Towards the range property for the lambda theory $\mathcal{H}$

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Abstract

A sketch of proof is given for the *range property for* $\mathcal{H}$: the range of a closed λ-term in the closed term model modulo β-conversion and equating unsolvable terms is either a singleton or infinite. The proof depends on one unresolved technical conjecture.

1. The range property

In this section we introduce the notion of 'range property' in λ-calculus and explain where it came from. Notations are as in Barendregt [1984].

1.1. Definition. Let $\mathcal{M}$ be a λ-algebra.

(i) The *range property* for $\mathcal{M}$ states that if F is a closed λ-term, then its range, considering F as a map $[\![F]\!] : \mathcal{M}{\to}\mathcal{M}$, has cardinality either 1 or $\mathrm{Card}(\mathcal{M})$. More explicitly, let

$$\mathrm{Range}^{\mathcal{M}}(F) = \{[\![F]\!]d \mid d \in \mathcal{M}\}.$$

Then the range property for $\mathcal{M}$ states that for all closed terms F the cardinality of the set $\mathrm{Range}^{\mathcal{M}}(F)$ is either 1 or $\mathrm{Card}(\mathcal{M})$.

(ii) Let T be a λ-theory. Then the range property is said to hold for T if it holds for $\mathcal{M}^o(\lambda T)$. Write $\mathrm{Range}^T(F)$ for $\mathrm{Range}^{\mathcal{M}^o(\lambda T)}(F)$.

1.2. Remark. For lambda theories T, S (seen as sets of equations) one has

$$T \subseteq S \;\Rightarrow\; \mathrm{Card}(\mathrm{Range}^S(F)) \leq \mathrm{Card}(\mathrm{Range}^T(F)),$$

since in S more terms are equated.

A hint for the validity of the range property for $\beta\boldsymbol{\eta}$ was given in Böhm [1968]. In that paper it was shown that if M, N are two different $\beta\boldsymbol{\eta}$-nfs are in the range of a closed λ-term F, then one could construct a third element L differing from both M and N.

1.3. Proposition (Böhm). *Let $M, N \in \mathit{Range}^{\beta\boldsymbol{\eta}}(F)$ be two distinct elements in nf. Then there exists an $L \in \mathit{Range}^{\beta\boldsymbol{\eta}}(F)/\{M, N\}$.*

Proof. Since M, N are in nf, they have $\beta\boldsymbol{\eta}$-nfs, by Corollary 15.1.5 in Barendregt [1984]. By assumption these $\beta\boldsymbol{\eta}$-nfs are distinct. Let $M = FP$, $N = FQ$, as $M, N \in \text{Range}(F)$. By the seperability theorem proved in Böhm [1968], there exists a closed term G such that $GM = Q$, $GN = P$. Take $L \equiv \mathsf{Y}(F \circ G)$. Claim: $L \in \text{Range}(F)$, $L \neq_{\beta\boldsymbol{\eta}} M$ and $L \neq_{\beta\boldsymbol{\eta}} N$. Indeed, $L = F(GL) \in \text{Range}(F)$. Moreover, if $L = M$, then

$$M = L = F(GL) = F(GM) = FQ = N,$$

a contradiction. If $L = N$, a similar contradiction follows. ■

The argument cannot be continued, however, since L does not need to have a nf. But the range property for $\beta\boldsymbol{\eta}$ can be proved by going over to codes, giving the proof of the constructive range theorem presented in Barendregt [1993].

1.4. Theorem. *Suppose F is a closed term and that $\mathcal{X} = \{M_0, \ldots, M_{n-1}\}$, with $n \geq 2$, are n distinct elements of the range of F in $\mathcal{M}^o(\lambda\beta\boldsymbol{\eta})$. Then there exists an element in Range$(F)/\mathcal{X}$.*

Proof. By assumption (and some notational abuse) we have $M_i \neq_{\beta\boldsymbol{\eta}} M_j$ for $i \neq j$. Let $FP_i = M_i$, for $i < n$. For a lambda term N, let $\#N \in \mathbb{N}$ be its code-number and let $\ulcorner N \urcorner = \mathbf{c}_{\#N}$ be the corresponding (Church) numeral. We claim that there exists a closed term G such that for all $i < n$ and all $N =_{\beta\boldsymbol{\eta}} M_i$

$$G\ulcorner N \urcorner = G\ulcorner M_i \urcorner = P_{i+1(\text{mod } n)}.$$

Indeed, define the partial computable function ψ such that

$$\begin{array}{rcll} \psi(k) & = & \#P_{i+1(\text{mod } n)}, & \text{if } \exists M \in \Lambda^o[M =_{\beta\boldsymbol{\eta}} M_i \ \&\ k = \#M] \\ & = & \uparrow \text{(undefined)}, & \text{else.} \end{array}$$

