Generating random weak orders and the probability of a Condorcet winner

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Abstract. We present an algorithm for generating a random weak order of m objects in which all possible weak orders are equally likely. The form of the algorithm suggests analytic expressions for the probability of a Condorcet winner both for linear and for weak preference orders.

1. Introduction

A classical problem in the theory of social choice is the determination of the probability of the occurrence of a Condorcet winner in an election. Consider a committee of n members to be called *voters*, faced with the choice between m alternatives. Suppose that each voter expresses an order of preference for these alternatives. Such a set of n preference orders is called a *profile*. An alternative c is said to be the *Condorcet winner* or, simply, *winner* of this profile if for every other alternative a the number of voters who prefer c to a is strictly larger than the number of voters who prefer a to c. Clearly, the occurrence of such a winner is a desirable situation in this form of group decision making. However, as is well known, a Condorcet winner need not exist; not even if the committee has an odd number of members, each having pronounced preferences. This regrettable fact is known as the no-winner form of Condorcet's paradox [Gehrlein 1983, p. 162] after the Marquis de Condorcet [1785, 1789] who is generally credited with its discovery [Riker, 1961, p. 901; Black, 1971, Chapter 18; Young, 1988; Van Deemen, 1997, Chapter 3]. See Riker [1961, p. 901] and Gehrlein [1983, p. 163] for references on the history and (re)discoveries of the paradox.

The problem to be addressed here is: determine the probability $P(n, m)$ (and, likewise, $P^*(n, m)$) for the occurrence of a Condorcet winner if the voters choose their preference orders independently and uniformly from the set of all possible linear (or weak) orders of the alternatives. Here, a *weak* order is a linear order on a partition of the alternatives into non-empty blocks, where the alternatives inside a single block are considered equally eligible by the voter who expresses the weak order in question.

Except for Jones, Radcliff, Taber and Timpone [1995] who study P as well as P^*, previous studies of these probabilities only consider P. Examples are: Garman & Kamien, 1968; Niemi & Weisberg, 1968; DeMeyer & Plott, 1970; May, 1971; Gehrlein &Fishburn, 1976; Gehrlein, 1983. All of these studies consider profiles of (random) linear orders that result from independent sampling without replacement of the m alternatives. Here we choose to assign *independent random scores* to the alternatives, and take the orders induced by these scores as (random) linear or weak orders of preference.

A profile can have only one Condorcet winner as here defined. There are two rules in the literature for designating an alternative c as to be *majority preferred* to an alternative a. Let $N(ac)$ denote the number of voters who prefer a to c. For c to be majority preferred to a, the *majority rule* (e.g., Sen [1970, p. 23]) requires $N(ca) \geq N(ac)$ whereas the *simple majority rule* (e.g., Fishburn [1973, p. 18]) requires this inequality to be strict. If n is even, applying the majority rule to all pairs of alternatives does not necessarily yield a unique Condorcet winner (provided there exists one) since it may happen that $N(ac) = N(ca)$ for some pairs $\{a, c\}$. Accordingly, studies on P and P^*, such as the ones just mentioned, only consider profiles with n odd. As our approach holds for odd and even n alike, we will use the simple majority rule. So, in contrast to, e.g., Kelly [1987, p. 15] and Van Deemen [1999, p. 172], we do not allow non-unique winners.

By simulating a million elections on a computer for various pairs (n, m), Jones et al. [1995] obtained a table of estimates of $P(n, m)$ and $P^*(n, m)$ which to our knowledge is the largest made so far. In their simulations they sampled linear orders by the algorithm of selection without replacement among the m alternatives. For the purpose of sampling weak orders of the m alternatives they drew up a list of all possible weak orders, and then chose elements from this list at random. Now, the number of weak orders on m objects behaves asymptotically as $m!/2(\log 2)^{m+1}$. (See [J.P. Barthelemy, 1980; Bailey, 1998]. The factor $\log 2$ is the radius of convergence of the generating function $F(z) = (2-e^z)^{-1}$ in (3).) Due to this fast increase as a function of m, the above listing procedure is only feasible for m up to about 15.

In this paper we present an algorithm for sampling weak orders on m alternatives which works for arbitrarily high values of m, and yields all possible weak orders with equal probabilities. It runs as follows. First we choose a 'maximum score' K according to a certain well-chosen probability distribution π_m on the natural numbers. Then we choose for every alternative *independently* a score from 1 to K according to the uniform distribution. The sample order is then the weak order of these scores.

An attractive feature of this algorithm is that the alternatives obtain *independent* scores. In fact, also for linear orders there exists an algorithm with this property: instead of selecting alternatives without replacement, we can allot to each of them *independently* a random score, uniformly distributed over the interval $[0, 1]$, and then take the (linear) order of these real-valued scores as our sample order. Generating preference orders by independent scores greatly facilitates the analysis of P and P^*.

A further simplification of the calculation of P and P^* results from the observation that the probability for the occurrence of a Condorcet winner equals m times the probability that a particular alternative wins. We choose one '*pivot*' alternative once and for all, and let the other $m-1$ alternatives compete with it independently. This approach enables us to give expressions for $P(n,m)$ and $P^*(n,m)$ in which m plays a fairly simple role, so that the study of asymptotic behavior for large m comes within reach.

