DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

ASYMPTOTIC INFERENCE FOR A NEARLY UNSTABLE SEQUENCE OF STATIONARY SPATIAL AR MODELS

Sándor Baran, Gyula Pap, Martien C.A. van Zuijlen

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

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SÁNDOR BARAN and GYULA PAP
Institute of Mathematics and Informatics, University of Debrecen
Pf. 12, H–4010 Debrecen, Hungary

MARTIEN C. A. VAN ZUIJLEN
Department of Mathematics, University of Nijmegen
Toernooiveld 1, 6525 ED Nijmegen, The Netherlands

Abstract

A nearly unstable sequence of stationary spatial autoregressive processes is investigated, where the autoregressive coefficients are equal, and their sum tends to one. It is shown that the limiting distribution of the least squares estimators for these coefficients are normal and, in contrast to the doubly geometric process, the typical rate of convergence is $n^{-5/4}$.

Keywords. Autoregressive model, asymptotic normality, martingale central limit theorem.

1 Introduction

The analysis of spatial models is of interest in many different fields such as geography, geology, biology and agriculture. See, e.g., Basu and Reinsel [3] for a discussion on these applications.

The only spatial autoregressive model for which nearly unstability has been studied is the so called doubly geometric spatial autoregressive process

$$X_{k,\ell} = \alpha X_{k-1,\ell} + \beta X_{k,\ell-1} - \alpha\beta X_{k-1,\ell-1} + \varepsilon_{k,\ell}$$

introduced by Martin [9]. It is, in fact, the simplest spatial model, since its nice product structure ensures that it can be considered as some kind of combination of two autoregressive processes on the line, and several properties can be derived by the analogy of one-dimensional autoregressive processes. This model has been used by

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Jain [8] in the study of image processing, by Martin [10], Cullis and Gleason [7], Basu and Reinsel [4] in agricultural trials and by Tjøstheim [13] in digital filtering.

In the stable case when $|\alpha|, |\beta| < 1$, asymptotic normality of several estimators $(\widehat{\alpha}_n, \widehat{\beta}_n)$ of (α, β) based on the observations $\{X_{k,\ell} : k, \ell = 1, \dots, n\}$ has been shown (e.g. Tjøstheim [12], [14], Basu and Reinsel [2], [3]), namely, $(n(\widehat{\alpha}_n - \alpha),\, n(\widehat{\beta}_n - \beta)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma_{\alpha,\beta})$ with some covariance matrix $\Sigma_{\alpha,\beta}$.

In the nearly unstable case when a sequence of stable models with $\alpha_n \to 1$, $\beta_n \to 1$ is considered, in contrast to the AR(1) model, the sequence of Gauss-Newton estimators $(\widehat{\alpha}_n, \widehat{\beta}_n)$ of (α_n, β_n) has been shown to be asymptotically normal (Bhattacharyya et. al. [6]), namely, $(n^{3/2}(\widehat{\alpha}_n - \alpha_n),\, n^{3/2}(\widehat{\beta}_n - \beta_n)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, \Sigma)$ with some covariance matrix Σ.

In the present paper we study asymptotic properties of the least squares estimator in a spatial model which can be considered as the simplest spatial model, that can not be reduced somehow to autoregressive models on the line (like the doubly geometric model). We will find a rather peculiar limiting behaviour of the covariance structure (see Proposition 2.1), and we show that the normalising factor in our unstable model differs from that in the doubly geometric model.

Our spatial autoregressive process $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ is a solution of the spatial stochastic difference equation

$$X_{k,\ell} = \alpha(X_{k-1,\ell} + X_{k,\ell-1}) + \varepsilon_{k,\ell}. \tag{1.1}$$

This model is stable (i.e., has a stationary solution) in case $|\alpha| < 1/2$ (see Whittle [15], Beasg [5], Basu and Reinsel [3]), and unstable if $|\alpha| = 1/2$. We remark, that in case $|\alpha| < 1/2$, a stationary solution can be given by

$$X_{k,\ell} = \sum_{(i,j)\in U_{k,\ell}} \binom{k+\ell-i-j}{k-i} \alpha^{k+\ell-i-j} \varepsilon_{i,j}, \tag{1.2}$$

where $U_{k,\ell} := \{(i,j) \in \mathbb{Z}^2 : i \le k \text{ and } j \le \ell\}$ and the convergence of the series is understood in L_2-sense.

