(X\) of \(\mathcal{N}\), the Cantor-Bendixson derivative \(X'\) of \(X\) is an analytic subset of Baire space \(\mathcal{N}\), and the frame of \(X\) and the frame of \(X'\) both are analytic subsets of the set \(\mathbb{N}\) of natural numbers.

(iv) For every subset \(X\) of \(\mathcal{N}\), if the frame of \(X\) is a positively Borel subset of \(\mathbb{N}\), then \(X'\) is a positively Borel subset of \(\mathcal{N}\).

**Proof.**

(i) Suppose that every co-enumerable subset of \(\{0\}\) is an enumerable subset of \(\{0\}\).

Then, for every \(a\), there exists \(\beta\) such that \(a = 0\) if and only if, for some \(n\), \(\beta(n) = 1\). Applying the Second Axiom of Continuous Choice, we find a function \(\gamma\) from \(\mathcal{N}\) to \(\mathbb{N}\) such that, for all \(a\), \(a = 0\) if and only if, for some \(n\), \((\gamma | a)(n) = 1\). Find \(n\) such that \((\gamma | 0)(n) = 1\). Find \(m\) such that \(\gamma^n(0m) = 2\) and for all \(j < m\), \(\gamma^n(0j) = 0\). Note that both \(0m \ast 1\) and \(\gamma | (0m \ast 1)\) are apart from \(0\). Contradiction.

(ii) Suppose that \(X\) is a countable union of spreads. Let \(X_0, X_1, \ldots\) be a sequence of spreads such that \(X = \bigcup_{n \in \mathbb{N}} X_n\), and, using the Second Axiom of Countable Choice, find \(\delta\) in \(\mathcal{N}\) such that, for all \(i, s\), \(s\) contains an element of \(X_i\) if and only if \(\delta^i(s) = 1\). Note that, for all \(a, a\) belongs to \(X'\) if and only if, for each \(n\), there exists \(p, i\) such that \(\delta^i(\overline{an} \ast (p)) = 1\) and \(a(n) \neq p\). We thus see that \(X'\) is \(\Pi^0_2\).

(iii) Note that, for every \(a, a\) belongs to \(X'\) if and only if \(\forall n \exists \beta \exists p \overline{an} = \overline{bn} \land a(p) \neq \beta(p) \land \beta \in X\). Using the Second Axiom of Countable Choice, we conclude that, for each \(a, a\) belongs to \(X'\) if and only if \(\exists \beta \forall n \overline{an} = \overline{b^{\alpha+1}n} \land a(\beta^0(n)) \neq b^{\alpha+1}(\beta^0(n)) \land \forall n [\beta^{\alpha+1} \in X]\).

Also note that, for each \(x\), \(s\) belongs to the frame of \(X\) if and only if \(\exists \alpha \exists n \overline{an} = s \land a \in X\), and \(s\) belongs to the frame of \(X'\) if and only if \(\exists \alpha \exists n \overline{an} = s \land a \in X'\).

We obtain the desired results by using some closure properties of the class of the analytic sets one finds in Veldman (2006a, theorem 9.9).
Note that, for all \( a, a' \) belongs to \( X' \) if and only if \( \forall n \exists p[\overline{a}n \ast (p) \in \text{frame of } X \land a(n) \neq p] \).

3.14. Note that Theorem 3.13(ii) applies to every enumerable subset of \( \mathcal{N} \) and to every spread.

The reader should be attentive to what is not said in Theorem 3.13. It does not seem possible to prove that, for every enumerable subset \( X \) of \( \mathcal{N} \), \( X'' \) is a positively Borel subset of \( \mathcal{N} \). In Veldman (2005c) examples are given of “simple” analytic sets that are not positively Borel. In the light of these examples, it is possible that one might derive a contradiction from the assumption that, for every spread \( X \), \( X'' \) is a positively Borel subset of \( \mathcal{N} \). This would be a result like Theorem 3.13(i). Because of the Brouwer-Kripke axiom one may not expect to be able to find an example of a subset \( X \) of \( \mathcal{N} \) such that \( X \) itself is either an enumerable set or a spread and \( X'' \) is not positively Borel. The Brouwer-Kripke-axiom would force the frame of \( X'' \) to be enumerable and thus \( X'' \) itself to be \( \Pi^0_1 \).

We now take up the main line of the paper. In the proof of the third item of the next theorem, it is important that we are restricting ourselves to hereditarily repetitive stumps.

3.15. Theorem:

(i) For every stump \( \sigma \), \( \text{Der}(\sigma, CB_\sigma) = \emptyset \).

(ii) For all stumps \( \sigma, \tau \), if \( \sigma \leq \tau \), then \( \text{Der}(\tau, CB_\sigma) = \emptyset \).

(iii) For all hereditarily repetitive stumps \( \tau, \sigma \), if \( \tau < \sigma \), then \( \emptyset \) belongs to \( \text{Der}(\tau, CB_\sigma) \).

Proof.

(i) We use induction on the set of stumps. Clearly, \( \text{Der}(1, CB_0) = CB_0 = \emptyset \). Assume that \( \sigma \) is a nonempty stump and that for each \( n \), \( \text{Der}(\sigma^n, CB_{\sigma^n}) = \emptyset \). Note that, according to Lemma 3.11(vi), for each \( n \), \( \text{Der}(\sigma^n, CB_{\sigma^n}) \cap \overline{\emptyset n \ast (1)} = \emptyset \) coincides with \( \emptyset \). It follows that \( \bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_{\sigma^n}) \) is a subset of the set \( \{\emptyset\} \) and, therefore, \( \text{Der}(\sigma, CB_\sigma) = (\bigcap_{n \in \mathbb{N}} \text{Der}(\sigma^n, CB_{\sigma^n}))' = \emptyset \).

(ii) This is an immediate consequence of (i) and Lemma 3.11(iv).

(iii) We use induction on the set of hereditarily repetitive stumps. Clearly, for every stump \( \sigma \), if \( \emptyset < \sigma \), then \( \sigma \) is nonempty and \( \emptyset \) belongs to \( \text{Der}(\emptyset, CB_\sigma) = CB_\sigma \). Assume that \( \tau \) is a nonempty hereditarily repetitive stump, and, for every \( n \), for every hereditarily repetitive stump \( \sigma \), if \( \tau^n < \sigma \), then \( \emptyset \) belongs to \( \text{Der}(\tau^n, CB_\sigma) \). Suppose that \( \sigma \) is a hereditarily repetitive stump such that \( \tau < \sigma \). Find \( m \) such that \( \tau \leq \sigma^m \), and note that, for each \( n \), \( \tau^n < \sigma^m \). Applying the induction hypothesis we find that \( \emptyset \) belongs to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\tau^n, CB_{\sigma^n}) \), and, using Lemma 3.11, that \( \emptyset m \ast (1) \ast \emptyset \) belongs to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\tau^n, CB_\sigma) \). There are infinitely many numbers \( p \) such that \( \sigma^m = \sigma^p \), and for each such number \( p \), \( \emptyset p \ast (1) \ast \emptyset \) belongs to \( \bigcap_{n \in \mathbb{N}} \text{Der}(\tau^n, CB_\sigma) \). It follows that \( \emptyset \) belongs to \( \text{Der}(\tau, CB_\sigma) = (\bigcap_{n \in \mathbb{N}} \text{Der}(\tau^n, CB_\sigma))' \). □

3.16. We may conclude from Theorem 3.15 that for all hereditarily repetitive stumps \( \sigma, \tau \), \( \text{Der}(\sigma, CB_\sigma) = \emptyset \) and, if \( \tau < \sigma \), then \( \text{Der}(\tau, CB_\sigma) \) is an inhabited set. In this sense, \( \sigma \) is the least stump \( \tau \) such that \( \text{Der}(\tau, CB_\sigma) = \emptyset \) and might be called the Cantor-Bendixson-rank of \( CB_\sigma \). Note however, that, as we observed in Subsection 2.5.4, the relations \( <, \leq \) are nondecidable relation on the set of stumps, and that we did not prove: for every hereditarily repetitive stump \( \tau \), if \( \text{Der}(\tau, CB_\sigma) = \emptyset \), then \( \sigma \leq \tau \).
We now intend to prove a very nonclassical counterpart to Theorem 3.15.

We introduce a (partial) binary operation \( \text{Perhaps} \) on the class of subsets of \( \mathcal{N} \). Given subsets \( X, Y \) of \( \mathcal{N} \) such that \( X \) is a subset of \( Y \), we let \( \text{Perhaps}(X, Y) \) be the set of all \( a \) such that there exists \( \beta \) in \( X \) with the property: if \( a \) is apart from \( \beta \), then \( a \) belongs to \( Y \).

From a classical point of view, the set \( \text{Perhaps}(X, Y) \) would be indistinguishable from the set \( X \cup Y \), as the statement: “if \( a \) is apart from \( \beta \), then \( a \) belongs to \( Y \)” would be equivalent to the statement: “\( a \) coincides with \( \beta \), or \( a \) belongs to \( Y \)”, and the statement: “\( a \) belongs to \( \text{Perhaps}(X, Y) \)” would be equivalent to the statement: “\( a \) belongs to \( X \), or \( a \) belongs to \( Y \)”. From our intuitionistic point of view, however, \( X \cup Y \) is a subset of \( \text{Perhaps}(X, Y) \), but the converse, in general, fails to be true. Even the set \( \text{Perhaps}(X, X) \) sometimes is a proper extension of the set \( X \).

It may be difficult for the reader to understand the meaning of the operation \( \text{Perhaps} \).

For this reason, let us consider the corresponding operation on subsets of \( \mathbb{R} \) and study an example.

For all subsets \( X, Y \) of \( \mathbb{R} \) such that \( X \) is a subset of \( Y \), we let \( \text{Perhaps}(X, Y) \) be the set of all \( x \) in \( \mathbb{R} \) such that there exists \( y \) in \( X \) with the property: if \( x \) is really apart from \( y \), then \( x \) belongs to \( Y \).

Let us now define \( X := [0, 1] \cup [1, 2] \). We claim that the set \( \text{Perhaps}(X, X) = \text{Perhaps}([0, 1] \cup [1, 2], [0, 1] \cup [1, 2]) \) coincides with the set \([0, 2]\), that is, with the real closure of the set \([0, 1] \cup [1, 2]\). It suffices to show that \([0, 2]\) is a subset of \( \text{Perhaps}(X, X) \).

Let \( x \) be an element of \([0, 2]\). We define: \( y = \inf(1, x) \), the infimum of the numbers 1, \( x \). We prefer this expression to the expression: the “least” of the numbers 1, \( x \), as, in general, we are unable to decide: \( y = 1 \) or \( y = x \). Note that \( y \) belongs to \([0, 1]\) and, therefore, to \( X \), and that, if \( x \) is really apart from \( y \), then \( x \) belongs to \([1, 2]\) and, therefore, to \( X \).

On the other hand, as we observed before (see Subsection 1.3), the set \( X \) is not a closed subset of \( \mathbb{R} \) and fails to coincide with the set \([0, 2]\). We may conclude that the set \( X \) is a proper subset of the set \( \text{Perhaps}(X, X) \), where we are using the expression “proper subset” in the sense explained in Subsection 1.3.

We might describe the above argument that \([0, 2]\) is a subset of \( \text{Perhaps}(X, X) \) as follows. Given a number \( x \) in \([0, 2]\), we make an attempt to prove that \( x \) belongs to \( X \) itself, that is, to \([0, 1] \cup [1, 2]\). Our first guess is that \( x \) belongs to \([0, 1]\). The statement that \( x \) belongs to \([0, 1]\) is equivalent to the statement that \( x \) really coincides with \( y = \inf(1, x) \), so we start checking if \( x \) really coincides with \( y \). In general, verifying if \( x \) really coincides with \( y \), is an infinite procedure. In the \( n \)-th step of this procedure we see whether the rational interval that is the \( n \)-th approximation of \( x \) partially covers the rational interval that is the \( n \)-th approximation of \( y \). If so, we continue with the next step of our procedure. If not, we discover that \( x \) is really apart from \( y \). We have to admit that our initial guess was wrong, but nevertheless, we may conclude that \( x \) belongs to \( X \) as it belongs to \([1, 2]\).

This example does not yet make it clear why we do not assume, that the set \( Y \), in the general definition of \( \text{Perhaps}(X, Y) \) above, is the same as the set \( X \). This is because we want to apply the operation \( \text{Perhaps} \) repeatedly, and intend to study, after \( X \) itself and \( \text{Perhaps}(X, X) \), also \( \text{Perhaps}(X, \text{Perhaps}(X, X)) \). We shall explain this in Subsection 3.18.

The sentence “\( a \) belongs to \( \text{Perhaps}(X, Y) \)” might be rendered into English by the words: “\( a \) belongs to \( X \); well, perhaps merely to \( Y \)”. We are thinking of a speaker who only after having stated: “\( a \) belongs to \( X \)” starts to consider the evidence he really has for his assertion, and then, interrupting himself, hesitates to maintain it in full force, and replaces it by a second, possibly somewhat weaker statement, intending something like: “If my plan to prove the first statement should fail, I at least will be able to prove this second
one.” Observe that, under this interpretation, also the statement “a belongs to X, perhaps merely to X” is a weaker one than the unconditional statement “a belongs to X”. If one makes the first statement one makes the following announcement: I have, well, not exactly a proof, but, in any case, a fairly good plan for verifying “a belongs to X”, and, be quiet, should this plan turn out to fail, after all, I am sure to find an undisputable proof of “a belongs to X”.

Let X be a subset of N. X will be called perhapsive if and only if X coincides with Perhaps(X, X). Waaldijk (1996) called perhapsive subsets of N weakly stable subsets of N.

Let X be a subset of N. X is called a stable subset of N if and only if X ^-^ coincides with X. This term was coined by van Dantzig (1947).

3.17. Theorem:

(i) For all subsets X, Y of N, if X is a subset of Y, then X is a subset of Perhaps(X, Y), and Perhaps(X, Y) is a subset of Y ^-^, and, if X is inhabited, then also Y is a subset of Perhaps(X, Y).

(ii) For all subsets X, Y, Z of N, if X is a subset of Y and Y is a subset of Z, then Perhaps(X, Y) is a subset of Perhaps(X, Z) and Perhaps(X, Z) is a subset of Perhaps(Y, Z).

(iii) Every Π^0_2-subset of N is perhapsive.

(iv) For all subsets X, Y of N, if Y is perhapsive and X reduces to Y, then X is perhapsive.

(v) Every stable subset of N is perhapsive.

Proof.

(i) Suppose X, Y are subsets of N and X is a subset of Y. Clearly, X is a subset of Perhaps(X, Y), because, for every α, if α belongs to X, then also: if α ≠ α, then α belongs to Y.

We now prove that Perhaps(X, Y) is a subset of Y ^-^. Assume that α belongs to Perhaps(X, Y). Determine β in X such that if α ≠ β, then α belongs to Y. If α ≠ β, then α belongs to Y. If ¬(α ≠ β), then α coincides with β, and α belongs to X and therefore also to Y. As ¬¬(α ≠ β), we conclude: ¬¬(α belongs to Y), that is, α belongs to Y ^-^.

The proof that, if X is inhabited, then Y ⊆ Perhaps(X, Y), is straightforward.

(ii) We leave the proof to the reader.

(iii) Suppose X is a subset of N and X belongs to Π^0_2.

Let Y_0, Y_1, . . . be a sequence of open subsets of N such that X = \( \cap_{n \in \mathbb{N}} Y_n \). Now assume that α belongs to Perhaps(X, X) and determine β in X such that, if α ≠ β, then α belongs to X. Let n be a natural number. Determine m such that every γ passing through \( \beta m \) belongs to \( Y_n \) and distinguish two cases. Either \( \overline{\alpha} m = \beta m \), and α belongs to \( Y_n \), or \( \overline{\alpha} m \neq \beta m \), therefore α ≠ β, and α belongs to X and therefore in particular to \( Y_n \). We conclude that Perhaps(X, X) is a subset of every set \( Y_n \) and, therefore, a subset of X. X is also a subset of Perhaps(X, X) and the two sets coincide.

(iv) Suppose X, Y are subsets of N, and γ is a function from N to N reducing X to Y, and Perhaps(Y, Y) is a subset of Y. Assume that α belongs to Perhaps(X, X) and determine β in X such that, if α ≠ β, then α belongs to X. Observe that γ | β belongs
to $Y$. Observe also that, if $y \# \gamma \# \beta$, then $\alpha \# \beta$, and, therefore, $\alpha$ belongs to $X$ and $y \# \alpha$ belongs to $Y$. We conclude that $y \# \alpha$ belongs to $\textit{Perhaps}(Y, Y)$ and so to $Y$, and, therefore, $\alpha$ belongs to $X$.

We conclude that $\textit{Perhaps}(X, X)$ is a subset of $X$. As $X$ is also a subset of $\textit{Perhaps}(X, X)$, the two sets coincide.

(v) This is an easy consequence of (i).

\[ \Box \]

3.18. We want to iterate the operation $\textit{Perhaps}$.

We define, for every subset $X$ of $\mathcal{N}$ and every stump $\sigma$, another subset of $\mathcal{N}$, called the $\sigma$-th perhapsive extension of $X$, notation $\mathcal{P}(\sigma, X)$, as follows, by induction on the set of stumps:

(i) $\mathcal{P}(0, X) := X$.

(ii) For every nonempty stump $\sigma$, $\mathcal{P}(\sigma, X) := \textit{Perhaps}(X, \bigcup_{n \in \mathbb{N}} \mathcal{P}(\sigma^n, X))$.

We want to make it clear, for the uninitiated reader, why it is natural to consider iterations of the operation $\textit{Perhaps}$.

We again make a small digression to subsets of $\mathbb{R}$.

Consider the set $X_0 := \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \} = \{0\} \cup \left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\}$.

The set $X_0$ is not a closed subset of $\mathbb{R}$.

The following example makes this clear. Let $d$ in $\mathcal{N}$ be the decimal expansion of the real number $\pi$. Let us consider, for instance, $x := \lim_{n \to \infty} x_n$, where $x_0 = 1$ and, for each $n$, either, for all $m \leq n$, there exists $i < 99$ such that $d(m+i) \neq 9$ and $x_{n+1} = \frac{1}{n+1}$, or, there exists $m \leq n$ such that, for all $i < 99$, $d(m+i) = 9$ and $x_{n+1} = x_n$.

Clearly, $x$ belongs to the closure $\overline{X}_0$ of $X_0$. Suppose that $x$ belongs to $X_0$. Then either $x = 0$ and there is no $m$ such that, for all $i < 99$, $d(m+i) = 9$, or we find $n$ such that $x + \frac{1}{n}$ and, therefore, for all $i < 99$, $d(n+i) = 99$. We have no proof of this statement.

Using Brouwer’s Continuity Principle one may obtain a contradiction from the assumption that $X_0$ is a subset of $X_0$. Brouwer’s Continuity Principle may be seen to imply the following:

For every binary relation $R \subseteq X_0 \times \mathbb{N}$, if, for every $x$ in $X_0$, there exists $m$ such that $xRm$, then, for every $x$ in $X_0$, there exist $m, n$ such that, for every $y$ in $X_0$, if $|x - y| < \frac{1}{n+1}$, then $yRm$.

(In order to prove this one may use the fact that the set $X_0$ really coincides with a spread).

Suppose now that $X_0$ is a subset of $X_0$. Applying the just-mentioned consequence of Brouwer’s Continuity Principle, we find $m, n$ such that, either $m = 0$ and, for every $y$ in $\overline{X}_0$, if $|y| < \frac{1}{n+1}$, then $y = 0$, or $m \neq 0$, and, for every $y$ in $\overline{X}_0$, if $|y| < \frac{1}{n+1}$, then $y = \frac{1}{m}$.

Both alternatives are absurd.

Note, however, for each $x$ in $\overline{X}_0$, if $x$ is really apart from 0, then one may calculate $n$ such that $x = \frac{1}{n+1}$, and $x$ belongs to $X$. It follows that $\overline{X}_0$ coincides with $\textit{Perhaps}(X_0, X_0)$.

Now consider the set $X_1 := \{0\} \cup \left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\} \cup \left\{\frac{1}{n+1} + \frac{1}{m+1} \mid n \in \mathbb{N}\right\}$.

Using Brouwer’s Continuity Principle one may prove that the closure $\overline{X}_1$ of $X_1$ does not coincide with $\textit{Perhaps}(X_1, X_1)$. For suppose it does. We then may determine $x$ in $X_1$ and $n$ in $\mathbb{N}$ such that, for each $y$ in $\overline{X}_1$, if $|y| < \frac{1}{n+1}$ and $y$ is really apart from $x$, then $y$ belongs to $X_1$.

Let us first assume: $x = 0$. Note that, now, every element of the closure of the set $\left\{\frac{1}{n+2} + \frac{1}{m+1} \mid m \in \mathbb{N}\right\}$ will belong to the set $\left\{\frac{1}{n+2} + \frac{1}{m+1} \mid m \in \mathbb{N}\right\}$ itself. One may conclude
from this that the set $\bar{X}_0$ is a subset of $X_0$. The latter statement just has been seen to be false.

Let us then assume: $x \neq 0$. Find $p$ such that $x > \frac{1}{p}$ and $p > n + 1$. Note that, now, every element of the closure of the set $\{\frac{1}{p+1} + \frac{1}{m+1} | m \in \mathbb{N}\}$ will belong to the set $\{\frac{1}{p+1} + \frac{1}{m+1} | m \in \mathbb{N}\}$ itself. One may conclude from this that the set $\bar{X}_0$ is a subset of $X_0$.

The latter statement just has been seen to be false.

It follows that the closure $\bar{X}_1$ of $X_1$ does not coincide with $\text{Perhaps}(X_1, X_1)$.

On the other hand, it is a nice exercise for the reader to prove that, for every $x$ in $\bar{X}_1$, if $x$ is really apart from $0$, one may determine $n_0$ such that $x$ belongs to $[\frac{1}{n_0+1}, \frac{1}{n_0}]$, and if now, in addition, $x$ is really apart from $n fi$, then one may determine $m$ such that $x$ really coincides with $\frac{1}{n_0+1} + \frac{1}{m+1}$, so $x$ belongs to $X_1$. This shows that $X$ really coincides with $\text{Perhaps}(X_1, \text{Perhaps}(X_1, X_1))$.

Let us now return to subsets of $\mathcal{N}$.

3.19. Theorem:

(i) For every subset $X$ of $\mathcal{N}$, for every stump $\sigma$, $X \subseteq \mathbb{P}(\sigma, X) \subseteq X^{\sim}$.

(ii) For every inhabited subset $X$ of $\mathcal{N}$, for all stumps $\sigma, \tau$, if $\sigma \leq \tau$, then $\mathbb{P}(\sigma, X) \subseteq \mathbb{P}(\tau, X)$.

(iii) For all subsets $X, Y$ of $\mathcal{N}$, if $X$ is a subset of $Y$, then, for every stump $\sigma$, $\mathbb{P}(\sigma, X)$ is a subset of $\mathbb{P}(\sigma, Y)$.

(iv) For all subsets $X, Y$ of $\mathcal{N}$, for every $s$, if $X$ is a subset of $Y$ and $X \cap s$ is inhabited, then $\text{Perhaps}(X, Y) \cap s$ coincides with $\text{Perhaps}(X \cap s, Y \cap s)$.

(v) For every subset $X$ of $\mathcal{N}$, for every $s$, if $X \cap s$ is inhabited, then, for every stump $\sigma$, $\mathbb{P}(\sigma, X) \cap s$ coincides with $\mathbb{P}(\sigma, X \cap s)$.

(vi) For every subset $X$ of $\mathcal{N}$, for every $s$, $\mathbb{P}(\sigma, s \ast X)$ coincides with $s \ast \mathbb{P}(\sigma, X)$.

Proof. (i), (ii), (iii): One proves this by induction on the set of stumps, using Theorem 3.17(i) and (ii).

(iv) Let $X, Y$ be subsets of $\mathcal{N}$ such that $X$ is a subset of $Y$ and let $s$ belong to $\mathbb{N}$. Clearly, $\text{Perhaps}(X \cap s, Y \cap s)$ is a subset of $\text{Perhaps}(X, Y)$ and thus of $\text{Perhaps}(X \cap s, Y \cap s)$. Now let $\alpha$ belong to $\text{Perhaps}(X, Y) \cap s$. Then $\alpha$ passes through $s$. Find $\beta$ in $X$ such that, if $\beta \neq \alpha$, then $\alpha$ belongs to $Y$. If $\beta$ passes through $s$, then $\alpha$ belongs to $\text{Perhaps}(X \cap s, Y \cap s)$. If $\beta$ does not pass through $s$, then $\alpha \neq \beta$, and, therefore, $\alpha$ belongs to $Y$, and certainly: if $\alpha \neq \gamma$, then $\alpha$ belongs to $\text{Perhaps}(X \cap s, Y \cap s)$.

Thus we see that $\text{Perhaps}(X, Y) \cap s$ coincides with $\text{Perhaps}(X \cap s, Y \cap s)$.

(v) One proves this by induction on the set of stumps, using (iv).

(vi) One proves this by induction on the set of stumps.

3.20. Theorem:

Let $X_0, X_1, \ldots$ be a sequence of inhabited spreads, that is, located and closed inhabited subsets of $\mathcal{N}$.

Let $\delta$ be an element of $\mathcal{N}$ such that for all $n, s$, $\delta^n(s) = 1$ if and only if $s$ contains an element of $X_n$.

Consider $X = \bigcup_{n \in \mathbb{N}} X_n$. 

(i) For all subsets $Y$ of $\mathbb{N}$ containing $X$, Perhaps($X, Y$) is the set of all $a$ such that, for some $n$, for all $m$, if $\delta^n(\bar{a}m) \neq 1$, then $a$ belongs to $Y$, that is: either $\delta^n(\bar{a}m) = 1$ or $a$ belongs to $Y$.

(ii) For all positively Borel subsets $Y$ of $\mathbb{N}$ containing $X$, Perhaps($X, Y$) is positively Borel.

(iii) For every stump $\sigma$, $\mathbb{P}(\sigma, X)$ is positively Borel.

Proof. First, let $a$ belong to Perhaps($X, Y$). Find $\beta$ in $X$ such that, if $a \neq \beta$, then $a$ belongs to $Y$. Find $n$ such that $\beta$ belongs to $X_n$. Note that, for every $m$, if $\delta^n(\bar{a}m) \neq 1$, then $\delta^n(\bar{a}m) = 1$ or $a$ belongs to $Y$.

Secondly, assume that $n$ is a natural number and let $a$ be such that, for all $m$, if $\delta^n(\bar{a}m) \neq 1$ then $a$ belongs to $Y$. Define $\gamma$ such that, for all $m$, if $\delta^n(\bar{m} * a(m)) = 1$, then $\gamma(m) = a(m)$, and, if not, then $\gamma(m)$ equals the least $p$ such that $\delta^n(\bar{m} * (p)) = 1$.

Note that $\gamma$ belongs to $X_n$ and, therefore, to $X$. Also observe that, if $\gamma \neq a$, then, for some $m$, $\delta^n(\bar{a}m) \neq 1$ and, therefore, $a$ belongs to $Y$. It follows that $a$ belongs to Perhaps($X, Y$).

(i) and (iii) easily follow from (i). □

3.21. Note that every singleton or one-element-set $\{a\}$ is an inhabited spread. For this reason we may apply Theorem 3.20 to every enumerable subset $X$ of $\mathbb{N}$. We should contrast this observation and Theorem 3.20 with the discussion in Subsection 3.14: the perhapsive extensions of an enumerable set always are positively Borel subsets of $\mathbb{N}$ but the Cantor-Bendixson derivatives probably are not.

Recall that we defined, in Subsection 2.5.1, for every stump $\sigma$, the successor of $\sigma$, notation: $S(\sigma)$ or $\sigma^+$, by: $\sigma^+(0) = 0$ and, for each $n$, $(\sigma^+)^n = \sigma$.

Note that, for each nonempty stump $\sigma$, for each $n$, $(\sigma^n)^+ \leq \sigma$. More generally, for all nonempty stumps $\sigma, \tau$, if $\sigma < \tau$, then $\sigma^+ \leq \tau$.

3.22. Theorem:

(i) For all hereditarily repetitive stumps $\sigma, \tau$, for all $r$, if $\bar{CB}_\sigma \cap \bar{r} \bar{\sigma}$ is a subset of $\mathbb{P}(\tau, C B_\sigma)$, then $\bar{CB}_\sigma$ is a subset of $\mathbb{P}(\tau, C B_\sigma)$.

(ii) For all nonhereditarily repetitive stumps $\tau, \sigma$, $\bar{CB}_\sigma$ is a subset of $\mathbb{P}(\tau, C B_\sigma)$ if and only if, for each $m$, there exists $n$ such that $\bar{CB}_\sigma^m$ is a subset of $\mathbb{P}(\tau^n, C B_\sigma^m)$.

(iii) For every stump $\sigma$, $\mathbb{P}(\sigma, C B_\sigma)$ coincides with $\bar{C B_\sigma} = (C B_\sigma)^{-\tau}$.

(iv) For all hereditarily repetitive stumps $\tau, \sigma$, if $\bar{C B}_\sigma$ is a subset of $\mathbb{P}(\tau, C B_\sigma)$, then $\sigma \leq \tau^+$.

Proof.

(i) This conclusion follows from Lemma 3.11 and Theorem 3.19.

(ii) Let us first assume that $\sigma, \tau$ are nonempty hereditarily repetitive stumps such that $\bar{C B}_\sigma$ is a subset of $\mathbb{P}(\tau, C B_\sigma)$. The latter set coincides with $\mathbb{P}(C B_\sigma, \bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma))$. Therefore, for every $a$ in $\bar{C B}_\sigma$, there exists $\beta$ in $C B_\sigma$ such that, if $a \neq \beta$, then $a$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma)$. Note that, for every $\beta$ in $\bar{C B}_\sigma$, either $\beta = 0$ or $\beta \neq 0$. Note that $\bar{C B}_\sigma$ is a spread containing $0$. Applying Brouwer's Continuity Principle we find $m, i$ such that either $i = 0$ and, for every $\alpha$ in $\bar{C B}_\sigma \cap \bar{m}i$, if $\alpha \neq 0$, then $\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma)$ or $i > 0$ and there exists $\beta$ passing through $\bar{0}i * (1)$ such that, if $\alpha \neq \beta$, then $\alpha$ belongs to $\bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma)$. Let $p$ be the greatest one of the numbers $m, i + 1$. Note that, in both cases, for every $q$ in $\mathbb{N}$, if $q \geq p$, then every $\alpha$ in $\bar{C B}_\sigma$ passing through $\bar{0}q * (1)$ belongs to
\[ \bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma) \], that is, \( \overline{C B_\sigma} \cap \bigcup_{q} q \cdot \{1\} \) is a subset of \( \bigcup_{n \in \mathbb{N}} \mathbb{P}(\tau^n, C B_\sigma) \). Now let \( m \) belong to \( \mathbb{N} \) and find \( q \geq p \) such that \( \sigma^q = \sigma^m \). Note that \( \overline{C B_\sigma} \cap \bigcup_{q} q \cdot \{1\} \) is a spread containing \( \overline{0q} \cdot \{1\} \). Applying Brouwer’s Continuity Principle again, we find \( n, \tau \) such that \( \overline{C B_\sigma} \cap \bigcup_{q} q \cdot \{1\} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \). Using Theorem 4.19, we conclude that \( \overline{C B_\sigma} \cap \bigcup_{q} q \cdot \{1\} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \), and also that \( \overline{C B_a q \tau} \cap \bigcup_{q} \tau \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \). Using (i), we conclude that \( \overline{C B_\sigma m} = \overline{C B_\sigma q} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \).

Conversely, assume that, for each \( m \), there exists \( n \) such that \( \overline{C B_\sigma m} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \). Note that, for every \( a \) in \( \overline{C B_\sigma} \), if \( a \neq 0 \), then there exist \( m, n \) in \( \mathbb{N} \), \( y \) in \( \mathbb{P}(\tau^n, C B_\sigma) \), such that \( a = \overline{0m} \cdot \{1\} \cdot y \). Therefore, for every \( a \) in \( \overline{C B_\sigma} \), if \( a \neq 0 \), then there exist \( m, n \) in \( \mathbb{N} \), \( y \) in \( \mathbb{P}(\tau^n, C B_\sigma) \), such that \( a = \overline{0m} \cdot \{1\} \cdot y \).