2. The number of weak orders

Let R be a binary relation on a finite set A. We recall that R is called *connected* if for all $a, b \in A$ we have either aRb or bRa, *transitive* if aRb and bRc imply aRc, and *antisymmetric* if aRb and bRa imply that $a = b$. A *weak order* on A is a binary relation on A that is both connected and transitive. A *linear order* is a weak order which is antisymmetric [Krantz, Luce, Suppes & Tversky, 1971, p. 14; Roberts 1979, p. 15; Michell, 1990, p. 167].

Interpreting A as comprising m alternatives, we let W be a weak order on A, and let $\sim$ denote the binary relation on A defined by

$$a \sim b \quad \text{iff} \quad aWb \text{ and } bWa \; .$$

In the case of a voter's preference order we interpret $a \sim b$ as meaning that the voter is indifferent with respect to a and b. Let $\widetilde{a} := \{\, b \in A \mid b \sim a \,\}$ denote the indifference class of a, and $A/\!\sim$ the set of indifference classes that W induces on A. Let k denote the cardinality of $A/\sim$. Clearly $1 \le k \le m$. The binary relation L on $A/\sim$ given by $\tilde{a}L\tilde{b}$ iff aWb, is a linear order [Roberts 1979, p. 31, Theorem 1.2.; Krantz et. al. 1971, p. 16]. Conversely every linear order on a partition of A into k nonempty subsets determines a unique weak order on A with $\#(A/\sim) = k$. Therefore, the number $w(m,k)$ of such weak orders is given by

$$w(m,k) = k!\ S(m,k) := \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^m \ ,$$

where $S(m,k)$, a Stirling number of the second kind, gives the number of partitions of a set of m elements into k nonempty subsets [Comtet, 1974, p.204; Van Lint and Wilson, 1993, p. 71, 105-6]. The total number W_m of weak orders on m objects is thus given by

$$W_m = w(m,1) + \cdots + w(m,m) = \sum_{k=1}^{m} k!\ S(m,k)\ . \tag{1}$$

The quickest way to find W_m is by use of the recursion

$$w(m,k) = k\big(w(m-1,k) + w(m-1,k-1)\big)\ , \tag{2}$$

which follows from the corresponding recursion for the Stirling numbers [Van Lint and Wilson, 1993, p. 105]. Equation (2) can be understood directly as follows. A weak order on $\{1,\ldots,m\}$ into k blocks can be obtained either by weakly ordering $\{1,\ldots,m-1\}$ into k blocks, and then adding m to one of the k blocks, or by weakly ordering $\{1,\ldots,m-1\}$ into $k-1$ blocks, and inserting the block $\{m\}$ in one of k ways. The calculation of $w(m,k)$ and, from these, W_m is illustrated in Table 1.

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For the sequel a second expression for W_m is relevant.

Theorem 1. *For all $m \in \mathbb{N}$,*

$$W_m = \sum_{k=0}^{\infty} \frac{k^m}{2^{k+1}}\ .$$

Proof. Define for $k \in \mathbb{N}$ and $z \in \mathbb{C}$ sufficiently small,

$$F_k(z) := \sum_{m=0}^{\infty} w(m,k) \frac{z^m}{m!}\ .$$

Then by (2) we have for $k \ge 1$:

$$\begin{aligned} F_k'(z) &= \sum_{m=1}^{\infty} w(m,k) \frac{z^{m-1}}{(m-1)!} = \sum_{m=0}^{\infty} w(m+1,k) \frac{z^m}{m!} \\ &= \sum_{m=0}^{\infty} k\big(w(m,k) + w(m,k-1)\big) \frac{z^m}{m!} = k\big(F_k(z) + F_{k-1}(z)\big)\ . \end{aligned}$$

The unique solution of this system of differential equations with boundary conditions $F_0(z) = 1$ for all z and $F_k(0) = 1$ for all k is

$$F_k(z) = (e^z - 1)^k \ .$$

Now put

$$F(z) := \sum_{m=0}^{\infty} W_m \frac{z^m}{m!} \ . \tag{3}$$

Then, because of (1) and noting that $w(m,k) = 0$ for $k > m$,

$$\begin{aligned} F(z) &= \sum_{k=0}^{\infty} F_k(z) \\ &= \frac{1}{1-(e^z-1)} = \frac{1}{2-e^z} = \frac{\frac{1}{2}}{1-\frac{1}{2}e^z} = \sum_{k=0}^{\infty} \tfrac{1}{2} \left(\tfrac{1}{2}e^z\right)^k \\ &= \sum_{k=0}^{\infty} \frac{e^{kz}}{2^{k+1}} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{m=0}^{\infty} \frac{k^m z^m}{m!} = \sum_{m=0}^{\infty} \frac{z^m}{m!} \left(\sum_{k=0}^{\infty} \frac{k^m}{2^{k+1}} \right) \ . \end{aligned}$$

The result follows from comparison with (3). We note that the power series for $F(z)$ has positive coefficients, and radius of convergence $\log 2$, so that on $[0, \log 2)$ the above interchange of summations is justified. □

3. Generating random weak orders

From Theorem 1 it follows that the sequence $\pi_m(1), \pi_m(2), \pi_m(3), \ldots$ of positive numbers given by

$$\pi_m(k) := \frac{1}{W_m} \cdot \frac{k^m}{2^{k+1}} \tag{4}$$

sums up to 1, and hence defines a probability distribution on the natural numbers. This distribution plays a crucial role in our algorithm. First we need a lemma.