We consider a nearly unstable sequence of stationary processes, i.e., for each $n \in \mathbb{N}$, we take a stationary solution $\{X^{(n)}_{k,\ell} : k, \ell \in \mathbb{Z}\}$ of equation (1.1) with parameter α_n converging to $1/2$, more precisely,

$$\alpha_n = \frac{1}{2} - \frac{\gamma_n}{n}, \quad \text{where} \quad \gamma_n > 0 \quad \text{and} \quad \gamma_n \to \gamma \ge 0 \quad \text{as} \quad n \to \infty. \tag{1.3}$$

For a set $H \subset \mathbb{Z}^2$, the least squares estimator $\widehat{\alpha}^{(n)}_H$ of α_n based on the observations $\{X^{(n)}_{k,\ell} : (k,\ell) \in H\}$ is obtained by minimizing the sum of squares

$$\sum_{(k,\ell)\in H} \left(X^{(n)}_{k,\ell} - \alpha(X^{(n)}_{k-1,\ell} + X^{(n)}_{k,\ell-1}) \right)^2$$

with respect to α, and it has the form

$$\widehat{\alpha}_H^{(n)} = \frac{\sum_{(k,\ell)\in H}(X_{k-1,\ell}^{(n)} + X_{k,\ell-1}^{(n)})X_{k,\ell}^{(n)}}{\sum_{(k,\ell)\in H}(X_{k-1,\ell}^{(n)} + X_{k,\ell-1}^{(n)})^2}.$$

Consider the triangles $T_{k,\ell} := \{(i,j) \in \mathbb{Z}^2 : i+j \geq 1,\ i \leq k \text{ and } j \leq \ell\}$ for $k, \ell \in \mathbb{Z}$. Note, that $T_{k,\ell} = \emptyset$ if $k + \ell \leq 0$.

1.1 Theorem. *For each $n \in \mathbb{N}$, let $\{X_{k,\ell}^{(n)} : k, \ell \in \mathbb{Z}\}$ be a stationary solution of equation (1.1) with parameter α_n given by (1.3), and with independent identically distributed random variables $\{\varepsilon_{k,\ell}^{(n)} : k, \ell \in \mathbb{Z}\}$ such that $\mathsf{E}\,\varepsilon_{0,0}^{(n)} = 0$, $\mathsf{Var}\,\varepsilon_{0,0}^{(n)} = 1$ and $\sup_{n\in\mathbb{N}} \mathsf{E}\left|\varepsilon_{0,0}^{(n)}\right|^4 < \infty$. Let (k_n) and (ℓ_n) be sequences of integers such that $k_n+\ell_n \uparrow \infty$ as $n \to \infty$. If*

$$\lim_{n\to\infty} \gamma_n^{1/2}(k_n + \ell_n)n^{-1/2} = \infty \tag{1.4}$$

then

$$\gamma_n^{-1/4}(k_n + \ell_n)n^{1/4}\left(\widehat{\alpha}_{T_{k_n,\ell_n}}^{(n)} - \alpha_n\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty.$$

Observe, that due to stationarity of the process $\{X_{k,\ell}^{(n)} : k, \ell \in \mathbb{N}\}$, the distribution of $\widehat{\alpha}_{T_{k_n,\ell_n}}^{(n)}$ equals the distribution of $\widehat{\alpha}_{T_{\widetilde{k}_n,\widetilde{\ell}_n}}^{(n)}$, where $\widetilde{k}_n := [(k_n + \ell_n)/2]$ and $\widetilde{\ell}_n := [(k_n + \ell_n + 1)/2]$. As $\widetilde{k}_n + \widetilde{\ell}_n = k_n + \ell_n$, in Theorem 1.1 we can substitute k_n by $\widetilde{k}_n$ and ℓ_n by $\widetilde{\ell}_n$. The sequence $(\widetilde{k}_n, \widetilde{\ell}_n)$ can be embedded into the sequence (k_n', ℓ_n'), where $k_n' := [n/2]$ and $\ell_n' := [(n+1)/2]$, i.e., there exists a strictly monotone increasing sequence (q_n) of positive integers such that $k_{q_n}' = \widetilde{k}_n$ and $\ell_{q_n}' = \widetilde{\ell}_n$, namely, $q_n = k_n + \ell_n$. Clearly $k_n' + \ell_n' = n$. Furthermore, let (r_n) be a monotone increasing sequence such that $r_{q_n} = n$. Condition (1.4) implies $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$. To prove Theorem 1.1, it is enough to show

$$\gamma_{r_n}^{-1/4}(k_n' + \ell_n')r_n^{1/4}\left(\widehat{\alpha}_{T_{k_n',\ell_n'}}^{(r_n)} - \alpha_{r_n}\right) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1) \qquad \text{as} \quad n \to \infty. \tag{1.5}$$