Using Theorem 4.19(iv), we conclude that, for each \( m \), \( \overline{C B_\sigma m} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \). Therefore, \( \overline{C B_\sigma} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \).

(iii) Note that \( \overline{C B_0} = \emptyset \) and that \( \mathbb{P}(0, \emptyset) = \emptyset \) coincides with \( \overline{\emptyset} = \emptyset \). Now prove the general statement by induction on the set of hereditarily repetitive stumps, using (ii).

(iv) We again use induction on the set of hereditarily repetitive stumps. Let us first consider \( \tau = 0 \). Suppose that, for some \( \sigma \), \( \overline{C B_\sigma} \) is a subset of \( \mathbb{P}(0, \overline{C B_\sigma}) = C B_\sigma \). Using Theorem 3.3(iv), we conclude that \( \overline{C B_\sigma} \) is a finite set. Now observe that either \( \sigma = 0 \) or \( \sigma \neq 0 \). Note that, if \( \sigma = 0 \), then \( \sigma \leq \{0\}^* = 1^* \). Suppose that \( \sigma \neq 0 \) and assume we find \( n \) such that \( \sigma^n \neq 0 \) and, therefore, \( 0 \) belongs to \( \overline{C B_\sigma} \). As \( \sigma \) is repetitive, it follows that \( \overline{C B_\sigma} \) is infinite. Contradiction. We conclude that, if \( \sigma \neq 0 \), then for each \( n, \sigma^n = 0 \) and, therefore, \( \sigma \leq \{0\}^+ \). Now assume that \( \tau \) is a nonempty hereditarily repetitive stump and that the statement has been proven for every one of its immediate substumps \( \tau^n \). Suppose that \( \sigma \) is a hereditarily repetitive stump and that \( \overline{C B_\sigma} \) is a subset of \( \mathbb{P}(\tau, C B_\sigma) \). Using (iii), we conclude that, for each \( m \), there exists \( n \) such that \( \overline{C B_\sigma m} \) is a subset of \( \mathbb{P}(\tau^n, C B_\sigma) \), and thus, by the induction hypothesis, \( \sigma^m \leq (\tau^n)^+ \leq \tau \). Therefore, \( \sigma \leq S(\tau) = \tau^+ \).

3.23. Let \( \sigma, \tau \) be hereditarily repetitive stumps. We define:

\[ \tau \text{ is the perhapsive rank of } C B_\sigma \]

if and only if

(i) \( \overline{C B_\sigma} \) is a subset of \( \mathbb{P}(\tau, C B_\sigma) \), and

(ii) for every stump \( \rho \), if \( \overline{C B_\sigma} \) is a subset of \( \mathbb{P}(\rho, C B_\sigma) \), then \( \tau \leq \rho \).

Let \( \sigma \) be a nonempty stump. \( \sigma \) will be called a successor stump if and only if there exists \( m \) such that, for every \( n \), \( \sigma^n \leq \sigma^m \). Note that, if \( \sigma \) is a successor stump, then there exists a stump \( \tau \) such that both \( \sigma \leq S(\tau) = \tau^+ \) and \( S(\tau) = \tau^+ \leq \sigma \). \( \sigma \) will be called a positive limit stump if and only if, for each \( m \), there exists \( n \) such that \( \sigma^m < \sigma^n \).

Constructively, it is of course far from true that every nonempty stump is either a successor stump or a positive limit stump.

3.24. Theorem:

(i) For all hereditarily repetitive stumps \( \sigma, \tau \), if the perhapsive rank of \( C B_\sigma \) is \( \tau \), then the perhapsive rank of \( C B_{S(\sigma)} \) is \( S(\tau) \).

(ii) \( \overline{C B_1} \) coincides with \( \mathbb{P}(0^*, C B_1) \).
(iii) For each \( n \), the perhapsive rank of \( CB_{S(n^*)} \) is \( n^* \).

(iv) For each hereditarily repetitive stump \( \sigma \) that is a positive limit stump, the perhapsive rank of \( CB_{\sigma} \) is \( \sigma \).

Proof.

(i) Using Theorem 3.22(iii), we observe: for all hereditarily repetitive stumps \( \rho, \sigma \),
\[ CB_{\rho} \text{ is a subset of } \mathcal{P}(\sigma, CB_{\rho}) \] if and only if \( CB_{S(\rho)} \) is a subset of \( \mathcal{P}(S(\sigma), CB_{S(\rho)}) \).
Is now easy to obtain the desired conclusion.

(ii) The statement is obvious.

(iii) It follows from Theorem 3.22(iv) that, for each \( m, n \), if \( CB_{n^*} \) is a subset of \( \mathcal{P}(m^*, CB_{n^*}) \), then \( m \leq n + 1 \). Using (ii) and complete induction, we conclude that, for each \( n \), the perhapsive rank of \( CB_{S(n^*)} \) is \( n^* \).

(iv) Let \( \sigma \) be a hereditarily repetitive stump and a positive limit stump. Note that, according to Theorem 3.22(iii), \( CB_{\sigma} \) is a subset of \( \mathcal{P}(\sigma, CB_{\sigma}) \). Now assume that \( \rho \) is a stump and \( CB_{\sigma} \) is a subset of \( \mathcal{P}(\rho, CB_{\sigma}) \). According to Theorem 3.22(ii), for each \( m, n \), there exists \( s \) such that \( CB_{S(m)} \) is a subset of \( \mathcal{P}(\rho^n, CB_{S(m)}) \), and thus, by Theorem 3.22(iv), \( \sigma^m \leq S(\rho^n) \). As \( \sigma \) is a positive limit stump, we may find, for each \( m, n \), natural numbers \( p, q \) such that \( \sigma^m < \sigma^p \leq S(\rho^n) \), and, therefore, \( \sigma^m < \rho^n \). Therefore, \( \sigma \leq \rho \). We thus see that the perhapsive rank of \( CB_{\sigma} \) is \( \sigma \). \( \square \)

Note that the classical mathematician would be tempted to conclude from Theorem 3.24:
for every infinite ordinal \( \sigma \) (he would identify such an ordinal with a stump greater than \( 0^*, 1^*, \ldots \)), the perhapsive rank of \( CB_{\sigma} \) is \( \sigma \), as he believes every such ordinal to be of the form \( S^0(\tau) \), where \( \tau \) is a limit ordinal. We know of course how to resist such temptations.

3.25. Theorem:

(i) For all subsets \( X, Y \) of \( N \), for every function \( \gamma \) from \( N \) to \( N \), if \( \gamma \) maps \( X \) into \( Y \), then for each stump \( \sigma \), \( \gamma \) maps \( \mathcal{P}(\sigma, X) \) into \( \mathcal{P}(\sigma, Y) \), and, in particular, if \( X \) is a subset of \( Y \), then \( \mathcal{P}(\sigma, X) \) is a subset of \( \mathcal{P}(\sigma, Y) \).

(ii) For all hereditarily repetitive stumps \( \sigma, \tau \), for every function \( \gamma \) from \( N \) to \( N \), if \( \sigma < \tau \) and \( \gamma \) maps \( CB_{\sigma} \) into \( CB_{\tau} \), then \( \gamma \) does not map surjectively the closure \( CB_{\sigma} \) of \( CB_{\sigma} \) onto the closure \( CB_{\tau} \) of \( CB_{\tau} \).

Proof.

(i) The proof is by induction on the set of stumps and left to the reader.

(ii) Observe that, according to (i), \( \gamma \) will map \( CB_{\sigma} = \mathcal{P}(\sigma, CB_{\sigma}) \) into \( \mathcal{P}(\sigma, CB_{\tau}) \) and, according to Theorem 3.22, the latter set is a of \( CB_{\tau} \). Therefore, \( \gamma \) does not map surjectively \( CB_{\sigma} \) onto \( CB_{\tau} \). \( \square \)

§4. Perhaps and Almost. We continue the study of the notion \textit{Perhaps} that we began in the previous section and we introduce a closely connected unary operation on subsets of \( N \) called \textit{Almost}. For every subset \( X \) of \( N \), we let \textit{Almost}(\( X \)) be the union of all perhapsive extensions of \( X \). It turns out that \textit{Almost}(\( X \)) is the least perhapsive set containing \( X \). If \( X \) is a Cantor-Bendixson-set, then \textit{Almost}(\( X \)) is itself a perhapsive extension of \( X \). If, on the other hand, \( X \) is a countable and dense subset of \( N \), \textit{Almost}(\( X \)) is \textit{not} a perhapsive extension of \( X \). We extend our considerations to countable unions of spreads and prove that also the set \textit{Almost}(\( E_2 \)) is not a perhapsive extension of \( E_2 \).
4.1. Let $X$ be a subset of $\mathcal{N}$. We let $\text{Almost}(X)$ be the set of all $\alpha$ in $\mathcal{N}$ such that for some stump $\sigma$, $\alpha$ belongs to $\mathcal{P}(\sigma, X)$. Observe that this definition involves a quantification on the set $\text{Stp}$ of stumps. The possibility of introducing the set $\text{Almost}(X)$ depends upon our acceptance of $\text{Stp}$ as a set and a domain of quantification.

4.2. Theorem:

(i) For all subsets $X$ of $\mathcal{N}$, $X$ is a subset of $\text{Almost}(X)$ and $\text{Almost}(X)$ is a subset of $X$.

(ii) For all subsets $X$ of $\mathcal{N}$, $\text{Perhaps}(X, \text{Almost}(X))$ coincides with $\text{Almost}(X)$, and, for each stump $\sigma$, $\mathcal{P}(\sigma, \text{Almost}(X))$ coincides with $\text{Almost}(X)$.

(iii) For all subsets $X, Y, Z$ of $\mathcal{N}$, if $X$ is a subset of $Y$ and $Y$ is a subset of $Z$, then $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$ is a subset of $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$.

(iv) For all subsets $X$ of $\mathcal{N}$, for every stump $\sigma$, $\text{Perhaps}(\mathcal{P}(\sigma, X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$.

(v) For all subsets $X$ of $\mathcal{N}$, $\text{Perhaps}(\text{Almost}(X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$, that is, $\text{Almost}(X)$ is perhapsive.

(vi) For all subsets $X, Y$ of $\mathcal{N}$, if $X$ is a subset of $Y$ and $Y$ is perhapsive, then $\text{Almost}(X)$ is a subset of $Y$.

Proof.

(i) is a direct consequence of Lemma 3.19(i).

(ii) Let $X$ be a subset of $\mathcal{N}$ and suppose that $\alpha$ belongs to $\text{Perhaps}(X, \text{Almost}(X))$. Find $\beta$ in $X$ such that, if $\alpha$ is apart from $\beta$, then $\alpha$ belongs to $\text{Almost}(X)$.

Using the Second Axiom of Countable Choice, we now build a nonempty stump $\sigma$ by specifying successively its immediate substumps $\sigma^0, \sigma^1, \sigma^2, \ldots$. For each $n$, if $\alpha(n) = \beta(n)$ or if there exists $p < n$ such that $\alpha(p) \neq \beta(p)$ then $\sigma^n$ is the empty stump $\emptyset$, and if $\alpha(n) \neq \beta(n)$ and there is no $p < n$ such that $\alpha(p) \neq \beta(p)$, we find a stump $\tau$ such that $\alpha$ belongs to $\text{Perhaps}(\tau, X)$, and define $\sigma^n := \tau$. We claim that $\alpha$ belongs to $\mathcal{P}(\sigma, X)$. For let $\beta$ be the sequence we just considered and observe: $\beta$ belongs to $X$, and if $\alpha$ is apart from $\beta$, and $n$ is the least $p$ such that $\alpha(p) \neq \beta(p)$, then $\alpha$ belongs to $\mathcal{P}(\sigma^n, X)$. We thus see that $\alpha$ belongs to $\text{Almost}(X)$.

Therefore, $\text{Perhaps}(X, \text{Almost}(X))$ is a subset of $\text{Almost}(X)$ and, as the converse is also true, see (i), the two sets coincide.

The second part of the statement now follows easily by induction on the set of stumps.

(iii) Let $X, Y, Z$ be subsets of $\mathcal{N}$ such that $X$ is a subset of $Y$ and $Y$ is a subset of $Z$. Assume that $\alpha$ belongs to $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$. We intend to show that $\alpha$ belongs to $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$. First determine $\beta$ in $\text{Perhaps}(X, Y)$ such that, if $\alpha$ is apart from $\beta$, then $\alpha$ belongs to $Z$. Then determine $\gamma$ in $X$ such that if $\beta$ is apart from $\gamma$, then $\beta$ belongs to $Y$. Now assume that $\alpha$ is apart from $\gamma$ and distinguish two cases: either $\alpha$ is apart from $\beta$ and, therefore, $\alpha$ belongs to $Z$, or $\gamma$ is apart from $\beta$ and, therefore, $\beta$ belongs to $Y$, and, therefore, $\alpha$ belongs to $\text{Perhaps}(Y, Z)$. In both cases $\alpha$ belongs to $\text{Perhaps}(Y, Z)$, so, if $\alpha$ is apart from $\gamma$, then $\alpha$ belongs to $\text{Perhaps}(Y, Z)$, and, therefore, $\alpha$ belongs to $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$.

We conclude that $\text{Perhaps}(\text{Perhaps}(X, Y), Z)$ is a subset of $\text{Perhaps}(X, \text{Perhaps}(Y, Z))$. 
(iv) Let $X$ be a subset of $\mathcal{N}$. We claim that, for each stump $\sigma$, the set $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$. In order to prove this, we use induction on the set of stumps.

As $\mathbb{P}(\bigcup \mathcal{X})$ coincides with $X$, we may conclude from (ii) that $\text{Perhaps}(\mathbb{P}(0, X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$. Now assume that $\sigma$ is a nonempty stump and that, for each $n$, $\text{Perhaps}(\mathbb{P}(\sigma^n, X), \text{Almost}(X))$ coincides with $\text{Almost}(X)$. Consider $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$. Observe that $\mathbb{P}(\sigma, X)$ coincides with $\text{Almost}(X), U_{n \in \mathcal{N}} \mathbb{P}(\sigma^n, X)$ and apply (iii) in order to conclude:

\[ \text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X)) \text{ is a subset of } \text{Perhaps}(\bigcup_{n \in \mathcal{N}} \mathbb{P}(\sigma^n, X), \text{Almost}(X)). \]

Note that $\text{Perhaps}(\bigcup_{n \in \mathcal{N}} \mathbb{P}(\sigma^n, X), \text{Almost}(X))$ coincides with $\bigcup_{n \in \mathcal{N}} \text{Perhaps}(\mathbb{P}(\sigma^n, X), \text{Almost}(X))$ and, therefore, with $\text{Almost}(X)$.

Therefore, $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$ is a subset of $\text{Perhaps}(X, \text{Almost}(X))$, and thus, by (ii), also of $\text{Almost}(X)$. Conversely, $\text{Almost}(X)$ is a subset of $\text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$, by Theorem 4.17(i). We thus see that the sets $\text{Almost}(X)$ and $\text{Perhaps}(X, \text{Almost}(X))$ coincide.

(v) Let $X$ be a subset of $\mathcal{N}$. Observe that $\text{Perhaps}(\text{Almost}(X), \text{Almost}(X))$ coincides with $\text{Perhaps}(\bigcup_{\sigma \in \text{Stp}} (\mathbb{P}(\sigma, X), \text{Almost}(X))$ and also with $\bigcup_{\sigma \in \text{Stp}} \text{Perhaps}(\mathbb{P}(\sigma, X), \text{Almost}(X))$, and therefore, according to (iv), with $\text{Almost}(X)$.

(vi) Let $X$ be a subset of $\mathcal{N}$, and suppose $Y$ is a perhapsive subset of $\mathcal{N}$ containing $X$. Note that, for each stump $\sigma$, $\mathbb{P}(\sigma, Y)$ coincides with $Y$. Using Theorem 3.19(iii), observe that, for each stump $\sigma$, $\mathbb{P}(\sigma, X)$ is a subset $\mathbb{P}(\sigma, Y)$ and thus of $Y$. It follows that $\text{Almost}(X)$ is a subset of $Y$. \hfill \Box

4.3. Let $X$ be a subset of $\mathcal{N}$. Because of Theorem 4.2(v) and (vi) we may call $\text{Almost}(X)$ the perhapsive closure of the set $X$: $\text{Almost}(X)$ is the least perhapsive set containing $X$.

4.4. Let $D$ be a subset of $\mathcal{N}$. $D$ is dense in itself if and only if $D$ is a subset of its derivative set $D'$, that is, for every $\alpha$ in $D$, for every $m$, there exists $\beta$ in $D$ such that $\beta$ is apart from $\alpha$ and $\alpha = \beta m \in D$. $D$ is discrete if and only if for all $\alpha, \beta$ in $D$ we may decide if $\alpha \neq \beta$ or $\alpha = \beta$.

Let $X$ be a subset of $\mathcal{N}$ and $Y$ a subset of $X$. $Y$ is a decidable subset of $X$ if and only if we may decide, for every $\alpha$ in $X$, if $\alpha$ belongs to $Y$ or not.

Recall, from Subsection 1.3, that $Y$ is a proper subset of $X$ if and only if the assumption that every element of $X$ is also an element of $Y$ leads to a contradiction.

In Subsection 1.3, we observed that, in general, if $Y$ is a proper subset of $X$, we are unable to find an element of $X$ that does not belong to $Y$, and that there are even many cases, where $Y$ is a proper subset of $X$, and we can prove that there is no element of $X$ that does not belong to $Y$.

The next theorem is about subsets $D$ of $\mathcal{N}$ satisfying: $D$ is enumerable and discrete and also dense in itself. An example of such a set is the set $\text{Fin} = \{ \alpha \in \mathcal{N} \mid \exists n \forall m > n \exists m \mid [\alpha(m) = 0] \}$ consisting of all $\alpha$ in $\mathcal{N}$ that assume only finitely many times a value different from 0. Second item of the next theorem is the statement that every Cantor-Bendixson set reduces to every such set.

The fourth item is a subtle extension of this fact that we need for proving the fifth item. The statement is that, for each Cantor-Bendixson set $C$, for each subset $D$ of $\mathcal{N}$ that is
enumerable and discrete and dense in itself, there is a continuous function from \( N \) to \( N \) reducing, for every stump \( \tau \), the \( \tau \)-th perhapsive extension \( \mathbb{P}(\tau, C) \) of \( C \) to the \( \tau \)-th perhapsive extension \( \mathbb{P}(\tau, D) \) of \( D \).

Let \( X \) be a subset of \( N \).

We define: \( X \) has bounded perhapsity if and only if there exists a stump \( \sigma \) such that, for every stump \( \tau \), if \( \sigma \leq \tau \), then \( \mathbb{P}(\tau, X) \) coincides with \( \mathbb{P}(\sigma, X) \). Every Cantor-Bendixson set has bounded perhapsity (see Theorem 3.22(iii) and Theorem 3.19(i) and (ii)).

We also define: \( X \) has (positively) unbounded perhapsity if and only if for all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma < \tau \), then \( \mathbb{P}(\sigma, X) \) is a proper subset of \( \mathbb{P}(\tau, X) \), (although there certainly is no element of \( \mathbb{P}(\tau, X) \) that does not belong to \( \mathbb{P}(\sigma, X) \)). The fifth item of the next theorem is the statement that every enumerable and discrete subset of \( N \) that is also dense in itself, has (positively) unbounded perhapsity.

4.5. **Theorem:** Let \( D \) be an enumerable and discrete subset of \( N \) that is also dense in itself.

(i) For each stump \( \sigma \), for every \( s \), if \( s \) contains an element of \( D \), then there exists an embedding of \( C B \sigma \) into \( N \cap s \) mapping \( C B \sigma \) itself onto a decidable subset of \( D \).

(ii) For each stump \( \sigma \), there exists a continuous function from \( N \) to \( N \) reducing the set \( C B \sigma \) to the set \( D \).

(iii) For no stump \( \sigma \), \( D \) reduces to \( C B \sigma \).

(iv) For each stump \( \sigma \), there exists a continuous function from \( N \) to \( N \) reducing, for each stump \( \tau \), the set \( \mathbb{P}(\tau, C B \sigma) \) to the set \( \mathbb{P}(\tau, D) \).

(v) \( D \) has (positively) unbounded perhapsity, that is, for all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma < \tau \), then \( \mathbb{P}(\sigma, D) \) is a proper subset of the set \( \mathbb{P}(\tau, D) \).

**Proof.**

(i) We use induction on the set of stumps. The statement is obviously true in case \( \sigma \) equals the empty stump \( \emptyset \). Now assume \( \sigma \) is a nonempty stump and for each \( n, s \) such that \( s \) contains an element of \( D \) there exists an embedding of \( C B \sigma \) into \( N \cap s \) mapping \( C B \sigma \) onto a decidable subset of \( D \). Let \( s \) be a natural number containing an element of \( D \). Find \( \alpha \) in \( N \) such that \( s \ast \alpha \) belongs to \( D \). We calculate two sequences \( k_0, k_1, \ldots \) and \( p_0, p_1, \ldots \) of natural numbers such that \( k_0 < k_1 < \cdots \) and, for each \( n, \) \( p_n \) is different from \( \alpha (k_n) \) and \( s \ast \alpha (k_n) \) contains an element of \( D \). For each \( n \) we construct an embedding \( \delta_n \) of \( C B \sigma \) into \( N \cap (s \ast \alpha (k_n) \ast \langle p_n \rangle) \) mapping \( C B \sigma \) onto a decidable subset of \( D \). Now let \( \gamma \) be an embedding of \( C B \sigma \) into \( N \cap s \) such that \( \gamma \mid \emptyset \) equals \( s \ast \alpha \) and, for each \( n, \) for each \( \beta \) in \( C B \sigma \), \( \gamma \mid (\langle n \rangle \ast (1) \ast \beta) \) equals \( s \ast \alpha (k_n) \ast \langle p_n \rangle \ast (\delta_n \ast \beta) \). Observe that \( \gamma \) maps \( C B \sigma \) itself onto a decidable subset of \( D \).

(ii) Let \( \sigma \) be a stump and, using (i), define an embedding \( \gamma \) from \( C B \sigma \) into \( N \) mapping \( C B \sigma \) itself onto a decidable subset of \( D \). Let \( \delta \) be an enumeration of \( D \), that is, \( D \) coincides with the set \( \{ \delta^0, \delta^1, \ldots \} \). Let \( E \) be the set of all \( s \) in \( N \) that contain an element of \( C B \sigma \). Observe that \( E \) is a decidable subset of \( N \). Let \( \zeta \) be a function from \( N \) to \( N \) such that for all \( \alpha \) in \( C B \sigma \), \( \zeta \mid \alpha \) coincides with \( \gamma \mid \alpha \), and for each \( \alpha, n, \) if \( \alpha \) does not belong to \( E \), then for each \( i, \) \( (\zeta \mid \alpha)(n + i) \) differs from \( \delta^i(n + i) \).

We claim that \( \zeta \) reduces \( C B \sigma \) to \( D \):

It will be clear that for every \( \alpha \), if \( \alpha \) belongs to \( C B \sigma \), then \( \zeta \mid \alpha \) belongs to \( D \). Assume now that \( \alpha \) is an element of \( N \) and \( \zeta \mid \alpha \)
belongs to \( D \). Then every initial part of \( \alpha \) belongs to \( CB_{\sigma} \) and \( \zeta \mid \alpha \) coincides with \( \gamma \mid \alpha \). We may decide if there exists \( \beta \) in \( CB_{\sigma} \) such that \( \gamma \mid \beta \) equals \( \gamma \mid \alpha \). Suppose that we decide there is no such \( \beta \), then in particular, \( \alpha \) does not belong to \( CB_{\sigma} \). However, as \( (CB_{\sigma})^{\sim} \) coincides with \( CB_{\sigma} \) (see Theorem 3.5(iii)), it is excluded that \( \alpha \) does not belong to \( CB_{\sigma} \), so there exists \( \beta \) in \( CB_{\sigma} \) such that \( \gamma \mid \beta \) equals \( \gamma \mid \alpha \). As \( \gamma \) is an embedding, we must have \( \alpha = \beta \), that is, \( \alpha \) belongs to \( CB_{\sigma} \).

This completes the proof of our claim that \( \zeta \) reduces \( CB_{\sigma} \) to \( D \).

(iii) is an easy consequence of (ii) and Theorem 3.9(vi).

(iv) Let \( \sigma \) be a stump and, as in the proof of (ii), let \( \zeta \) be a function from \( N \) to \( N \) embedding \( CB_{\sigma} \) onto a decidable subset of \( D \) and reducing \( CB_{\sigma} \) to \( D \). We claim that for each stump \( \tau \), \( \zeta \) reduces the set \( \mathbb{P}(\tau, CB_{\sigma}) \) to the set \( \mathbb{P}(\tau, D) \) and prove this claim by induction on the set of stumps. The statement is obviously true if \( \tau \) equals the empty stump \( 0 \). Now assume that \( \tau \) is a nonempty stump and, for each \( n \), \( \zeta \) reduces the set \( \mathbb{P}(\tau^{n}, CB_{\sigma}) \) to the set \( \mathbb{P}(\tau^{n}, D) \). Suppose that \( \alpha \) belongs to \( \mathbb{P}(\tau, CB_{\sigma}) \). Find \( \beta \) in \( CB_{\sigma} \) such that, if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, CB_{\sigma}) \).

Observe that, if \( \zeta \mid \alpha \) is apart from \( \zeta \mid \beta \), then \( \alpha \) is apart from \( \beta \), and \( \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, CB_{\sigma}) \), and \( \zeta \mid \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, D) \). As \( \zeta \mid \beta \) belongs to \( D \), this shows that \( \zeta \) maps \( \mathbb{P}(\tau, CB_{\sigma}) \) into \( \mathbb{P}(\tau, D) \).

Now assume that \( \alpha \) belongs to \( N \) and \( \zeta \mid \alpha \) belongs to \( \mathbb{P}(\tau, D) \). Find \( \beta \) in \( D \) such that, if \( \zeta \mid \alpha \) is apart from \( \beta \), then \( \zeta \mid \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, D) \). Now find out if there exists \( \delta \) in \( CB_{\sigma} \) such that \( \zeta \mid \delta \) equals \( \beta \) and distinguish two cases.

First Case. There exists \( \delta \) in \( CB_{\sigma} \) such that \( \zeta \mid \delta \) equals \( \beta \), say \( \delta_{0} \). Observe that both \( \alpha \) and \( \delta_{0} \) belong to \( CB_{\sigma} \) and that, if \( \alpha \) is apart from \( \delta_{0} \), then \( \zeta \mid \alpha \) is apart from \( \zeta \mid \delta_{0} = \beta \), therefore \( \zeta \mid \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, D) \) and \( \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, CB_{\sigma}) \). As \( \delta_{0} \) belongs to \( CB_{\sigma} \), this shows that \( \alpha \) belongs to \( \mathbb{P}(\tau, CB_{\sigma}) \).

Second Case. There is no \( \delta \) in \( CB_{\sigma} \) such that \( \zeta \mid \delta \) equals \( \beta \). As \( \beta \) belongs to \( D \) and \( D \) is discrete, this implies that, for every \( \delta \) in \( CB_{\sigma} \), \( \beta \) is apart from \( \zeta \mid \delta \). So, for every \( \delta \) in \( CB_{\sigma} \) there exists \( n \) such that \( \delta \neq \beta n \). Using Theorem 3.7(i) we conclude that there exists \( n \) such that, for every \( \delta \) in \( CB_{\sigma} \), \( \delta \neq \beta n \). It follows that, for every \( \delta \) in the closure \( CB_{\sigma} \) of \( CB_{\sigma} \), \( \beta \) is apart from \( \zeta \mid \delta \).

Now observe that \( \zeta \mid \alpha \) belongs to \( \mathbb{P}(\tau, D) \) and therefore to \( D^{\sim} \). As \( \zeta \) reduces \( CB_{\sigma} \) to \( D \), \( \zeta \) also reduces \( (CB_{\sigma})^{\sim} \) to \( D^{\sim} \). We conclude that \( \alpha \) itself belongs to \( (CB_{\sigma})^{\sim} = CB_{\sigma} \), therefore \( \beta \) is apart from \( \zeta \mid \alpha \), and \( \zeta \mid \alpha \) belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, D) \) and \( \alpha \) itself belongs to \( \bigcup_{n \in N} \mathbb{P}(\tau^{n}, CB_{\sigma}) \), and therefore also to \( \mathbb{P}(\tau, CB_{\sigma}) \).

We have shown that, for every \( \alpha \), if \( \zeta \mid \alpha \) belongs to \( \mathbb{P}(\tau, D) \), then \( \alpha \) belongs to \( \mathbb{P}(\tau, CB_{\sigma}) \) and conclude; \( \zeta \) reduces \( \mathbb{P}(\tau, CB_{\sigma}) \) to \( \mathbb{P}(\tau, D) \).

(v) Let \( \sigma, \tau \) be hereditarily repetitive stumps such that \( \sigma < \tau \). We conclude from Theorem 3.22 that \( \mathbb{P}(S(\tau), CB_{S(\tau)}) \) coincides with \( CB_{S(\tau)} \) and \( \mathbb{P}(\sigma, CB_{S(\tau)}) \) does not, and therefore, in view of Theorem 3.19(ii), \( \mathbb{P}(\sigma, CB_{S(\tau)}) \) is a proper subset of \( \mathbb{P}(S(\tau), CB_{S(\tau)}) \).

Let \( \zeta \) be a function from \( N \) to \( N \) embedding \( CB_{S(\tau)} \) onto a decidable subset of \( D \) and reducing \( CB_{S(\tau)} \) to \( D \). We constructed such a function in the proof of (ii).

We also saw, in our proof of (iv), that \( \zeta \) also reduces \( \mathbb{P}(\sigma, CB_{S(\tau)}) \) to \( \mathbb{P}(\sigma, D) \) and \( \mathbb{P}(S(\tau), CB_{\tau}) \) to \( \mathbb{P}(S(\tau), D) \). Assume that \( \mathbb{P}(S(\tau), D) \) is a subset of \( \mathbb{P}(\sigma, D) \). It follows that \( \mathbb{P}(S(\tau), CB_{S(\tau)}) \) is a subset of \( \mathbb{P}(\sigma, CB_{S(\tau)}) \) and we obtain a contradiction.
We conclude that \( \mathbb{P}(S(\tau), D) \) is not a subset of \( \mathbb{P}(\sigma, D) \) and that \( \mathbb{P}(\sigma, D) \) is a proper subset of \( \mathbb{P}(S(\tau), D) \).

It follows that \( \mathbb{P}(\sigma, D) \) is also a proper subset of \( \mathbb{P}(\tau, D) \). For suppose that \( \mathbb{P}(\sigma, D) \) coincides with \( \mathbb{P}(\tau, D) \) and thus with \( \mathbb{P}(S(\sigma), D) \). As \( S(\sigma) \leq \tau \), \( \mathbb{P}(S(\sigma), D) \) is a subset of \( \mathbb{P}(\tau, D) \), and also \( \mathbb{P}(S(\tau), D) \) is a subset of \( \mathbb{P}(\tau, D) \), and the sets \( \mathbb{P}(S(\tau), D) \) and \( \mathbb{P}(\sigma, D) \) coincide. Contradiction. \( \square \)

4.6. Let \( X \) be a nonempty enumerable subset of \( \mathcal{N} \). An element \( \delta \) of \( \mathcal{N} \) is called an enumeration of \( X \) if and only if \( X = \{\delta_0, \delta_1, \ldots\} \). We let \( \text{Almost}^*(X) \) be the set of all \( \alpha \) in \( \mathcal{N} \) such that, for each enumeration \( \delta \) of \( X \), for each \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \delta_\beta(\beta(n)) \). We could perhaps say that \( \text{Almost}^*(X) \) is the set of all \( \alpha \) in \( \mathcal{N} \) for which we see that every attempt to give evidence that \( \alpha \) is apart from every element of \( X \) will fail (positively): given an enumeration \( \delta \) of \( X \), such evidence would consist in an element \( \beta \) of \( \mathcal{N} \) with the property that, for all \( n \), \( \alpha(\beta(n)) \neq \delta_\beta(\beta(n)) \).

For every \( \delta \), we let \( \text{Ens}_\delta \) be the enumerable set \( \{\delta_0, \delta_1, \ldots\} \).

4.7. Theorem: Let \( X \) be an enumerable subset of \( \mathcal{N} \).

(i) \( X \) is a subset of \( \text{Almost}^*(X) \).