Lemma 2. *Let $l \in \mathbb{N}$. Then*

$$\sum_{k=l}^{\infty} \frac{1}{2^{k+1}} \binom{k}{l} = 1 \ .$$

Proof. In an infinite sequence of tosses with a fair coin the probability that head comes up for the $(l+1)$-th time in the $(k+1)$-th toss is

$$\frac{1}{2^{k+1}} \binom{k}{l} \ .$$

The lemma expresses the fact that the $(l+1)$-th head is certain to come up eventually. □

Theorem 3. *Let A be a set of m elements, $m \geq 1$. Let a stochastic weak order R on A be generated by the following algorithm:*

(i) *Draw an integer-valued random variable K according to the probability distribution π_m. (See the instruction below).*

(ii) *To each $a \in A$ assign a random score X_a according to the uniform distribution on $\{1, \ldots, K\}$.*

(iii) *Put aRb iff $X_a \leq X_b$.*

Then all of the W_m possible weak orders on A are obtained with the same probability $1/W_m$.

Proof. Consider a fixed weak order R on A. As indicated in Section 2, R is completely determined by a number $l \geq 1$ and a partition of A into l nonempty disjoint blocks, $B_1, B_2, \ldots, B_l$, by the prescription:

$$aRb \quad \text{iff} \quad a \in B_i \text{ and } b \in B_j \text{ with } i \leq j \ .$$

Therefore the score assignment $X : A \to \{1, 2, \ldots, K\} : a \mapsto X_a$ induces the weak order R in step (iii) iff X takes a constant value on each block B_i of R which is a strictly increasing function of i. Now, any strictly increasing function $\{1, \ldots, l\} \to \{1, \ldots, K\}$ is completely determined by its range, so for fixed K there are $\binom{K}{l}$ functions which induce the weak order R on A. (We take $\binom{K}{l}$ to be 0 if $K < l$.)
Since the number of possible assignments $X : A \to \{1, 2, \ldots, K\}$ is K^m, it follows that a fixed weak order R having l blocks is produced by the algorithm with the following probability

$$\begin{aligned}
\mathbb{P}[R \text{ is produced}] &= \sum_{k=l}^{\infty} \mathbb{P}[K = k] \cdot \mathbb{P}[R \text{ is produced} | K = k] \\
&= \sum_{k=l}^{\infty} \left(\frac{1}{W_m} \cdot \frac{k^m}{2^{k+1}} \right) \cdot \left(\frac{1}{k^m} \binom{k}{l} \right) \\
&= \frac{1}{W_m} \sum_{k=l}^{\infty} \frac{1}{2^{k+1}} \binom{k}{l} = \frac{1}{W_m} \ ,
\end{aligned}$$

where in the last equality sign Lemma 2 is used. □

Remark. The above reasoning yields an independent proof of Theorem 1: First *define* W_m as the infinite sum in Theorem 1, and then use the proof of Theorem 3 to show that

$$\sum_R \frac{1}{W_m} = 1 \ .$$

It then follows that W_m is the number of weak orders of A.

Instruction. In order to draw repeatedly a random variable K with distribution π_m, as required in step (i), one may proceed as follows.

1. Before the start of the simulations:

   (1a.) Calculate W_m using the recursion (2).

(1b.) Choose a small number δ such that $1/\delta$ is of the order of the total number of weak orders to be generated, and find $N \in \mathbb{N}$ so large that

$$W_m - \sum_{k=1}^{N} \frac{k^m}{2^{k+1}} < \delta \, .$$

(1c.) Fill an array with the partial sums $S_0, S_1, S_2, \ldots, S_N$ given by

$$S_k := \sum_{j=0}^{k} \frac{j^m}{2^{j+1}} \, , \qquad k = 0, 1, \ldots, N-1 \, ; \qquad S_N := W_m \, .$$

2. For each of the weak orders to be sampled:

(2a.) Let $Y := W_m \cdot \mathrm{RND}(1)$, where RND(1) produces a random number uniformly over [0,1].

(2b.) Let K be the least integer for which $S_K \geq Y$.

Then for all $k \in \{1, \ldots, N-1\}$:

$$\mathbb{P}[K = k] = \mathbb{P}[S_{k-1} < Y \leq S_k] = \frac{1}{W_m}(S_k - S_{k-1}) = \frac{1}{W_m} \frac{k^m}{2^{k+1}} = \pi_m(k) \, .$$

With probability less than δ the random variable K takes the value N where actually it should take a larger value.

4. Probabilities of Condorcet winners from profiles produced by independent scores

As a spin-off from the algorithm in Section 3 we are enriched with a general idea. Apparently we can generate random weak orders from independent random scores $(X_a)_{a \in A}$ with identical discrete probability distributions. Obviously we can also generate linear orders in this way if only we take the probability distribution of the scores continuous, thus excluding indifference.