To simplify notation in what follows we omit the prime from our index sequences and assume that $k_n = [n/2]$ and $\ell_n = [(n+1)/2]$. We can write

$$\widehat{\alpha}_{T_{k_n,\ell_n}}^{(r_n)} - \alpha_{r_n} = A_n/B_n$$

with

$$A_n := \sum_{(i,j)\in T_{k_n,\ell_n}} (X_{i-1,j}^{(r_n)} + X_{i,j-1}^{(r_n)})\varepsilon_{i,j}^{(r_n)},$$

$$B_n := \sum_{(i,j)\in T_{k_n,\ell_n}} (X_{i-1,j}^{(r_n)} + X_{i,j-1}^{(r_n)})^2.$$

Hence, the statement of Theorem 1.1 is a consequence of the following two propositions, where (r_n) is a monotone increasing sequence of positive integers.

1.2 Proposition. *If* $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$, *then*

$$\gamma_{r_n}^{1/2} n^{-2} r_n^{-1/2} B_n \xrightarrow{\mathsf{P}} 1, \qquad \textit{as} \quad n \to \infty.$$

1.3 Proposition. *If* $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$, *then*

$$\gamma_{r_n}^{1/4} n^{-1} r_n^{-1/4} A_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \qquad \textit{as} \quad n \to \infty.$$

The proofs of Propositions 1.2 and 1.3 are provided in Sections 3 and 4, respectively. Section 2 is devoted to the limiting behavior of the covariance structure of the sequence of random fields $\{X_{k,\ell}^{(r_n)} : k, \ell \in \mathbb{Z}\}$, $n \in \mathbb{N}$.

2 Covariance structure

Let $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ be a stationary solution of equation (1.1) with parameter α. Clearly $\mathsf{Cov}(X_{i_1,j_1}, X_{i_2,j_2}) = \mathsf{Cov}(X_{i_1-i_2,j_1-j_2}, X_{0,0})$ for all $i_1, j_1, i_2, j_2 \in \mathbb{Z}$. Let $R_{k,\ell} := \mathsf{Cov}(X_{k,\ell}, X_{0,0})$ for $k, \ell \in \mathbb{Z}$.

2.1 Lemma. *If* $k, \ell \in \mathbb{Z}$ *with* $k\ell \le 0$, *then*

$$0 \le R_{k,\ell} = \frac{1}{\sqrt{1-4\alpha^2}} \left(\frac{1-\sqrt{1-4\alpha^2}}{2\alpha} \right)^{|k|+|\ell|}. \tag{2.1}$$

If $k, \ell \in \mathbb{Z}$ *with* $k\ell \ge 0$, *then*

$$0 \le R_{k,\ell} = R_{0,|k-\ell|} - \sum_{i=0}^{|k|\wedge|\ell|-1} \binom{|k-\ell|+2i}{i} \alpha_n^{|k-\ell|+2i}. \tag{2.2}$$

Proof. Representation (1.2) of $X_{k,\ell}$ immediately implies

$$R_{k,\ell} = \mathsf{E}\, X_{k,\ell} X_{0,0} = \mathsf{E}\, X_{0,0} X_{k,\ell} = R_{-k,-\ell}.$$

Hence, it is sufficient to prove the lemma for $k, \ell \in \mathbb{Z}$ with $k \ge 0$. It is easy to show that $R_{0,0} = \mathsf{Var}\, X_{0,0} = (1-4\alpha_n^2)^{-1/2}$. For $k \ge 1$, equation (1.1) and the stationarity of $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ implies

$$R_{k,\ell} = \alpha(R_{k-1,\ell} + R_{k,\ell-1}). \tag{2.3}$$

For $k \ge 1$ and $\ell \le -1$ we also have

$$R_{k,\ell} = \alpha(R_{sk+1,\ell} + R_{k,t\ell+1}). \tag{2.4}$$

By solving the system of difference equations (2.3)–(2.4), we immediately obtain (2.1) (see, e.g., Basu and Reinsel [3]).

