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# Squeezing enhanced control

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**Abstract**

We study an open system in contact with its environment, the electromagnetic field. The information gained by measuring a quadrature of the field is used to send control pulses to the system. Goal is to fix the unknown state of the system in time. We show that in the special case of an essentially commutative interaction this goal can be achieved. In dealing with spontaneous decay we approximate the essentially commutative situation by bringing the field in a squeezed state. We show that when squeezing goes to infinity, the state can again be kept fixed.

## 1 Introduction

The last decade there have been rapid developments in quantum information theory, initiated mainly by some fundamental papers [36], [17] showing the increased possibilities when quantum features are exploited in computations. However, implementation of the proposed algorithms on real physical qubits still poses a great challenge. One of the problems is the interaction with the environment, i.e. the electromagnetic field, and the decoherence that goes along with it. Dealing with this problem motivates the development of theory and methods for coherently manipulating, or controlling, quantum systems.

Decoherence is a result of ignoring information lost from an open quantum system to its environment via their interaction. However, the lost information can be retrieved, at least partially, by observing the environment, i.e. by performing measurements on it. The decoherence can be combatted by using the retrieved information in a scheme for controlling the quantum system, see also [21].

Since the electromagnetic field and the open system are in interaction, information on the system itself is gained when measuring some observables of the field. Hence conditioning on the obtained measurement results provides a back-action of the measurement in the field on the open system. One of the pioneers in this area is Belavkin who extended many ideas in classical filtering theory, cf. [39], to the quantum regime [6], [10]. Quantum filtering theory [9], [10] explains how the state, conditioned on the result of a continuous time measurement in the environment, evolves in time. Note that since the result of the continuous time measurement are random, the conditioned state is also a random state. Quantum filtering theory provides a stochastic differential equation, the *Belavkin equation*, for the state evolution in which the measurement process is one of its driving terms [9], [10].

Another approach to the back-action due to conditioning, is via quantum trajectory theory as developed in quantum optics in the late 1980's and early 1990's [13], but already envisioned by Davies [38], [15] in the 1970's. In this approach photon counting measurements are analysed to obtain a continuous time evolution of the open system interrupted by jumps the moments at which photons are detected. Differentiation of the trajectory evolution leads to a *stochastic Schrödinger equation* [14], which is a stochastic differential equation for the evolution of the state conditioned on the outcomes of the counting experiment. A diffusive limit of photon counting in which the jumps in the state space decrease in size but become increasingly frequent, makes it possible to incorporate homodyne and heterodyne detection schemes into quantum trajectory theory [5], [13], [44]. The stochastic Schrödinger equations encountered in quantum optics are equivalent to the Belavkin equations from quantum filtering theory [10], [12].

The result of the continuous time measurement in the field can be used to exert control over the system. The solution to the quantum filtering problem [6], [9] makes it possible to directly carry over many ideas in classical control theory [39], [29], [30] to the quantum regime, [7], [8], [9], [11], [16], [41]. Coming from the quantum trajectory approach, other pionering work in quantum control was done by Wiseman and Milburn in the first half of the 1990's, see [43], [42], [45]. Two different objectives in control problems can be distinguished, one where the state is controlled in order to let it follow a certain path in time [11], [16], and one where the semigroup describing the dissipative evolution of the open system, i.e. the channel itself, is being controlled [32], [21], [2], [3].

In this paper a problem of the second type is considered. The question addressed here is how to keep an unknown state of an open system fixed in time, i.e. how to keep its dynamical semigroup as close to identity as possible. In this article we will not be concerned with optimality results. The main issue is to find or engineer situations where the control is perfect, in which case the control scheme is said to *restore quantum information* [21]. Furthermore, we will not be concerned here with encoding our system into the code space of a larger system and then protecting just this code space [4], [3].

The control scheme consists of two parts. The first part is an evolution over a period of $\tau$ time units in which a quadrature of the field is observed. This evolution is governed by the Belavkin equation corresponding to this measurement. In the second part the result of the measurement is used to construct a laser pulse designed, if at all possible, to take the system through a Rabi cycle that corrects the evolution of the past $\tau$ time units. This scheme is studied in the limit for very small $\tau$, i.e. the control pulses are sent at very high frequency.

In general the above control scheme will not be able to restore quantum information. Since the interaction of the field and the system is studied in the weak coupling limit, the field acts as two classical noises. However these two noises, represented by two different quadratures of the field, do not commute with each other. Therefore only one of these noises can be observed and its disturbing effect on the system corrected. An idealised interaction of system and field in which there is only one instead of two classical noises present is called *essentially commutative* [28]. In the essentially commutative case it will turn out that the above control scheme restores quantum information.

For the more realistic situation where both noises are present our strategy will be to manipulate the state of the field in order to approximate the essentially commutative case. This is done by putting the field in a squeezed state, i.e. one quadrature's variance increases while the other quadrature's variance decreases, [18], [19], [25]. The idea is to measure the noise with the large variance and correct its disturbing effect on the system. It will turn out that when squeezing goes to infinity the control scheme described above will restore quantum information.

This paper is organised as follows. Section 2 describes the dissipative evolution of the open system within the Markov approximation. The joint evolution of system and field is given by unitaries satisfying a quantum stochastic differential equation in the sense of [23]. In the next section a brief exposition of quantum stochastic calculus [23] is given. This enables us to make sense of the quantum stochastic differential providing the unitaries of section 2. Sections 2 and 3 describe dilation theory and quantum stochastic calculus in a nutshell. Section 4 is a brief exposition of quantum filtering theory. It contains a derivation of the Belavkin equation for field quadrature measurement.