Proposition 4. *Let A be a finite set of m elements, $m \geq 1$. Let a stochastic order R on A be generated by the following algorithm:*

(i) *To each $a \in A$ assign a random score X_a according to the uniform distribution on $[0, 1]$.*

(ii) *Put aRb iff $X_a \leq X_b$.*

Then with probability 1 the order R is a linear order and each of the $m!$ possible linear orders on A occurs with equal probability $1/m!$.

This idea can be used to simulate elections. However, computer simulations to estimate probabilities are typically run for want of an analytic expression. So, let us first see what can be done analytically with this insight. We are interested in *profiles* $(R_1, R_2, \cdots, R_n)$ of linear or weak orders on the set A of alternatives. We say that $a \in A$ is *majority preferred* to $b \in A$ (and we write aMb) if

$$\#\{\, j \leq n \mid aR_jb \,\} < \#\{\, j \leq n \mid bR_ja \,\} \, .$$

(Note that aR_jb stands for 'voter j prefers b to a'.) An alternative $c \in A$ is called the *Condorcet winner* of the profile $(R_1, R_2, \cdots, R_n)$ if

$$\forall_{a\in A\setminus\{c\}} : \quad cMa \; . \tag{5}$$

It was observed by Niemi and Weisberg [1968, p. 321] that, since there can be at most one winner, and the alternatives all have equal probabilities to win, the probability for a majority winner to occur equals m times the probability that any given alternative wins. So let us pick out a *pivot* alternative $c \in A$ and consider the probability that (5) is the case. The different events $[cMa]$ and $[cMb]$ with $a \neq b$ are in general statistically dependent. Indeed, taking for example $n = 1$, $A = \{a, b, c\}$, and linear preference orders, we find that

$$\mathbb{P}[cMa] = \mathbb{P}[cMb] = \tfrac{1}{2} \; ,$$

whereas

$$\mathbb{P}[cMa \text{ and } cMb] = \tfrac{1}{3} \neq \tfrac{1}{2} \times \tfrac{1}{2} \; .$$

Of course this statistical dependence greatly complicates the calculation of the probability for (5) to occur. However, when independent scores are used to generate the profile, the events $[cMa]$ and $[cMb]$ for different a, b and c become independent when conditioned on the scores $X_c^1, X_c^2, \cdots, X_c^n$ of the pivot (and, in the weak case, also on the maximum scores K_1, K_2,..., K_n).

So, considering linear orders, for any $(x_1, x_2, \cdots, x_n) \in [0,1]^n$ let $\mathbb{P}_{(x_1,x_2,\cdots,x_n)}(E)$ denote the probability of an event E conditioned on $X_c^1 = x_1, \ldots, X_c^n = x_n$. Then,

$$\begin{aligned}
\mathbb{P}[\forall_{a\in A\setminus\{c\}} : cMa] &= \int_0^1 \cdots \int_0^1 \mathbb{P}_{(x_1,x_2,\cdots,x_n)}\left[\forall_{a\in A\setminus\{c\}} : cMa\right] dx_1 dx_2 \cdots dx_n \; , \\
&= \int_0^1 \cdots \int_0^1 \left(\mathbb{P}_{(x_1,x_2,\cdots,x_n)}[cMa]\right)^{m-1} dx_1 dx_2 \cdots dx_n \; ,
\end{aligned}$$

where a is an arbitrary alternative different from c. So we only have to calculate the probability $\mathbb{P}_{(x_1,x_2,\cdots,x_n)}[cMa]$ for some $a \neq c$. However, since the preference orders $R_1, R_2, \cdots, R_n$ are independent, this latter probability is equal to

$$\sum_{\substack{\omega\in\{1,-1\}^n \\ \sum_j \omega(j)>0}} \prod_{j=1}^n \mathbb{P}_{(x_1,x_2,\cdots,x_n)}[cR_j^{\omega(j)}a] \; ,$$

where $[cR_j^1 a]$ denotes the event that voter j prefers c to a, and $[cR_j^{-1}a]$ the event that voter j prefers a to c, and the sum is over all sequences ω of 1's and -1's whose sum $\sum_j \omega(j)$ is positive, meaning that there are more voters who prefer c to a (these are indicated by the 1's in ω) than there are who prefer a to c (indicated by the -1's). Now, given that $X_c^j = x_j$ for $j = 1, \ldots, n$, the probability that voter j prefers c to a is $\mathbb{P}[X_a^j < x_j] = x_j$, while the probability that he prefers a to c is $1 - x_j$.

Collecting results we conclude:

Theorem 5. *The probability $P(n,m)$ for the ocurrence of a Condorcet winner when n voters express linear preference orders on m alternatives is given by*

$$P(n,m) = m \int_0^1 \cdots \int_0^1 S_n(x_1, x_2, \cdots, x_n)^{m-1} dx_1 dx_2 \cdots dx_n \ ,$$

where

$$S_n(x_1, x_2, \cdots, x_n) := \sum_{\substack{\omega \in \{1,-1\}^n \\ \sum_j \omega(j) > 0}} \prod_{j=1}^{n} f_{\omega(j)}(x_j) \ ,$$

and

$$f_1(x) := x \quad \text{and} \quad f_{-1}(x) := 1 - x \ .$$

In an analogous way we obtain the corresponding theorem for weak orderings.