Now, let $k, \ell \in \mathbb{Z}$ with $k \geq 0$, $\ell \geq 0$. From the representation (1.2) of $X_{k,\ell}$ we get

$$R_{k,\ell} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{s+t+i+j}{s+i}\binom{i+j}{i}\alpha^{s+t+2i+2j} = \sum_{i=0}^{\infty}\binom{s+t+2i}{s+i}\alpha^{s+t+2i} \geq 0.$$

On the other hand, equation (1.1) and the stationarity of $\{X_{k,\ell} : k, \ell \in \mathbb{Z}\}$ implies

$$R_{k,\ell} = \alpha(R_{k+1,\ell} + R_{k,\ell+1}) + \binom{k+\ell}{k}\alpha^{k+\ell}. \tag{2.5}$$

From (2.3) and (2.5) we obtain

$$R_{k,\ell} = R_{k+1,\ell+1} + \binom{k+\ell}{k}\alpha^{k+t\ell}.$$

Using this recursion we can simultaneously reduce both indices until one of them reaches zero. Thus, we get (2.2) and the proof is complete. $\square$

Consider now a nearly unstable sequence of stationary processes $\{X^{(n)}_{k,\ell} : k, \ell \in \mathbb{Z}\}$, $n \in \mathbb{N}$, described in Theorem 1.1. For each $n \in \mathbb{N}$, let us introduce the piecewise constant random field

$$X^{(n)}(s,t) := r_n^{-1/4} X^{(r_n)}_{[ns]+1,[nt]+1}, \qquad s, t \in \mathbb{R},$$

where, again, (r_n) is an arbitrary monotone increasing sequence of positive integers such that $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$.

2.2 Proposition. *Let $s_1, t_1, s_2, t_2 \in \mathbb{R}$. Then*

$$\lim_{n\to\infty} \gamma_{r_n}^{1/2} \operatorname{Cov}\Big(X^{(n)}(s_1,t_1), X^{(n)}(s_2,t_2)\Big) = 0, \qquad \text{if } \ s_1 - s_2 \neq t_1 - t_2, \tag{2.6}$$

$$\limsup_{n\to\infty} \gamma_{r_n}^{1/2} \Big|\operatorname{Cov}\Big(X^{(n)}(s_1,t_1), X^{(n)}(s_2,t_2)\Big)\Big| \leq \frac{1}{2}, \qquad \text{if } \ s_1 - s_2 = t_1 - t_2. \tag{2.7}$$

Proof. According to the relationship of $s_1 - s_2$ and $t_1 - t_2$ we have three different cases.

CASE 1. Let $s_1 - s_2 \geq 0 \geq t_1 - t_2$ (or $s_1 - s_2 \leq 0 \leq t_1 - t_2$) and $s_1 - s_2 \neq t_1 - t_2$. Then, for all $n \in \mathbb{N}$, we have $[ns_1] - [ns_2] \geq 0 \geq [nt_1] - [nt_2]$. Lemma 2.1 implies that

$$\operatorname{Cov}\Big(X^{(n)}(s_1,t_1),\, X^{(n)}(s_2,t_2)\Big) \leq (2+4\alpha_{r_n})^{-1/2}\gamma_{r_n}^{-1/2}\left(1-\frac{1}{\beta_n}\right)^{\frac{n}{2}(|s_1-s_2|+|t_1-t_2|)} \tag{2.8}$$

for sufficiently large $n \in \mathbb{N}$, where

$$\beta_n := \frac{2\alpha_{r_n}}{2\alpha_{r_n} - 1 + \sqrt{1-4\alpha_{r_n}^2}}.$$

It is not difficult to see that $\beta_n \to \infty$, as $n \to \infty$. Hence,

$$\left(1-\frac{1}{\beta_n}\right)^n = \left(1-\frac{1}{\beta_n}\right)^{\beta_n\left(\frac{2\gamma_{r_n}}{r_n}+\frac{\sqrt{2+4\alpha_{r_n}}\sqrt{\gamma_{r_n}}}{\sqrt{r_n}}\right)\frac{n}{2\alpha_{r_n}}} \to 0 \quad \text{as} \quad n \to \infty$$

because of the condition $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$. This implies (2.6).
Case 2. Let $s_1 - s_2 > 0$, $t_1 - t_2 > 0$ (or $s_1 - s_2 < 0$, $t_1 - t_2 < 0$) and $s_1 - s_2 \neq t_1 - t_2$. For all $n \in \mathbb{N}$, $[ns_1] - [ns_2] \geq 0$ and $[nt_1] - [nt_2] \geq 0$. Lemma 2.1 implies that