(ii) For all \( \delta, \zeta \), if \( \text{Ens}_\delta \) is a subset of \( \text{Ens}_\zeta \), and, for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \delta_\beta(\beta(n)) \), then, for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \zeta_\beta(\beta(n)) \).

(iii) \( \text{Almost}(X, \text{Almost}^*(X)) \) coincides with \( \text{Almost}^*(X) \).

(iv) \( \text{Almost}^*(X) \) is perhapsive, that is, \( \text{Perhaps} (\text{Almost}^*(X), \text{Almost}^*(X)) = \text{Almost}^*(X) \).

(v) \( \text{Almost}(X) \) is a subset of \( \text{Almost}^*(X) \).

Proof.

The proof of (i) is left to the reader.

(ii) Suppose that \( \text{Ens}_\delta \) is a subset of \( \text{Ens}_\zeta \). Using the Second Axiom of Countable Choice, we find \( \gamma \) such that, for each \( n \), \( \delta^n = \zeta^{-\gamma(n)} \). Assume that, for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \delta^n(\beta(n)) \). Then, for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \zeta^{-\gamma(n)}(\beta(n)) \). Therefore, for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \zeta^{-\gamma(n)}(\beta(n)) \), and for all \( \beta \), there exists \( n \) such that \( \alpha(\beta(n)) = \zeta^n(\beta(n)) \).

(iii) Let \( \delta \) be an enumeration of \( X \), so \( X = \text{Ens}_\delta = \{\delta_0, \delta_1, \ldots\} \). Assume that \( \alpha \) belongs to \( \text{Perhaps}(X, \text{Almost}^*(X)) \), and let \( \delta^n \) be an element of \( X \) such that, if \( \alpha \) is apart from \( \delta^n \), then \( \alpha \) belongs to \( \text{Almost}^*(X) \). Let \( \beta \) belong to \( \mathcal{N} \) and distinguish two cases: \( \text{either } \alpha(\beta(n)) \text{ equals } \delta^n(\beta(n)) \text{ or } \alpha(\beta(n)) \text{ is different from } \delta^n(\beta(n)) \). In the latter case \( \alpha \) is apart from \( \delta^n \), therefore \( \alpha \) belongs to \( \text{Almost}^*(X) \) and there exists \( m \) such that \( \alpha(\beta(m)) \text{ equals } \delta^n(\beta(m)) \). So in both cases there exists \( m \) such that \( \alpha(\beta(m)) = \delta^n(\beta(m)) \). We conclude, using (ii), that \( \alpha \) belongs to \( \text{Almost}^*(X) \).

(iv) Let \( \delta \) be an enumeration of \( X \), so \( X = \{\delta_0, \delta_1, \ldots\} \). Assume that \( \alpha \) belongs to \( \text{Perhaps} (\text{Almost}^*(X), \text{Almost}^*(X)) \) and let \( \gamma \) be an element of \( \text{Almost}^*(X) \) such that, if \( \alpha \) is apart from \( \gamma \), then \( \alpha \) belongs to \( \text{Almost}^*(X) \). Let \( \beta \) belong to \( \mathcal{N} \) and find \( n \) such that \( \gamma(\beta(n)) = \delta^n(\beta(n)) \), and distinguish two cases: \( \text{either } \alpha(\beta(n)) \text{ equals } \gamma(\beta(n)) \text{ or } \alpha(\beta(n)) \text{ is different from } \gamma(\beta(n)) \). In the latter case \( \alpha \) is apart from \( \gamma \), therefore \( \alpha \) belongs to \( \text{Almost}^*(X) \), and there exists \( m \) such that \( \alpha(\beta(m)) = \delta^n(\beta(m)) \).
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\[ \overline{\varphi}^{m} \beta(m) \]. So in both cases there exists \( m \) such that \( \overline{\varphi}(\beta(m)) = \overline{\varphi}^{m}(\beta(m)) \). We conclude that \( \alpha \) belongs to \( \text{Almost}^{*}(X) \).

(v) This follows from (iv) and Theorem 4.2(vi).

4.8. Let \( \sigma \) be a stump, and suppose \( \delta, \alpha \) belong to \( \mathcal{N} \). \( \sigma \) secures \( \alpha \) with respect to \( \delta \) if and only if for every \( \beta \) there exists \( n \) such that \( \overline{\varphi}^{m} \beta(n) \) belongs to \( \alpha \) and there exists \( m < n \) such that \( \overline{\varphi}(\beta(m)) = \overline{\varphi}^{m}(\beta(m)) \).

The following statement follows from Brouwer’s Thesis on bars (see Subsection 2.6.2):

For every \( \delta, \alpha \), if \( \alpha \) belongs to \( \text{Almost}^{*}(\mathcal{E}_{\alpha}^{\delta}) \), then there exists a stump \( \sigma \) that secures \( \alpha \) with respect to \( \delta \).

4.9. Theorem:

(i) For every stump \( \sigma \), for every \( \alpha \), for every \( \delta \), if \( \sigma \) secures \( \alpha \) with respect to \( \delta \), then \( \alpha \) belongs to \( \mathbb{P}(\sigma, \mathcal{E}_{\alpha}^{\delta}) \).

(ii) (Using Brouwer’s Thesis on bars:)

For every enumerable subset \( X \) of \( \mathcal{N} \), \( \text{Almost}^{*}(X) \) coincides with \( \text{Almost}(X) \) and \( \text{Almost}^{*}(X) \) is a subset of \( X^{\neg\neg} \).

Proof.

(i) We use induction on the set of stumps.

Note that the statement is true if \( \sigma \) is the empty stump \( \emptyset \), as there are no \( \alpha, \delta \) such that \( \emptyset \) secures \( \alpha \) with respect to \( \delta \). Now assume that \( \sigma \) is a nonempty stump and that, for each \( n \), the statement holds for \( \sigma^{n} \). Suppose \( \alpha, \delta \) are such that \( \sigma \) secures \( \alpha \) with respect to \( \delta \). Assume that \( \alpha \) is apart from \( \delta^{0} \) and find \( m \) such that \( \overline{\varphi}^{1} \beta(m) \) does not belong to \( \sigma \). Now let \( \beta \) belong to \( \mathcal{N} \) and consider the infinite sequence \( \langle m \rangle \ast \beta \). Note that there exists \( n \) such that \( \langle m \rangle \ast \beta \) belongs to \( \sigma \) and, for some \( i < n \), \( \overline{\varphi}(\beta(i)) = \overline{\varphi}^{1}(\beta(i)) \). Also note that, if \( \langle m \rangle \ast \beta \) belongs to \( \sigma \), then \( \beta \) belongs to \( \sigma^{m} \). Define \( \xi \) in \( \mathcal{N} \) such that, for each \( i \) and \( \xi^{i} = \delta^{i+1} \) and observe that \( \sigma^{m} \) secures \( \alpha \) with respect to \( \mathcal{E}_{\alpha}^{\xi} \). It follows that \( \alpha \) belongs to \( \mathbb{P}(\sigma^{m}, \mathcal{E}_{\alpha}^{\xi}) \) and, as \( \mathcal{E}_{\alpha}^{\xi} \) is a subset of \( \mathcal{E}_{\alpha}^{\delta} \), also to \( \mathbb{P}(\sigma^{m}, \mathcal{E}_{\alpha}^{\delta}) \) (see Theorem 3.19(iii)). Therefore, if \( \alpha \) is apart from \( \delta^{0} \), there exists \( n \) such that \( \sigma \) belongs to \( \mathbb{P}(\sigma^{n}, \mathcal{E}_{\alpha}^{\delta}) \). As \( \delta^{0} \) belongs to \( D_{\delta} \), we may conclude that \( \alpha \) belongs to \( \mathbb{P}(\sigma, \mathcal{E}_{\alpha}^{\delta}) \).

(ii) Find \( \delta \) such that \( X = \mathcal{E}_{\alpha}^{\delta} \). Note that, for every \( \alpha \), if \( \alpha \) belongs to \( \text{Almost}^{*}(X) \), then, as we observed in Subsection 4.8, there is a stump \( \sigma \) that secures \( \alpha \) with respect to \( \delta \), and thus, by (i), \( \alpha \) belongs to \( \mathbb{P}(\sigma, \mathcal{E}_{\alpha}^{\delta}) \) and also to \( \text{Almost}(X) \). According to Theorem 4.7(v), \( \text{Almost}(X) \) is also a subset of \( \text{Almost}^{*}(X) \). Theorem 4.2(i) now implies that \( \text{Almost}^{*}(X) \) is a subset of \( X^{\neg\neg} \).

\[ \square \]

4.10. Theorem 4.9 consists of two statements concerning enumerable subsets of \( \mathcal{N} \). We now want to extend these results to the more general class of all countable unions of inhabited spreads. One of the reasons for this extension is that we want to prove that also \( E_{2} \), the leading set of the class \( \Sigma_{2}^{0} \), is a set of unbounded perhapsity.

According to the definition in Subsection 2.3.2, a subset \( X \) of \( \mathcal{N} \) is a spread if \( X \) is (sequentially) closed and there exists \( \gamma \) such that, for every \( s, \gamma(s) = 1 \) if and only if (the finite sequence of natural numbers coded by \( s \) contains an element of \( X \). Not every closed set is a spread (see Subsection 1.3). One may also verify that not every countable union
of closed sets coincides with a countable union of spreads. Still, the class of all countable unions of inhabited spreads is a large and important subclass of $\Sigma^0_2$.

Observe that, if $X$ is a spread and $s$ contains an element of $X$, then the set of all natural numbers $n$ such that $s \tau n\langle n \rangle$ contains an element of $X$ is an inhabited and decidable subset of $\mathbb{N}$.

Let $\gamma$ belong to $\mathcal{N}$. We let $Z_\gamma$ be the set of all natural numbers $n$ such that either $\gamma (0) = n$ or $\gamma (n + 1) = 1$. Observe that $Z_\gamma$ is an inhabited and decidable subset of $\mathbb{N}$. Conversely, if $X$ is an inhabited and decidable subset of $\mathbb{N}$, we may find $\gamma$ such that $X$ coincides with $Z_\gamma$. Thus we obtain a nice survey of all inhabited and decidable subsets of $\mathbb{N}$. We want to obtain a similar survey of all spreads.

Let $\gamma$ belong to $\mathcal{N}$. We let $S_\gamma$ be the set of all $a$ such that, for each $n$, either $\gamma (\langle an\rangle, 0) = a(n)$ or $\gamma (\langle an\rangle, a(n) + 1) = 1$. Observe that $S_\gamma$ is an inhabited spread, and that, for each $s$, for each $n$, $s \tau n\langle n \rangle$ contains an element of $S_\gamma$ if and only if $s$ contains an element of $S_\gamma$ and either $\gamma (\langle s\rangle, 0) = n$ or $\gamma (\langle s\rangle, n + 1) = 1$. Conversely, if $X$ is an inhabited subset of $\mathcal{N}$ and a spread, we may find $\gamma$ such that $X$ coincides with $S_\gamma$.

Let $\delta$ be a subset of $\mathcal{N}$ such that $Cus_\delta$ is a subset of $\delta$, we let $Perhaps^\delta (\delta, Y)$ be the set of all $\alpha$ such that, for some $n$, for all $m$, if $\overline{\alpha}^n m$ does not contain an element of $S_{\overline{\alpha}^n m}$, then $\alpha$ belongs to $Y$.

For every $\delta$, we define a subset $\mathbb{P}^\delta (\sigma, \delta)$ of $\mathcal{N}$, as follows, by induction on the set of stumps:

(i) $\mathbb{P}^\delta (1, \delta) = Cus_\delta$, and,

(ii) for every nonempty stump $\sigma$, $\mathbb{P}^\delta (\sigma, \delta)$ coincides with $Perhaps^\delta (\delta, \bigcup_{n \in \mathbb{N}} \mathbb{P}^\delta (\sigma^n, Cus_\delta))$.

For every $\delta$, we let $Almost^\delta (\delta)$ be the set of all $\alpha$ such that, for every $\beta$, there exists $n$ with the property: $\overline{\beta} (\beta (n))$ contains an element of $S_{\overline{\alpha}^n \beta}$, Intuitively, $\alpha$ belongs to $Almost^\delta (\delta)$ if we positively know there is no $\beta$ effectively proving that $\alpha$ does not belong to $Cus_\delta$ in the strong sense that, for every $n$, $\overline{\beta} (\beta (n))$ does not contain an element of $S_{\overline{\alpha}^n \beta}$.

Let $\sigma$ be a stump, and suppose $\delta, \alpha$ belong to $\mathcal{N}$. $\sigma$ secures $^\delta \alpha$ with respect to $\delta$ if and only if for every $\beta$ there exists $n$ such that $\beta n$ belongs to $\sigma$ and there exists $m < n$ such that $\overline{\beta} (\beta (m))$ contains an element of $S_{\overline{\alpha}^n \beta}$.

The following statement follows from Brouwer’s Thesis on bars:

For every $\delta, \alpha$, if $\alpha$ belongs to $Almost^\delta (\delta)$, then there exists a stump $\sigma$ that secures $^\delta \alpha$ with respect to $\delta$.

The main items of the next theorem are (vii) and (viii). These items extend the conclusions we found for enumerable subsets of $\mathcal{N}$ in Theorem 4.9(ii) to countable unions of spreads. The other items are about characterizing the perhapsive extensions and the perhapsive closure of a countable union of spreads.

4.11. Theorem:

(i) For every $\delta$, for every subset $Y$ of $\mathcal{N}$ such that $Cus_\delta$ is a subset of $Y$, $Perhaps^\delta (\delta, Y)$ coincides with $Perhaps(Cus_\delta, Y)$. 

(ii) For every \( \delta \), for every stump \( \sigma \), \( \mathbb{P}^\triangle(\sigma, \delta) \) coincides with \( \mathbb{P}(\sigma, Cus_\delta) \).

(iii) For every \( \delta \), for every \( \epsilon \), if \( Cus_\delta \) is a subset of \( Cus_\epsilon \), then, for every subset \( Y \) of \( \mathcal{N} \) such that \( Cus_\epsilon \) is a subset of \( Y \), \( \text{Perhaps}(\delta, Y) \) is a subset of \( \text{Perhaps}(\epsilon, Y) \), and, for every stump \( \sigma \), \( \mathbb{P}^\triangle(\sigma, \delta) \) is a subset of \( \mathbb{P}^\triangle(\sigma, \epsilon) \).

(iv) For every \( \delta \), for every \( \epsilon \), if \( Cus_\delta \) is a subset of \( Cus_\epsilon \), then \( \text{Almost}^\triangle(\delta) \) is a subset of \( \text{Almost}^\triangle(\epsilon) \), and thus, if \( Cus_\delta \) coincides with \( Cus_\epsilon \), then \( \text{Almost}^\triangle(\delta) \) coincides with \( \text{Almost}^\triangle(\epsilon) \).

(v) For every \( \delta \), the set \( \text{Almost}^\triangle(\delta) \) is perhapsive.

(vi) For every \( \delta \), for every stump \( \sigma \), \( \mathbb{P}^\triangle(\sigma, \delta) \) is a subset of \( \text{Almost}^\triangle(\delta) \).

(vii) (Using Brouwer’s Thesis on bars:)

For every \( \delta \), the set \( \text{Almost}^\triangle(\delta) \) coincides with \( \bigcup_{\sigma \in \text{sup}} \mathbb{P}^\triangle(\sigma, \delta) \), and thus with \( \text{Almost}^\triangle(\text{Cus}_\delta) \).

(viii) (Using Brouwer’s Thesis on bars:)

For every \( \delta \), the set \( \text{Almost}^\triangle(\delta) \) is a subset of the set \( (\text{Cus}_\delta)^- \).

Proof.

(i) Let \( \delta, \alpha \) belong to \( \mathcal{N} \) and assume that \( \alpha \) belongs to \( \text{Perhaps}^\triangle(\delta, Y) \). Find \( n \) such that for all \( m \), if \( \overline{\alpha}m \) does not contain an element of \( S_\theta^\varphi \), then \( \alpha \) belongs to \( Y \). Now define \( \beta \) such that, for all \( m \), if \( \overline{\beta}m \ast (\alpha(m)) \) contains an element of \( S_\varphi^\psi \), then \( \beta(m) = \alpha(m) \), and, if not, then \( \beta(m) = \text{the least } k \text{ such that } \overline{\beta}m \ast (k) \text{ contains an element of } S_\varphi^\psi \). Observe that \( \beta \) belongs to \( S_\varphi^\psi \) and therefore to \( Cus_\delta \) and that, if \( \alpha \) is apart from \( \beta \), then, for some \( m \), \( \overline{\alpha}m \) does not contain an element of \( S_\varphi^\psi \), and, therefore, \( \alpha \) belongs to \( Y \). We conclude that \( \alpha \) belongs to \( \text{Perhaps}(Cus_\delta, Y) \).

Conversely, assume that \( \alpha \) belongs to \( \text{Perhaps}(Cus_\delta, Y) \). Find \( \beta \) in \( Cus_\delta \) such that, if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( Y \). Find \( n \) such that \( \beta \) belongs to \( S_\varphi^\psi \). Note that, if \( \overline{\alpha}m \) does not contain an element of \( S_\varphi^\psi \), then \( \alpha \) is apart from \( \beta \), and \( \alpha \) belongs to \( Y \). We conclude that \( \alpha \) belongs to \( \text{Perhaps}^\triangle(\delta, Y) \).

(ii) We use (i) and straightforward induction on the set of stumps.

(iii) This is an immediate consequence of (i) and (ii), Theorem 3.17(ii) and Theorem 3.19(iii).

(iv) Assume that \( Cus_\delta \) is a subset of \( Cus_\epsilon \) and let \( \alpha \) belong to \( \text{Almost}^\triangle(\delta) \). We determine \( \gamma \) such that, for each \( n \), for each \( m \), if \( \overline{\gamma}n^\varphi \ast (\alpha(m)) \) contains an element of \( S_\varphi^\psi \), then \( \gamma(n)(m) = \alpha(m) \), and, if not, then \( \gamma(n)(m) = \text{the least } k \text{ such that } \overline{\gamma}n^\varphi \ast (k) \text{ contains an element of } S_\varphi^\psi \). Observe that \( \gamma \) belongs to \( S_\varphi^\psi \) and therefore to \( Cus_\delta \) and that, if \( \alpha \) is apart from \( \gamma \), then, for some \( n \), \( \overline{\alpha}n \) does not contain an element of \( S_\varphi^\psi \), and, therefore, \( \alpha \) belongs to \( Y \). We conclude that \( \alpha \) belongs to \( \text{Perhaps}(Cus_\delta, Y) \).

Conversely, assume that \( \alpha \) belongs to \( \text{Perhaps}(Cus_\delta, Y) \). Find \( \beta \) in \( Cus_\delta \) such that, if \( \alpha \) is apart from \( \beta \), then \( \alpha \) belongs to \( Y \). Find \( n \) such that \( \beta \) belongs to \( S_\varphi^\psi \). Note that, if \( \overline{\alpha}n \) does not contain an element of \( S_\varphi^\psi \), then \( \alpha \) is apart from \( \beta \), and \( \alpha \) belongs to \( Y \). We conclude that \( \alpha \) belongs to \( \text{Perhaps}^\triangle(\delta, Y) \).

(v) Let \( \delta \) be an element of \( \mathcal{N} \) and let \( \gamma \) belong to \( \text{Almost}^\triangle(\delta) \). Let \( \alpha \) be an element of \( \mathcal{N} \) and suppose that, if \( \alpha \) is apart from \( \gamma \), then \( \alpha \) belongs to \( \text{Almost}^\triangle(\delta) \). We have to show that \( \alpha \) belongs to \( \text{Almost}^\triangle(\delta) \). Let \( \beta \) belong to \( \mathcal{N} \). Find \( n \) such that
\( \overline{\gamma}(\beta(n)) \) contains an element of \( S_\varphi \). Now distinguish two cases. Either \( \overline{\alpha}(\beta(n)) = \overline{\gamma}(\beta(n)) \) and thus \( \overline{\alpha}(\beta(n)) \) contains an element of \( S_\varphi \) or \( \overline{\alpha}(\beta(n)) \neq \overline{\gamma}(\beta(n)) \).

In the latter case \( \alpha \) is apart from \( \gamma \) and thus belongs to \( \text{Almost}^\wedge(\delta) \). In particular, there exists \( k \) such that \( \overline{\alpha}(\beta(k)) \) contains an element of \( S_\varphi \). We thus see that \( \alpha \) belongs to \( \text{Almost}^\wedge(\delta) \).

(vi) Using (ii), we conclude that \( \mathbb{P}^\wedge(\sigma, \delta) \) is a subset of \( \text{Almost}(\text{Cus}_\delta) \) and thus of the least perhapsive set containing \( \text{Cus}_\delta \) (see Theorem 4.2(vi)). As, by (v), \( \text{Almost}^\wedge(\delta) \) is perhapsive, \( \mathbb{P}^\wedge(\sigma, \delta) \) is a subset of \( \text{Almost}^\wedge(\delta) \).

(vii) We first prove, by induction on the set of stumps, that, for every stump \( \sigma \), for all \( \alpha, \delta \), if \( \sigma \) secures \( \wedge^\wedge \) \( \alpha \) with respect to \( \delta \), then \( \alpha \) belongs to \( \mathbb{P}^\wedge(\sigma, \delta) \).

Observe that this statement holds if \( \alpha \) is the empty stump \( \varnothing \), as there are no \( \alpha, \sigma \) such that \( \varnothing \) secures \( \wedge^\wedge \) \( \alpha \) with respect to \( \sigma \).

Now assume that \( \sigma \) is a nonempty stump and that, for each \( n \), the statement holds for \( \sigma \langle n \rangle \). Suppose that \( \alpha \), \( \sigma \) are such that \( \sigma \) secures \( \wedge^\wedge \) \( \alpha \) with respect to \( \delta \). Assume that \( m \) is a natural number with the property that \( \alpha \langle m \rangle \) does not contain an element of \( S_{\varphi+1} \). Also note that, if \( \langle m \rangle \ast \beta n \) belongs to \( \sigma \), then \( \beta n \) belongs to \( \sigma^m \).

Define \( \zeta \) in \( \mathcal{N} \) such that, for each \( i \), \( \zeta^i = \delta^{i+1} \) and observe that \( \sigma^m \) secures \( \wedge^\wedge \) \( \alpha \) with respect to \( \zeta \). Therefore \( \alpha \) belongs to \( \mathbb{P}^\wedge(\sigma^m, \zeta) \) and, as \( \text{Cus}_\zeta \) is a subset of \( \text{Cus}_\delta \), also to \( \mathbb{P}^\wedge(\sigma^m, \delta) \).

Therefore, if, for some \( m \), \( \alpha m \) does not contain an element of \( S_{\varphi+1} \), there exists \( n \) such that \( \alpha \) belongs to \( \mathbb{P}^\wedge(\sigma^n, \delta) \). It follows that \( \alpha \) belongs to \( \mathbb{P}^\wedge(\sigma, \delta) \).

Using Brouwer’s Thesis we conclude, that, for every \( \alpha \), if \( \alpha \) belongs to \( \text{Almost}^\wedge(\text{Cus}_\delta) \), then there exists a stump \( \sigma \) securing \( \wedge^\wedge \) \( \alpha \) with respect to \( \delta \) and, therefore, \( \alpha \) belongs to \( \mathbb{P}^\wedge(\sigma, \delta) \).

(viii) This follows from (vii), (ii) and Theorem 4.2(i). \( \square \)

4.12. \( E_2 \) is the set of all \( \alpha \) such that, for some \( n \), \( \alpha^n = \varnothing \), and \( A_2 \) is the set of all \( \alpha \) such that, for every \( n \), there exists \( m \) with the property \( \alpha^n(m) \neq 0 \). We introduced these sets in Veldman (2008a, subsection 2.10) and we proved that these sets are complete elements of the classes \( \Sigma_2^0 \) and the class \( \Pi_2^0 \), respectively, that is, \( \Sigma_2^0 \) is the class of all subsets of \( \mathcal{N} \) reducing to \( E_2 \) and \( \Pi_2^0 \) is the class of all subsets of \( \mathcal{N} \) reducing to \( A_2 \) (see Veldman, 2008a, theorem 2.11). Note that \( E_2 \) is a countable union of inhabited spreads. We let \( \eta \) be an element of \( \mathcal{N} \) such that, for each \( n \), \( S_{\varphi} \) is the set of all \( \alpha \) such that \( \alpha^n = \varnothing \). Observe that \( E_2 \) coincides with \( \text{Cus}_\eta \). We define \( \text{Almost}^\wedge(E_2) := \text{Almost}^\wedge(\eta) \).

For every subset \( X \) of \( \mathcal{N} \) we let \( X^c \), the constructive complement of \( X \), be the set of all \( \alpha \) in \( \mathcal{N} \) such that, for every \( \beta \) in \( X \), \( \alpha \) is apart from \( \beta \).

Note that every element of \( A_2 \) is apart from every element of \( E_2 \).

We let \( \text{Fin} \) be the set of all \( \alpha \) in Cantor space \( C \) such that there exists \( n \) such that, for all \( j > n \), \( \alpha(j) = 0 \). An element \( \alpha \) of \( C \) belongs to \( \text{Fin} \) if and only if \( \alpha \) is the characteristic function of a finite subset of the set \( \mathbb{N} \) of natural numbers.

\( \text{Fin} \) is an enumerable and discrete subset of \( \mathcal{N} \) that is dense in itself. For each stump \( \sigma \), \( \text{CB}_\sigma \) is a decidable subset of \( \text{Fin} \).

It follows from Theorem 4.5 that, for every stump \( \sigma \), \( \text{CB}_\sigma \) reduces to \( \text{Fin} \).

We let \( \text{Inf} \) be the set of all \( \alpha \) in Cantor space \( C \) such that, for all \( n \), there exists \( j \) such that \( j > n \) and \( \alpha(j) = 1 \). An element \( \alpha \) of \( C \) belongs to \( \text{Inf} \) if and only if \( \alpha \) is the characteristic function of an infinite subset of the set \( \mathbb{N} \) of natural numbers.
Note that every element of Inf is apart from every element of Fin.
The reader should compare the sixth item of the following theorem with Theorem 5.4(iv)
in Section 5.

4.13. Theorem:
(i) The set $(E_2)^c$ coincides with the set $A_2$ and the set Fin$^c \cap C$ coincides with the set Inf.
(ii) The set $(A_2)^c$ coincides with the set Almost$^\triangleleft(E_2)$ and the set Inf$^c \cap C$ coincides with the set Almost$^*$ (Fin).
(iii) The set Almost$^\triangleleft(E_2)$ coincides with the set of all $\alpha$ such that, for every $\gamma$, there exists $n$ such that $a^n(\gamma(n)) = 0$ and the set Almost$^*$ (Fin) coincides with the set of all $\alpha$ such that, for every $\gamma$, if, for every $n$, $\gamma(n) < \gamma(n+1)$, then there exists $p$ such that $a(\gamma(p)) = 0$.
(iv) There exists a function $\gamma$ from $\mathbb{N}$ to $\mathbb{N}$ reducing, for each stump $\sigma$, the set $\mathbb{P}(\sigma, \text{Fin})$ to the set $\mathbb{P}(\sigma, E_2)$.
(v) The set $E_2$ is of (positively) unbounded perhapsity, that is, for all stumps $\sigma, \tau$, if $\sigma < \tau$, then $\mathbb{P}(\sigma, E_2)$ is a proper subset of $\mathbb{P}(\tau, E_2)$.
(vi) The sets Almost$^*$ (Fin) and Almost$^\triangleleft(E_2)$ reduce to each other.

Proof.
(i) Every element of $A_2$ is apart from every element of $E_2$ and, therefore, $A_2$ is a subset of $(E_2)^c$. Now let $\alpha$ belong to $(E_2)^c$ and $n$ to $\mathbb{N}$. Let $\beta$ be an element of $\mathcal{N}$ satisfying $\beta^n = 0$ and, for all $p$, if $p \neq n$, then $\beta^p = \alpha^p$, and $\beta(0) = \alpha(0)$. Note that $\beta$ belongs to $E_2$. As $\alpha$ is apart from $\beta$, there exists $m$ such that $a^n(m) \neq 0$. Thus we see that $(E_2)^c$ is a subset of $A_2$.

Every element of Inf is apart from every element of Fin and, therefore, Inf is a subset of Fin$^c$. Now let $\alpha$ belong to Fin$^c \cap C$ and $n$ to $\mathbb{N}$. As $\alpha$ is apart from $\alpha n \ast 0$, there exists $m > n$ such that $a(m) = 1$. Thus we see that Fin$^c \cap C$ is a subset of Inf.

(ii) Let $\alpha$ belong to $(A_2)^c$. We have to prove that $\alpha$ belongs to Almost$^\triangleleft(E_2)$, that is, we have to show that, for every $\beta$, there exists $n$ such that $a^n(\beta(n)) = 0(\beta(n))$.

Let $\beta$ belong to $\mathcal{N}$. We define $\gamma$ in $\mathcal{N}$ such that $\gamma(()) = a(())$ and, for each $n$, if $\gamma^n(\beta(n)) \neq 0(\beta(n))$, then $\gamma^n = a^n$, and, if $\gamma^n(\beta(n)) = 0(\beta(n))$, then $\gamma^n = 1$. Note that $\gamma$ belongs to $A_2$, and, therefore, $\alpha \# \gamma$. Note that $\gamma(()) = a(())$ and find $n, i$ such that $\gamma((), i) \neq a((), i)$. Conclude that $\gamma^n(\beta(n)) = 0(\beta(n))$.

Let $\alpha$ belong to Almost$^\triangleleft(E_2)$. We have to prove that $\alpha$ belongs to $(A_2)^c$, that is, for every $\beta$ in $A_2$, $\alpha \# \beta$. Let $\beta$ belong to $A_2$. Find $\gamma$ in $\mathcal{N}$ such that, for each $n$, $\beta^n(\gamma(n)) \neq 0$. Find $n$ such that $\gamma^n((\gamma(n) + 1) = 0((\gamma(n) + 1))$. Conclude that $\beta^n(\gamma(n)) \neq a^n(\gamma(n))$, so $\alpha \# \beta$.

Let $\alpha$ belong to Inf$^c \cap C$. We have to prove that $\alpha$ belongs to Almost$^*$ (Fin), that is, for every enumeration $\delta$ of Fin, for each $\beta$ in $\mathcal{N}$, there exists $n$ such that $\alpha(\beta(n)) = \overline{\delta}^n(\beta(n))$. Let $\delta$ be an enumeration of Fin, and let $\beta$ belong to $\mathcal{N}$. Find $c$ in $\mathcal{N}$ such that, for each $n$, $\alpha n \ast 0$ coincides with $\delta^c(n)$. Note that, for each $n$, if $\delta^c(n)(\beta(c(n))) \neq \overline{\alpha}(\beta(c(n)))$, then there exists $m > n$ such that $\alpha(m) = 1$. Let $\gamma$ be an element of $\mathcal{N}$ such that, for each $n$, if for every $i \leq n$, $\delta^c(i)(\beta(c(i))) \neq \overline{\alpha}(\beta(c(i)))$, then $\gamma(n) = \alpha(n)$, and if there exists $i < n$ such that $\delta^c(i)(\beta(c(i))) \neq \overline{\alpha}(\beta(c(i)))$, then $\gamma(n) = 1$. Note that $\gamma$ belongs to Inf and find $n$ such that $\alpha(n) \neq \gamma(n)$. Clearly, for some $j$, $\overline{\alpha}(\beta(j)) = \delta^j(\beta(j))$. 
Let $\alpha$ belong to $\text{Almost}^* (\text{Fin})$. We have to prove that $\alpha$ belongs to $\text{Inf} \cap C$, that is, for every $\beta$ in $\text{Inf}$, $\alpha \neq \beta$. Let $\beta$ belong to $\text{Inf}$. Let $\delta$ be an enumeration of $\text{Fin}$. Find $\epsilon$ in $\mathcal{N}$ such that, for each $n$, $\overline{\delta}(\epsilon(n)) \neq \overline{\beta}(\epsilon(n))$. Find $n$ such that $\overline{\delta}(\epsilon(n)) = \overline{\beta}(\epsilon(n))$ and conclude: $\overline{\epsilon} \neq \overline{\beta}(\epsilon(n))$ and: $\alpha \neq \beta$.

(iii) In view of (ii), it suffices to show: for every $\alpha$, $\alpha$ belongs to $(A_2)^c$ if and only if, for each $\gamma$, there exists $n$ such that $\sigma^n(\gamma(n)) = 0$.