Theorem 6. *The probability $P^*(n,m)$ for the ocurrence of a Condorcet winner when n voters express weak preference orders with respect to m alternatives is given by*

$$P^*(n,m) = \frac{m}{W_m^n} \sum_{k_1=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left(\frac{1}{2}\right)^{k_1 + \cdots + k_n + n} \sum_{x_1=1}^{k_1} \cdots \sum_{x_n=1}^{k_n} S_n^*(k_1, \cdots, k_n; x_1, \cdots, x_n)^{m-1} \ ,$$

where

$$S_n^*(k_1, k_2, \cdots, k_n; x_1, x_2, \cdots, x_n) := \sum_{\substack{\omega \in \{-1,0,1\}^n \\ \sum_j \omega(j) > 0}} \prod_{j=1}^{n} g_{\omega(j)}(k_j, x_j) \ ,$$

and

$$g_1(k,x) := x - 1, \quad g_0(k,x) := 1, \quad \text{and} \quad g_{-1}(k,x) := k - x \ .$$

Proof. We suppose that each voter j determines his preference order by first choosing K_j according to the distribution π_m and, then, for each alternative $a \in A$ picks a score $X_a^j \in \{1, \ldots, K_j\}$ at random. Let $\underline{k} := (k_1, k_2, \cdots, k_n)$ and $\underline{x} := (x_1, x_2, \cdots, x_n)$, and for an event E let $\mathbb{P}_{\underline{k},\underline{x}}(E)$ denote the probability that E occurs given that $K_1 = k_1$, $K_2 = k_2, \ldots, K_n = k_n$, and $X_c^1 = x_1$, $X_c^2 = x_2, \ldots, X_c^n = x_n$. We write $a >_j b$ or $aR_j^1 b$ if voter j prefers a to b, $a <_j b$ or $aR_j^{-1} b$ if voter j prefers b to a, and $a \sim_j b$ or $aR_j^0 b$ if voter j is indifferent with respect to a and b. Then we have for $a, c \in A$ with $a \neq c$,

$$\mathbb{P}_{\underline{k},\underline{x}}[c >_j a] = \mathbb{P}[x_j > X_a^j | K_j = k_j] = \frac{x_j - 1}{k_j} \ ,$$

$$\mathbb{P}_{\underline{k},\underline{x}}[c \sim_j a] = \mathbb{P}[x_j = X_a^j | K_j = k_j] = \frac{1}{k_j} \ ,$$

$$\mathbb{P}_{\underline{k},\underline{x}}[c <_j a] = \mathbb{P}[x_j < X_a^j | K_j = k_j] = \frac{k_j - x_j}{k_j} \ .$$

It follows that

$$\mathbb{P}_{\underline{k},\underline{x}}[cMa] = \sum_{\substack{\omega\in\{-1,0,1\}^n \\ \sum_j \omega(j)>0}} \mathbb{P}[cR_1^{\omega(1)}a \text{ and } cR_2^{\omega(2)}a \text{ and } \ldots \text{ and } cR_n^{\omega(n)}a]$$

$$= \sum_{\substack{\omega\in\{-1,0,1\}^n \\ \sum_j \omega(j)>0}} \prod_{j=1}^{n} \frac{g_{\omega(j)}(k_j,x_j)}{k_j} = \frac{S^*(\underline{k},\underline{x})}{k_1\cdots k_n} .$$

Since the scores X_a^j are all independent given $K_1, K_2, \cdots, K_n$, we have

$$P^*(n,m) = m\mathbb{P}[\forall_{a\in A\setminus\{c\}} : cMa]$$

$$= m\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} \pi_m(k_1)\cdots\pi_m(k_n) \sum_{x_1=1}^{k_1}\cdots\sum_{x_n=1}^{k_n} \frac{1}{k_1}\cdots\frac{1}{k_n}\mathbb{P}_{\underline{k},\underline{x}}[\forall_{a\in A\setminus\{c\}} : cMa]$$

$$= \frac{m}{W_m^n}\sum_{k_1=1}^{\infty}\cdots\sum_{k_n=1}^{\infty} \frac{k_1^m}{2^{k_1+1}}\cdots\frac{k_n^m}{2^{k_n+1}} \frac{1}{k_1\cdots k_n}\sum_{x_1=1}^{k_1}\cdots\sum_{x_n=1}^{k_n}\left(\frac{S^*(\underline{k},\underline{x})}{k_1\cdots k_n}\right)^{m-1} .$$

In this expression all the factors k_j cancel, and the result is obtained. $\square$

5. Some particular cases

In this section we demonstrate the use of the formulas in Theorems 5 and 6 by calculating $P(n,m)$ and $P^*(n,m)$ in certain particular cases.