$$\mathsf{Cov}\left(X^{(n)}(s_1,t_1), X^{(n)}(s_2,t_2)\right) \leq r_n^{-1/2} R^{(r_n)}_{0,|[ns_1]-[ns_2]-[nt_1]+[nt_2]|},$$

and (2.6) can be derived as in Case 1.
Case 3. Finally, let $s_1-s_2 = t_1-t_2$. It is easy to see that $\big|[ns_1]-[ns_2]-[nt_1]+[nt_2]\big| \leq 2$. From (2.2) follows, that

$$\mathsf{Cov}\left(X^{(n)}(s_1,t_1), X^{(n)}(s_2,t_2)\right) \leq r_n^{-1/2}(1-4\alpha_{r_n}^2)^{-1/2}\left(1-\frac{1}{\beta_n}\right)^{|[ns_1]-[ns_2]-[nt_1]+[nt_2]|},$$

where the last term of the right-hand side tends to 1 as $n \to \infty$. This implies (2.7).
$\square$

In order to estimate covariances and moments we make use the following lemma which is a natural generalization of Lemma 2.6 of Baran et. al. [1].

2.3 Lemma. *Let $\xi_1, \xi_2, \ldots$ be independent random variables with $\mathsf{E}\,\xi_i = 0$, $\mathsf{E}\,\xi_i^2 = 1$ for all $i \in \mathbb{N}$, and $\widetilde{M}_4 := \sup_{i\in\mathbb{N}} \mathsf{E}\,\xi_i^4 < \infty$. Let $a_1, a_2, \ldots, b_1, b_2, \ldots \in \mathbb{R}$ such that $a_i b_i \geq 0$ for all $i \in \mathbb{N}$ and $\sum_{i=1}^\infty a_i^2 < \infty$, $\sum_{j=1}^\infty b_j^2 < \infty$. Let $X := \sum_{i=1}^\infty a_i\xi_i$, $Y := \sum_{j=1}^\infty b_j\xi_j$, where the convergence of the infinite sums understood in L_2-sense. Then*

$$0 \leq \mathsf{Cov}(X^2, Y^2) \leq 2\widetilde{M}_4\,\mathsf{Cov}(X,Y)^2, \qquad \mathsf{E}\,X^2Y^2 \leq 3\widetilde{M}_4\,\mathsf{E}\,X^2\,\mathsf{E}\,Y^2.$$

3 Proof of Proposition 1.2

During the proof of Propositions 1.2 and 1.3 we will use notation

$$M_4 := \sup_{i\in\mathbb{N}} \mathsf{E}\,|\varepsilon_{0,0}|^4.$$

To prove Proposition 1.2 it is enough to show that

$$\gamma_{r_n}^{1/2} n^{-2} r_n^{-1/2}\,\mathsf{E}\,B_n \to 1, \tag{3.1}$$

$$\gamma_{r_n} n^{-4} r_n^{-1}\,\mathsf{Var}\,B_n \to 0. \tag{3.2}$$

Equation (1.1), representation (1.2) and the stationarity of $\{X_{k,\ell}^{(n)} : k, \ell \in \mathbb{Z}\}$ imply

$$\mathsf{E}\, B_n = \alpha_{r_n}^{-2} \sum_{(i,j)\in T_{k_n,\ell_n}} \mathsf{Var}\left(X_{i,j}^{(r_n)} - \varepsilon_{i,j}^{(r_n)}\right) = \alpha_{r_n}^{-2} \sum_{(i,j)\in T_{k_n,\ell_n}} \left(\mathsf{Var}\, X_{i,j}^{(r_n)} - 1\right)$$
$$= \alpha_{r_n}^{-2} \frac{(k_n+\ell_n)(k_n+\ell_n+1)}{2} \left(\mathsf{Var}\, X_{0,0}^{(r_n)} - 1\right) = \alpha_{r_n}^{-2} \frac{n(n+1)}{2} \left(\frac{1}{\sqrt{1-4\alpha_{r_n}^2}} - 1\right),$$

thus, we obtain (3.1).