Let $\alpha$ belong to $(A_2)^c$ and $\gamma$ in $\mathcal{N}$. Define $\beta$ in $\mathcal{N}$ as follows. For each $n$, $\beta(n, \gamma(n)) = \max(1, \alpha(n, \gamma(n)))$, and, for each $m$, if there is no $n$ such that $m = n$, $\beta(m) = \alpha(m)$. Note that $\beta$ belongs to $A_2$ and find $m$ such that $\alpha(m) \neq \beta(m)$. Find $n$ such that $m = n$, $\gamma(n))$ and conclude: $\sigma^n(\gamma(n)) = \alpha(n, \gamma(n)) = 0$.

Conversely, suppose that $\alpha$ belongs to $\mathcal{N}$, and, for each $\gamma$, there exists $n$ such that $\sigma^n(\gamma(n)) = 0$. Let $\beta$ belong to $A_2$. Find $\gamma$ in $\mathcal{N}$ such that, for each $n$, $\sigma^n(\gamma(n)) \neq 0$. Find $n$ such that $\sigma^n(\gamma(n)) = 0$ and conclude: $\beta \neq \alpha$.

As to the second statement, it suffices to show, in view of (ii): for every $\alpha$, $\alpha$ belongs to $(\text{Inf})^c$ if and only if, for each strictly increasing $\gamma$, there exists $n$ such that $\alpha(\gamma(n)) = 0$.

Let $\alpha$ belong to $(\text{Inf})^c$ and let $\gamma$ be an element of $\mathcal{N}$ such that, for each $n$, $\gamma(n) < \gamma(n + 1)$. Define $\beta$ in $\mathcal{N}$ as follows. For each $n$, $\beta(\gamma(n)) = \max(1, \alpha(\gamma(n)))$, and, for each $m$, if there is no $n$ such that $m = \gamma(n)$, $\beta(m) = \alpha(m)$. Note that $\beta$ belongs to $\text{Inf}$ and find $m$ such that $\alpha(m) \neq \beta(m)$. Find $n$ such that $m = \gamma(n)$ and conclude: $\alpha(\gamma(n)) = 0$.

Conversely, suppose that $\alpha$ belongs to $\mathcal{N}$, and, for each $\gamma$, there exists $n$ such that $\alpha(\gamma(n)) = 0$. Let $\beta$ belong to $\text{Inf}$. Find $\gamma$ in $\mathcal{N}$ such that, for each $n$, $\beta(\gamma(n)) \neq 0$. Find $n$ such that $\beta(\gamma(n)) = 0$ and conclude: $\beta \neq \alpha$.

(iv) We use induction on the set of stumps. Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$ in $\mathcal{C}$, for every $m, n$, if, for every $i \leq m = n$, $\alpha(i) \leq 1$, then $(\gamma \alpha)^m(n) = \alpha(m + n)$, and, if, for some $i \leq m = n$, $\alpha(i) > 1$, then, for every $m \geq n$, $(\gamma \alpha)^m(n) = 2$. Clearly, $\gamma$ reduces $\text{Fin} = \mathbb{P}(1, \text{Fin})$ to $E_2 = \mathbb{P}(1, E_2)$. Now assume that $\sigma$ is a nonempty stump and that, for each $n$, $\gamma$ reduces $\mathbb{P}(\sigma^n, \text{Fin})$ to $\mathbb{P}(\sigma^n, E_2)$.

Let $\alpha$ belong to $\mathbb{P}(\sigma^n, \text{Fin})$. Find $\beta$ in $\text{Fin}$ such that, if $\alpha$ is apart from $\beta$, then, for some $n$, $\alpha$ belongs to $\mathbb{P}(\sigma^n, \text{Fin})$. Observe that $\gamma | \beta$ belongs to $E_2$ and that, if $\gamma | \alpha$ is apart from $\gamma | \beta$, then $\alpha$ is apart from $\beta$, and for some $n$, $\alpha$ belongs to $\mathbb{P}(\sigma^n, \text{Fin})$, and, by the induction hypothesis, $\gamma | \alpha$ belongs to $\mathbb{P}(\Delta(\sigma^n, E_2))$. Clearly, $\gamma$ maps $\mathbb{P}(\sigma, \text{Fin})$ into $\mathbb{P}(\Delta(\sigma, E_2))$.

Conversely, let $\alpha$ be an element of $\mathcal{N}$ such that $\gamma | \alpha$ belongs to $\mathbb{P}(\Delta(\sigma, E_2))$. Note that $\gamma | \alpha$ belongs to $(E_2)^\infty$ and, therefore, $\alpha$ belongs to $\text{Fin}^\infty$ and, for each $n$, $\alpha(n) \leq 1$. Find $n$ in $\mathbb{N}$ such that, for all $j$, if $(\gamma | \alpha)^n(j) \neq 0$, then, for some $p$, $\gamma | \alpha$ belongs to $\mathbb{P}(\Delta(\sigma^p, E_2))$. Consider $\beta = \overline{\alpha} \ast 0$ and note that $\beta$ belongs to $\text{Fin}$. If $\beta$ is apart from $\beta$, then there exists $m \geq n$ such that $\alpha(m) \neq 0$ and $\gamma | \alpha)(m - n) \neq 0$, and, therefore, for some $p$, $\gamma | \alpha$ belongs to $\mathbb{P}(\Delta(\sigma^p, E_2))$, and, thus, by the induction hypothesis, $\alpha$ belongs to $\mathbb{P}(\sigma^p, \text{Fin})$. Thus we see that, for every $\alpha$, if $\gamma | \alpha$ belongs to $\mathbb{P}(\sigma, E_2)$, then $\alpha$ belongs to $\mathbb{P}(\sigma, \text{Fin})$, that is, $\gamma$ reduces $\mathbb{P}(\sigma, \text{Fin})$ to $\mathbb{P}(\Delta(\sigma, E_2))$.

(v) Let $\sigma, \tau$ be stumps such that $\sigma < \tau$ and $\mathbb{P}(\sigma, E_2)$ coincides with $\mathbb{P}(\tau, E_2)$. As $\sigma < \sigma^+ \leq \tau$, it follows from Theorem 3.19(ii) that $\mathbb{P}(\sigma^+, E_2)$ coincides with $\mathbb{P}(\sigma, E_2)$, that is, $\text{Perhaps}(E_2, \mathbb{P}(\sigma, E_2))$ coincides with $\mathbb{P}(\sigma, E_2)$. Using Theorem 3.19(ii),
one may prove, by induction on the set of stumps, that, for every stump $\rho$, $P(\rho, E_2)$ is a subset of $P(\sigma, E_2)$:

Note that $P(\emptyset, E_2) = E_2$ is a subset of $P(\sigma, E_2)$. Assume that $\rho$ is a nonempty stump and that, for each $n$, $P(\rho^n, E_2)$ is a subset of $P(\sigma, E_2)$. Then $P(\rho, E_2) = \text{Perhaps}(E_2, \bigcup_{n \in \mathbb{N}} P(\rho^n, E_2)$ is a subset of $\text{Perhaps}(E_2, P(\sigma, E_2)) = P(\sigma^+, E_2)$ and thus of $P(\sigma, E_2)$.

Therefore, $\text{Almost}^\Delta(E_2)$ is a subset of $P(\sigma, E_2)$ and the two sets coincide. It follows from Theorem 4.11(v) that $P(\sigma, E_2)$ is perhapsive. According to Theorem 3.17(iv), also $P(\sigma, \text{Fin})$ is perhapsive. But $\text{Almost}(\text{Fin})$ is the least perhapsive set containing $\text{Fin}$ and does not coincide with $P(\sigma, \text{Fin})$ (see Theorems 4.2(v) and 4.5(v)). Contradiction. Thus we see that $E_2$, like $\text{Fin}$, is of unbounded perhapsity.

(vi) According to Brouwer’s Thesis, the set $\text{Almost}^*(\text{Fin})$ coincides with $\bigcup_{\sigma \in \text{Stp}} P(\sigma, \text{Fin})$ and the set $\text{Almost}^\Delta(E_2)$ coincides with $\bigcup_{\sigma \in \text{Stp}} P(\sigma, E_2)$ (see Theorem 4.9 and Theorem 4.11). As in the proof of (iv), we let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, for every $m, n$, if, for every $i \leq m + n$, $\alpha(i) \leq 1$, then $(\gamma | \alpha)^m(n) = \alpha(m + n)$, and, if, for some $i \leq m + n$, $\alpha(i) > 1$, then, for every $m \geq n$, $(\gamma | \alpha)^m(n) = 2$. It now follows from (iv) that $\gamma$ reduces $\text{Almost}^*(\text{Fin})$ to $\text{Almost}^\Delta(E_2)$. We may also prove this directly, that is, without applying Brouwer’s Thesis, as follows.

Let $\alpha$ belong to $\text{Almost}^*(\text{Fin})$. Let $\beta$ be an element of $\mathcal{N}$. Let $\delta$ be an infinite sequence of natural numbers such that, for every $p$, $\delta(p) + \beta(\delta(p)) < \delta(p + 1) + \beta(\delta(p + 1))$. Find $p$ such that $\alpha(\delta(p) + \beta(\delta(p))) = 0$ and observe: $(\gamma | \alpha)^\beta(\delta(p))) = 0$, so there exists $n$ such that $(\gamma | \alpha)^\beta(\delta(n))) = 0$. We thus see that, for every $\alpha$, if $\alpha$ belongs to $\text{Almost}^*(\text{Fin})$, then $\gamma | \alpha$ belongs to $\text{Almost}^\Delta(E_2)$.

Conversely, let $\alpha$ belong to $\mathcal{N}$ and assume that $\gamma | \alpha$ belongs to $\text{Almost}^\Delta(E_2)$. Let $\beta$ be an element of $\mathcal{N}$ such that, for each $n$, $\beta(n) < \beta(n + 1)$. Note that, for each $n$, $\beta(n) \geq n$ and determine $\delta$ in $\mathcal{N}$ such that, for each $n$, $\beta(n) = n + \delta(n)$. Now find $n$ such that $(\gamma | \alpha)^\delta(\delta(n))) = 0$ and note that $\alpha(\beta(n))) = 0$. We thus see that, for every $\alpha$, if $\gamma | \alpha$ belongs to $\text{Almost}^\Delta(E_2)$, then $\alpha$ belongs to $\text{Almost}^*(\text{Fin})$.

We now prove that, conversely, the set $\text{Almost}^\Delta(E_2)$ reduces to the set $\text{Almost}^*(\text{Fin})$. This is a surprising fact, because, as we shall see, in Section 5 (see Theorem 5.4(ii)), the set $E_2$ does not reduce to the set $\text{Fin}$. We let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $n$, $(\delta | \alpha)(n)$ is the greatest number $j \leq n$ such that for, every $i < j$, there exists $k < n$ such that $\alpha^i(k) \neq 0$. We let $e$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, $(e | \alpha)(n) = 1$ if $(\delta | \alpha)(n + 1) > (\delta | \alpha)(n)$, and $(e | \alpha)(n) = 0$ if not $(\delta | \alpha)(n + 1) > (\delta | \alpha)(n)$. We claim that $e$ reduces the set $\text{Almost}^\Delta(E_2)$ to the set $\text{Almost}^*(\text{Fin})$, and we establish this claim as follows.

First, let $\alpha$ belong to $\text{Almost}^\Delta(E_2)$. Let $\beta$ be a sequence of natural numbers such that, for each $n$, $\beta(n) < \beta(n + 1)$. Note that, for each $n$, $\beta(n) \geq n$ and, if, for each $i < n$, $(e | \alpha)(\beta(i))) = 1$, then, for every $j < n$, there exists $k < \beta(n)$ such that $\alpha^j(k) \neq 0$. Let $\zeta$ be a sequence of natural numbers such that, for every $n$, if, for each $i \leq n$, $(e | \alpha)(\beta(i))) = 1$, then $\alpha^n(\zeta(n))) \neq 0$. Now find $n$ such that $\alpha^n(\zeta(n))) = 0$ and note that, for some $i \leq n$, $(e | \alpha)(\beta(i))) = 0$. We thus see that, for every $\alpha$, if $\alpha$ belongs to $\text{Almost}^\Delta(E_2)$, then $e | \alpha$ belongs to $\text{Almost}^*(\text{Fin})$.
Conversely, let $\alpha$ belong to $N$ and assume that $e|\alpha$ belongs to $\text{Almost}^*(\text{Fin})$. Let $\beta$ be a sequence of natural numbers. Note that, for every $p$, if, for all $i \leq p + 1$, $a^i(\beta(i)) \neq 0$ then there exists $q > p$ such that $(\delta|\alpha)(q) > (\delta|\alpha)(p)$ and $(e|\alpha)(q) = 1$. Let $\zeta$ be a strictly increasing sequence of natural numbers such that, for each $n$, if, for all $i \leq \zeta(n) + 1$, $a^i(\beta(i)) \neq 0$, then $(e|\alpha)(\zeta(n + 1)) = 1$. Now find $n$ such that $(e|\alpha)(\zeta(n)) = 0$ and conclude that, for some $i \leq n + 1$, $a^i(\beta(i)) = 0$. We thus see that, for every $\alpha$, if $e|\alpha$ belongs to $\text{Almost}^*(\text{Fin})$, then $\alpha$ belongs to $\text{Almost}^{\Delta^0}(E_2)$. □

4.14. Markov's Principle, in its original form, states that, for every primitive-recursive sequence $\alpha$ in $N$, if the assumption that there is no $n$ such that $\alpha(n) = 1$ leads to a contradiction, then there exists $n$ such that $\alpha(n) = 1$. The generalized Principle of Markov extends this statement to all sequences of natural numbers, and does not require that sequences are given by some kind of algorithm. We do not want to propose the original or the generalized principle of Markov as an axiom for intuitionistic analysis. There does not seem to be a good reason for adopting this axiom. Nevertheless, the following theorem seems to be of some importance.

4.15. Theorem:
The following statements are equivalent:
(i) The generalized Principle of Markov: The set $(E_1)^{\sim\sim}$ is a subset of the set $E_1$, that is, for every $\alpha$,
\[\neg\exists n[\alpha(n) = 1], \text{then } \exists n[\alpha(n) = 1].\]
(ii) The set $\text{Fin}^{\sim\sim}$ is a subset of the set $\text{Fin}^{\geq} = \text{Almost}^*(\text{Fin})$.
(iii) For every enumerable subset $X$ of $N$, the set $X^{\sim\sim}$ is a subset of the set $\text{Almost}^*(X)$.
(iv) The set $(E_2)^{\sim\sim}$ is a subset of the set $(E_2)^{\geq} = \text{Almost}^{\Delta^0}(E_2)$.
(v) For every $\delta$, the set $(\text{Cus}^\delta\delta)^{\sim\sim}$ is a subset of the set $\text{Almost}^{\Delta^0}(\delta)$.

Proof. We shall prove (i) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (i), and then (i)$\Rightarrow$ (v), (v)$\Rightarrow$ (iv), and (iv) $\Rightarrow$ (i).

(i) $\Rightarrow$ (iii). Let $\delta$ belong to $N$ and assume that $\beta$ belongs to $(E_n\delta)^{\sim\sim}$. Note that $\neg\neg\exists n[\beta = \delta^0]$, and therefore, for every $\gamma$, $\neg\neg\exists n[\delta^0(\gamma(n)) = \beta(\gamma(n))]$, and thus, by the generalized Principle of Markov: $\exists n[\delta^0(\gamma(n)) = \beta(\gamma(n))].$ It follows that $\beta$ belongs to $\text{Almost}^*(X)$.

(iii) $\Rightarrow$ (ii). Obvious.

(ii) $\Rightarrow$ (i). Let $\alpha$ belong to $N$ and assume $\neg\neg\exists n[\alpha(n) = 1].$ Let $\beta$ in $N$ be such that, for each $n$, $\beta(n) = 0$ if and only if, for some $m < n, a(m) = 1$. Note that $\beta$ belongs to $\text{Fin}^{\sim\sim}$ and thus to $\text{Almost}^*(\text{Fin})$. Applying Theorem 4.13(iii) and considering the function $\gamma$ such that for each $n, \gamma(n) = n$, find $p$ such that $\beta(p) = 0$. It follows that there exists $q > p$ such that $\alpha(q) = 1$.

(i) $\Rightarrow$ (v). Let $\delta$ belong to $N$ and assume that $\beta$ belongs to $(\text{Cus}^\delta\delta)^{\sim\sim}$. Then $\neg\neg\exists n \forall m[\beta m$ contains an element of $S_{\bar{\varphi}}]$, that is, $\neg\neg\exists n \forall m[\varphi^0(0) = \beta m \uparrow \delta^n(\beta m + 1) = 1]$. Therefore, for every $\gamma$, $\neg\neg\exists n[\beta(\gamma(n))$ contains an element of $S_{\bar{\varphi}}]$, and thus, by the generalized Principle of Markov, $\exists n[\beta(\gamma(n))$ contains an element of $S_{\bar{\varphi}}].$ It follows that $\beta$ belongs to $\text{Almost}^{\Delta^0}(\delta)$.

(v) $\Rightarrow$ (iv). Obvious.

(iv) $\Rightarrow$ (i). Let $\alpha$ belong to $N$ and assume $\neg \exists n[\alpha(n) = 1].$ Let $\beta$ in $N$ be such that, for each $n$, $\beta(n) = 0$ if and only if, for some $m < n, a(m) = 1$. Note that $\neg\neg\exists n \forall m >
Let $\delta$ be such that, for all $n, p, \delta^n(p) = \beta(n + p)$. Now $\delta$ belongs to $(E_2)^{<\omega}$ and thus to $\text{Almost}^*(E_2)$. Using Theorem 4.13(iii) and considering the function $\gamma$ such that, for all $n, \gamma(n) = 0$, we find $n$ such that $\delta^n(0) = 0 = \beta(n)$, and, therefore, there exists $m < n$ such that $\alpha(m) = 1$.

§5. Finite unions of closed sets. In order to study the class of finite unions of closed subsets of $\mathcal{N}$, we introduce a binary operation called disjunction on the class of all subsets of $\mathcal{N}$. We prove that a finite union of closed sets is not always a closed set itself, and that its closure coincides with its double complement. We also show that a finite union of spreads, that is, located closed sets, always has finite perhapsity. We indicate how to obtain countably many sets $X$ with the property: $CBr^{*} \subseteq X \subseteq (CBr^{*})^\cap$. Finally, we bring in an operation called conjunction, and we study finite intersections of finite unions of closed sets.

5.1. In Subsection 4.12, we introduced the set $\text{Inf}$ consisting of all $\alpha$ in Cantor space $C$ such that, for each $n$, there exists $j > n$ such that $\alpha(j) = 1$. An element $\alpha$ of $\mathcal{N}$ belongs to $\text{Inf}$ if and only if $\alpha$ is the characteristic function of an infinite subset of the set of natural numbers. $\text{Inf}$ is a countable intersection of open sets and thus belongs to the class $\Pi^0_2$. We now prove that $\text{Inf}$ is a complete element of the class $\Pi^0_2$.

5.2. Theorem: Every $\Pi^0_2$-subset of $\mathcal{N}$ reduces to $\text{Inf}$.

Proof. Let $X$ be a $\Pi^0_2$-subset of $\mathcal{N}$ and assume that $Y_0, Y_1, \ldots$ is an infinite sequence of open subsets of $\mathcal{N}$ such that $X = \bigcap_{n \in \mathbb{N}} Y_n$. Let $C_0, C_1, \ldots$ be a sequence of decidable subsets of $\mathbb{N}$ such that for each $\alpha$, for each $n, \alpha$ belongs to $Y_n$ if and only if some initial part of $\alpha$ belongs to $C_n$. We define a function $\gamma$ from $\mathcal{N}$ to $\mathcal{N}$ such that for each $\alpha$, for each $n, (\gamma|\alpha)(n)$ belongs to $\{0, 1\}$ and $(\gamma|\alpha)(n) = 1$ if and only if the least $i < n + 1$ such that for every $j < i$ some initial part of $\alpha(n + 1)$ belongs to $C_j$ is greater than the least $i < n$ such that for every $j < i$ some initial part of $\alpha(n)$ belongs to $C_j$. One verifies without difficulty that $\gamma$ reduces $X$ to $\text{Inf}$. □

5.3. We shall see soon that the set $\text{Fin}$ is not a complete element of the class $\Sigma^0_2$, and thus thwart an expectation one might form after Theorem 5.2.

We define a binary operation $D$ on the class of subsets of Baire space $\mathcal{N}$. For all subsets $X, Y$ of $\mathcal{N}$ we let $D(X, Y)$ be the set of all $\alpha$ such that either $\alpha^0$ belongs to $X$ or $\alpha^1$ belongs to $Y$. We call the set $D(X, Y)$ the disjunction of the sets $X$ and $Y$.

Observe that, for all subsets $X, Y, Z$ of $\mathcal{N}$, $Z$ reduces to $D(X, Y)$ if and only if there exist subsets $Z_0, Z_1$ of $\mathcal{N}$ such that $Z = Z_0 \cup Z_1$ and $Z_0$ reduces to $X$ and $Z_1$ reduces to $Y$.

For every subset $X$ of $\mathcal{N}$ we denote $D(X, X)$ by $D^2(X)$. We define a subset $A_1$ of $\mathcal{N}$: $A_1$ is the set of all $\alpha$ such that, for every $n, \alpha(n) = 0$. So the sequence $\langle 0 \rangle$ is the one and only element of $A_1$.

Observe that, for every subset $X$ of $\mathcal{N}$, $X$ reduces to $A_1$ if and only if $X$ is closed and $X$ reduces to $D^2(A_1)$ if and only if there exist closed sets $X_0, X_1$ such that $X = X_0 \cup X_1$.

Observe that the sequential closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ is a spread containing $\emptyset$. The first item of the next theorem implies that the set $D^2(A_1)$ is not sequentially closed, although it is the union of two spreads.

5.4. Theorem:

(i) For each $n$, $\overline{D^2(A_1)} \cap \overline{\emptyset n}$ is not a subset of $D^2(A_1)$.

(ii) The closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ coincides with Perhaps($D^2(A_1), D^2(A_1))$ and with $(D^2(A_1))^\cap$. 
(iii) $D^2(A_1)$ is not perhapsive and does not belong to $\Pi^0_2$.

(iv) $D^2(A_1)$ belongs to $\Sigma^0_2$ but does not reduce to $\text{Fin}$, and, therefore, also the set $E_2$ does not reduce to $\text{Fin}$.

(v) $\text{Fin}$ does not reduce to $D^2(A_1)$.

Proof.

(i) Let $n$ be a natural number and suppose that $\overline{D^2(A_1)} \cap \overline{\mathbb{Q}}n$ is a subset of $D^2(A_1)$. For every $a$ in the spread $\overline{D^2(A_1)} \cap \overline{\mathbb{Q}}n$ we may decide either $a^0 = 0$ or $a^1 = 0$. Applying the Continuity Principle we find $m$ such that $m > n$ and either, for every $a$ in $D^2(A_1)$ passing through $\overline{\mathbb{Q}}m$, $a^0 = 0$ or, for every $a$ in $D^2(A_1)$ passing through $\overline{\mathbb{Q}}m$, $a^1 = 0$. This is absurd, as for each $m$, there exist $a, b$ in $D^2(A_1)$ passing through $\overline{\mathbb{Q}}m$ such that $a^0$ is apart from $0$ and $b^1$ is apart from $0$.

We conclude that $\overline{D^2(A_1)} \cap \overline{\mathbb{Q}}n$ is not a subset of $D^2(A_1)$.

(ii) Clearly, Perhaps($D^2(A_1), D^2(A_1)$) is a subset of $(D^2(A_1))$ and $D^2(A_1)$ is included in $D^2(A_1)$. We now show that $D^2(A_1)$ is a subset of Perhaps($D^2(A_1), D^2(A_1)$). Assume that $a$ belongs to $D^2(A_1)$. Define $p$ in $\mathbb{N}$ such that $p^0 = 0$ and for each $n$, if there exists no $p$ such that $n = (0, p)$, then $p(n) = a(n)$. Observe that $\beta$ belongs to $D^2(A_1)$, and, if $a$ is apart from $\beta$, then $a^0$ is apart from $0$ and consequently $\beta^1$ coincides with $0$, so $a$ belongs to $D^2(A_1)$. Therefore, $a$ belongs to Perhaps($D^2(A_1), D^2(A_1)$).

(iii) Follows from (ii) and Theorem 3.17(iii).

(iv) $D^2(A_1)$ obviously belongs to $\Sigma^0_2$. Assume now that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $D^2(A_1)$ to $\text{Fin}$. Let $B_0$ be the set of all $a$ in $\mathcal{N}$ such that $a^0 = 0$ and let $B_1$ be the set of all $a$ in $\mathcal{N}$ such that $a^1 = 0$. Observe that $B_0, B_1$ are spreads and that $D^2(A_1) = B_0 \cup B_1$. For every $a$ in $D^2(A_1)$ there exists $m$ such that, for every $i > m$, $(\gamma | a)(i) = 0$. Applying the Continuity Principle two times, we find $n, m$ such that, for every $a$ from $B_0 \cup B_1$, if $an = \overline{\mathbb{Q}}m$, then, for every $i > m$, $(\gamma | a)(i) = 0$. We now prove that, for every $a$ in the set $D^2(A_1) \cap \overline{\mathbb{Q}}n$, for every $i > m$, $(\gamma | a)(i) = 0$:

Let $a$ belong to $D^2(A_1) \cap \overline{\mathbb{Q}}n$ and suppose $i > m$. Find $p > n$ such that, for every $\beta$ in $D^2(A_1)$, if $\beta$ passes through $\overline{\mathbb{Q}}p$, then $(\gamma | \beta)(i) = (\gamma | a)(i)$. Let $\beta$ be an element of $D^2(A_1)$ passing through $\overline{\mathbb{Q}}p$ and observe: $(\gamma | a)(i) = (\gamma | \beta)(i) = 0$.

It follows that $\gamma$ maps $D^2(A_1) \cap \overline{\mathbb{Q}}n$ into $\text{Fin}$. Therefore $D^2(A_1) \cap \overline{\mathbb{Q}}n$ is a subset of $D^2(A_1)$, and this contradicts (i).

The second statement now follows from the first one, as the set $\text{Fin}$ belongs to the class $\Sigma^0_2$ and the set $E_2$ is a complete element of the class $\Sigma^0_2$ (see Subsection 4.12).

(v) Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $\text{Fin}$ to $D^2(A_1)$. Note that, for each $m$, $\overline{\mathbb{Q}}m * \emptyset$ belongs to $\text{Fin}$, and, therefore, $\gamma | (\overline{\mathbb{Q}}m * \emptyset)$ belongs to $D^2(A_1)$. It follows that $\gamma | \emptyset$ belongs to the closure $\overline{D^2(A_1)}$ of $D^2(A_1)$ and therefore, in view of (ii), to $D^2(A_1)^\infty$. So $\emptyset$ belongs to $\text{Fin}^\infty$. But $\emptyset$ does not belong to $\text{Fin}$. □

5.5. Let $X$ be a subset of $\mathcal{N}$ and $n$ a natural number.

We define a subset of $\mathcal{N}$, the $n$-fold disjunction of $X$, notation $D^n(X)$. $D^n(X)$ is the set of all $a$ in $\mathcal{N}$ such that, for some $k < n$, $a^k$ belongs to $X$.

Note that $D^0(X)$ is the empty set $\emptyset$. 

Observe that, for every subset $Z$ of $\mathbb{N}$, $Z$ reduces to $D^n(X)$ if and only if there exist subsets $Z_0, Z_1, \ldots, Z_{n-1}$ of $\mathbb{N}$, each of them reducing to $X$, such that $Z = Z_0 \cup Z_1 \cup \cdots \cup Z_{n-1}$. Let $\gamma$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that, for each $\alpha$, $(\gamma | \alpha)^0 = \alpha^0$ and, for each $i$, $(\gamma | \alpha)^{i+1} = \alpha^i$. It is easily seen that for each subset $X$ of $\mathbb{N}$, for every positive $n$, $\gamma$ reduces $D^n(X)$ to $D^{n+1}(X)$.

Observe that, for each positive $n$, the closure $D^{n+1}(A_1)$ of $D^n(A_1)$ is a spread containing $\emptyset$. For every $a$, for each positive $n$, $a$ belongs to $D^{n+1}(A_1)$ if and only if, for each $k$, the sequence $0$ passes through one of $a^0k, a^1k, \ldots, a^{n-1}k$.

Recall that we defined a special sequence $0^*, 1^*, \ldots$ of stumps in Subsection 2.5.1. Note that for every subset $X$ of $\mathbb{N}$, $P(0^*, X) = X$ and, for every $n$, $P((n + 1)^*, X)$ equals $\text{Perhaps}(X, P(n^*, X))$.

5.6. Theorem:

(i) For each $n$, the closure $D^{n+1}(A_1)$ of the set $D^n(A_1)$ coincides with its $n$-th perhapsive extension $P(n^*, D^n(A_1))$ and with its double complement $(D^n(A_1))^-$. 

(ii) For all $n, r$, for each stump $\sigma$, if $D^{n+1}(A_1) \cap \overline{\sigma r}$ is a subset of $P(\sigma, D^{n+1}(A_1))$, then $n^* \leq \sigma$.

(iii) For each $n$, for each function $\gamma$ from $\mathbb{N}$ to $\mathbb{N}$, if $\gamma$ maps $D^n(A_1)$ into $D^{n+1}(A_1)$, then $\gamma$ does not map surjectively the closure $D^n(A_1)$ of $D^n(A_1)$ onto the closure $D^{n+1}(A_1)$ of $D^{n+1}(A_1)$.

(iv) For each $n$, the set $D^n(A_1)$ reduces to the set $D^{n+1}(A_1)$, but $D^{n+1}(A_1)$ does not reduce to $D^n(A)$.

Proof.

(i) We use induction. Note that $P(0^*, D^1(A_1))$ coincides with $D^1(A_1)$, and that $D^1(A_1)$ coincides with its closure $D^1(A_1)$ and with its double complement $(D^1(A_1))^-$. Therefore, $D^1(A_1)$ coincides with $P(0^*, D^1(A_1))$ and with its double complement $(D^1(A_1))^-$. Suppose that $a$ belongs to $D^{n+1}(A_1)$. Define $\beta$ such that $\beta^0 = 0$ and for each $p$, if there does not exist $i$ such that $p = (n + 1) \star i$, then $\beta(p) = \alpha(p)$. Observe that $\beta$ belongs to $D^{n+1}(A_1)$, and, if $a \neq \beta$, then $a^{n+1} \neq 0$ and $a$ belongs to $D^{n+1}(A_1)$ and so to $P(n^*, D^n(A_1))$, and thus also to $P(n^*, D^{n+1}(A_1))$ as, according to Lemma 3.19(iii), $P(n^*, D^{n+1}(A_1))$ is a subset of $P(n^*, D^{n+2}(A_1))$. Therefore, $a$ belongs to $P(0^*, D^{n+2}(A_1))$.

This shows that $D^{n+1}(A_1)$ is a subset of $P((n + 1)^*, D^{n+2}(A_1))$. Conversely, it follows from Lemma 3.19(i) that $P((n + 1)^*, D^{n+2}(A_1))$ is a subset of $D^{n+2}(A_1)$.

(ii) We use induction. The statement to be proven is trivially true if $n = 0$. Now let $n$ be a natural number and assume that, for each stump $\sigma$, for each $r$, if $D^{n+1}(A_1) \cap \overline{\sigma r}$ is a subset of $P(\sigma, D^{n+1}(A_1))$, then $n^* \leq \sigma$. Let $\sigma$ be a stump and $r$ a natural number such that $D^{n+2}(A_1) \cap \overline{\sigma r}$ is a subset of $P(\sigma, D^{n+2}(A_1))$. For each $\alpha$ in $D^{n+2}(A_1) \cap \overline{\sigma r}$ we can find $\beta$ in $D^{n+2}(A_1)$ such that, if $a \neq \beta$, then $a$ belongs to $\bigcup_{k \in \mathbb{N}} P(\sigma^k, D^{n+2}(A_1))$. For every $\beta$ in $D^{n+2}(A_1)$ we can find $i < n + 2$ such that $\beta^i = 0$. Note that $D^{n+2}(A_1) \cap \overline{0r}$ is a spread containing $0$. We apply the Continuity Principle and find $m, i$ such that $m > r$ and $i < n + 2$ and for all $\alpha$ in $D^{n+2}(A_1) \cap \overline{0m}$ there exists $\beta$ such that $\beta^i = 0$ and, if $a \neq \beta$, then $a$ belongs to
\( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1)) \). Without endangering generality, we may assume \( i = n + 1 \).