5.1. Single voter case

Obviously $P(1,m) = 1$ for all m since a single linear order always has a top element. As $S_1(x) = x$, this is just what Theorem 5 ascertains:

$$m\int_0^1 x^{m-1}dx = x^m\Big|_0^1 = 1 .$$

For a single weak preference order the situation is already nontrivial: the top cluster may contain more than one alternative. (The voter hesitates as to which alternative is the best). It is not difficult to see, however, that

$$P^*(1,m) = \frac{mW_{m-1}}{W_m} .$$

Indeed, there are m ways to choose a winning alternative, and then W_{m-1} ways to weakly order the remaining ones below it. As an illustration we calculate this result now from Theorem 6: since $S_1^*(k,x) = x-1$,

$$P^*(1,m) = \frac{m}{W_m}\sum_{k=1}^{\infty}\left(\frac{1}{2}\right)^{k+1}\sum_{x=1}^{k}(x-1)^{m-1} = \frac{m}{W_m}\sum_{u=0}^{\infty}u^{m-1}\sum_{k=u+1}^{\infty}\left(\frac{1}{2}\right)^{k+1}$$

$$= \frac{m}{W_m}\sum_{u=0}^{\infty}\frac{u^{m-1}}{2^{u+1}} = \frac{mW_{m-1}}{W_m} .$$

It is of interest to note that $P^*(1,m)$ is *not* decreasing in m and that the asymptotic behavior of W_m is such that

$$\lim_{m\to\infty} P^*(1,m) = \log 2 \ .$$

5.2. Two voters

Two voters with linear preference orders yield a Condorcet winner iff they put the same cadidate on top of their lists. The probability for this to happen is

$$P(2,m) = \frac{1}{m} \ .$$

This is obtained from Theorem 5 by realizing that $S_2(x,y) = xy$, so that

$$P(2,m) = m\left(\int_0^1 x^{m-1}dx\right)\left(\int_0^1 y^{m-1}dy\right) = \frac{m}{m^2} = \frac{1}{m} \ .$$

The weak case is more complicated than this but can still be greatly simplified since $S_2^*(\underline{k},\underline{x})$ does not depend on $\underline{k}$ (see the proof of Lemma 7). For $n=2$ we denote $\underline{k}$ by (k,l), and $\underline{x}$ by (x,y).

Lemma 7. *For all* $m \geq 1$:

$$P^*(2,m) = \frac{m}{W_m^2}\left((-1)^{m-1} + 4\sum_{l=1}^{m-1}\binom{m-1}{l}(-1)^{m-l-1}W_l^2\right) \ . \tag{6}$$

Proof. To find $S_2^*(k,l;x,y)$ we must sum over $\omega \in \{(1,1),(1,0),(0,1)\}$. So

$$S_2^*(k,l;x,y) = (x-1)(y-1) + (x-1) + (y-1) = xy - 1 \ .$$

We calculate:

$$\begin{aligned}
P^*(2,m) &= \frac{m}{W_m^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\left(\frac{1}{2}\right)^{k+l+2}\sum_{x=1}^{k}\sum_{y=1}^{l}(xy-1)^{m-1} \\
&= \frac{m}{W_m^2}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}(xy-1)^{m-1}\left(\sum_{k=x}^{\infty}\left(\frac{1}{2}\right)^{k+1}\right)\left(\sum_{l=y}^{\infty}\left(\frac{1}{2}\right)^{l+1}\right) \\
&= \frac{m}{W_m^2}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}(xy-1)^{m-1}\frac{1}{2^x\cdot 2^y} \\
&= \frac{m}{W_m^2}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\sum_{l=0}^{m-1}\binom{m-1}{l}(-1)^{m-l-1}\frac{x^l y^l}{2^x 2^y} \\
&= \frac{m}{W_m^2}\left(\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}(-1)^{m-1}\frac{1}{2^x 2^y} + \sum_{l=1}^{m-1}\sum_{x=1}^{\infty}\sum_{y=1}^{\infty}\binom{m-1}{l}(-1)^{m-l-1}\frac{x^l y^l}{2^x 2^y}\right)
\end{aligned}$$

The result follows from the fact that

$$\sum_{x=1}^{\infty} \frac{x^l}{2^x} = 2W_l \quad \text{for} \quad l \geq 1 \ .$$

$\square$

Let us check the formula for $P^*(2, m)$. It is easily verified that b is the winning alternative in 3 of the $W_2^2 = 9$ ordered pairs of weak orders on $\{a, b\}$. Hence, $P^*(2, 2) = 2 \times 3/9$. This concurs with (6). Examining the W_m^2 ordered pairs of weak orders on $\{a_1, \ldots, a_m\}$, it turns out that a_1 appears as Condorcet winner in 29, 579, 19997 of these pairs for $m = 3, 4, 5$, respectively. These frequencies also result from the part within the brackets in (6). So, $P^*(2, 3) = 3 \times 29/13^2 = .515, P^*(2, 4) = 4 \times 579/75^2 = .412, P^*(2, 5) = 5 \times 19997/541^2 = .342$.