Applying Lemma 2.3 and representation (1.2), we get that

$$\mathsf{Var}\, B_n = \alpha_{r_n}^{-4} \sum_{(i_1,j_1)\in T_{k_n,\ell_n}} \sum_{(i_2,j_2)\in T_{k_n,\ell_n}} \mathsf{Cov}\left((X_{i_1,j_1}^{(r_n)} - \varepsilon_{i_1,j_1}^{(r_n)})^2, (X_{i_2,j_2}^{(r_n)} - \varepsilon_{i_2,j_2}^{(r_n)})^2\right)$$
$$\leq \frac{2M_4}{\alpha_{r_n}^4} \sum_{(i_1,j_1)\in T_{k_n,\ell_n}} \sum_{(i_2,j_2)\in T_{k_n,\ell_n}} \mathsf{Cov}\left(X_{i_1,j_1}^{(r_n)}, X_{i_2,j_2}^{(r_n)}\right)^2.$$

From the stationarity of $\{X_{k,\ell}^{(n)} : k, \ell \in \mathbb{Z}\}$ it follows that the triangle T_{k_n,ℓ_n} may be replaced by $T_{n,0}$. Hence,

$$\gamma_{r_n} n^{-4} r_n^{-1}\, \mathsf{Var}\, B_n \leq \frac{M_4}{2\alpha_{r_n}^4} \iint_T \iint_T \gamma_{r_n} \mathsf{Cov}\Big(X^{(n)}(s_1,t_1), X^{(n)}(s_2,t_2)\Big)^2 \, \mathrm{d}s_1 \mathrm{d}t_1 \mathrm{d}s_2 \mathrm{d}t_2,$$

where $T := \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq 1, -s \leq t \leq 0\}$. As the area of the triangle T is finite and the integrand is uniformly bounded on $T \times T$, Fatou's lemma and Proposition 2.2 imply (3.2). $\square$

4 Proof of Proposition 1.3

For a given $n \in \mathbb{N}$ and $1 \leq m \leq n$, let

$$A_{n,m} := \sum_{(p,q)\in T_{k_m,\ell_m}} (X_{p-1,q}^{(r_n)} + X_{p,q-1}^{(r_n)})\varepsilon_{p,q}^{(r_n)},$$

where $A_{n,0} := 0$. Let $\mathcal{F}_m^n$ denote the σ-algebra generated by the random variables

$$\left\{\varepsilon_{p,q}^{(r_n)} : (p,q) \in U_{k_m,\ell_m}\right\}.$$

Obviously, $A_{n,n} = A_n = \sum_{m=1}^n (A_{n,m} - A_{n,m-1})$. First we show that $(A_{n,m} - A_{n,m-1}, \mathcal{F}_m^n)$ is a square integrable martingale difference. Let $R_m := T_{k_m,\ell_m} \setminus T_{k_{m-1},\ell_{m-1}}$, where $R_1 := T_{k_1,\ell_1}$. By the representation (1.2),

$$A_{n,m} - A_{n,m-1} = \alpha_{r_n}^{-1} \sum_{(p,q)\in R_m} \varepsilon_{p,q}^{(r_n)} \sum_{\substack{(i,j)\in U_{p,q} \\ (i,j)\neq(p,q)}} \binom{p+q-i-j}{p-i} \alpha_{r_n}^{p+q-i-j} \varepsilon_{i,j}^{(r_n)}.$$

Collecting first the terms containing only $\varepsilon_{i,j}^{(r_n)}$ with $(i,j) \in R_m$, and then the rest, we obtain the decomposition

$$A_{n,m} - A_{n,m-1} = A_{n,m,1} + \sum_{(k,\ell)\in R_m} \varepsilon_{s,t}^{(r_n)} A_{n,m,2,s,t}, \tag{4.1}$$

where

$$A_{n,m,1} := \alpha_{r_n}^{-1} \sum_{(p,q)\in R_m} \varepsilon_{p,q}^{(r_n)} \sum_{\substack{(i,j)\in U_{p,q}\setminus U_{k_{m-1},\ell_{m-1}} \\ (i,j)\neq(p,q)}} \binom{p+q-i-j}{p-i} \alpha_{r_n}^{p+q-i-j} \varepsilon_{i,j}^{(r_n)},$$

$$A_{n,m,2,p,q} := \alpha_{r_n}^{-1} \sum_{(i,j)\in U_{p,q}\cap U_{k_{m-1},\ell_{m-1}}} \binom{p+q-i-j}{p-i} \alpha_{r_n}^{p+q-i-j} \varepsilon_{i,j}^{(r_n)}.$$