Now consider the set \( B \) consisting of all \( x \) in \( D^{n+2}(A_1) \) such that \( \exists m = \exists m \) and \( a^{n+1} = a^m \). Observe that \( B \) is a spread and a subset of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+2}(A_1)) \).

As for each \( x \) in \( B \), \( a^{n+1} \neq 0 \), one may prove, for every stump \( \tau \), for each \( a \) in \( B \), if \( a \) belongs to \( \mathbb{P}(\tau, D^{n+2}(A_1)) \), then \( a \) belongs to \( \mathbb{P}(\tau, D^{n+1}(A_1)) \). So \( B \) is a subset of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). In fact, the set \( D^{n+1}(A_1) \cap \exists m \) is a subset of \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \).

The set \( D^{n+1}(A_1) \cap \exists m \) is a spread containing \( 0 \). We apply the Continuity Principle a second time and find \( q, k \) such that \( q > m \) and every \( a \) in \( D^{n+1}(A_1) \cap \exists q \) belongs to \( \mathbb{P}(n^*, D^{n+1}(A_1)) \), and, therefore, to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). It follows that also \( a \) belongs to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \).

The set \( D^{n+1}(A_1) \cap \exists m \) is a spread containing \( 0 \). We apply the Continuity Principle a second time and find \( q, k \) such that \( q > m \) and every \( a \) in \( D^{n+1}(A_1) \cap \exists q \) belongs to \( \mathbb{P}(n^*, D^{n+1}(A_1)) \), and, therefore, to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). It follows that also \( a \) belongs to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \).

The set \( D^{n+1}(A_1) \cap \exists m \) is a spread containing \( 0 \). We apply the Continuity Principle a second time and find \( q, k \) such that \( q > m \) and every \( a \) in \( D^{n+1}(A_1) \cap \exists q \) belongs to \( \mathbb{P}(n^*, D^{n+1}(A_1)) \), and, therefore, to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \). It follows that also \( a \) belongs to \( \bigcup_{k \in \mathbb{N}} \mathbb{P}(\sigma^k, D^{n+1}(A_1)) \).

(iii) Let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) mapping \( D^n(A_1) \) into \( D^{n+1}(A_1) \). According to Theorem 3.25(i), \( \gamma \) will map \( D^n(A_1) = \mathbb{P}(n^*, D^n(A_1)) \) into \( \mathbb{P}(n^*, D^{n+1}(A_1)) \), and according to (ii), the latter set is a proper subset of \( D^{n+1}(A_1) \).

(iv) Let \( n \) be a natural number and let \( \gamma \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( a \), for every \( i < n \), \( (\gamma | a)^i = a^i \), and \( (\gamma | a)^n = 1 \). Clearly, \( \gamma \) reduces \( D^n(A_1) \) to \( D^{n+1}(A_1) \).

Note that \( D^1(A_1) \) does not reduce to \( D^0(A_1) = 0 \). Now assume that \( n \) is positive and \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( D^{n+1}(A_1) \) to \( D^n(A_1) \). For each \( i < n + 1 \), let \( B_i \) be the set of all \( a \) such that \( a^i = 0 \). Observe that each \( B_i \) is a spread containing \( 0 \) and that \( \gamma \) maps \( \bigcup_{i < n+1} B_i \) into \( \bigcup_{i < n} B_i \). Applying the Continuity Principle \( n + 1 \) times we find natural numbers \( p_0, p_1, \ldots, p_n \) and \( k_0, k_1, \ldots, k_n \) such that for each \( i < n + 1, k_i < n \) and for each \( a \) in \( B_i \), passing through \( \exists p_i \), \( (\gamma | a)^i \) will belong to \( B_{k_i} \). Without loss of generality we may assume \( k_0 = k_1 = 0 \). Let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( a \), \( (\delta | a)^0 = \exists p_0 \) and \( (\delta | a)^1 = \exists p_1 \). Observe that \( a \) belongs to \( D^2(A_1) \) if and only if \( \delta | a \) belongs to \( D^2(A_1) \) and only if \( (\gamma | (\delta | a)^0 = 0 \). Therefore \( D^2(A_1) \) reduces to \( A_1 \). But, as we saw in Theorem 5.4, \( D^2(A_1) \) does not reduce to \( A_1 \).

□

5.7. The facts reported in the last few theorems have their counterparts in the domain of the real numbers.

Let \( X, Y \) be subsets of \( \mathbb{R} \), and let \( X \) be a real subset of \( Y \). As in Subsection 3.16, we let \( \text{Perhaps}(X, Y) \) be the set all real numbers \( x \) such that, for some element \( y \) of \( X \), if \( x \) is really apart from \( y \), then \( x \) belongs to \( Y \). We call a subset \( X \) of \( \mathbb{R} \) \text{perhapsive} if \( \text{Perhaps}(X, X) \) really coincides with \( X \).

5.8. Theorem:

(i) The union \([0, 1] \cup [1, 2]\) of the closed real intervals \([0, 1]\) and \([1, 2]\) is not closed and not perhapsive and does not belong to \( \Pi^0_2 \).
The closed real interval \([0, 2]\) is the least closed set containing both \([0, 1]\) and \([1, 2]\), and also the least perhapsive set containing both \([0, 1]\) and \([1, 2]\).

\([0, 2]\) coincides with \((\{0\} \cup [1, 2], [0, 1] \cup [1, 2])\).

(ii) For all open subsets \(Y_0, Y_1\) of \(\mathbb{R}\), the intersection \(Y_0 \cap Y_1\) is an open subset of \(\mathbb{R}\), and the closed set \(\mathbb{R} \setminus (Y_0 \cap Y_1)\) really coincides with the real closure \((\mathbb{R} \setminus Y_0) \cup (\mathbb{R} \setminus Y_1)\) of the set \((\mathbb{R} \setminus Y_0) \cup (\mathbb{R} \setminus Y_1)\) and with its double complement \(((\mathbb{R} \setminus Y_0) \cup (\mathbb{R} \setminus Y_1))^-\).

(iii) For every finite sequence \(X_0, X_1, \ldots, X_n\) of really closed subsets of \(\mathbb{R}\), the real closure \(\bigcup_{k \leq n} X_k\) of the set \(\bigcup_{k \leq n} X_k\) really coincides with its double complement \((\bigcup_{k \leq n} X_k)^-\).

(iv) The open real interval \((0, 1)\) does not coincide with any finite union of closed sets.

Proof. We discussed the results mentioned in (i) in Subsection 3.16. We leave the proofs of the other statements to the reader. As to (iv), note that \([0, 1] = (0, 1)\) does not coincide with \((0, 1)^-\). □

5.9. We return to Baire space \(N\). We give the new name \(T\) to the set \(CB_2^*\) that we introduced in Section 3.1, so \(T\) coincides with \(\{0\} \cup \{0\} \ast \{1\} \ast \{n \in \mathbb{N}\}\).

Observe that \(T\) contains all elements of Cantor space \(C\) that assume the value 1 either not at all or exactly one time. Its closure \(\overline{T}\) consists of all elements of Cantor space \(C\) that do not assume the value 1 two times, that is, they assume the value 1 at most one time.

We let \(E_1\) be the set of all \(\alpha\) such that, for some \(n, \alpha(n) \neq 0\).

For every nonempty spread \(X\) we let \(r_X\) be a function from \(N\) to \(N\) such that, for each \(\alpha\), for each \(n\), if \(\overline{\alpha}n\) contains an element of \(X\), then \(r_X|\alpha\) passes through \(\overline{\alpha}n\), and, if \(\overline{\alpha}n\) does not contain an element of \(X\), then \(r_X|\alpha\) passes through \((r_X|\alpha)(n - 1) \ast (p)\), where \(p\) is the least \(i\) such that \((r_X|\alpha)(n - 1) \ast (i)\) contains an element of \(X\). Note that \(r_X\) maps \(N\) onto \(X\), and that, for each \(\alpha\) in \(X\), \(r_X|\alpha\) coincides with \(\alpha\). \(r_X\) is called the canonical retraction of \(N\) onto \(X\).

5.10. Theorem:

(i) For all closed sets \(X, Y\), the sequential closure \(\overline{X \cup Y}\) of the set \(X \cup Y\) is a closed subset of \(N\) and coincides with the set \((X \cup Y)^-\).

(ii) For all spreads \(X, Y\), the set \(\overline{X \cup Y}\) coincides with \((X \cup Y, X \cup Y)\).

(iii) For each positive \(n\), for every \(n\)-sequence \(X_0, \ldots, X_{n-1}\) of closed sets, the set \(X_0 \cup \cdots \cup X_{n-1}\) coincides with the set \((X_0 \cup \cdots \cup X_{n-1})^-\).

(iv) For each positive \(n\), for every \(n\)-sequence \(X_0, \ldots, X_{n-1}\) of spreads, the set \(X_0 \cup \cdots \cup X_{n-1}\) coincides with \(P(n^*, X_0 \cup \cdots \cup X_{n-1})\).

(v) For each positive \(n\), the set \(E_1\) does not reduce to the set \(D^n(A_1)\).

(vi) The set \(T\) reduces to the set \(\text{Fin}\) and the set \(D^2(A_1)\) does not reduce to the set \(T\).

(vii) For each positive \(n\), the set \(T\) does not reduce to the set \(D^n(A_1)\).

(viii) For each positive \(n\), the closure \(\overline{T}\) of \(T\) is a subset of \(D^n(A_1)\) but not of \(D^n(A_1)\).

Proof.

(i) Let \(X, Y\) be closed sets. Let \(C, D\) be decidable subsets of \(\mathbb{N}\) such that every \(\alpha, \alpha\) belongs to \(X\) if and only if, for each \(n, \overline{\alpha}n\) belongs to \(C\), and \(\alpha\) belongs to \(Y\) if and only if, for each \(n, \overline{\alpha}n\) belongs to \(D\). Note that, for all \(\alpha, \alpha\) belongs to \(X \cup Y\) if and only if, for each \(n\), either, for each \(m \leq n, \overline{\alpha}m\) belongs to \(C\) or, for each \(m \leq n, \overline{\alpha}m\) belongs to \(D\). Therefore, \(X \cup Y\) is a closed subset of \(N\) in the sense of Subsection 1.2 (see also Subsection 2.3.2).
Assume that $\alpha$ belongs to $X \cup Y$, then for each $n$, either for every $m \leq n$, $\bar{\alpha} m$ belongs to $C$, or for every $m \leq n$, $\bar{\alpha} m$ belongs to $D$. Observe that if there exists $n$ such that $\bar{\alpha} n$ does not belong to $C$, then for every $n$, $\bar{\alpha} n$ belongs to $D$. Therefore, also if $\neg\neg$ (there exists $n$ such that $\bar{\alpha} n$ does not belong to $C$), then for every $n$, $\bar{\alpha} n$ belongs to $D$, that is $\alpha$ belongs to $Y$. So if $\alpha \not\in X$, then $\alpha \in Y$, and consequently $\neg\neg(\alpha \in X \lor \alpha \in Y)$.

We thus see that $X \cup Y$ is a subset of $(X \cup Y)^{\neg\neg}$.

Clearly, if $\alpha$ belongs to $(X \cup Y)^{\neg\neg}$, then, for each $n$, either, for each $m \leq n$, $\bar{\alpha} m$ belongs to $C$ or, for each $m \leq n$, $\bar{\alpha} m$ belongs to $D$, and, therefore, $\alpha$ belongs to $X \cup Y$.

(ii) Let $X$ and $Y$ be spreads, and let $r_X$ be the canonical retraction of $\mathcal{N}$ onto $X$, as we defined it just before this theorem. Note that, if $\alpha$ belongs to $X \cup Y$ and $\alpha$ is apart from $r_X|\alpha$, then there exists $n$ such that $\bar{\alpha} n$ does not contain an element of $X$, and thus, for every $m \geq n$, $\bar{\alpha} m$ contains an element of $Y$. This shows that $X \cup Y$ is a subset of $(X \cup Y, X \cup Y)$. Conversely, $(X \cup Y, X \cup Y)$ is a subset of $(X \cup Y)^{\neg\neg}$, by Theorem 4.17(i), and thus also of $X \cup Y$, by (i).

(iii) and (iv) are straightforward extensions of (i) and (ii), respectively, and the proofs are left to the reader.

(iv) Assume that $n$ is a natural number.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $E_1$ to $D^n(A_1)$. Note that for every $\alpha$ in $\mathcal{N} = E_1$, $\gamma|\alpha$ belongs to $D^n(A_1) = (D^n(A_1))^{\neg\neg}$. Also, for every $\alpha$, $\alpha$ belongs to $E_1$ if and only if $\gamma|\alpha$ belongs to $D^n(A_1)$ and thus $\neg\neg(\alpha$ belongs to $E_1)$ if and only if $\neg\neg(\gamma|\alpha$ belongs to $D^n(A_1))$. It follows that, for every $\alpha$, $\neg\neg(\alpha$ belongs to $E_1)$.

Note that $0$ does not belong to $E_1$.

(v) Note that $T$ coincides with $CB_{2^*}$ and that $\textbf{Fin}$ is an enumerable and discrete subset of $\mathcal{N}$ that is also dense in itself. The first half of the statement thus follows from Theorem 4.5(ii). The second half of the statement now is an easy consequence of Theorem 5.4(iv).

(vi) We use induction. We have seen, in Theorem 3.7, that the set $T = CB_{2^*}$ does not belong to $\Pi_2^0$, so $T$ is not closed and does not reduce to $D^1(A_1)$. Let $n$ be a natural number and assume that $T$ does not reduce to $D^n(A_1)$. Suppose that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $T$ to $D^{n+1}(A_1)$. Calculate $i$ such that $i < n + 1$ and $(\gamma|0)^n = 0$. Without endangering generality we assume $i = n$, that is $(\gamma|0)^n = 0$.

Using the First Axiom of Countable Choice, determine $\alpha$ in $\mathcal{N}$ such that, for each $j$, $\alpha(j) < n + 1$ and $(\gamma|0j * (1) * 0)^n = 0$. We claim that it is possible to decide, for each $j$, if there exists $k > j$ such that $\alpha(k) = n$ or not, in the following way.

Let $j$ be a natural number and let $\beta$ be the element of $C$ such that, for each $p$, $\beta(p) = 1$ if and only if $p$ is the least $k > j$ such that $\alpha(k) = n$. We claim that $(\gamma|\beta)^n = 0$ and we prove this claim as follows:

First, assume that there is no $k > j$ such that $\alpha(k) = n$. Then $\beta = 0$ and $(\gamma|\beta)^n = 0$.

Secondly, assume such $k$ exists. Let $k_0$ be the least such $k$. Note that $\beta = 0k_0 * (1) * 0$, and, as $\alpha(k_0) = n$, also in this case $(\gamma|\beta)^n = 0$.

Thirdly, note that, in any case, $\neg\neg(\text{there is no such } k \text{ or there exists such } k)$. Therefore $\neg\neg((\gamma|\beta)^n = 0)$, and thus, in any case, $(\gamma|\beta)^n = 0$.

It follows that $\gamma|\beta$ belongs to $D^{n+1}(A_1)$, and thus $\beta$ belongs to $T$, so either $\beta = 0$ or $\beta \not= 0$. In the first case, if $\beta = 0$, then there does not exist $k > j$ such that $\alpha(k) = n$, and in the second case, if $\beta \not= 0$, then there exists such $k$. 
Assume now that \( j \) is a natural number and there is no \( k > j \) such that \( \alpha(k) = n \). Observe that the sequence \( \beta \) belongs to the closure of the set \( \{ \beta \in \mathcal{N} | \exists i < n[(\gamma | \beta)^i = 0] \} \). As the set \( \{ \beta \in \mathcal{N} | \exists i < n[(\gamma | \beta)^i = 0] \} \) is a finite union of closed sets, we use (iii) and conclude \( \neg \exists i < n[(\gamma | \beta)^i = 0] \).

Assume we find \( i < n \) such that \( (\gamma | \beta)^i = 0 \). Let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) satisfying the following conditions:

1. for every \( \beta \), for every \( n \), the function \( \delta \) maps the sequence \( \mathcal{O}n \ast (1) \ast \beta \) onto the sequence \( \gamma | (\mathcal{O}(n + j) \ast (1) \ast \beta) \), and,
2. for every \( \beta \), for every \( n \), if \( \beta n \) does not contain an element of \( T \), then, for some \( p \), \( (\gamma | \beta)|p \) does not contain an element of \( D^\alpha(A_1) \).

Observe that \( \delta \) reduces \( T \) to \( D^\alpha(A_1) \). According to the induction hypothesis, \( T \) does not reduce to \( D^\alpha(A_1) \) so there is no \( i < n \) such that \( (\gamma | \beta)^i = 0 \).

We thus see that the assumption that there is no \( k > j \) such that \( \alpha(k) = n \) leads to a contradiction. We conclude that we always find ourselves in the second one of the above two cases that is, for every \( j \), there exists \( k > j \) such that \( \alpha(k) = n \).

We let \( \zeta \) in \( \mathcal{N} \) be a strictly increasing sequence such that, for each \( n \), \( \alpha(\zeta(n)) = n \).

We let \( \delta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) satisfying the following conditions:

1. for every \( \beta \), for every \( n \), the function \( \delta \) maps the sequence \( \mathcal{O}n \ast (1) \ast \beta \) onto the sequence \( \gamma | (\mathcal{O}(n + j) \ast (1) \ast \beta) \), and,
2. for every \( \beta \), if \( \beta(0) \) differs from both 0, 1, then \( \delta|\beta = \gamma | \beta \).

Observe that, for every \( \beta \), \( \beta \) belongs to \( T \) if and only if \( (\delta|\beta)^n = 0 \).

Therefore \( T \) is closed. Contradiction. We conclude that \( T \) does not reduce to \( D^\alpha+1(A_1) \) and thus complete the inductive step of our argument.

(vii) The proof is left to the reader. \( \square \)

5.11. It follows from Theorem 4.5(v) and Theorem 3.19(i) that, for every enumerable and discrete subset \( D \) of \( \mathcal{N} \) that is also dense in itself there are uncountably many sets \( X \) with the property \( D \subseteq X \subseteq \mathcal{X} \), although \( \mathcal{T} \) coincides with \( (T, T) \) and \( T \) has ity 1*. We need some preparations.

Let \( f \) be a function from Cantor space \( \mathcal{C} \) to itself such that for every \( \alpha \), for every \( n \), \((f | \alpha)(n) = 1 \) if and only if there exists \( m \leq n \) such that \( n = \overline{\alpha}m \). Observe that for all \( \alpha \) in \( \mathcal{C} \) there exist infinitely many \( j \) such that \((f | \alpha)(j) = 1 \) and also infinitely many \( j \) such that \((f | \alpha)(j) = 0 \), and that for all \( \alpha, \beta \) in \( \mathcal{C} \), if \( \alpha \neq \beta \), then there are only finitely many \( j \) such that \((f | \alpha)(j) = (f | \beta)(j) = 1 \).

Let \( \dot{+} \) be the binary operation on \( \mathcal{C} \) that is defined by: for all \( \alpha, \beta \) in \( \mathcal{C} \), for all \( n \), \((\alpha \dot{+} \beta)(n) = 1 \) if and only if \( \alpha(n) = 1 \) and \( \beta(n) = 0 \).

Let \( \mathrm{Max} \) be the binary operation on \( \mathcal{C} \) that is defined by: for all \( \alpha, \beta \) in \( \mathcal{C} \), for all \( n \), \((\mathrm{Max}(\alpha, \beta))(n) = 1 \) if and only if \( \alpha(n) = 1 \) or \( \beta(n) = 1 \).

For every \( \alpha \) in \( \mathcal{C} \), we define subsets \( \mathcal{T}(\alpha) \) and \( \mathcal{U}(\alpha) \) of \( \mathcal{C} \), as follows:

\[
\mathcal{T}(\alpha) := \{0\} \cup \{\mathcal{O}n \ast (1) \ast \beta | n \in \mathbb{N} \text{ and } \alpha(n) = 1\}, \text{ and } \mathcal{U}(\alpha) := \mathcal{T}(f | \alpha) \cup \mathcal{T}(\mathcal{O} \dot{+} (f | \alpha)).
\]
Note that, for each $\alpha$ in $C$, $T$ coincides with $T(\alpha) \cup T(1 - \alpha)$. Also observe that, for every $\alpha$ in $C$ the closure $\overline{T(\alpha)}$ of $T(\alpha)$ is a spread containing $0$. Finally, note that, for all $\alpha$, $\gamma$ in $C$, $\gamma$ belongs to $\overline{T(\alpha)}$ if and only if, for each $n$, if $\gamma(n) = 1$, then $\alpha(n) = 1$.

5.12. Theorem:

(i) For all $\alpha$, $\beta$, $\gamma$ in $C$, if $\overline{T(\alpha)}$ is a subset of $\overline{T(\beta)} \cup \overline{T(\gamma)}$, then either there exists $n$ such that, for every $j > n$, if $\alpha(j) = 1$, then $\beta(j) = 1$ or there exists $n$ such that, for every $j > n$, if $\alpha(j) = 1$, then $\gamma(j) = 1$.

(ii) For each $\alpha$ in $C$, the set $\overline{T(\alpha)}$ coincides with the set $\overline{(T(\alpha))}^{-\infty}$.

(iii) For all $\alpha$, $\beta$ in $C$, $\overline{T(\alpha)} \cup \overline{T(\beta)} = \overline{T(\text{Max}(\alpha, \beta))}$.

(iv) For all $\alpha$, $\beta$ in $C$, if $\overline{U(\alpha)}$ is a subset of $\overline{U(\beta)}$, then $\alpha = \beta$.

(v) For each $\alpha$ in $C$, $T \subseteq \overline{U(\alpha)} \subseteq T^{-\infty}$, and $\overline{U(\alpha)}$ does not coincide with either $T$ or $T^{-\infty}$, but $\overline{(U(\alpha))}^{-\infty} = \overline{U(\alpha)}$ coincides with $T^{-\infty} = \overline{T}$.

(vi) For each $\alpha$ in $C$, the set $D^2(A_1)$ reduces to the set $\overline{U(\alpha)}$.

(vii) There is no set $C$ such that $T \subseteq C \subseteq T^{-\infty}$ and the set $D^3(A_1)$ reduces to the set $C$.

(viii) For each positive $n$ there exists a set $C$ such that $T \subseteq C \subseteq T^{-\infty}$ and $C$ is a union of $n + 1$ closed sets not coinciding with any union of $n$ closed sets.

(ix) $T$ is a countable union of closed sets not coinciding with any finite union of closed sets.

Proof.

(i) Suppose that $\overline{T(\alpha)}$ is a subset of $\overline{T(\beta)} \cup \overline{T(\gamma)}$. Using the Continuity Principle we find $n$ such that either every $\delta$ in $\overline{T(\alpha)}$ passing through $0^*n$ belongs to $\overline{T(\beta)}$ or every $\delta$ in $\overline{T(\alpha)}$ passing through $0^*n$ belongs to $\overline{T(\gamma)}$. If the first alternative obtains, then for all $j > n$, $\alpha(j) = 1$ entails $\beta(j) = 1$, and if the second one obtains, then for all $j > n$, $\alpha(j) = 1$ entails $\gamma(j) = 1$.

(ii) The proof is similar to the proof of Theorem 3.5(iii).

(iii) Note that, for every $\gamma$ in $C$, $\gamma$ belongs to $\overline{T(\alpha)} \cup \overline{T(\beta)}$ if and only if, for each $n$, if $\gamma(n) = 1$, then $\alpha(n) = 1$ or $\beta(n) = 1$, if and only if $\gamma$ belongs to $\overline{T(\text{Max}(\alpha, \beta))}$.

(iv) Suppose that $\overline{U(\alpha)}$ is a subset of $\overline{U(\beta)}$. In particular, $\overline{T(\text{Max}(f |a ))}$ is a subset of $\overline{T(f |b )} \cup \overline{T(0 - (f |a ))}$. Observe that there is no $n$ such that, for every $j > n$, $(f |a)(j) = 0$ entails $(f |\beta)(j) = 1$, and, therefore, by (i), there exists $n$ such that for every $j > n$, $(f |a)(j) = 0$ entails $(f |\beta)(j) = 0$, and thus $\alpha = \beta$.

(v) Clearly, for every $\alpha$ in $C$, $T \subseteq \overline{U(\alpha)} \subseteq T^{-\infty}$. It is now an easy consequence of (iv) that, for every $\alpha$ in $C$, $\overline{U(\alpha)}$ does not coincide with either $T$ or $T^{-\infty}$.

(vi) Let $\alpha$ belong to $C$. We show how to reduce $D^2(A_1)$ to $\overline{U(\alpha)} = \overline{T(f |a )} \cup \overline{T(0 - (f |a ))}$. We let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for all $\beta$, for all $n$,

(i) if $\beta^0n = \overline{0n}$, then $(\gamma |\beta)(n) = 0$, and,

(ii) if $\beta^0n = \overline{0n}$ and $\beta^1n \neq \overline{0n}$ and $(\gamma |\beta)n = \overline{0n}$, then $(\gamma |\beta)(n) = (f |a)(n)$, and,

(iii) if $\beta^0n = \overline{0n}$ and $\beta^1n \neq \overline{0n}$ and $(\gamma |\beta)n \neq \overline{0n}$, then $(\gamma |\beta)(n) = 0$, and
(iv) if \( \beta^0 n \neq \overline{0} n \) and \( \beta^1 n = \overline{0} n \) and \( (\gamma | \beta)n = \overline{0} n \), then \( (\gamma | \beta)(n) = 1 - (f | \alpha)(n) \), and,

(v) if \( \beta^0 n \neq \overline{0} n \) and \( \beta^1 n = \overline{0} n \) and \( (\gamma | \beta)n \neq \overline{0} n \), then \( (\gamma | \beta)(n) = 0 \), and

(vi) if \( \beta^0 n \neq \overline{0} n \) and \( \beta^1 n \neq \overline{0} n \), then \( (\gamma | \beta)(n) = 2 \).

Note that, for all \( \beta \), if \( \beta^0 = 0 \), then \( \gamma | \beta \) assumes at most one time a value different from 0, and, for all \( n \), if \( (\gamma | \beta)(n) \neq 0 \), then \( (\gamma | \beta)(n) = 1 - (f | \alpha)(n) = 1 \).

Similarly, for all \( \beta \), if \( \beta^1 = 0 \), then \( \gamma | \beta \) assumes at most one time a value different from 0, and, for all \( n \), if \( (\gamma | \beta)(n) = 0 \), then \( (\gamma | \beta)(n) = 1 - (f | \alpha)(n) = 1 \).

We claim that, for all \( \beta \) in \( C \), \( \beta^0 = 0 \) if and only if \( \gamma | \beta \) belongs to \( T(f | \alpha) \). We prove this claim as follows:

It will be clear that if \( \beta^0 = 0 \), then \( \gamma | \beta \) belongs to \( T(f | \alpha) \).

Observe that, for every \( \beta \) in \( C \), if \( \beta^0 \neq 0 \) and \( \beta^1 = 0 \), then \( \gamma | \beta \) belongs to \( T(1 - (f | \alpha)) \) and not to \( T(f | \alpha) \), and if both \( \beta^0 \neq 0 \) and \( \beta^1 \neq 0 \), then \( \gamma | \beta \) does not belong to \( T(f | \alpha) \) and also not to \( T(1 - (f | \alpha)) \).

Now assume that \( \beta \) is an element of \( C \) and that \( \gamma | \beta \) belongs to \( T(f | \alpha) = (T(f | \alpha))^- \). Note that both \( -(\beta^0 \neq 0 \) and \( \beta^1 = 0 \) \) and \( -(\beta^0 \neq 0 \) and \( \beta^1 \neq 0 \) \). We may conclude \( -(\beta^0 \neq 0 \), that is, \( \beta^0 = 0 \).

For similar reasons, we have that \( \beta^1 = 0 \) if and only if \( \gamma | \beta \) belongs to \( T(1 - (f | \alpha)) \). So \( \gamma \) reduces \( D^2(A_1) \) to \( U(\alpha) \).

Observe that \( D^2(A_1) \) does not reduce to \( T \) (see Theorem 5.10(vi)) and also not to the closed set \( T^- \) (see Theorem 5.4), so we have a second argument proving that \( U(\alpha) \) is different from both \( T \) and \( T^- \).

(vii) Suppose \( C \) is a set such that \( T \subseteq C \subseteq T^- \) and \( \gamma \) is a function from \( N \) to \( N \) reducing \( D^3(A_1) \) to \( C \). Note that \( T^- \) is a subset of \( D^3(A_1) \). We claim that \( \gamma \) maps every \( a \) in \( T^- \) onto \( 0 \). We prove this claim as follows:

Assume \( a \) belongs to \( T^- \) and \( |a| \neq 0 \). Let \( m \) be the least \( p \) such that \( (\gamma | a)(p) \neq 0 \) and calculate \( n \) such that, for each \( \beta \) passing through \( \overline{a} \), \( (\gamma | a)(m + 1) = \overline{m} * (1) \), otherwise \( \gamma | a \) would not belong to \( T \), and \( a \) itself would not belong to \( D^3(A_1) \). There are two elements \( i \) of \( \{0, 1, 2\} \) such that \( a^i n = \overline{m} \), and without loss of generality we may assume \( a^0 n = a^1 n = \overline{0} n \). Define a function \( \delta \) from \( N \) to \( N \) such that, for every \( \beta \), \( (\delta | \beta)^1 = \overline{m} * \beta^0 \), and \( (\delta | \beta)^1 = \overline{0} n * \beta^1 \), and \( (\delta | \beta)^2 \neq 0 \). Observe that for every \( \beta \), \( \beta \) belongs to \( D^2(A_1) \) if and only if \( (\delta | \beta) \) equals \( \overline{m} * (1) * 0 \), therefore \( D^2(A_1) \) is closed. According to Theorem 5.4, however, \( D^2(A_1) \) is not closed.

Therefore \( \gamma \) indeed maps \( T^- \) onto \( \{0\} \) and \( T^- \) is a subset of \( D^3(A_1) \). Using Brouwer’s Continuity Principle we find \( m \) and \( i < 3 \) such that, for every \( a \) in \( T^- \), if \( a \) passes through \( \overline{m} \), then \( a^i = 0 \). This is clearly false.
(viii) For each positive \( n \), for each \( i < n + 1 \), we let \( E_n^i \) be the closure of the set \( \{0\} \cup \{0(k(n + 1) + i) \mid k \in \mathbb{N}\} \), and we define \( C_n := \bigcup_{i < n + 1} E_n^i \). It will be clear that \( C_n \) is a union of \( n + 1 \) closed sets and that \( T \subseteq C_n \subseteq T^{\supseteq} \). Suppose that we find closed sets \( F_0, F_1, \ldots, F_{n-1} \) such that \( C_n \) coincides with \( \bigcup_{i<n} F_i \). Applying the Continuity Principle \( n + 1 \) times we find for each \( i < n \) numbers \( m_i, n_i \) such that every \( a \) in \( E_n^i \) passing through \( m_i \) belongs to \( F_{n_i} \). Without endangering generality we may assume \( n_0 = n_1 = 0 \). It easily follows that the closure of \( E_0^0 \cup E_1^1 \) forms part of \( \bigcup_{i<n} F_i \) and therefore of \( C_n \). Using once more the Continuity Principle, we obtain a contradiction.

(ix) Clearly, \( T \) is a countable union of singletons, and, therefore, a countable union of closed sets. Assume that \( C_0, C_1, \ldots, C_{n-1} \) is a finite sequence of closed sets such that \( T \) coincides with \( \bigcup_{i<n} C_i \). Using Brouwer’s Continuity Principle we find \( m, i_0 \) such that \( T \cap m \) is a subset of \( C_{i_0} \). So \( T \cap m \) is a closed set, but it is not, according to Theorem 3.3(iv).

\[ \square \]

5.13. For each positive \( n \), there exists a subset of \( \mathbb{R} \) that is a union of \( n + 1 \) closed sets and does not coincide with any union of \( n \) closed sets. Inspired by Theorem 5.12(vii), we define such sets as follows. We let \( U \) be the set consisting of the real numbers really coinciding with one of the rational numbers \( 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \). For each positive \( n \), for each \( i < n + 1 \), we let \( F_n^i \) be the real closure of the set \( \{0\} \cup \{k(n+1) + i \mid k \in \mathbb{N}\} \), and we define \( D_n := \bigcup_{i<n+1} F_n^i \). It will be clear that \( D_n \) is a union of \( n + 1 \) closed sets and that \( U \subseteq D_n \subseteq U^{\supseteq} \). The proof that \( D_n \) is not a union of \( n \) closed sets is similar to the proof of Theorem 5.12(vii) and left to the reader.