Then we can take G as the λ-defining term for ψ. Now take L such that $L = F(G\ulcorner L \urcorner)$, by applying the second fixed-point theorem to $(F \circ G)$. We claim that L is the required element. As before $L \in \text{Range}(F)$. Moreover, suppose $L \in \mathcal{X}$, i.e. $L =_{\beta\boldsymbol{\eta}} M_i$ for some $i < n$. Then

$$M_i =_{\beta\boldsymbol{\eta}} L = F(G\ulcorner L \urcorner) = F(G\ulcorner M_i \urcorner) = FP_{i+1(\text{mod } n)} = M_{i+1(\text{mod } n)};$$

a contradiction with the assumption that the M_i are all distinct. ■

2. Validity of the range property

In this section the know versions of the Range theorem are summarized, including an abstract version, due to Statman, in terms of Ershov numerations.

2.1. Proposition. *Let T be any λ-theory, i.e. set of equations between closed terms closed under derivation. Then the range property holds for the open term model $\mathcal{M}(T)$.*

The essence is to distinguish whether for a closed term F the free variable x occurs in all terms $M =_T Fx$. If this is the case, then F has an infinite range (remember that a λ-theory consists of a set of equations between closed terms). If in some $M =_T Fx$ the variable x has disappeared, then the range of F is a singleton. For details of the proof see Barendregt [1984], Proposition 20.2.4.

2.2. PROPOSITION. *Let T be a ce*[1] *λ-theory. Then the range property holds for the closed term model $\mathcal{M}^o(T)$.*

This follows directly from the validity of Theorem 1.4 generalized to any ce theory T.

2.3. PROPOSITION (Wadsworth). *Let $\mathcal{M}$ be a λ-algebra satisfying*

$$\mathcal{B} = \{M = N \mid M, N \in \Lambda^o \;\&\; BT(M) = BT(N)\}.$$

Then the range property holds for $\mathcal{M}$.

The proof, from Barendregt [1984] Theorem 20.2.6, resembles that of Proposition 2.1, but now one distinguishes wether or not x is a free variable in $BT(Fx)$.

If so, then by the Böhm-out technique, Barendregt [1984] Corollary 10.3.9 one has for some P, Q

$$Fx\vec{P} = x\vec{Q}.$$

Then the cardinality of the range of F is Card($\mathcal{M}$).

If not, then F and $\lambda x.F\mathsf{I}$ have the same BT and hence are equal, so Card(Range(F))=1.

For the final result we will give a light introduction to numbered sets of Ershov, see Visser [1980] or Eršov [1973].

2.4. DEFINITION. (i) A *numeration* is $\langle\mathbb{N}, \sim\rangle$, with $\sim$ an equivalence relation.

(ii) A *morphism* $f{:}\langle\mathbb{N}, \sim_1\rangle \to \langle\mathbb{N}, \sim_2\rangle$ is a total computable map $f{:}\mathbb{N}\to\mathbb{N}$ with

$$\forall n, m \in \mathbb{N}.[n \sim_1 m \;\Rightarrow\; f(n) \sim_2 f(m)].$$

(iii) $\langle\mathbb{N}, \sim\rangle$ is called *pre-complete* if every partial unary computable function $\psi{:}\mathbb{N}\to\mathbb{N}$ can be made total modulo $\sim$, that is:

$$\exists f \text{ total and computable } \forall n \in \mathbb{N}.[\psi(n)\downarrow \;\Rightarrow\; f(n) \sim \psi(n)].$$

(iv) $\langle\mathbb{N}, \sim\rangle$ is called *positive* if $\sim$ is ce.

2.5. PROPOSITION (Statman). *Let $f : \langle\mathbb{N}, \sim_1\rangle \to \langle\mathbb{N}, \sim_2\rangle$ be a morphism. Suppose that $\langle\mathbb{N}, \sim_1\rangle$ is pre-complete and $\langle\mathbb{N}, \sim_2\rangle$ is positive. Then the range of f is either a singleton or infinite.*

For the proof see Barendregt [1993] Corollary 5.6, making use of the ADN theorem in Visser [1980].

Proposition 2.2 follows directly from this result.

[1] Computably enumerable; previously called *recursively enumerable*: re.

3. Steps towards the range property for $\mathcal{H}$

3.1. Definition. Let $\mathcal{H}$ be the λ-theory axiomatized by (β-conversion and) $\{M = N \mid M, N \in \Lambda^o \ \&\ M, N \text{ unsolvable}\}$.