The term $A_{n,m,1}$ is a quadratic form of the variables $\{\varepsilon_{i,j}^{(r_n)} : (i,j) \in R_m\}$, hence $A_{n,m,1}$ is independent of $\mathcal{F}_{m-1}^n$. The terms $A_{n,m,2,p,q}$ are linear combinations of the variables $\{\varepsilon_{i,j}^{(r_n)} : (i,j) \in U_{k_{m-1},\ell_{m-1}}\}$, thus they are measurable with respect to $\mathcal{F}_{m-1}^n$. Hence,

$$\mathsf{E}(A_{n,m} - A_{n,m-1} \mid \mathcal{F}_{m-1}^n) = \mathsf{E}\, A_{n,m,1} + \sum_{(p,q)\in R_m} A_{n,m,2,p,q}\, \mathsf{E}(\varepsilon_{p,q}^{(r_n)} \mid \mathcal{F}_{m-1}^n) = 0.$$

By the Martingale Central Limit Theorem (see, e.g., Shiryayev [11, Theorem 4, p. 511]), the statement is a consequence of the following two propositions, where $\mathbb{1}_H$ denotes the indicator function of the set H.

4.1 Proposition. *If* $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$ *then*

$$\gamma_{r_n}^{1/2} n^{-2} r_n^{-1/2} \sum_{m=1}^{n} \mathsf{E}\left((A_{n,m} - A_{n,m-1})^2 \,\middle|\, \mathcal{F}_{m-1}^n\right) \xrightarrow{\mathsf{P}} 1 \qquad \text{as } n \to \infty.$$

4.2 Proposition. *If* $\lim_{n\to\infty} n\gamma_{r_n}^{1/2} r_n^{-1/2} = \infty$ *then for all* $\delta > 0$,

$$\gamma_{r_n})^{1/2} n^{-2} r_n^{-1/2} \sum_{m=1}^{n} \mathsf{E}\left((A_{n,m} - A_{n,m-1})^2 \mathbb{1}_{\left\{|A_{n,m}-A_{n,m-1}|\geq \delta\gamma_{r_n}^{-1/4} n r_n^{1/4}\right\}} \,\middle|\, \mathcal{F}_{m-1}^n\right)$$

converges to 0 *in probability as* $n \to \infty$.

Proof of Proposition 4.1. Using decomposition (4.1), from the measurability of $A_{n,m,2,p,q}$ with respect to the σ-algebra $\mathcal{F}_{m-1}^n$, from the independence of $A_{n,m,1}$ and $\{\varepsilon_{p,q}^{(r_n)} : (p,q) \in R_m\}$ from $\mathcal{F}_{m-1}^n$, and from $\mathsf{E}((A_{n,m,1}\varepsilon_{p,q}^{(r_n)}) = 0$, one obtains

$$\mathsf{E}\left((A_{n,m} - A_{n,m-1})^2 \,\middle|\, \mathcal{F}_{m-1}^n\right) = \mathsf{E}\, A_{n,m,1}^2 + \sum_{(p,q)\in R_m} A_{n,m,2,p,q}^2 =: V_m^n.$$

The statement will follow from

$$\gamma_{r_n}^{1/2} n^{-2} r_n^{-1/2} \sum_{m=1}^{n} \mathsf{E}\, V_m^n \to 1, \tag{4.2}$$

$$\gamma_{r_n} n^{-4} r_n^{-1} \operatorname{Var}\left(\sum_{m=1}^{n} V_m^n \right) \to 0. \tag{4.3}$$

As $A_{n,m} - A_{n,m-1}$ is a martingale difference with respect to $\mathcal{F}_m^n$, we have

$$\mathsf{E}\, V_m^n = \mathsf{E}\left(\mathsf{E}(A_{n,m}^2 \,|\, \mathcal{F}_{m-1}^n) - A_{n,m-1}^2 \right) = \mathsf{E}\, A_{n,m}^2 - \mathsf{E}\, A_{n,m-1}^2.$$

Thus, using the independence of the terms $\varepsilon_{p,q}^{(r_n)}$ and representation (1.2) we get $\sum_{m=1}^{n} \mathsf{E}\, V_m^n = \mathsf{E}\, A_n^2 = \sum_{(p,q)\in T_{k_n,\ell_n}} \mathsf{E}\left(X_{p-1,q}^{(r_n)} + X_{p,q-1}^{(r_n)} \right)^2 = \mathsf{E}\, B_n$. In this way, (4.2) directly follows from (3.1).