5.3. Three or more voters

Our approach does not permit treatment of large numbers of voters since the polynomials S_n and S_n^* become forbiddingly complicated for large n. So, we confine ourselves to considering $P(n, m)$ for $n = 3, 4, 5, 6$, and $m = 3, \ldots, 10$. We first associate to S_n the functions $T_j(x_1, \ldots, x_n)$ to denote the sum of all j-th order products of the arguments of $S_n, j < n$. Now, using Theorem 5 we have, for $n = 3$,

$$\begin{aligned} S_3(x, y, z) &= xyz + xy(1 - z) + xz(1 - y) + yz(1 - x) = xy + xz + yz - 2xyz \\ &= T_2(x, y, z) - 2xyz \end{aligned}$$

so that

$$P(3, m) = m \int_0^1 \int_0^1 \int_0^1 (T_2(x, y, z) - 2xyz)^{m-1} dxdydz \ .$$

Since, for $n = 4$, a strict majority comprises 4 or 3 voters,

$$S_4(x, y, z, t) = xyzt + xyz(1 - t) + xyt(1 - z) + xzt(1 - y) + yzt(1 - x)$$

so that

$$P(4, m) = m \int_0^1 \int_0^1 \int_0^1 \int_0^1 (T_3(x, y, z, t) - 3xyzt)^{m-1} dxdydzdt$$

We similarly obtain, omitting the arguments of the T_j for brevity,

$$P(5, m) = m \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (T_3 - 3T_4 + 6xyztu)^{m-1} dxdydzdtdu,$$

$$P(6, m) = m \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (T_4 - 4T_5 + 10xyztuv)^{m-1} dxdydzdtdudv.$$

The exact calculation of $P(n, m)$ for $n = 3, 4, 5, 6$, and m up to about 20 is a matter of seconds for an algebraic program such as Maple. Table 2 gives the results in the exact form of the rational fractions resulting from these calculations as well as in real numbers.

Garman & Kamien [1968, p. 314] computed $1 - P(n, m)$ as multinomial probabilities for some small and odd n and some small m. The rational fractions they report are the complements of the corresponding ones in Table 2. The same holds for Gillett [1977] who computed $1 - P(n, 3)$ also for some even n. The estimates of $P(n, m)$ that Jones et al [1995] obtained by simulations are confined to odd n and values of m up to 15. The estimates they report for $n = 3, 5$ deviate at most 0.002 from the corresponding ones in Table 2. Finally, let us check by way of example two of our results for $P(n$ odd, m even) for the linear dependence on the $P(n, j)$ with j odd and $< m$ as established by Gehrlein & Fishburn [1976, p. Theorem 2]. Taking the coefficients α_j^m from their Table III [o.c., 1976, p. 24], we find

$$\begin{aligned} P(3,6) &= \alpha_0^6 + \alpha_3^6 P(3,3) + \alpha_5^6 P(3,5) = 3 - 5 \cdot \frac{17}{18} + 3 \cdot \frac{21}{25} = \frac{359}{450}, \\ P(5,8) &= \alpha_0^8 + \alpha_3^8 P(5,3) + \alpha_5^8 P(5,5) + \alpha_7^8 P(5,7) \\ &= -17 + 28 \cdot \frac{67}{72} - 14 \cdot \frac{32019}{40000} + 4 \cdot \frac{608721061}{864360000} = \frac{767419}{1152480}, \end{aligned}$$

which agrees with the calculated values in Table 2. Although for large n the polynomials S_n and S_n^* become more and more complicated, asymptotics of $P(n, m)$ and $P^*(n, m)$ for large m may well be feasible. Using the above formulas for $P(n, m)$ it can be shown that $P(3, m)$, and hence $P(n, m)$ for all n, tends to 0 as m tends to ∞, $P(3, m)$ being strictly decreasing as a function of m. These results were proved before by May [1971] and Fishburn, Gehrlein & Maskin [1979, Theorem 1], respectively.

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TABLE 2 HERE

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We checked the formula for $P^*(n, m)$ in Theorem 6 only for the case $(n = 3, m = 3)$. An examination of the 13^3 ordered triples of weak orders on $\{a, b, c\}$ shows that 486 of those triples have c as Condorcet winner. Hence, $P^*(3, 3) = 3 \times 486/13^3 = .664$. This concurs with Jones et al [1995, Table 3], who report the complement. We programmed the formula for $P^*(3, 3)$ in Fortran, putting $\delta = .001$ and, thus, $N = 23$ (see the *Instruction* in Section 3), and taking 23 as upper limit for $k_i, i = 1, 2, 3$. The summations yield 485.8889 and tend to 486 in the limit for increasing upper limits for the k_i. So, the formula agrees with the above. However, the number of terms S^* that were added to obtain 485.8889 was 21,024,576. Clearly, the formula is not very well suited for computational purposes.

6. Discussion

Going by independent scores we obtained an algorithm for generating random weak orders, and formulas for $P(n, m)$ and $P^*(n, m)$. We have confined ourselves to profiles that satisfy the Impartial Culture (IC) condition, an assumption made in almost all studies on this subject. What progress is in these contributions? We evaluate this question with respect to the present state of the study of the development of closed-form expressions for these probabilities.