5.14. We want to study the class of all subsets of \( \mathcal{N} \) that are the union of finitely many closed subsets of \( \mathcal{N} \). This class is closed under the operation of (finite) union, but also, as we are to see in a moment, under the operation of (finite) intersection.

Let \( X, Y \) be subsets of \( \mathcal{N} \). We let the conjunction of \( X \) and \( Y \), notation: \( C(X, Y) \), be the set of all \( a \) such that \( a^0 \) belongs to \( X \) and \( a^1 \) belongs to \( Y \).

More generally, let \( n \) be a positive natural number and let \( X_0, \ldots, X_{n-1} \) be subsets of \( \mathcal{N} \). We let the \( n \)-fold conjunction of \( X_0, \ldots, X_{n-1} \), notation \( C(X_0, \ldots, X_{n-1}) \) or \( C_{i=0}^n(X_i) \), be the set of all \( a \) such that, for all \( j < n \), \( a^j \) belongs to \( X_j \).

Observe that, for every positive \( n \), for all subsets \( Z, X_0, \ldots, X_{n-1} \) of \( \mathcal{N} \), \( Z \) reduces to \( C(X_0, \ldots, X_{n-1}) \) if and only if there exist subsets \( Z_0, \ldots, Z_{n-1} \) of \( \mathcal{N} \) such that \( Z = Z_0 \cap \ldots \cap Z_{n-1} \) and for each \( j < n \), \( Z_j \) reduces to \( X_j \).

Let \( X \) be a subset of \( \mathcal{N} \) and \( n \) a natural number. We let the \( n \)-fold conjunction of \( X \), notation: \( C^n(X) \), be the set of all \( a \) such that for all \( j < n \), \( a^j \) belongs to \( X \).

Note that \( C^0(X) = \mathcal{N} \).

Observe that for every positive \( n \), for all subsets \( Z, X \) of \( \mathcal{N} \), \( Z \) reduces to \( C^n(X) \) if and only if there exist subsets \( Z_0, \ldots, Z_{n-1} \) of \( \mathcal{N} \), each of them reducing to \( X \), such that \( Z = Z_0 \cap \ldots \cap Z_{n-1} \).

Recall that, for every natural number \( m \), there exists a natural number \( k = \text{length}(m) \) and natural numbers \( m(0), \ldots, m(k-1) \) such that \( m = (m(0), \ldots, m(k-1)) \). For all natural numbers \( m, n \) we define: \( m \) \text{ bows } \text{ to } \text{ } n, \text{ notation: } m < n, \text{ if and only if } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{ } \text{
For all natural numbers \( n \), we let \( Q_n \) be the (finite) union of all sets \( P_m \) such that \( m \) bows to \( n \). Observe that, for each \( n, k \), if \( k = \text{length}(n) \), then \( Q_n \) coincides with \( C(D^{(0)}(A_1), \ldots, D^{(k-1)}(A_1)) \).

Observe that \( Q_0 = Q(\emptyset) = N \) and, for each \( n \), if there exists \( i < \text{length}(n) \) such that \( n(i) = 0 \), then \( Q_n = \emptyset \).

For all \( n, j, p \) such that \( j < \text{length}(n) \) and \( p > 1 \) we let \( c(n, j, p) \) be the natural number \( c \) satisfying the following conditions: \( \text{length}(c) = \text{length}(n) \) and, for all \( i < \text{length}(n) \), if \( i \neq j \), then \( c(i) = n(i) \), and \( c(j) \) is the greatest natural number \( q \) such that \( p \cdot q \leq n(j) \).

We also define, for all \( n, j \) such that \( j < \text{length}(n) \), \( c(n, j, 1) = n \).

The first item of the next theorem shows that the class of finite unions of closed sets is closed under the operation of intersection. The other items help us to decide, for certain members \( X, Y \) of this class, if \( X \) reduces to \( Y \) or not.

5.15. **Theorem:**

(i) For all positive natural numbers \( p, q \), the set \( C(D^p(A_1), D^q(A_1)) \) reduces to the set \( D^{p \cdot q}(A_1) \).

(ii) For all positive natural numbers \( p, m, n \), if \( p > 1 \), then the set \( Q(\langle p \rangle \cdot m) \) reduces to the set \( Q_n \) if and only if there exists \( j < \text{length}(n) \) such that the set \( Q_m \) reduces to the set \( Q_{c(n, j, p)} \).

(iii) For all positive natural numbers \( n \), the set \( C((A_1, D^n(A_1)) \) reduces to the set \( D^n(A_1) \).

(iv) For each positive natural number \( m \), the set \( Q(\langle 1 \rangle \cdot m) \) reduces to the set \( Q_m \).

**Proof.**

(i) Let \( p, q \) be positive natural numbers. Let \( \gamma \) be a function from \( N \) to \( N \) such that for all \( i < p, j < q \), for all \( n \), the number \( \langle \gamma | a \rangle^{p,j+i}(n) \) equals the number \( \langle \text{Max}(a^{0,i}, a^{1,j}) \rangle(n) \). The function \( \gamma \) reduces the set \( C(D^p(A_1), D^q(A_1)) \) to the set \( D^{p \cdot q}(A_1) \).

(ii) Let \( p, m, n \) be positive natural numbers such that \( p > 1 \) and let \( \gamma \) be a function from \( N \) to \( N \) reducing \( Q(\langle p \rangle \cdot m) \) to \( Q_n \). Using the Continuity Principle a finite number of times we find \( s \) in \( N \) and a function \( F \) from the set of numbers bowing to \( (p) \cdot m \) to the set of numbers bowing to \( n \) such that, for every \( t < (p) \cdot m \), the function \( \gamma \) maps every \( a \) in \( Pt \) passing through 0s into the set \( PF(t) \).

We claim that there exists \( j < \text{length}(n) \) such that for all \( t, u \) bowing to \( (p) \cdot m \), if \( t(0) \neq u(0) \), then \( (F(t))(j) \neq (F(u))(j) \) and we prove this claim as follows:

Suppose there is no such \( j \). Let \( X \) be the set of all \( a \) in Cantor space \( C \) passing through \( 0s \) such that \( a^{0} \) assumes at most one time the value 1 and, for each \( i \), if there is no \( q \) such that \( i = (0, q) \), then \( a(i) = 0 \). Observe that \( X \) is a spread containing \( 0 \). Remark also that for every \( a \) in \( X \), for every \( j < \text{length}(n) \), the sequence \( \langle \gamma | a \rangle^{j} \) belongs to \( D^{(j)}(A_1) \). Indeed, let \( j \) be a natural number and let \( t, u \) be numbers bowing to \( (p) \cdot m \) such that \( t(0) \neq u(0) \) and \( k := (F(t))(j) = (F(u))(j) \). Observe that for every \( a \), if \( a \) belongs to \( P_t \cup P_u \), then \( \langle \gamma | a \rangle^{j,k} = 0 \). As \( X \) forms part of \( (P_t \cup P_u)^{-\infty} \), also for every \( a \), if \( a \) belongs to \( X \), then \( \langle \gamma | a \rangle^{j,k} = 0 \).

We conclude that \( \gamma \) maps \( X \) into \( Q_n \), therefore \( X \) forms part of \( Q(\langle p \rangle \cdot m) \) and, in particular, for every \( a \) in \( X \), there exists \( k \) such
that \( \alpha^{0,k} = 0 \). Using the Continuity Principle we find \( t, k \) such that \( t \geq s \) and every \( \alpha \) in \( X \) passing through \( \bar{0}t \) has the property \( \alpha^{0,k} = 0 \). This is false, and our claim holds true.

Now choose \( j < \text{length}(n) \) such that for all \( t, u \) bowing to \( \langle p \rangle \ast m \), if \( t(0) \neq u(0) \), then \( (F(t))(j) \neq (F(u))(j) \). For each \( k < p \) we let \( C_k \) be the set of all numbers \( (F(t))(j) \) where \( t \) is some number bowing to \( \langle p \rangle \ast m \) such that \( t(0) = k \). Observe that for all \( k, \ell < p \), if \( k \neq \ell \), then \( C_k \) and \( C_\ell \) are disjoint subsets of the set \( \{0, 1, \ldots, n(j) - 1\} \). We now determine \( k < p \) such that for every \( \ell < p \), the number of elements of \( C_k \) does not exceed the number of elements of \( C_\ell \). Observe that the number of elements of \( C_k \) does not exceed the greatest natural number \( q \) such that \( p \ast q < n(j) \).

In order to see that \( Q_m \) reduces to \( Q_{c(n,j,p)} \) we define a function \( \delta \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( \alpha \), \( \delta(\alpha) \) passes through \( \bar{0}s \) and \( (\delta|\alpha)^{0,k} = 0 \) and for all \( j < \text{length}(m) \), for all \( q \), \( (\delta|\alpha)^{j+1,q} = \bar{0}s \ast \alpha^j \).

Note that, for every \( \alpha \), \( \alpha \) belongs to \( Q_m \) if and only if \( \gamma | (\delta|\alpha) \) belongs to \( Q_n \) and, in addition, there is some \( i \) in \( C_k \) such that \( (\gamma | (\delta|\alpha))^j \ast i = 0 \).

We conclude that, if \( Q_{\langle p \rangle \ast m} \) reduces to \( Q_n \), then there exists \( j < \text{length}(n) \) such that \( Q_m \) reduces to \( Q_{c(n,j,p)} \).

Conversely, assume that \( p, m, n \) are natural numbers such that for some \( j < \text{length}(n) \), the set \( Q_m \) reduces to the set \( Q_{c(n,j,p)} \). Using (i) one may prove that \( Q_{\langle p \rangle \ast m} \) reduces to \( Q_n \).

We leave the proof of (iii) and (iv) to the reader.

\[ \square \]

5.16. Theorem 5.15 enables us, given any \( m, n \), to decide in finitely many steps if the set \( Q_m \) reduces to the set \( Q_n \) or not.

One may prove, for instance, that the sets \( C^3(D^2(A_1)) \) and \( C^2(D^3(A_1)) \) do not reduce to each other.

The next question would be to extend this algorithm to the class of sets of the form \( D(Q_n(0), \ldots, Q_n(k-1)) \), where \( n \) is a natural number and \( k = \text{length}(n) \). We have no answer to this question.

§6. Forming limits and finding more hierarchies. We consider various upper bounds for a given sequence of subsets of \( \mathcal{N} \). Taking such limits repeatedly, we erect new hierarchies, similar to the Cantor-Bendixson-hierarchy discussed in Section 3. In order to prove the two final results of this section, we have to make a small excursion into the art of handling transfinite inductions intuitionistically.

6.1. For all subsets \( X, Y \) of \( \mathcal{N} \), we let the (disjoint) sum of \( X \) and \( Y \), notation \( X \oplus Y \), be the set \( \langle 0 \rangle \ast X \cup \langle 1 \rangle \ast Y \).

For every infinite sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), we let the (countable) (disjoint) sum of the sequence \( X_0, X_1, \ldots \), notation \( \bigoplus_{n \in \mathbb{N}} X_n \), be the set \( \bigcup_{n \in \mathbb{N}} \langle n \rangle \ast X_n \).

It is not difficult to verify (see Veldman, 2008a, theorem 2.9) that, for all subsets \( X, Y, Z \) of \( \mathcal{N} \), the set \( X \oplus Y \) reduces to \( Z \) if and only if both \( X \) and \( Y \) reduce to \( Z \), and, for every infinite sequence \( X_0, X_1, \ldots \) of subsets of \( \mathcal{N} \), every subset \( Z \) of \( \mathcal{N} \), the set \( \bigoplus_{n \in \mathbb{N}} X_n \) reduces to \( Z \) if and only if, for each \( n \), \( X_n \) reduces to \( Z \).

The reducibility relation \( \preceq \) between subsets of \( \mathcal{N} \) thus behaves like the partial ordering belonging to a countably complete upper semilattice.

Let \( n \) be a positive natural number and let \( X_0, \ldots, X_{n-1} \) be subsets of \( \mathcal{N} \). We let the disjunction of \( X_0, \ldots, X_{n-1} \), notation \( D(X_0, \ldots, X_{n-1}) \) or \( D_{i=0}^{n-1}(X_i) \), be the set of all
\( \alpha \) such that, for some \( i < n, \alpha^i \) belongs to \( X_i \). As in Subsection 5.14, we let the conjunction of \( X_0, \ldots, X_{n-1} \), notation \( C(X_0, \ldots, X_{n-1}) \) or \( C_{i=0}^{n-1}(X_i) \), be the set of all \( \alpha \) such that, for all \( i < n, \alpha^i \) belongs to \( X_i \).

Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathcal{N} \). We call the sequence \( X_0, X_1, \ldots \) increasing in complexity if and only if, for each \( n \), \( X_n \) reduces to \( X_{n+1} \). We call the sequence strictly increasing in complexity if and only if, for each \( n \), \( X_n \) reduces to \( X_{n+1} \) and \( X_{n+1} \) does not reduce to \( X_n \). We call the sequence \( X_0, X_1, \ldots \) disjunctively closed if and only if, for all \( n \), for all \( s \) such that \( \text{length}(s) = n \), there exists \( p \) such that \( D_{i=0}^{n-1}(X_{s(i)}) \) reduces to \( X_p \). Note that, if the sequence \( X_0, X_1, \ldots \) is disjunctively closed, then, in particular, for all \( n, p \) there exists \( q \) such that \( D^n(X_p) \) reduces to \( X_q \). We call the sequence \( X_0, X_1, \ldots \) conjunctively closed if and only if, for all \( n \), for all \( s \) such that \( \text{length}(s) = n \), there exists \( p \) such that \( C_{i=0}^{n-1}(X_{s(i)}) \) reduces to \( X_p \). Note that, if the sequence \( X_0, X_1, \ldots \) is conjunctively closed, then, in particular, for all \( n, p \) there exists \( q \) such that \( C^n(X_p) \) reduces to \( X_q \). Also note that, according to Theorem 5.15, the sequences \( A_1, D^2(A_1), D^3(A_1), \ldots \) and \( D^2(A_1), C^2(D^2(A_1)), C^3(D^2(A_1)), \ldots \), are, both of them, disjunctively and conjunctively closed and strictly increasing in complexity.

For every finite sequence \( a_0, a_1, \ldots, a_{n-1} \) of elements of \( \mathcal{N} \), we let \( (a_0, a_1, \ldots, a_{n-1}) \) be the element \( y \) of \( \mathcal{N} \) such that \( y(0) = 0 \) and, for each \( i < n, y^i = a_i \), and, for each \( i \geq n, y^i = 0 \).

6.2. Theorem:

(i) Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathcal{N} \), increasing in complexity and disjunctively closed. The set \( D_{i=0}^{n}(X_n) \) reduces to the set \( D_n \). Proof.

(ii) Let \( X_0, X_1, \ldots \) be an infinite sequence of subsets of \( \mathcal{N} \), increasing in complexity and conjunctively closed. The set \( C_{i=0}^{n}(X_n) \) reduces to the set \( C_n \).

Proof. (i) Using the First Axiom of Countable Choice we determine \( \delta \) such that, for all \( m, n \), the set \( D_{i=0}^{n}(X_{\delta(m,n)}) \) reduces to the set \( X_{\delta(m,n)} \), and, for all \( n, \delta(n) < \delta(n + 1) \). Using the Second Axiom of Countable Choice, we then determine \( \epsilon \) such that, for all \( m, n, \epsilon(m,n) \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( D_{i=0}^{n}(X_{\delta(m,n)}) \) to \( X_{\epsilon(m,n)} \). We now determine \( \gamma \) such that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \), and, for all \( \alpha, \beta \), for all \( m, n, \gamma \) maps \( (m)*\alpha, (n)*\beta \) onto \( (\delta(m,n))* (\epsilon(m,n)| (\alpha, \beta)) \}. Finally, we let \( \eta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \alpha, \eta|\alpha = \gamma| (\alpha^0, \alpha^1) \). One easily verifies that \( \eta \) reduces the set \( D_{i=0}^{n}(X_{\eta(m,n)}) \) to the set \( D_n \).

(ii) We can use the same proof as for (i), only replacing “\( D \)” by “\( C \)”.

6.3. Let \( X_0, X_1, \ldots \) be a sequence of subsets of \( \mathcal{N} \). We consider a sample of six from the many sets \( Y \) there are with the property that each set \( X_n \) reduces to \( Y \). We define:

\[
0 - \bigcup_{n \in \mathbb{N}} X_n := \bigoplus_{n \in \mathbb{N}} X_n = \bigcup_{n \in \mathbb{N}} \langle n \rangle \ast X_n,
\]

\[
1 - \bigcup_{n \in \mathbb{N}} X_n := \bigcup_{n \in \mathbb{N}} \langle 0, n \rangle \ast (1) \ast X_n,
\]

\[
2 - \bigcup_{n \in \mathbb{N}} X_n := \{ \alpha \in \mathcal{N} \} \text{ Either } \alpha = 0 \text{ or } \alpha \text{ belongs to } 1 - \bigcup_{n \in \mathbb{N}} X_n).
\]
3 - $\bigcup_{n \in \mathbb{N}} X_n := \{ \alpha \in \mathcal{N} \mid \text{If } \alpha \neq \emptyset \text{, then } \alpha \text{ belongs to } 1 - \bigcup_{n \in \mathbb{N}} X_n \}$,

4 - $\bigcup_{n \in \mathbb{N}} X_n := \{ \alpha \in \mathcal{N} \mid \text{Either } \alpha^0 = \emptyset \text{ or there exists } n \text{ such that } \overline{\alpha}n = \overline{0}n \text{ and } \alpha(n) \neq 0 \text{ and } \alpha^{n+1} \text{ belongs to } X_n \}$, and

5 - $\bigcup_{n \in \mathbb{N}} X_n := \{ \alpha \in \mathcal{N} \mid \text{If } \alpha^0 \neq \emptyset \text{, then there exists } n \text{ such that } \overline{\alpha}n = \overline{0}n \text{ and } \alpha(n) \neq 0 \text{ and } \alpha^{n+1} \text{ belongs to } X_n \}$

Note that 1 - $\bigcup_{n \in \mathbb{N}} X_n$ is a subset of 2 - $\bigcup_{n \in \mathbb{N}} X_n$ and that 2 - $\bigcup_{n \in \mathbb{N}} X_n$ is a subset of 3 - $\bigcup_{n \in \mathbb{N}} X_n$. Also, 4 - $\bigcup_{n \in \mathbb{N}} X_n$ is a subset of 5 - $\bigcup_{n \in \mathbb{N}} X_n$.

Observe that, for each sequence $X_0, X_1, X_2, \ldots$ of subsets of $\mathcal{N}$, the set 2 - $\bigcup_{n \in \mathbb{N}} X_n$ is of the same degree of reducibility as the set 4 - $\bigcup_{n \in \mathbb{N}} X_n$ and the set 3 - $\bigcup_{n \in \mathbb{N}} X_n$ is of the same degree of reducibility as the set 5 - $\bigcup_{n \in \mathbb{N}} X_n$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, $(\gamma | \alpha)^0 = \alpha$, and, for each $n$, if $n$ is the least $k$ such that $\alpha(k) \neq 0$, then, if $\alpha(n) = 1$, $(\gamma | \alpha)^{n+1}$ equals the sequence $\beta$ such that $\alpha = \overline{0}n * (1) * \beta$, and, if $\alpha(n) \neq 1$, then $(\gamma | \alpha)^{n+1}$ does not belong to $X_n$. Note that $\gamma$ reduces 2 - $\bigcup_{n \in \mathbb{N}} X_n$ to 4 - $\bigcup_{n \in \mathbb{N}} X_n$ and 3 - $\bigcup_{n \in \mathbb{N}} X_n$ to 5 - $\bigcup_{n \in \mathbb{N}} X_n$.

Let $\delta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for each $\alpha$, for each $n$, if $\alpha^0$ passes through $\overline{0}n$, then $\delta | \alpha$ passes through $\overline{0}n$, and, if $n$ is the least $k$ such that $\alpha^0(k) \neq 0$, then $\delta | \alpha = \overline{0}n * (1) * \alpha^{n+1}$. Note that $\delta$ reduces 4 - $\bigcup_{n \in \mathbb{N}} X_n$ to 2 - $\bigcup_{n \in \mathbb{N}} X_n$ and 5 - $\bigcup_{n \in \mathbb{N}} X_n$ to 3 - $\bigcup_{n \in \mathbb{N}} X_n$.

Let us now consider some examples.

6.3.1. First, suppose that, for each $n$, the set $X_n$ coincides with the set $A_1 = \{ \emptyset \}$. For each $i < 4$, we define $V_i := i - \bigcup_{n \in \mathbb{N}} X_n$.

We now observe the following:

$V_0$ is a closed set and reduces to $A_1$ itself. $V_0$ also reduces to each one of the sets $V_1, V_2, V_3, V_4$. $V_1$, however, is not closed and does not reduce to $V_0$. $V_2$ coincides with the set $T = CB_2\mathcal{N}$. We have seen, in Theorem 3.3(iv), that this set is not sequentially closed and, therefore, does not reduce to $V_0$.

$V_1$ does not reduce to $V_2$. For suppose that $\gamma$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $V_1$ to $V_2$. Note that $\gamma | \emptyset$ will belong to $\overline{T} = T^{-\infty}$. Therefore, $\emptyset$ does not belong to $V_1$ while, at the same time, not: $(\gamma | \emptyset$ does not belong to $V_2)$.
Neither does \( V_2 \) reduce to \( V_1 \). For suppose that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( V_2 \) to \( V_1 \). Find \( m \) such that \( \gamma \upharpoonright \mathbb{0} = \mathbb{0}m \ast \{1\} \ast \mathbb{0} \). Find \( p \) such that, for every \( \alpha \), if \( \alpha \) passes through \( \mathbb{0}p \), then \( \gamma \upharpoonright \alpha \) passes through \( \mathbb{0}m \ast \{1\} \ast \mathbb{0} \). It follows that, for every \( \alpha \) in \( V_2 \), if \( \alpha \) passes through \( \mathbb{0}p \), then \( \gamma \upharpoonright \alpha = \mathbb{0}m \ast \{1\} \ast \mathbb{0} \). Therefore, also for every \( \alpha \) in the closure \( \overline{V}_2 \) of \( V_2 \), if \( \alpha \) passes through \( \mathbb{0}p \), then \( \gamma \upharpoonright \alpha = \mathbb{0}m \ast \{1\} \ast \mathbb{0} \). We conclude that \( \overline{V}_2 \cap \overline{\mathbb{0}}p \) coincides with \( \overline{V}_2 \cap \overline{\mathbb{0}}p \), but this is false, according to Theorem 3.3(iv). \( V_3 \) is easily seen to coincide with the closure \( \overline{T} \) of \( T = CB^{2*} \) and reduces to \( V_0 \). Note that we might also study \( V_1 \oplus V_2 \) as an upper bound of the sequence \( X_0, X_1, \ldots \).

6.3.2. Next, suppose that, for each \( n \), the set \( X_n \) coincides with the set \( D^{n+1}(A_1) \). Note that we proved, in Theorem 5.6(iv), that, for each \( n \), the set \( X_n \) strictly reduces to the set \( X_{n+1} \), that is, \( X_n \) reduces to \( X_{n+1} \), but, conversely, the set \( X_{n+1} \) does not reduce to the set \( X_n \). For each \( i < 3 \), we define \( W_i := i - \bigsqcup_{n \in \mathbb{N}} X_n \).

We now observe the following:

Note that, for each \( n \), the set \( X_n \) reduces to the set \( W_0 \), and, therefore, for each \( n \), the set \( W_0 \) does not reduce to the set \( X_n \). It follows from Theorem 6.2 that the set \( D(W_0, W_0) \) reduces to the set \( W_0 \), whereas, as an easy consequence of Theorem 5.6(iv), for each \( n \), the set \( D(X_n, X_n) \) does not reduce to the set \( X_n \). One may prove that the set \( W_2 \) is not closed and that its closure \( \overline{W}_2 \) coincides with its double complement \( (W_2)^\sim \). It then follows easily that \( W_1 \) does not reduce to \( W_2 \), as any function reducing \( W_1 \) to \( W_2 \) would map \( \mathbb{0} \) into \( \overline{W}_2 = (W_2)^\sim \) and force us to the conclusion that \( \mathbb{0} \) belongs to \( (W_1)^\sim \). Note that \( \mathbb{0} \) is an element of \( (W_1)^\sim \).

Neither does \( W_2 \) reduce to \( W_1 \). For suppose that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( W_2 \) to \( W_1 \). Find \( n \) such that \( \gamma \upharpoonright \mathbb{0} \) passes through \( \mathbb{0}n \ast \{1\} \). Find \( p \) such that, for every \( \alpha \), if \( \alpha \) passes through \( \mathbb{0}p \), then \( \gamma \upharpoonright \alpha \) passes through \( \mathbb{0}n \ast \{1\} \). It follows that the set \( W_2 \cap \overline{\mathbb{0}}p \) reduces to the set \( X_n \). It also is not difficult to see that, for each \( q \), \( X_q \) reduces to \( W_2 \cap \overline{\mathbb{0}}p \). Let \( m \) be the greatest of the two numbers \( n + 1, p + 1 \). Note that \( X_m \) reduces to \( W_2 \cap \overline{\mathbb{0}}p \) but not to \( X_n \). Contradiction.

The set \( W_2 \) does not reduce to the set \( W_3 \). For suppose that \( \gamma \) is a function from \( \mathcal{N} \) to \( \mathcal{N} \) reducing \( W_2 \) to \( W_3 \). Suppose that \( \gamma \upharpoonright \mathbb{0} \) is apart from \( \mathbb{0} \). It then follows, as in the argument from the previous paragraph, that, for some \( n, p \), the set \( W_2 \cap \overline{\mathbb{0}}p \) reduces to the set \( X_n \), and this turned out to be false. Therefore, \( \gamma \upharpoonright \mathbb{0} \) coincides with \( \mathbb{0} \). Consider the set \( T \) consisting of all \( \alpha \) with the property that, for all \( m, n \), if \( \alpha(m) \neq 0 \) and \( \alpha(n) \neq 0 \), then \( \alpha(m) = \alpha(n) = 1 \) and \( m = n \). Note that, for all \( \alpha \) in \( T \), if \( \gamma \upharpoonright \alpha \) is apart from \( \mathbb{0} \), then \( \alpha \) is apart from \( \mathbb{0} \) and there exists \( n \) such that \( \alpha = \mathbb{0}n \ast \{1\} \ast \mathbb{0} \), so \( \alpha \) belongs to \( W_2 \) and, therefore, \( \gamma \upharpoonright \alpha \) belongs to \( W_3 \). It follows that, for each \( \alpha \) in \( T \), \( \gamma \upharpoonright \alpha \) belongs to \( W_3 \), and, therefore, \( \alpha \) itself belongs to \( W_3 \). Using Brouwer's Continuity Principle we find \( p, n \) such that either, for every \( \alpha \) in the spread \( T = T^{\sim} \), if \( \alpha \) passes through \( \overline{\mathbb{0}}p \), then \( \alpha \) is \( \mathbb{0} \), or, for every \( \alpha \) in the spread \( T = T^{\sim} \), if \( \alpha \) passes through \( \overline{\mathbb{0}}p \), there exists \( \beta \) in \( X_n \) such that \( \alpha = \mathbb{0}n \ast \{1\} \ast \beta \). Both alternatives are clearly false.
The set $W_3$ does not reduce to the set $W_2$. The proof of this fact requires some preparations.

We let $S$ be the set of all $\alpha$ such that, if $\alpha$ is apart from 0, then there exist $n, i$ such that $i < n$ and $\alpha$ passes through $0n * (i)$. Note that the set $S$ is a spread and that 0 belongs to $S$. We let $\gamma$ be a function from $S$ to $\mathcal{N}$ such that for all $\alpha$ in $S$, for all $n, i$, if $\alpha$ passes through $0n * (i)$, then $\gamma | \alpha$ passes through $0n * (1)$, and, for all $p$, either there is no $j$ such that $p = (i, j)$ and $(\gamma | \alpha)(n + 1 + p) = \alpha(n + 1 + p)$ or there is $j$ such that $p = (i, j)$ and $(\gamma | \alpha)(n + 1 + p) = 0$. Note that, for every $\alpha$ in $S$, for all $n, i$, if $\alpha$ passes through $0n * (i)$, then there exists $\beta$ in $D^n(A_1)$ such that $\gamma | \alpha = \overline{0n * (1)} * \beta$, so $\gamma | \alpha$ belongs to $W_3$. One may verify that $\gamma$ maps the spread $S$ onto the set $W_3$ and that $\gamma | 0 = 0$. Now assume that $\delta$ is a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $W_3$ to $W_2$. Applying Brouwer’s Continuity Principle we find $p, n$ such that either, for every $\alpha$ in $S$, if $\alpha$ passes through $0p$, then $\delta | (\gamma | \alpha) = 0$, or for every $\alpha$ in $S$, if $\alpha$ passes through $0p$, then $\delta | (\gamma | \alpha)$ passes through $0n * (1)$. Both alternatives are false. In the first case, one may verify that, for every $\alpha$, if $\alpha$ belongs to $W_3 \cap 0p$, then $\gamma | \alpha = 0$. So $\gamma$ reduces the set $W_3 \cap 0p$ to the set $A_1 = \{0\}$. Note that the set $D^2(A_1)$ reduces to the set $W_3 \cap 0p$ and that the set $D^2(A_1)$ is not closed (see Theorem 5.4(i)). Contradiction. In the second case, we find that the set $W_3 \cap 0p$ reduces to the set $X_n$. Note that the set $X_{n+1}$ reduces to the set $W_3 \cap 0p$ and not to the set $X_n$. Contradiction.

Note that we might also study $W_1 \oplus W_2$, $W_1 \oplus W_3$, $W_2 \oplus W_3$ and $(W_1 \oplus W_2) \oplus W_3$ as upper bounds for the sequence $X_0, X_1, \ldots$.

6.3.3. It will be clear that every infinite sequence of subsets of $\mathcal{N}$ admits of a wide variety of upper bounds.

Let $X$ be a subset of $\mathcal{N}$. $X$ will be called \textit{disjunctively productive} if and only if, for each $n$, the set $D^{n+1}(X)$ does not reduce to the set $D^n(X)$. Note that, for every subset $X$ of $\mathbb{N}$, for every positive $n$, the set $D^n(X)$ reduces to the set $D^{n+1}(X)$. We have seen (see Theorem 5.6(iv)) that the set $A_1$ is disjunctively productive. The next theorem makes it clear that there are many more sets with this property.

Let $n$ be a positive natural number and let $X_0, \ldots, X_{n-1}$ be subsets of $\mathcal{N}$. As in Subsection 6.1, we let the \textit{disjunction} of $X_0, \ldots, X_{n-1}$, notation $D(X_0, \ldots, X_{n-1})$ or $D_{i=0}^{n-1}(X_i)$, be the set of all $\alpha$ such that, for some $i < n$, $\alpha^i$ belongs to $X_i$.

6.4. Theorem: Let $X_0, X_1, X_2, \ldots$ be a disjunctively closed sequence of subsets of $\mathcal{N}$ such that $A_1$ reduces to $X_0$ and, for each $m$, there exists $p > m$ such that $X_p$ does not reduce to $X_m$.

Let $W$ be $2 - \bigcup_{n \in \mathbb{N}} X_n$, that is, the set of all $\alpha$ such that either $\alpha = 0$ or, for some $n$, for some $\beta$ in $X_n$, $\alpha = \overline{0n * (1)} * \beta$.

(i) For each $k$, for each $l$, the set $D(D^{k+1}(A_1), D^l(W))$ does not reduce to the set $D(D^k(A_1), D^l(W))$.

(ii) For each $l$, the set $D^{l+1}(W)$ does not reduce to the set $D^l(W)$, that is, $W$ is disjunctively productive.
Proof. (i) Note that, according to Theorem 5.6(iv), for each $k$, the set $D^{k+1}(A_1)$ does not reduce to the set $D^k(A_1)$. Also note that $D^0(W) = \emptyset$ and thus, for each $k$, the set $D(D^k(A_1), D^0(W))$ is of the same degree of reducibility as the set $D^k(A_1)$. The statement (i) thus is true if $l = 0$.