Considering the variances, from Lemma 2.3, from the definition of the terms $A_{n,m,2,p,q}$ and from the representation (1.2) of $X_{p,q}^{(r_n)}$ follows, that

$$\operatorname{Var}\left(\sum_{m=1}^{n} V_m^n \right) \le \frac{2M_4}{\alpha_{r_n}^4} \sum_{(p_1,q_1)\in T_{k_n,\ell_n}} \sum_{(p_2,q_2)\in T_{k_n,\ell_n}} \operatorname{Cov}(X_{p_1,q_1}^{(r_n)}, X_{p_2,q_2}^{(r_n)})^2,$$

hence one can derive (4.3) as (3.2) has been derived. $\square$

Proof of Proposition 4.2. To prove Proposition 4.2 it suffices to show

$$\gamma_{r_n} n^{-4} r_n^{-1} \sum_{m=1}^{n} \mathsf{E}\left((A_{n,m} - A_{n,m-1})^4 \,\middle|\, \mathcal{F}_{m-1}^n \right) \xrightarrow{\mathsf{P}} 0. \tag{4.4}$$

From the decomposition (4.1) of $A_{n,m} - A_{n,m-1}$ it follows that

$$(A_{n,m} - A_{n,m-1})^4 \le 2^3 A_{n,m,1}^4 + 2^3 \left(\sum_{(p,q)\in R_m} \varepsilon_{s,t}^{(r_n)} A_{n,m,2,p,q} \right)^4.$$

By the independence of $A_{n,m,1}$ and $\mathcal{F}_{m-1}^n$, we have $\mathsf{E}\left(A_{n,m,1}^4 \,\middle|\, \mathcal{F}_{m-1}^n \right) = \mathsf{E}\, A_{n,m,1}^4$. Using measurability of $A_{n,m,2,p,q}$ with respect to $\mathcal{F}_{m-1}^n$, we obtain

$$\mathsf{E}\left(\left(\sum_{(p,q)\in R_m} \varepsilon_{p,q}^{(r_n)} A_{n,m,2,p,q} \right)^4 \,\middle|\, \mathcal{F}_{m-1}^n \right) \le \left((M_4 - 3)^+ + 3 \right) \left(\sum_{(p,q)\in R_m} A_{n,m,2,p,q}^2 \right)^2.$$

Hence, in order to prove (4.4), it suffices to show that

$$\gamma_{r_n} n^{-4} r_n^{-1} \sum_{m=1}^{n} \mathsf{E}\, A_{n,m,1}^4 \to 0, \tag{4.5}$$

$$\gamma_{r_n} n^{-4} r_n^{-1} \sum_{m=1}^{n} \mathsf{E}\left(\sum_{(p,q)\in R_m} A_{n,m,2,p,q}^2 \right)^2 \to 0. \tag{4.6}$$

As $k_m = [m/2]$ and $\ell_m = [(m+1)/2]$, for even values of m

$$A_{n,m,1} = \alpha_{r_n}^{-1} \sum_{p=-\ell_m+1}^{k_m} \sum_{i=-\infty}^{p-1} \alpha_{r_n}^{p-i} \varepsilon_{p,\ell_m}^{(r_n)} \varepsilon_{i,\ell_m}^{(r_n)},$$

while for the odd values we have

$$A_{n,m,1} = \alpha_{r_n}^{-1} \sum_{q=-k_m+1}^{\ell_m} \sum_{j=-\infty}^{q-1} \alpha_{r_n}^{q-j} \varepsilon_{k_m,q}^{(r_n)} \varepsilon_{k_m,j}^{(r_n)}.$$

Since $k_m + \ell_m = m$, similarly to the proof of Lemma 2.7 of [1] one can show that $\mathsf{E}\, A_{n,m,1}^4 = O(m^2)$, as $n \to \infty$, which implies (4.5).

Lemma 2.3, the definition of $A_{n,m,2,p,q}$ and representation (1.2) imply

$$\mathsf{E} \left(\sum_{(p,q)\in R_m} A_{n,m,2,p,q}^2 \right)^2 \le \frac{3M_4}{\alpha_{r_n}^4} \left(\sum_{(p,q)\in R_m} \mathsf{Var}\, X_{p,q}^{(r_n)} \right)^2 .$$

As by Lemma 2.1 $\mathsf{Var}\, X_{p,q}^{(r_n)} = (1-4\alpha_{r_n}^2)^{-1/2}$, we have

$$\mathsf{E} \left(\sum_{(p,q)\in R_m} A_{n,m,2,p,q}^2 \right)^2 \le 3M_4 \alpha_{r_n}^{-4} (1-4\alpha_{r_n}^2)^{-1} m^2 \le 3M_4 \alpha_{r_n}^{-4} \frac{r_n m^2}{\gamma_{r_n}},$$

hence, we obtain (4.6). $\square$

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