For a fixed order of the $m!$ possible linear orders of m alternatives, let $\underline{n} = (n_1, \ldots, n_{m!})$ characterize a profile with n_j voters expressing the j-th linear order of preference, and let $\underline{p} = (p_1, \ldots, p_{m!})$ comprise the probabilities of a voter expressing the j-th linear order, $j = 1, \ldots, m!$. Listing the closed-form expressions of $P(n, m)$ in the literature, Gehrlein [1983, p. 170] finds the most tractable, essentially multinomial, representation of $P(n, 3|\underline{p})$ and $P(n, 4|\underline{p})$ in Gehrlein & Fishburn [1976]. Taking $\underline{p} = (1/m!, \ldots, 1/m!)$ in accord with IC, and determining the constraints on $\underline{n}$ for which a profile of linear orders enjoys a Condorcet winner, these latter authors obtained computable expressions for $P(n, 3)$ and $P(n, 4)$ which they were able to evaluate for odd n up to 49. Gehrlein & Fishburn [1976] concentrated on $P(n, m)$ with m odd since they showed [o.c., 1976, Theorem 2] that, for all even $m \geq 4$ and n odd, $P(n, m)$ can be written as a linear combination of the $P(n, j)$ with j odd and less than m, and with coefficients being independent of n. In developing their expression of $P(n, 4)$, they made use of the remark of Niemi & Weisberg [1968, p. 213] that $P(n, m)$ equals m times the probability that any particular alternative wins. Using this remark they also obtained an expression for $P(n, 5)$, and noted [o.c., p. 25] that, more generally, expressions for $P(n, 7), P(n, 9), \ldots$ can similarly be obtained. Gehrlein & Fishburn [1979] obtained computable expressions for $P(n, 7)$ and $P(n, 9)$. These expressions are growingly complex for increasing m, and increasingly difficult to evaluate for larger n. In fact, Gehrlein [1983, Table 1] evaluates these formulas for the (odd n, odd m) pairs ($\leq$ 49, 3), ($\leq$ 35, 5), ($\leq$ 9, 7), ($\leq$ 9, 9), ($\leq$ 7, 11), ... , ($\leq$ 7, 17), ($\leq$ 5, 19), ... , ($\leq$ 5, 25), and uses an approximation for evaluations in the pairs where m or n exceeds a limit as here indicated.

In the approach taken here, these roles of n and m have in a sense been reversed. The expression of $P(n, m)$ in Theorem 5 is hardly more difficult to evaluate for larger m, even allows study of asymptotic behavior as $m \to \infty$, but increases in complication quickly for larger n, to become intractable for values of n from about 11 onwards. On the other hand, it is valid for odd and even values of n and m alike. To the best of our knowledge, calculations of $P(4, m)$ and $P(6, m)$ [Table 2] are not in the literature.

As regards weak orders, our expression in Theorem 6 for $P^*(n, m)$ is mainly of theoretical interest. It is hardly computable except for the smallest values of n and m, and even in these cases its evaluation is more involved than a direct check of the W_m^n possible profiles for a Condorcet winner. We can only point to the small progress on the case $n = 2$ in Section 5.2.

We regard as main contributions of this essay (i) the algorithm for generating random weak orders, and (ii) the approach by independent scores. The algorithm facilitates simulations of weak orders and, thereby, numerical estimates of $P^*(n, m)$. For instance, the algorithm takes less than one second for generating 1001 random weak orders of 6 alternatives and, thus, considerably improves the procedure of Jones et al. [1995] who needed quite some computer time for simulating an election with $n = 1001$ and $m = 6$ [o.c., 1995, footnote 10]. Also, the algorithm almost naturally prompts the approach by independent scores that leads to new expressions for $P(n,m)$ and $P^*(n,m)$. Unfortunately, all of these developments only pertain to profiles that satisfy the condition of IC from which we did not escape.

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TABLE 1

$w(m, k)$ and W_m

$m \downarrow$	$k \rightarrow$ 1	2	3	4	5	6	7	W_m
1	1	0	0	0	0	0	0	1
2	1	2	0	0	0	0	0	3
3	1	6	6	0	0	0	0	13
4	1	14	36	24	0	0	0	75
5	1	30	150	240	120	0	0	541
6	1	62	540	1560	1800	720	0	4683
7	1	126	1806	8400	16800	15120	5040	47293

Table 2

$P(n, m)$

$n \downarrow$ $m \rightarrow$	3	4	5	6	7	8	9	10
3	$\frac{17}{18}$	$\frac{8}{9}$	$\frac{21}{25}$	$\frac{359}{450}$	$\frac{33569}{44100}$	$\frac{536}{735}$	$\frac{13913}{19845}$	$\frac{67079}{99225}$
	.944	.889	.840	.798	.761	.729	.701	.676
4	$\frac{4}{9}$	$\frac{197}{576}$	$\frac{1107}{4000}$	$\frac{1043}{4500}$	$\frac{15359}{77175}$	$\frac{91745}{526848}$	$\frac{5499323}{35562240}$	$\frac{705967}{5080320}$
	.444	.342	.277	.232	.199	.174	.155	.139
5	$\frac{67}{72}$	$\frac{31}{36}$	$\frac{32019}{40000}$	$\frac{269513}{360000}$	$\frac{608721061}{864360000}$	$\frac{767419}{1152480}$	$\frac{1574336347}{2489356800}$	$\frac{37525387727}{62233920000}$
	.931	.861	.800	.749	.704	.666	.632	.603
6	$\frac{989}{1944}$	$\frac{1037}{2592}$	$\frac{472549}{1440000}$	$\frac{1078499}{3888000}$	$\frac{13057391131}{54454680000}$	$\frac{1837328467}{8712748800}$	$\frac{9553049400803}{50812751001600}$	$\frac{23921196935141}{141146530560000}$
	.509	.400	.328	.277	.240	.211	.188	.169