Now let $k, l$ be natural numbers and suppose that $l > 0$ and the set $D(D^{k+1}(A_1), D^l(W))$ reduces to the set $D(D^k(A_1), D^l(W))$. Let $\gamma$ be a function from $\mathbb{N}$ to $\mathbb{N}$ reducing $D(D^{k+1}(A_1), D^l(W))$ to $D(D^k(A_1), D^l(W))$.

For each $m$, for each $a$ we let $c(a, m)$ be the number of elements of the set $\{i < l | a^i m = \bar{0}_m\}$. We claim the following:

For each $j < l$, for each $a$ in $D(D^{k+1}(A_1), D^l(W))$, if, for each $m$, $c(a^1, m) > j$, then, for each $m$, $c(\gamma | a)^1, m) > j$.

We establish this claim by induction.

First, suppose that $a$ belongs to $D(D^{k+1}(A_1), D^l(W))$, and, for each $m$, $c(a^1, m) > 0$, and, for some $m$, $c((|a|^1, m) = 0$. Find $n_0, n_1, \ldots, n_{l-1}$ such that, for each $i < l$, $(|a|^1)^i$ passes through $\bar{n}_i \ast \langle 1 \rangle$. Find $r$ such that, for all $\beta$, if $\beta$ passes through $\bar{n}_r$, then, for each $i < l$, $(|\gamma | \beta)^1, i$ passes through $\bar{n}_i \ast \langle 1 \rangle$. Using the fact that the sequence $X_0, X_1, X_2, \ldots$ is disjunctively closed, find $m$ such $D(D^k(A_1), D(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}))$ reduces to $X_m$ and then find $p$ such that $p > r$ and $X_p$ does not reduce to $X_m$. Note that $X_p$ also does not reduce to $D(D^k(A_1), D(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}))$. As, for all $m$, $c(a^1, m) > 0$, we may assume, without loss of generality, that $a^1, 0$ passes through $\bar{0}_r$. Now let $\zeta$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that, for all $\beta$, (i) $\zeta | \beta$ passes through $\bar{n}_r$, and (ii) for all $i \leq k$, $(\zeta | \beta)^0, i \# \bar{0}_i$, and (iii) $(\zeta | \beta)^1, 0 = \bar{0}_p \ast \langle 1 \rangle \ast \beta$, and (iv) for each $i, 0 < i < l$, then $(\zeta | \beta)^1, i$ does not belong to $W$.

Note that, for every $\beta$, $\beta$ belongs to $X_p$ if and only if $\zeta | \beta$ belongs to $D(D^{k+1}(A_1), D^l(W))$. Also observe that, for each $\beta$, for each $i < l$, $(\gamma | \zeta | \beta)^1, i$ passes through $\bar{n}_i \ast \langle 1 \rangle$. Let $\eta$ be a function from $\mathbb{N}$ to $\mathbb{N}$ such that, for each $i < k$, $(|\gamma | \beta)^0, i = (\gamma | \zeta | \beta)^0, i$ and, for each $i < l$, $(\gamma | (\zeta | \beta)^1, i = \bar{n}_i \ast \langle 1 \rangle \ast (\eta | \beta)^1, i$. Note that $\eta$ reduces $X_p$ to $D(D^k(A_1), D(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}))$. Contradiction.

We conclude that, for each $a$ in $D(D^{k+1}(A_1), D^l(W))$, if, for each $m$, $c(a^1, m) > 0$, then, for each $m$, $c((\gamma | a)^1, m) > 0$.

Next, assume that $j < l - 1$ and that

for each $a$ in $D(D^{k+1}(A_1), D^l(W))$, if, for each $m$, $c(a^1, m) > j$, then, for each $m$, $c((\gamma | a)^1, m) > j$.

We want to prove:

for each $a$ in $D(D^{k+1}(A_1), D^l(W))$, if, for each $m$, $c(a^1, m) > j + 1$, then, for each $m$, $c((\gamma | a)^1, m) > j + 1$.

Suppose that $a$ belongs to $D(D^{k+1}(A_1), D^l(W))$, and, for each $m$, $c(a^1, m) > j + 1$.

We determine $p$ such that, for every $\beta$, if $\beta$ passes through $\bar{a} p$, then $c((\gamma | \beta)^1, m) = c((\gamma | a)^1, m) \leq j$. We may assume, without loss of generality, that for each $i \leq j$, $a^{1, i} m = \bar{0}_m$, and for all $i$, if $j < i < l$, then $a^{1, i} m \neq \bar{0}_m$. We now consider the set $T,$
introduced in Subsections 3.1 and 5.9, and its closure \( \overline{T} \). \( T \) itself is the set of all \( \alpha \) in \( C \) that assume the value 1 either not at all or exactly one time, and \( \overline{T} \) is the set of all \( \alpha \) in \( C \) that assume the value 1 at most one time. Note that \( \overline{T} \) is a spread containing \( \emptyset \). We let \( \zeta \) be a function from \( T \) to \( N \) such that, for every \( \beta \) in \( T \), (i) for each \( i \leq k \), \( (\zeta |\beta)|^0,i = 0 \) and (ii) for each \( i < j \), \( (\zeta |\beta)|^1,i = 0 \), and (iii) for each \( i \), if \( j < i < l \), then \( (\zeta |\beta)|^1,i \) does not belong to \( W \). Note that, for each \( \beta \), \( \beta \) belongs to \( D^{j+1}(W) \) if and only if \( (\zeta |\beta) \) belongs to \( D(D^{k+1}(A_1), D^l(W)) \). Also note that, for all \( \beta \) in \( T \), either \( \beta^j = 0 \) or \( \beta^l = 0 \), that \( T \) is a subset of \( D^2(A_1) \). As \( \emptyset \) belongs to \( W \), it follows that, for all \( \beta \) in \( T \), \( (\zeta |\beta) \) belongs to \( D(D^{k+1}(A_1), D^l(W)) \). Moreover, for every \( \beta \) in \( T \), for every \( n \), at most one of the finite sequences \( \beta^0n, \beta^0n, \ldots, \beta^l \) differs from \( \emptyset n \), and, therefore, \( (c, \beta(n) < j \). We conclude that, for every \( \beta \) in \( T \), for every \( n \), also \( c((\zeta |\beta), n) > j \). It follows, in view of the induction hypothesis, that, for every \( \beta \) in \( T \), for every \( n \), \( c((\zeta |\beta), n) > j \). As, for every \( \beta \) in \( T \), for all \( i \), if \( j < i < l \), then \( (\gamma |(\zeta |\beta)|^1,i, m \neq \emptyset m \), we conclude that, for every \( \beta \) in \( T \), \( (\gamma |(\zeta |\beta)|^1,i, 0 \). As our functions are continuous, it follows that, for every \( \beta \) in the closure \( \overline{T} \) of \( T \), \( (\gamma |(\zeta |\beta)|^1,i, 0 \), so \( (\gamma |(\zeta |\beta) \) belongs to \( D(D^{k+1}(A_1), D^l(W)) \), and \( (\zeta |\beta) \) belongs to \( D(D^{k+1}(A_1), D^l(W)) \), and \( \beta \) itself belongs to \( D^{j+1}(W) \). We thus see that \( T \) is a subset of \( D^{j+1}(W) \). Therefore, for every \( \beta \) in \( T \), one may determine \( j < k \) such that either \( \beta^j = 0 \) or \( \beta^l \# 0 \). Using Brouwer's Continuity Principle, we find \( p, j \) such that \( j < k \) and either, for every \( \beta \) in \( T \) passing through \( \emptyset p, \beta^j = 0 \), or, for every \( \beta \) in \( T \) passing through \( \emptyset p, \beta^j \# 0 \). Both alternatives are false.

We conclude that, for each \( \alpha \) in \( D(D^{k+1}(A_1), D^l(W)) \), if, for each \( m \), \( c(\alpha^1, m) > j + 1 \), then, for each \( m \), \( c(\gamma |(\zeta |\beta), 1, m) = j + 1 \).

We now are sure that, for each \( j < l \), for each \( \alpha \) in \( D(D^{k+1}(A_1), D^l(W)) \), if, for each \( m \), \( c(\alpha^1, m) > j \).

Note that \( \emptyset \) belongs to \( D(D^{k+1}(A_1), D^l(W)) \) and that, in view of the result we just proved, for every \( i < l \), \( (\gamma |(\zeta |\beta)|^1,i, 0 \). For each \( i < k + 1 \), we let \( P_i \) be the set of all \( \alpha \) such that \( \alpha^0,i = 0 \), and, for each \( i < l \) let \( P_{k+1+l} \) be the set of all \( \alpha \) such that \( \alpha^1,i = 0 \). Note that, for each \( i < k + 1 + l \), \( P_i \) is a spread containing \( \emptyset \). Also note that, for each \( i < k + 1 + l \), \( P_i \) is a subset of \( D(D^{k+1}(A_1), D^l(W)) \). Finally observe that for each \( \alpha \) in \( D(D^{k+1}(A_1), D^l(W)) \), either there exists \( i < k + 1 + l \) such that \( i \neq k + 1 \) and \( \alpha \) belongs to \( P_i \), or, there exists \( i < l \) such that \( \alpha^1,i \# 0 \) and, therefore, for some \( m \), \( c(\alpha^1, m) < l \).

Observing that \( \gamma \) maps \( \bigcup \limits_{i<k+1+l} P_i \) into \( D(D^{k+1}(A_1), D^l(W)) \), we apply Brouwer's Continuity Principle \( k + l + 1 \) times. For each \( i < k + 1 + l \) we determine numbers \( m_i, q_i, \) \( n_i \) such that \( n_i < k + 1 < n_i < k + 1 + l \) and, either: \( m_i = 0 \) and, for every \( \alpha \) in \( P_i \cap \emptyset q_i, \gamma |\alpha \) belongs to \( P_{n_i} \), or: \( m_i = 1 \) and, for every \( \alpha \) in \( P_i \cap \emptyset q_i \), for some \( m \), \( c(\gamma |(\zeta |\beta), 1, m) < l \).

The second of these two alternatives, however, is excluded, as, for every \( i < l \), \( (\gamma |(\zeta |\beta)|^1,i, 0 \), and thus, for every \( m \), \( c(\gamma |(\zeta |\beta), 1, m) < l \). It follows that, for every \( i < k + 1 + l \), \( m_i = 0 \). Moreover, there will exists \( i, j \) such that \( i < j < k + 1 + l \) and \( n_i = n_j \). We consider the case \( n_0 = n_{k+1} = 0 \), leaving the other very similar cases to the reader. Let \( q \) be the greatest of the two numbers \( q_0, q_{k+1} \). We let \( \zeta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \beta \), (i) \( (\zeta |\beta) \) passes through \( \emptyset q \), and (ii) \( (\zeta |\beta)|^0,0 = \emptyset q \), and (iii) for each \( i \), if \( 0 < i < k + 1 \), then \( (\zeta |\beta)|^0,i \# 0 \), and (iv) for each \( n \), if \( \beta^n = \emptyset n \), then \( (\zeta |\beta)|^1,0(q + n) = \emptyset(q + n) \) and, if \( n \) is the least \( j \) such that \( \beta^j(j) \neq 0 \). Then there exists \( e, \)
not belonging to $X_n$, such that $(\zeta | \beta)^{1,0} = \overline{0}(n + q) * \{1\} * \epsilon$, and $(\nu)$ for each $i$, if $0 < i < l$, then $(\zeta | \beta)^{1,1}$ does not belong to $W$.

Note that, for every $\beta$, if $\beta$ belongs to $D^2(A_1)$, then $\zeta | \beta$ belongs to either $P_0$ or $P_{k+1}$ and, as $\zeta | \beta$ passes through $\overline{0}q$, in both cases, $(\gamma | (\zeta | \beta))^{0,0} = \overline{0}$. Conversely, for every $\beta$, if $(\gamma | (\zeta | \beta))^{0,0} = \overline{0}$, then $\gamma | (\zeta | \beta)$ belongs to $D(D^k(A_1), D^l(W))$ and $\zeta | \beta$ belongs to $D(D^{k+1}(A_1), D^l(W))$, and so, in view of the definition of $\zeta$, either $(\zeta | \beta)^{0,0} = \overline{0}$ or $(\zeta | \beta)^{1,0} = \overline{0}$, that is, either $\beta^0 = \overline{0}$ or $\beta^1 = \overline{0}$, so $\beta$ belongs to $D^2(A_1)$. We thus see that $D^2(A_1)$ reduces to $A_1$ and is a closed set. According to Theorem 5.4(i), however, the set $D^2(A_1)$ is not closed.

We conclude that, for all $k, l$, the set $D(D^{k+1}(A_1), D^l(W))$ does not reduce to the set $D(D^k(A_1), D^l(W))$.

(ii) Note that the set $A_1$ reduces to the set $W$. According to (i), for each $l$, the set $D(A_1, D^l(W))$ does not reduce to the set $D^l(W)$, and a fortiori, therefore, also $D^{l+1}(W)$ does not reduce to $D^l(W)$. □

6.5. Theorem: Let $X_0, X_1, X_2, \ldots$ be a conjunctively closed sequence of inhabited subsets of $N$ such that, for each $m$, for each $r$, there exists $p > r$ such that $X_p$ does not reduce to $X_m$.

Let $W$ be $2 - \bigcup_{n \in \mathbb{N}} X_n$, that is, the set of all $\alpha$ such that either $\alpha = \overline{0}$ or, for some $n$, for some $\beta$ in $X_n$, $\alpha = \overline{0}n * \{1\} * \beta$.

(i) $W$ is not a closed subset of $\mathcal{N}$.
(ii) For each $l$, the set $C^{l+1}(W)$ does not reduce to the set $C^l(W)$, that is, $W$ is conjunctively productive.

Proof.

(i) Let $LPO$ be the set of all $\alpha$ in $\mathcal{N}$ such that either $\alpha \not\in \overline{0}$ or $\alpha = \overline{0}$. (We use the abbreviation $LPO$ because of the limited principle of omniscience considered by Bishop & Bridges (1985). This principle is the (false) statement that the set $LPO$ coincides with $\mathcal{N}$.)

We prove that $LPO$ reduces to $W$.

Using the Second Axiom of Countable Choice, we first determine $\delta$ in $\mathcal{N}$ such that, for each $n$, $\delta^n$ belongs to $X_n$.

We let $\zeta$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, for every $n$, if $\alpha n = \overline{0}n$, then $(\zeta | \alpha)n = \overline{0}n$, and, if $n$ is the least $i$ such that $\alpha(i) \neq 0$, then $\zeta | \alpha = \overline{0}n * \{1\} * \delta^n$.

Note that, for every $\alpha$, either $\alpha = \overline{0}$ or $\alpha \not\in \overline{0}$ if and only if either $\zeta | \alpha = \overline{0}$ or there exists $n$ such that $\zeta | \alpha = \overline{0}n * \{1\} * \delta^n$, and, therefore, $\alpha = \overline{0}$ or $\alpha \not\in \overline{0}$ if and only if $\zeta | \alpha$ belongs to $W$.

Now assume that $W$ is a closed subset of $\mathcal{N}$. Then also $LPO$ is a closed subset of $\mathcal{N}$ and, consequently, $LPO$ coincides with $\mathcal{N}$. Using Brouwer’s Continuity Principle we find $m$ such that, either, for every $\alpha$, if $\alpha$ passes through $\overline{0}m$, then $\alpha \not\in \overline{0}$, or, for every $\alpha$, if $\alpha$ passes through $\overline{0}m$, then $\alpha = \overline{0}$. Both alternatives are false.

We conclude that $W$ is not a closed subset of $\mathcal{N}$.

(ii) Note that $W$ does not reduce to $\mathcal{N}$, and, therefore, $C^1(W)$ does not reduce to $C^0(W)$.

Now let $l$ be a positive natural number and suppose that $C^{l+1}(W)$ reduces to $C^l(W)$.

Let $\gamma$ be a function from $\mathcal{N}$ to $\mathcal{N}$ reducing $C^{l+1}(W)$ to $C^l(W)$. For each $m$, for each
We establish this claim by induction.

First, suppose that \( \alpha \) belongs to \( C^{l+1}(W) \) and that, for all \( m \), \( c(\alpha, m) > 0 \), and, for some \( m \), \( c(\gamma | \alpha, m) = 0 \). Find numbers \( n_0, n_1, \ldots, n_{l-1} \), such that, for each \( i < l \), \( (\gamma | \alpha)^i(n_i) \neq 0 \) and, for every \( k < n_i \), \( (\gamma | \alpha)^i(k) = 0 \). Note that, for each \( i < l \), \( (\gamma | \alpha)^i(n_i) = 1 \). Find \( r \) such that, for every \( \beta \), if \( \beta \) passes through \( \alpha r \), then, for each \( i < l \), \( (\gamma | \beta)^i(n_i + 1) = \overline{0n_i} + 1 \). Find \( m \) such that \( C(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}) \) reduces to \( X_m \) and then find \( p \) such that \( p > r \) and \( X_p \) does not reduce to \( X_m \). Note that \( X_p \) also does not reduce to \( C(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}) \). Note that \( c(\alpha, p) > 0 \). It follows that there exists \( i < l + 1 \) such that \( a^i p = \overline{0p} \). Without loss of generality, we may assume that \( \alpha^0 p = \overline{0p} \). Let \( \zeta \) be a function from \( N \) to \( N \) such that, for all \( \beta \), (i) \( \zeta | \beta \) passes through \( \alpha r \), and (ii) \( \zeta | \beta \) passes through \( \alpha r \), and (iii) for all \( i \), if \( 0 < i < l + 1 \), then \( (\zeta | \beta)^i = \alpha^i \). Recall that \( \alpha \) belongs to \( C^{l+1}(W) \), and, therefore, for every \( \beta \), for every \( i \), if \( 0 < i < l + 1 \), then \( (\zeta | \beta)^i \) belongs to \( W \). Let \( \eta \) be a function from \( N \) to \( N \) such that, for every \( \beta \), for every \( i < l \), \( (\gamma | (\zeta | \beta)^i) = \overline{0n_i} + 1 \). Observe that, for every \( \beta \), \( \beta \) belongs to \( X_p \) if and only if \( \zeta | \beta \) belongs to \( C^{l+1}(W) \) if and only if \( (\zeta | \beta)^i \) belongs to \( W \), whereas we have chosen \( X_p \) in such a way that \( X_p \) does not reduce to \( C(X_{n_0}, X_{n_1}, \ldots, X_{n_{l-1}}) \). Contradiction.

Therefore, for each \( \alpha \) in \( C^{l+1}(W) \), if, for each \( m \), \( c(\alpha, m) > 0 \), then, for each \( m \), \( c((\gamma | \alpha), m) > 0 \).

Next, assume that \( j < l - 1 \) and that 

\[
\text{for each } \alpha \text{ in } C^{l+1}(W), \text{ if, for each } m, \ c(\alpha, m) > j, \text{ then, for each } m, \ c((\gamma | \alpha), m) > j.
\]

We want to prove:

\[
\text{for each } \alpha \text{ in } C^{l+1}(W), \text{ if, for each } m, \ c(\alpha, m) > j + 1, \text{ then, for each } m, \ c((\gamma | \alpha), m) > j + 1.
\]

Suppose that \( \alpha \) belongs to \( C^{l+1}(W) \) and that, for all \( m \), \( c(\alpha, m) > j + 1 \), and, for some \( m \), \( c((\gamma | \alpha), m) \leq j + 1 \). Find \( m \) such that \( c((\gamma | \alpha), m) \leq j + 1 \). The induction hypothesis now guarantees \( c((\gamma | \alpha), m) = j + 1 \). Without loss of generality, we may assume that, for all \( i < l \), if \( i \leq j \), then \( (\gamma | \alpha)^i m = \overline{0m} \) and, if \( i > j \), then \( (\gamma | \alpha)^i m \neq \overline{0m} \). Find numbers \( n_{j+1}, n_{j+2}, \ldots, n_{l-1} \) such that, for each \( i \), if \( j < i < l - 1 \), then \( n_i < m \) and \( (\gamma | \alpha)^i \) passes through \( \overline{0n_i} + 1 \). Find \( r \) such that, for all \( \beta \), if \( \beta \) passes through \( \alpha r \), then, for each \( i < l \), \( (\gamma | \beta)^i m = (\gamma | \alpha)^i m \). Find \( n \) such that \( C_i^{l-1}(X_{n_i}) \) reduces to \( X_n \) and then find \( p \) such that \( p > r \) and \( X_p \) does not reduce to \( X_n \). Note that \( X_p \) also does not reduce to \( C_i^{l-1}(X_{n_i}) \). Note that \( c(\alpha, p) > j + 1 \). Without loss of generality, we may assume that, for all \( i \), if \( i \leq j + 1 \) then \( a^i p = \overline{0p} \). Let \( \zeta \) be a function from \( N \) to \( N \) such that, for all \( \beta \), (i) \( \zeta | \beta \) passes through \( \alpha r \), and (ii) for all \( i \), if \( i < j + 1 \) or \( j + 1 < i < k + 1 \), then \( (\zeta | \beta)^i = \alpha^i \), and (iii) \( (\zeta | \beta)^{j+1} = \overline{0p} + 1 \). Note that, for every \( m \), for every \( \beta \), \( c((\zeta | \beta), m) \geq c(\alpha, m) - 1 \), and, therefore, \( c((\zeta | \beta), m) > j \). The induction hypothesis now implies that, for every \( m \), for every \( \beta \), \( c((\zeta | \beta), m) > j \) and therefore, for all \( i \),
if \( i \leq j \), then \( (\gamma |\zeta |\beta)^i = 0 \), as, for every \( i \), if \( j < i < k \), then \( (\gamma |\zeta |\beta)^i \neq 0 \). Let \( \eta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \beta \), for every \( i \), if \( i < l - j - 1 \), then \( (\gamma |\zeta |\beta)^i = 0 \). Observe that, for every \( \beta \), belongs to \( \mathcal{X}_p \) if and only if \( \eta |\beta \) belongs to \( C_{l - j + 1}^1(X_n) \). Therefore, \( X_p \) reduces to \( C_{l - j + 1}^1(X_n) \), whereas we have chosen \( X_p \) in such a way that \( X_p \) does not reduce to \( C_{l - j + 1}^1(X_n) \). Contradiction.

Therefore, for each \( \alpha \) in \( C_{l + 1}^1(W) \), if, for each \( m \), \( c(\alpha, m) > j + 1 \), then, for each \( m \), \( c(\gamma |\alpha, m) > j + 1 \). We thus see that, for each \( \alpha \) in \( C_{l + 1}^1(W) \), if, for each \( m \), \( c(\alpha, m) > l \), then, for each \( m \), \( c(\gamma |\alpha, m) > l \). Let \( \zeta \) be a function from \( \mathcal{N} \) to \( \mathcal{N} \) such that, for every \( \beta \), for each \( i < l \), \( (\zeta |\beta)^i = 0 \), and \( (\zeta |\beta)^i = \beta \). Note that for every \( \beta \), \( \beta \) belongs to \( W \) if and only if \( \zeta |\beta \) belongs to \( C_{l + 1}^1(W) \) if and only if for each \( i \), if \( i < l \), then \( (\gamma |\zeta |\beta)^i = 0 \). We thus see that \( W \) is a closed subset of \( \mathcal{N} \), whereas, according to (i), \( W \) is not a closed subset of \( \mathcal{N} \).

We conclude that, for each \( l \), the set \( C_{l + 1}^1(W) \) does not reduce to the set \( C_l(W) \). □

6.6. Theorems 6.4 and 6.5 enable us to find, given any sequence \( X_0, X_1, \ldots \) of sets that strictly increases in complexity, to find an upper bound for the sequence that itself is the first element of another sequence of sets that strictly increases in complexity. It follows, for instance, that the set \( W_0 := 2 - \bigcup_{n \in \mathbb{N}} D^n(A_1) \) is both disjunctively and conjunctively productive. Starting from this \( W_0 \), we may define a sequence \( W_0, W_1, \ldots \) of sets by: for each \( m \),

\[
W_{m+1} := 2 - \bigcup_{n \in \mathbb{N}} D^n(W_m).
\]

We then may go on and define a sequence \( W_0, W_{o+1}, \ldots \) by: \( W_0 := 2 - \bigcup_{n \in \mathbb{N}} W_n \) and, for each \( m \), \( W_{o+m+1} := 2 - \bigcup_{n \in \mathbb{N}} D^n(W_{o+m}) \).

Alternatively, we might define a sequence \( U_0, U_1, \ldots \) of sets by: \( U_0 := W_0 \) and, for each \( m \), \( U_{m+1} := 2 - \bigcup_{n \in \mathbb{N}} C^n(U_m) \). We then may go on and define a sequence \( U_0, U_{o+1}, \ldots \) by: \( U_0 := 2 - \bigcup_{n \in \mathbb{N}} U_n \) and, for each \( m \), \( U_{o+m+1} := 2 - \bigcup_{n \in \mathbb{N}} C^n(U_{o+m}) \).

Observe that all these sets belong to the class \( \Sigma_2 \).

In the spirit of the above examples, we introduce two classes of subsets of \( \mathcal{N} \). We first define, for every stump \( \sigma \), a subset \( DCB_\sigma \) of \( \mathcal{N} \), as follows, by transfinite induction:

(i) \( DCB_0 := \{0\} = A_1 \)

(ii) For every nonempty stump \( \sigma \), \( DCB_\sigma := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{0 \} + (1) + \bigcup_{i=0} D^n(DCB_{\sigma_i}) = 2 - \bigcup_{n \in \mathbb{N}} D^n(DCB_{\sigma_i}) \).

We have added the letter \( D \) of Disjunction to the letters \( C, B \) of Cantor and Bendixson. We sometimes want to call the sets \( DCB_\sigma \), where \( \sigma \) is a stump, the disjunctive-Cantor-Bendixson-sets.

We also introduce, for every stump \( \sigma \), a subset \( CCB_\sigma \) of \( \mathcal{N} \), as follows, by transfinite induction:

(i) \( CCB_0 := D^2(A_1) \)
(ii) For every nonempty stump $\sigma$,
\[
CCB_\sigma := \{0\} \cup \bigcup_{n \in \mathbb{N}} \{0n \ast (1) \ast C^n_{i=0}(CCB_{\sigma^i}) = 2 - \bigcap_{n \in \mathbb{N}} C^n_{i=0}(CCB_{\sigma^i})
\]

We have added the letter $C$ of Conjunction to the letters $C$, $B$ of Cantor and Bendixson. We sometimes want to call the sets $CCB_\sigma$, where $\sigma$ is a stump, the conjunctive-Cantor-Bendixson-sets.

Note that every disjunctive-Cantor-Bendixson-set and every conjunctive-Cantor-Bendixson-set belongs to the class $\Sigma^0_2$. In order to prove some properties of these two classes of sets, we need a special principle of induction on the set of stumps, that is of some value in itself. We now introduce this principle. Some other variants of induction on the set of stumps are mentioned in Veldman (2004).

Let $(\sigma_0, \ldots, \sigma_{m-1})$ and $(\tau_0, \ldots, \tau_{n-1})$ be finite sequences of stumps. We call the sequence $(\tau_0, \ldots, \tau_{n-1})$ a simplification of the sequence $(\sigma_0, \ldots, \sigma_{m-1})$ if and only if there exists $i < m$ and a positive natural number $k$ such that $\sigma_i$ is nonempty and $n = m + (k - 1)$ and for each $j < i$, $\sigma_j = \tau_j$, and for each $j < k$, $\tau_{i+j} = (\sigma_i)^j$ and, for each $j < m - i - 1$, $\tau_{i+k+j} = \sigma_{i+j+1}$. So the sequence $(\tau_0, \ldots, \tau_{n-1})$ results from the sequence $(\sigma_0, \ldots, \sigma_{m-1})$ if one replaces $(\sigma_i)$ by $((\sigma_i)^0, \ldots, (\sigma_i)^{k-1})$.

We denote the set of finite sequences of stumps by $Stp^*$.

6.7. Theorem: (A Principle of Induction on finite sequences of stumps:)

Let $P$ be a subset of $Stp^*$ such that every nonempty finite sequence of stumps belongs to $Stp^*$ as soon as each one of its simplifications belongs to $Stp^*$.

Then every nonempty finite sequence of stumps belongs to $Stp^*$.

6.8. The proof of Theorem 6.7 will occupy us until Subsection 6.18. We then will apply Theorem 6.7 in Theorems 6.19 and 6.20.

Let $\sigma$ be a stump. We let $B(\sigma)$ be the set of all natural numbers $s$ belonging to $\sigma$, that is such that $\sigma(s) = 1$ (see Subsection 2.5.3).

One should think of $B(\sigma)$ as a set of (code numbers of) finite sequences of natural numbers. One may prove, by induction on the set of stumps (see Subsection 2.5.3), that, for every stump $\sigma$,

(i) $B(\sigma)$ is a decidable subset of $\mathbb{N}$,

(ii) for every $s$, for every $n$, if $s \ast \langle n \rangle$ belongs to $B(\sigma)$, then $s$ belongs to $B(\sigma)$, and

(iii) for every $a$ in $\mathbb{N}$ there exists $n$ such that $\overline{an}$ does not belong to $B(\sigma)$.

We define a binary relation $<^\#$ on $\mathbb{N}$:

for all $m, n$ in $\mathbb{N}$, $m <^\# n$ if and only if there exists $p$ such that $m = n \ast \langle p \rangle$, that is, $m$ is, as a finite sequence, an immediate extension of $n$.

Let $A$ be a subset of $\mathbb{N}$ and let $<_0$ be a binary relation on $A$.

Let $P$ be a subset of $A$. $P$ is called $<_0$-hereditary if and only if, for every $a$ in $A$, $a$ belongs to $P$ as soon as every $b$ in $A$ such that $b <_0 a$ belongs to $P$. $<_0$ is called inductive on $A$ if and only if every $<_0$-hereditary subset of $A$ coincides with $A$.

6.9. Lemma: For every nonempty stump $\sigma$, the relation $<^\#$ is inductive on $B(\sigma)$.
Proof. We use induction on the set of stumps. Assume that \( \sigma \) is a nonempty stump and that the statement holds for every immediate substant \( \sigma^n \) of \( \sigma \) such that \( \sigma^n \) is nonempty. Let \( P \) be a \(<_0\)-hereditary subset of \( \mathbb{B}(\sigma) \). Note that, for every \( n \), if \( \langle n \rangle \) belongs to \( \mathbb{B}(\sigma) \), then \( \sigma^n \) is nonempty, and the set \( \{ s \in \mathbb{N} | \langle n \rangle \ast s \in P \} \) is a \(<_0\)-hereditary subset of \( \mathbb{B}(\sigma^n) \), and, therefore, by the induction hypothesis, this set coincides with \( \mathbb{B}(\sigma^n) \), and in particular, \( \langle \rangle \) belongs to this set, and, therefore, \( \langle n \rangle \) belongs to \( P \). As \( P \) is \(<_0\)-hereditary, also \( \langle \rangle \) belongs to \( P \), and \( P \) coincides with \( \mathbb{B}(\sigma) \). □

6.10. Let \( A \) be a subset of \( \mathbb{N} \) and let \(<_0\) be a binary relation on \( A \).

We define another binary relation on \( A \), called: \textit{the transitive closure of \(<_0\), notation: \( (<_0)_{tc} \)}, as follows:

\[
\text{for all } m, n \in A, m (<_0)_{tc} n \text{ if and only if there exist } s, k \text{ such that } k = \text{length}(s), \text{ and } k > 1, \text{ and } s(0) = m \text{ and } s(k - 1) = n, \text{ and, for all } j < k - 1, s(j + 1) <_0 s(j).
\]

Note that \( (<_0)_{tc} \) is a \textit{transitive} relation on \( A \), that is: for all \( m, n, p \), if \( m (<_0)_{tc} n \) and \( n (<_0)_{tc} p \), then \( m (<_0)_{tc} p \).

Also note that for every transitive relation \( <_1 \) on \( A \), if, for all \( m, n \), if \( m <_0 n \), then \( m <_1 n \), then also, for all \( m, n \), if \( m (<_0)_{tc} n \), then \( m <_1 n \). In this sense, \( (<_0)_{tc} \) is the least transitive relation containing the relation \( <_0 \).

The following observation seems to be of some value.

6.11. Lemma: Let \( A \) be a subset of \( \mathbb{N} \) and let \(<_0\) be a binary relation on \( A \).

(i) If \(<_0\) is inductive on \( A \), then \( (<_0)_{tc} \) is inductive on \( A \).