In this section we sketch a possible path of proof for the range property for $\mathcal{H}$.

We first sketch a difficulty encountered in trying to prove this result. Let F be a possible counterexample, i.e. $1 < \text{Card}(\text{Range}^{\mathcal{H}}(F)) < \aleph_0$. By Remark 1.2 one has $\text{Card}(\text{Range}^{\mathcal{B}}(F)) \leq \text{Card}(\text{Range}^{\mathcal{H}}(F)) \leq \text{Card}(\text{Range}^{\beta}(F))$. Therefore, by Propositions 2.2 and 2.2, $\text{Range}^{\mathcal{H}}(F)$ must be a singleton in any model $\mathcal{M}_B$ equating terms with equal Böhm-like trees and infinite in the term model $\mathcal{M}^o(\lambda\beta)$. Then $x \notin \text{BT}(Fx)$, but $x \in \text{FV}(M)$ for all $M =_{\mathcal{H}} Fx$. This means that during the growth of the BT the free variable x is 'pushed into infinity'. If some trace of x towards infinity occurs in a context $xP_1 \ldots P_n$ with n maximal, then the range of F is infinite by considering $F(\lambda x_1 \ldots x_n.\mathbf{c}_k)$. The case that is left is that in Fx the free variable x is pushed into infinity and gets more and more arguments to eat. An example of this situation is an F such that

$$Fx =_\beta \lambda z.z(F(x\Omega)z).$$

Then

$$Fx = \lambda z.z(F(x\Omega)z) = \lambda z.z^2(F(x\Omega\Omega)z) = \ldots = \lambda z.z^k(F(x\Omega^{\sim k})z) = \ldots .$$

In this case $\text{Range}^{\mathcal{H}}(F)$ has cardinailty 1, as sooner or later $M\Omega^{\sim k} =_{\mathcal{H}} \Omega$. The difficulty is that in general x, while being pushed to infinity, may get an infinite sequence $P_1, P_2, P_3, \ldots$ as arguments (possibly containing the x) and that it is not clear which arguments M can 'eat themselves through' this sequence. (We saw that through the sequence $\Omega, \Omega, \Omega, \ldots$ of cumulative arguments, no M can eat its way, i.e. eventually becomes unsolvable). It is not decidable which terms can eat themselves through a given infinite sequence.

Following a different strategy, we believe that the following statements are correct and hence the range property for $\mathcal{H}$ is valid.

3.2. Conjecture. *Let $J_Z \equiv WWZ$, with $W \equiv \lambda wzxy.x(wwzy)$. This is a parametrized version of Wadsworth's infinite $\boldsymbol{\eta}$-expansion of I. Let $F \in \Lambda^{\emptyset}$ and suppose that for all $A, B \in \Lambda^{\emptyset}$ one has $FA =_{\mathcal{B}} FB$. Then*

(i) $\forall n, m \in \mathbb{N}.[n \neq m \ \Rightarrow\ J_{\mathbf{c}_n} \neq_H J_{\mathbf{c}_m}]$.

(ii) $F\Omega \neq_{\mathcal{H}} FA \ \Rightarrow\ \forall n \in \mathbb{N}.F\Omega \neq_{\mathcal{H}} F(J_{\mathbf{c}_n}A)$.

(iii) $F\Omega \neq_{\mathcal{H}} FA \ \Rightarrow\ \forall n, m \in \mathbb{N}.[n \neq m \ \Rightarrow\ F(J_{\mathbf{c}_n}A) \neq_{\mathcal{H}} F(J_{\mathbf{c}_m}A)]$.

From Conjecture 3.2(iii) the range property for $\mathcal{H}$ follows easily.

References

Barendregt, Henk [1993]. Constructive proofs of the range property in lambda calculus, *Theoret. Comput. Sci.* **121**(1-2), pp. 59–69. A collection of contributions in honour of Corrado Böhm on the occasion of his 70th birthday.

Barendregt, Henk P. [1984]. *The Lambda Calculus: its Syntax and Semantics*, revised edition, North-Holland, Amsterdam.

Böhm [1968]. Alcune proprietà delle forme normali nel K calcolo, *Technical Report 696*, INAC, Roma.

Eršov, Ju. L. [1973]. Theorie der Numerierungen. I, *Z. Math. Logik Grundlagen Math.* **19**, pp. 289–388. Translated from the Russian and edited by G. Asser and H.-D. Hecker.

Visser, Albert [1980]. Numerations, λ-calculus & arithmetic, *To H. B. Curry: essays on combinatory logic, lambda calculus and formalism*, Academic Press, London, pp. 259–284.