(ii) If \( (<_0)_{tc} \) is inductive on \( A \), then \(<_0\) is inductive on \( A \).

Proof.

(i) Suppose that \(<_0\) is inductive on \( A \), and assume that \( P \) is a \( (<_0)_{tc} \)-hereditary subset of \( A \). Let \( Q \) be the set of all \( m \) in \( A \) such that, for all \( n \), if either \( n = m \) or \( n (<_0)_{tc} m \), then \( n \) belongs to \( P \). Observe that \( Q \) is a \(<_0\)-hereditary subset of \( A \) and, therefore, coincides with \( A \). As \( Q \) is a subset of \( P \), also \( P \) coincides with \( A \).

(ii) Suppose that \( (<_0)_{tc} \) is inductive on \( A \), and assume that \( P \) is a \(<_0\)-hereditary subset of \( A \). Observe that \( P \) is also a \( (<_0)_{tc} \)-hereditary subset of \( A \) and thus coincides with \( A \). □

6.12. Let \( A \) be a subset of \( \mathbb{N} \) and let \(<_0\) be a binary relation on \( A \). A natural number \( s \) is or codes a \(<_0\)-decreasing sequence in \( A \) if and only if, for each \( j < \text{length}(s) \), \( s(j) \) belongs to \( A \), and, for each \( j < \text{length}(s) - 1 \), \( s(j + 1) <_0 s(j) \).

Let \( \sigma \) be a stump. We say that \( \sigma \) \textit{captures} \(<_0\) on \( A \) if and only if every finite \(<_0\)-decreasing sequence in \( A \) belongs to \( \sigma \). \(<_0\) is called a \textit{stumpy} relation on \( A \) if and only if some stump captures \(<_0\) on \( A \).

Also the following observation seems to be of some value.

6.13. Lemma:

(i) For every subset \( A \) of \( \mathbb{N} \), for every binary relation \(<_0\) on \( A \), if \(<_0\) is stumpy on \( A \), then \(<_0\) is inductive on \( A \).
(ii) For every decidable subset $A$ of $\mathbb{N}$, for every decidable binary relation $<_0$ on $A$, if $<_0$ is inductive on $A$, then $<_0$ is stumpy on $A$.

Proof. (i) We prove, by induction on the set of stumps, that, for every stump $\sigma$,

(*) for every subset $A$ of $\mathbb{N}$, for every binary relation $<_0$ on $A$, if $\sigma$ captures $<_0$ on $A$, then $<_0$ is inductive.

Note that, if $\sigma$ is the empty stump, then the statement (*) is true.

Assume that $\sigma$ is a nonempty stump and that the statement holds for every one of its immediate substumps $\sigma^m$. Suppose that $A$ is a subset of $\mathcal{N}$ and $<_0$ is a binary relation on $A$ and that $\sigma$ captures $<_0$ on $A$. Let $P$ be a $<_0$-hereditary subset of $A$.

Let $n$ be a natural number such that $n$ belongs to $A$. Note that $\sigma^n$ captures $<_0$ on the set of all elements $a$ from $A$ with the property: $a \ (<_0)^{\mathcal{N}} \ n$. It follows from the induction hypothesis that all such elements of $A$ belong to $P$, and, in particular, that, for each $m$ in $A$, if $m <_0 n$ then $m$ belongs to $P$. Therefore, $n$ itself belongs to $P$.

We thus see that every $<_0$-hereditary subset $P$ of $A$ coincides with $A$ and that the statement (*) holds for $\sigma$.

(ii) Suppose that $A$ is a decidable subset of $\mathbb{N}$, and that $<_0$ is a binary relation on $A$ that is inductive on $A$. Let $P$ be the set of all $m$ in $A$ such that, for some stump $\alpha$, for every $s$, if $(m) * s$ is a finite $<_0$-decreasing sequence $s$ in $A$, then $s$ belongs to $\alpha$. We prove that $P$ is $<_0$-hereditary:

Suppose that $a$ belongs to $A$ and that, for all $m$ in $A$, if $m <_0 a$, then $m$ belongs to $P$. Using the Second Axiom of Countable Choice, we find $\beta$ in $\mathcal{N}$ such that $\beta(\langle \rangle) = 1$ and, for each $m$, $\beta^m$ is a stump, and, if $m$ belongs to $A$ and $m <_0 a$, then, for every $s$, if $(a, m) * s$ is a finite $<_0$-decreasing sequence in $A$, then $s$ belongs to $\beta^m$. Observe that $\beta$ is a stump and that, for every $t$, if $(a) * t$ is a finite $<_0$-decreasing sequence in $A$, then $t$ belongs to $\beta^m$. We thus see that $a$ itself belongs to $P$.

We conclude that $A$ coincides with $P$. Using the Second Axiom of Countable Choice, we find $\beta$ in $\mathcal{N}$ such that $\beta(\langle \rangle) = 0$ and, for each $m$, $\beta^m$ is a stump, and, if $m$ belongs to $A$, then, for every $s$, if $(m) * s$ is a finite $<_0$-decreasing sequence $s$ in $A$, then $s$ belongs to $\beta^m$. It follows that $\beta$ captures $<_0$ on $A$ and that $<_0$ is stumpy. □

6.14. Let $A$, $B$ be subsets of $\mathbb{N}$ and let $<_0, <_1$ be binary relations on $A$, $B$, respectively. We let $A \times B$ be the subset of $\mathbb{N}$ consisting of all numbers of the form $\langle a, b \rangle$ where $a$ belongs to $A$ and $b$ to $B$. We define a relation $<_2$ on the set $A \times B$ as follows. For all $a_0, a_1$ in $A$, $b_0, b_1$ in $B$, $\langle a_0, b_0 \rangle <_2 \langle a_1, b_1 \rangle$ if and only if either $a_0 <_0 a_1$ and $b_0 = b_1$ or $a_0 = a_1$ and $b_0 < b_1$. We call the relation $<_2$ the interweaving of the relations $<_0, <_1$ and denote the structure $(A \times B, <_2)$ by $(A, <_0) \bullet (B, <_1)$.

Let $n$ be a positive natural number and let $A_0, A_1, \ldots, A_{n-1}$ be subsets of $\mathbb{N}$ and let $<_0, <_1, \ldots, <_{n-1}$ be binary relations on $A_0, A_1, \ldots, A_{n-1}$, respectively. We let $\Pi_{i=0}^{n-1} A_i$ be the set of all $s$ in $\mathbb{N}$ such that $\text{length}(s) = n$ and, for each $i < n$, $s(i)$ belongs to $A_i$. We define a relation $\otimes^{n-1}_{i=0} <_i$ on the set $\Pi_{i=0}^{n-1} A_i$ as follows. For all $s, t$ in $\Pi_{i=0}^{n-1} A_i$, $s(\otimes^{n-1}_{i=0} <_i) t$ if and only if, for some $j < n$, $s(j) <_j t(j)$, and, for all $i < n$, if $i \neq j$, then $s(i) = t(i)$. The relation $\otimes^{n-1}_{i=0} <_i$ is called the interweaving of the relations $<_0, <_1, \ldots, <_{n-1}$.
6.15. Lemma:

(i) Let $A, B$ be subsets of $\mathbb{N}$ and let $<_0, <_1$ be binary relations on $A, B$, respectively. If $<_0$ is inductive on $A$ and $<_1$ is inductive on $B$, then their interweaving $<_0 \otimes <_1$ is inductive on $A \times B$.

(ii) Let $n$ be a positive natural number and let $A_0, A_1, \ldots, A_{n-1}$ be subsets of $\mathbb{N}$ and let $<_0, <_1, \ldots, <_{n-1}$ be binary relations on $A_0, A_1, \ldots, A_{n-1}$, respectively. If, for each $i < n$, the relation $<_i$ is inductive on $A_i$, then their interweaving $\otimes_{i=0}^{n-1} <_i$ is inductive on $\prod_{i=0}^{n-1} A_i$.

Proof.

(i) Let $P$ be a $<_0 \otimes <_1$-hereditary subset of $A \times B$. Let $Q$ be the set of all $a$ in $A$ such that, for all $b$ in $B$, $(a, b)$ belongs to $P$. Note that, for every $a$ in $A$, if every $a'$ in $A$ with the property: $a' <_0 a$ belongs to $Q$, then the set of all $b$ in $B$ such that $(a, b)$ belongs to $P$ is a $<_1$-hereditary subset of $B$ and, therefore, $a$ itself belongs to $Q$. We thus see that $Q$ is a $<_0$-hereditary subset of $A$ and conclude that $Q$ coincides with $A$. It follows that $P$ coincides with $A \times B$.

(ii) The argument is a straightforward extension of the argument for (i), using induction. □

6.16. Let $A$ be a subset of $\mathbb{N}$. We let $A^*$ be the set of all natural numbers coding a finite sequence of elements of $A$, that is, for all $s$, $s$ belongs to $A^*$ if and only if, for all $i < \text{length}(s)$, $s(i)$ belongs to $A$.

Let $A$ be a set and $<_0$ a binary relation on $A$. We introduce a binary relation on the set $A^*$, calling it $(<_0)^*$, the sequential extension of the relation $<_0$:

For all $s, t, m, n$ in $\mathbb{N}$, if $s, t$ belong to $A^*$ and $\text{length}(s) = m$ and $\text{length}(t) = n$, we define: $t (<_0)^* s$, or: $t$ is a $<_0$-simplification of $s$,

if and only if there exist $i < m$ and a positive natural number $k$ such that $n = m + (k - 1)$, and for each $j < i$, $t(j) = s(i)$, and for each $j < k$,

$t(i + j) <_0 s(i)$, and for each $j < m - i - 1$, $t(i + k + j) = s(i + j + 1)$.

So the sequence $t = (t(0), \ldots, t(n - 1))$ is obtained from the sequence $s = (s(0), \ldots, s(m - 1))$ if one replaces its subsequence $(s(i))$ by $(t(i), \ldots, t(i + k - 1))$ where, for each $j < k$, $t(i + j) <_0 s(i)$.

Let $\sigma$ be a nonempty stump. For each $s$ in $(B(\sigma))^*$ we let $(B(\sigma))^* \upharpoonright s$ be the set of all $t$ in $(B(\sigma))^*$ such that either $t = s$ or $t ((<_0)^*)^c s$.

Note that, for every $s, t$ is a $<_0$-simplification of the finite sequence $(t)$, or: $s (<_0)^* (t)$, if and only if $\text{length}(s) > 0$ and, and there exists $m$ in $\mathbb{N}$ such that $m = \text{length}(t) = \text{length}(s)$ and, for each $i < m$, $s(i) = (t(i))$, so $s = (t(0), t(1), \ldots, t(m - 1))$, and, for each $i < m$, $s(i)$ belongs to $\sigma$.

For each natural number $n$, we let $F_n$ be a mapping from $\mathbb{N}$ to $\mathbb{N}$ such that, for each $s$, for each $m$, if $\text{length}(s) = m$, then $F_n(s) = (s(0) * (n), s(1) * (n), \ldots, s(m - 1) * (n))$.

For each natural number $w$, we define another natural number, called: the expansion of $w$, notation: $\text{Exp}(w)$, as follows, by induction on $\text{length}(w)$:

(i) $\text{Exp}(()) = ()$

(ii) For each $w$, for each $n$, $\text{Exp}(w * (n)) = \text{Exp}(w) * n$.

Note that, for each $m, w$ such that $m = \text{length}(w)$, $\text{Exp}(w) = w(0) * w(1) * \ldots * w(m - 1)$. 

Note that, for every \( m, t \) such that \( m = \text{length}(t) \), for every \( s \), the following statements are equivalent:

(i) \( s((t(0)), \langle t(1), \ldots, t(m-1) \rangle) \).

(ii) there exist \( v, w \) such that \( m = \text{length}(v) = \text{length}(w) \), and, for each \( i < m \), \( v(i) \neq \langle \rangle \) and \( w(i) = F_{t(i)}(v(i)) \) and \( s = \text{Exp}(w) \).

For each \( m \), we let \( \langle \rangle^m \) be the element \( s \) of \( \mathbb{N} \) such that \( \text{length}(s) = m \), and, for each \( i < m \), \( s(i) = \langle \rangle \). Note that \( \langle \rangle^1 = \langle \rangle \).

6.17. Theorem:

For every stump \( \sigma \), the sequential extension \( \langle \# \rangle^* \) of the relation \( \langle \# \rangle \) is an inductive relation on \( (\mathbb{B}(\sigma))^* \uparrow \langle \rangle \).

Proof. We use induction on the set of stumps. The statement of the lemma is trivially true if \( \sigma \) is the empty stump. Now assume that \( \sigma \) is nonempty and that, for each \( n \), the relation \( \langle \# \rangle^* \) is inductive on \( (\mathbb{B}(\sigma^n))^* \uparrow \langle \rangle \). We have to prove that \( \langle \# \rangle^* \) is also inductive on \( (\mathbb{B}(\sigma))^* \uparrow \langle \rangle \).

Let \( P \) be a \( \langle \# \rangle^* \)-hereditary subset of \( (\mathbb{B}(\sigma))^* \uparrow \langle \rangle \).

Let \( t, m \) belong to \( \mathbb{N} \) and suppose \( m = \text{length}(t) > 0 \). Let \( s \) be an element of \( \mathbb{N} \) such that \( \text{length}(s) = m \) and, for each \( i < m \), \( s(i) = (t(i)) \). We want to prove that \( (\mathbb{B}(\sigma))^* \uparrow s \) is a subset of \( P \).

Note that, for every \( u \) in \( (\mathbb{B}(\sigma))^* \uparrow s \) there exist \( v, w \) such that \( m = \text{length}(v) = \text{length}(w) \), and for each \( i < m \), \( v(i) \) belongs to \( (\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle \), and \( w(i) = F_{t(i)}(v(i)) \), and \( u = \text{Exp}(w) \).

For each \( v \) in \( \Pi_{i=0}^{m-1}(\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle^m \) we define the result of \( v \), notation \( \text{Res}(v) \), as the element \( u \) of \( \mathbb{N} \) such that, for some \( w \), \( m = \text{length}(w) \), and, for each \( i < m \), \( w(i) = F_{t(i)}(v(i)) \), and \( u = \text{Exp}(w) \).

We let \( Q \) be the subset of \( \Pi_{i=0}^{m-1}(\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle^m \) consisting of all \( v \) such that \( \text{length}(v) = m \), and for all \( i < m \), \( v(i) \) belongs to \( (\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle \), and \( \text{Res}(v) \) belongs to \( P \).

Note that \( Q \) is a \( \bigotimes_{i=0}^{m-1}(\langle \# \rangle^* \)-hereditary subset of \( \Pi_{i=0}^{m-1}(\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle^m \). Using Lemma 6.15 and the induction hypothesis we conclude that \( \Pi_{i=0}^{m-1}(\mathbb{B}(\sigma^i))^* \uparrow \langle \rangle^m \) coincides with \( Q \). It follows that \( (\mathbb{B}(\sigma))^* \uparrow s \) is a subset of \( P \).

Note that \( s = \text{Res}(\langle \rangle^m) \). We thus see that every \( \langle \# \rangle \)-simplification \( s \) of \( \langle \rangle \) belongs to \( P \) and may conclude that also \( \langle \rangle \) itself belongs to \( P \). Therefore, \( (\mathbb{B}(\sigma))^* \uparrow \langle \rangle \) is a subset of \( P \).

We thus see that \( \langle \# \rangle^* \) is an inductive relation on \( (\mathbb{B}(\sigma))^* \uparrow \langle \rangle \). □

6.18. Proof of Theorem 6.7. Let \( P \) be a subset of \( \text{Stp}^* \) such that every nonempty finite sequence of stumps belongs to \( P \) as soon as each one of its simplifications belongs to \( P \).

Let \( (\sigma_0, \ldots, \sigma_{n-1}) \) be a nonempty finite sequence of stumps. Let \( \tau \) be some stump such that for each \( i < n \), \( \tau^i = \sigma_i \). For each \( t \) in \( \mathbb{B}(\tau) \) we define a stump \( \tau^t \), the substump of \( \tau \) at \( t \), as follows. The definition is by induction on \( \text{length}(t) \). We define: \( \langle \rangle \tau := \tau \) and for each \( t, i \), if \( t \ast i \) belongs to \( \mathbb{B}(\tau) \), then \( \tau^t \ast i := (\tau^t)^i \).

Observe that, for all nonempty elements \( a = \langle a(0), \ldots, a(m-1) \rangle \) and \( b = \langle b(0), \ldots, b(n-1) \rangle \) of \( \mathbb{B}(\tau) \), if \( \langle b(0), \ldots, b(n-1) \rangle \) is a \( \langle \# \rangle \)-simplification of \( \langle a(0), \ldots, a(m-1) \rangle \), then the finite sequence of stumps \( \langle b(0) \tau, \ldots, b(n-1) \tau \rangle \) is a simplification of the finite sequence of stumps \( \langle a(0) \tau, \ldots, a(m-1) \tau \rangle \) in the sense of Subsection 6.6.
Let $Q$ be the set of all elements $(a(0), \ldots, a(m-1))$ of $(B(\tau))^* \uparrow \{\tau\}$ such that $(a(0)\tau, \ldots, a(m-1)\tau)$ belongs to $P$. Note that $Q$ is a $(<^*\tau)$-hereditary subset of $(B(\tau))^* \uparrow \{\tau\}$ and conclude by Theorem 6.17 that $Q$ coincides with $(B(\tau))^* \uparrow \{\tau\}$. In particular, $(0), (1), \ldots, (n-1)$ belongs to $Q$, and, therefore, $(0)\tau, \ldots, (n-1)\tau$ belongs to $P$, that is, $(\sigma_0, \ldots, \sigma_{n-1})$ belongs to $P$.

6.19. Theorem:

(i) For each stump $\sigma$, the set $DCB_{\sigma}$ belongs to the class $\Sigma^0_2$, and its closure $\overline{DCB_{\sigma}}$ coincides with its double complement $(DCB_{\sigma})^{\overline{\overline{\cdot}}}$.

(ii) For all hereditarily repetitive stumps $\sigma$, $\tau$, if $\sigma \leq \tau$ then the set $DCB_{\sigma}$ reduces to the set $DCB_{\tau}$.

(iii) For every stump $\sigma$, for every $n$, the set $DCB_{\sigma}$ reduces to the set $DCB_{\sigma} \cap \check{\Omega}_n$.

(iv) For every finite sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ of stumps, the set $D(A_1, D^{n-1}_{i=0}(DCB_{\sigma}_i))$ does not reduce to the set $D^{n-1}_{i=0}(DCB_{\sigma}_i)$.

(v) For all hereditarily repetitive stumps $\sigma$, $\tau$, if $\sigma < \tau$, then the set $DCB_{\tau}$ does not reduce to the set $DCB_{\sigma}$.

(vi) For each stump $\sigma$, for each $n$, the set $D^{n+1}(DCB_{\sigma})$ does not reduce to the set $D^n(DCB_{\sigma})$.

Proof.

(i) We leave the proof to the reader as it is similar to the proof of Theorem 3.5(iii).

(ii) We use induction on the set of hereditarily repetitive stumps. It is obvious that for each stump $\tau$, the set $DCB_{\tau}$ reduces to the set $DCB_{\tau}$. Now assume that $\sigma$, $\tau$ are hereditarily repetitive stumps, $\sigma$ is nonempty and $\sigma \leq \tau$. Using our Axioms of Countable Choice we find a strictly increasing $\alpha$ such that, for each $m$, $\sigma^m \leq \tau^\alpha(m)$ and also $\gamma$ such that, for each $m$, $\gamma^m$ is a function from $N$ to $N$ reducing the set $DCB_{\sigma^m}$ to the set $DCB_{\tau^\alpha(m)}$. We leave it to the reader to define $\delta$ in such a way that, for each $m$, $\delta^m$ is a function from $N$ to $N$ reducing the set $D^{n+1}_{i=0}(DCB_{\sigma^m})$ to the set $D^n(DCB_{\tau^\alpha(m)})$. Finally, we construct a function $\zeta$ from $N$ to $N$ such that for every $m$, for every $e$, the sequence $\zeta^e(0m * (1) * e)$ equals the sequence $0(\alpha(m)) * (1) * (\delta^m | e)$. It will be clear that $\zeta$ maps $0$ onto $0$ and reduces $DCB_{\sigma}$ to $DCB_{\tau}$.

(iii) We leave the proof to the reader who should keep in mind that, for each nonempty stump $\sigma$, for each $n$, the set $D^{n+1}_{i=0}(DCB_{\sigma}$) reduces to the set $D^n(DCB_{\sigma})$.

(iv) We intend to use the Principle of Induction on the set $Stp^*$ of finite sequences of stumps expressed in Theorem 6.7. Note that the statement we want to prove is true for the empty sequence of stumps. Let $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ be a nonempty finite sequence of stumps and assume that the statement has been proved for every finite sequence of stumps that is a simplification of the sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$. (Observe that, if the sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ has no simplifications, then, for each $i < n$, $\sigma_i$ is the empty stump, and the statement to be proved is equivalent to the statement: the set $D^{n+1}_{i=0}(A_1)$ does not reduce to the set $D^n(A_1)$. We have proven this in Theorem 5.6(iv), but the argument we are about to explain furnishes another proof of this special case.) Let us assume that $\gamma$ is a function from $N$ to $N$ reducing the set $D(A_1, D^{n-1}_{i=0}(DCB_{\sigma}_i))$ to the set $D^{n-1}_{i=0}(DCB_{\sigma}_i)$. Consider the sets $B_0, B_1, \ldots, B_n$ which are defined as follows: $B_0 := \{\alpha | \alpha \in N | \alpha^0 = 0\}$ and, for each $i < n$, $B_{i+1} := \{\alpha | \alpha \in N | \alpha^1 = 0\}$. Observe that every one of these sets
is a spread containing 0 and forming part of $D(A_1, D_i^n (DCB_{\sigma_i}))$. Applying the Continuity Principle $n + 1$ times we find for each $i \leq n$ natural numbers $m_i$ and $k_i$ such that, for every $\alpha$ in $B_i$, passing through $\hat{0}m_i$, the sequence $(\gamma | \alpha)^{m_i}$ belongs to $DCB_{\sigma_i}$. Without loss of generality we assume $k_0 = k_1 = 0$.

Applying the Continuity Principle two more times we find $p_0, p_1, n_0, n_1, t_0, t_1$, such that for every $i < 2$ either $t_i = 0$ and for every $\alpha$ in $B_i$ passing through $\hat{0}p_i$, the sequence $(\gamma | \alpha)^{p_i}$ coincides with $0$, or $t_i = 1$ and for every $\alpha$ in $B_i$ passing through $\hat{0}p_i$, there exists $\beta$ such that $(\gamma | \alpha)^{p_i}$ equals $\hat{0}m_i * (1) * \beta$ and $\beta$ belongs to $D_{i=0}^{p_i} (DCB_{\sigma_0})$. Let us first assume $t_0 = t_1 = 0$. Observe that now for every $\alpha$ in $B_0$ passing through $\hat{0}p_0$ the sequence $(\gamma | \alpha)^{p_0}$ coincides with $0$, and also for every $\alpha$ in $B_1$ passing through $\hat{0}p_1$, the sequence $(\gamma | \alpha)^{p_0}$ coincides with $0$. We now see that the set $D^2(A_1)$ reduces to the set $A_1$, as follows. We let $p$ be the greatest of the two numbers $p_0, p_1$. We construct a function $\delta$ from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, (1) $(\delta|\alpha)^{p_0} = \hat{0}p$, and (2) $(\delta|\alpha)^{p_0} = \hat{0}p * \alpha^{0}$ and (3) if $\alpha^1 = 0$, then $(\delta|\alpha)^{1, i} = \hat{0}$, but (4) if $\alpha^1 \neq 0$, then $(\delta|\alpha)^{1, i}$ does not belong to $DCB_{\sigma_0}$, and (5) for each $i$, if $0 < i < n - 1$, then $(\delta|\alpha)^{1, i}$ does not belong to $DCB_{\sigma_0}$. It is not difficult to verify that for each $\alpha$, $\alpha$ belongs to $D^2(A_1)$ if and only if $\delta|\alpha$ belongs to $D(A_1, D_i^n (DCB_{\sigma_i}))$ if and only if $(\gamma | \alpha)^{0}$ passes through $\hat{0}p_0 * (1)$. Let $e$ be a function from $\mathcal{N}$ to $\mathcal{N}$ such that, for every $\alpha$, the sequence $(\gamma | (\hat{0}q * \alpha)^{0})^0$ equals $\hat{0}n_0 * (1) * (e|\alpha)$. Observe that, for every $\alpha$, $\hat{0}q * \alpha$ belongs to $D(A_1, D_i^n (DCB_{\sigma_i}))$ if and only if either $e|\alpha$ belongs to $D_{i=0}^{n_0} (DCB_{\sigma_0})$ or for some positive $i < n$, the sequence $(\gamma | \hat{0}q * \alpha)^i$ belongs to $DCB_{\sigma_0}$. Using (iii) we conclude that $D(A_1, D_i^n (DCB_{\sigma_i}))$ reduces to $D(D_{i=0}^{n_0} (DCB_{\sigma_0}), D_{i=0}^{n_2} (DCB_{\sigma_1})))$. Observe that $D_{i=0}^{n_0} (DCB_{\sigma_0})$ reduces to $DCB_{\sigma_0}$ and therefore $D(A_1, D_i^n (DCB_{\sigma_i}))$ reduces to $D(D_{i=0}^{n_0} (DCB_{\sigma_0}), D_{i=0}^{n_2} (DCB_{\sigma_1})))$ and therefore to $D(D_i^n (DCB_{\sigma_0}), D_{i=0}^{n_2} (DCB_{\sigma_1})))$. We now obtain a contradiction by our induction hypothesis, as the finite sequence $(\sigma_0^0, \ldots, \sigma_0^{n_0-1}, \sigma_1, \ldots, \sigma_{n-1})$ is a simplification of the finite sequence $(\sigma_0, \sigma_1, \ldots, \sigma_n)$.

(v) This easily follows from (iv). Let $\sigma, \tau$ be hereditarily repetitive stumps such that $\sigma < \tau$. Calculate $m$ such that $\sigma \leq \tau^m$. Observe that $DCB_{\sigma}$ reduces to $DCB_{\tau}$.

On the other hand $D_i^{n_0+1} (DCB_{\tau})$ reduces to $DCB_{\tau}$ but not to $DCB_{\tau^m}$. Therefore $DCB_{\tau}$ does not reduce to $DCB_{\sigma}$.

(vi) This follows immediately from (iv).

6.20. Theorem:

(i) For each stump $\sigma$, the set $CCB_{\sigma}$ belongs to the class $\Sigma_0^0$ and its closure $\overline{CCB_{\sigma}}$ coincides with its double complement $(CCB_{\sigma})^{--}$.

(ii) For all hereditarily repetitive stumps $\sigma, \tau$, if $\sigma \leq \tau$, then the set $CCB_{\sigma}$ reduces to the set $CCB_{\tau}$.

(iii) For every stump $\sigma$, for every $n$, the set $CCB_{\sigma}$ reduces to the set $CCB_{\sigma} | \hat{\sigma}n$.

(iv) For every finite sequence $(\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$ of stumps, the set $C(D^2(A_1), C_i^{n-1} (CCB_{\sigma_i}))$ does not reduce to the set $C_i^{n-1} (CCB_{\sigma_i})$. □
(v) For all hereditarily repetitive stumps \( \sigma, \tau \), if \( \sigma < \tau \), then the set \( \text{CCB}_\tau \) does not reduce to the set \( \text{CCB}_\sigma \).

(vi) For each stump \( \sigma \), for each \( n \), the set \( \text{C}_{n+1}(\text{CCB}_\sigma) \) does not reduce to the set \( \text{C}_n(\text{CCB}_\sigma) \).

**Proof.** We only prove (iv) and leave it to the reader to prove the remaining statements of the theorem.

We again want to use the Principle of Induction on the set of finite sequences of stumps expressed in Theorem 6.7.

Let us assume that \( (\sigma_0, \ldots, \sigma_{n-1}) \) is a nonempty finite sequence of stumps and that the statement has been proved for every simplification of the finite sequence \( (\sigma_0, \ldots, \sigma_{n-1}) \). Suppose we find a function \( \gamma \) from \( \mathcal{N} \) to \( \mathcal{N} \) reducing the set \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})) \) to the set \( C_{i=0}^{n-1}(\text{CCB}_{\sigma_i}) \). Note that, for every stump \( \sigma \), the sequence \( \overline{0} \) is an element of \( \text{CCB}_\sigma \).

For every \( i \) in \( \{0, 1\} \), we let \( B_i \) be the set of all \( \alpha \) in \( \mathcal{N} \) such that, both \( \alpha^{0,i} \) coincides with \( \overline{0} \) and, for every \( j \leq n - 1 \), the sequence \( \alpha^{1,j} \) coincides with \( \overline{0} \). Observe that \( B_0, B_1 \) are spreads containing \( \overline{0} \) and subsets of \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})) \).

Repeatedly applying the Continuity Principle we find, for every \( i \) in \( \{0, 1\} \), for each \( j < n - 1 \), natural numbers \( n = n(i, j) \) and \( t = t(i, j) \) such that either \( t = 0 \) and for every \( \alpha \) in \( B_i \) passing through \( \overline{0}n \), the sequence \( (\gamma | \alpha)^j \) coincides with \( \overline{0} \) or \( t = 1 \) and for every \( \alpha \) in \( B_i \) passing through \( \overline{0}n \), the sequence \( (\gamma | \alpha)^j \) is apart from \( \overline{0} \).

We claim that there must exist \( i \) in \( \{0, 1\} \) and \( j < n - 1 \) such that \( t = t(i, j) = 1 \). For suppose not. We then calculate \( N = \max\{n(i, j)|i \in \{0, 1\}, j < n - 1\} \) and construct a function \( \delta \) from \( \mathcal{N} \) to \( \mathcal{N} \) such that for every \( i < 2 \), the sequence \( (\delta | \sigma)^0,j \) coincides with \( \overline{0}N \ast \alpha^i \) and for every \( j < n - 1 \), the sequence \( (\delta | \sigma)^1,j \) coincides with \( \overline{0} \), and \( (\delta | \sigma)(0) = 0 \).

Observe that, for every \( \alpha, \alpha \) belongs to \( D^2(A_1) \) if and only if, for every \( j \leq n - 1 \), the sequence \( (\gamma | (\delta | \alpha)^j \) coincides with \( \overline{0} \). Therefore, the set \( D^2(A_1) \) reduces to the set \( A_1 \), and the set \( D^2(A_1) \) is closed, but it is not, according to Theorem 5.4(i).

Without loss of generality we may assume that \( t = t(0, 0) = 1 \). We now determine \( p, q \) such that for every \( \alpha \) passing through \( \overline{0}p \) the sequence \( (\gamma | \alpha)^0 \) passes through \( \overline{0}q \ast (1) \).

As in the proof of Theorem 6.19 we may conclude that the set \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})) \) reduces to the set \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})), C_{i=0}^{n-2}(\text{CCB}_{\sigma_{i+1}})) \).

But then also the set \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})), C_{i=0}^{n-2}(\text{CCB}_{\sigma_{i+1}})) \) will reduce to the set \( C(D^2(A_1), C_{i=0}^{n-1}(\text{CCB}_{\sigma_i})), C_{i=0}^{n-2}(\text{CCB}_{\sigma_{i+1}})) \), and we obtain a contradiction, as the finite sequence \( ((\sigma_0)^0, \ldots, (\sigma_0)^2, \sigma_1, \ldots, \sigma_{n-1}) \) is a simplification of the finite sequence \( (\sigma_0, \ldots, \sigma_{n-1}) \). \( \square \)

6.21. In this section, we restricted our attention to the class \( \Sigma^0_2 \). In the other additive classes of the intuitionistic Borel hierarchy similar things will happen. A strengthening of Theorem 3 in the introduction of this paper states (see Veldman, 2008a, corollary 8.10):

*Let \( (X, Y) \) be a special complementary pair of leading positively Borel sets. Then, for each \( k \), the set \( D^{k+1}(Y) \) does not reduce to the set \( D^k(Y) \).*

This fact enables us to apply Theorem 6.4 in the class consisting of all sets of the form \( \bigcup_{n \in \mathbb{N}} Y_n \), where each set \( Y_n \) reduces to the set \( Y \). And, of course, this class will also allow its own variants of the other results in this chapter.
BIBLIOGRAPHY


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