Sections 5 and 6 deal with controlling the state of an open system in the essentially commutative case and the more realistic situation of spontaneous decay of a two-level system, respectively. Here we show that for the essentially commutative case it is possible to restore quantum information. For spontaneous decay, however, problems are encountered motivating the investigation in the remainder of the paper.

Section 7 shows how to describe the interaction of system and field when the field is in a squeezed state. To do this we have to do quantum stochastic calculus in the GNS-representation space of the squeezed state. In the last section the Belavkin equation for measuring a quadrature of a squeezed field is given and a control scheme based on this measurement is presented. It turns out that when squeezing goes to infinity, the scheme restores quantum information.

## 2 The dilation

Let $\mathcal{B} := M_n$ stand for the algebra of observables of an $n$-dimensional quantum system. On this algebra $\{T_t\}_{t\geq 0}$ is a semigroup of completely positive identity preserving operators. It represents the irreversible time evolution of the system in the Heisenberg picture. Lindblad's theorem [31] asserts that $T_t = \exp(tL)$ where the generator $L$ is given by

$$L(X) = i[H, X] + \sum_{j=1}^{k} V_j^* X V_j - \frac{1}{2}\{V_j^* V_j, X\}, \qquad X \in \mathcal{B},$$

with $H$ and the $V_j's$ fixed elements of $\mathcal{B}$, $H$ being selfadjoint. The notation $\{X, Y\}$ stands for the anticommutator $XY + YX$. For simplicity, we take $k = 1$ and $H = 0$, i.e.

$$L(X) = V^* X V - \frac{1}{2}\{V^* V, X\}. \tag{2.1}$$

This paper deals mainly with two special cases of the above situation. In the first special case we have either $V = V^*$ or $V = -V^*$. This case is called *essentially commutative* [28], see section 5. In the second special case we have

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \qquad (2.2)$$

Then the semigroup $T_t$ describes spontaneous decay to the ground state of a two-level system, see sections 6 and 8.

The system $\mathcal{B}$ and its environment, the electromagnetic field, evolve reversibly in time. The irreversible evolution $T_t$ of $\mathcal{B}$ is the result after tracing out the field. Up to section 7 the electromagnetic field to which the system $\mathcal{B}$ is coupled, will be taken in the vacuum state or a coherent state. Then, see section 7 for more details, a decay channel in the field can be modelled by the *bosonic* or *symmetric Fock space* over the Hilbert space $L^2(\mathbb{R})$ of square integrable wave functions on the real line, i.e.

$$\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R})^{\otimes_s n}.$$

The algebra generated by the field observables on $\mathcal{F}$ contains all bounded operators and it is denoted by $\mathcal{W}$.

For future convenience we already distinguish two decay channels in the field, i.e. we rewrite $L$ as

$$L(X) = V_f^* X V_f - \frac{1}{2}\{V_f^* V_f, X\} + V_s^* X V_s - \frac{1}{2}\{V_s^* V_s, X\}, \qquad X \in \mathcal{B}, \qquad (2.3)$$

where $V_f = \kappa_f V$, $V_s = \kappa_s V$ and $\kappa_f, \kappa_s \in \mathbb{R}$ such that $|\kappa_f|^2 + |\kappa_s|^2 = 1$. The subscripts $f$ and $s$ stand for *forward* and *side channel*, respectively. On the forward channel in the field we will put a laser with which we want to control the system, while in the side channel of the field we are going to perform a measurement. The decay rates into the forward and side channel are given by $|\kappa_f|^2$ and $|\kappa_s|^2$, respectively. Since the field is modelled by these two decay channels, we need two copies of the algebra $\mathcal{W}$, denoted $\mathcal{W}^f \otimes \mathcal{W}^s$.

The free evolution of a channel in the field is given by the unitary group $S_t$, the second quantization of the left shift $s(t)$ on $L^2(\mathbb{R})$, i.e. $s(t): \ f \mapsto f(\cdot + t)$. In the Heisenberg picture the evolution on $\mathcal{W}^f \otimes \mathcal{W}^s$ is

$$W \mapsto (S_t \otimes S_t)^* W (S_t \otimes S_t) := \mathrm{Ad}[S_t \otimes S_t](W), \qquad W \in \mathcal{W}^f \otimes \mathcal{W}^s.$$

The system $\mathcal{B}$ and field together form a closed system, thus their joint evolution is given by a one-parameter group $\{\hat{T}_t\}_{t\in\mathbb{R}}$ of *-automorphisms on $\mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s$

$$X \mapsto \hat{U}_t^* X \hat{U}_t := \mathrm{Ad}[\hat{U}_t](X), \qquad X \in \mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s.$$

The group $\hat{U}_t$ is a perturbation of the free evolution without interaction. We describe this perturbation by the family of unitaries $U_t := (S_{-t} \otimes S_{-t})\hat{U}_t$ for all $t \in \mathbb{R}$ satisfying the *cocycle* identity

$$U_{t+s} = (S_{-s} \otimes S_{-s}) U_t (S_s \otimes S_s) U_s, \qquad \text{for all} \ \ t, s \in \mathbb{R}.$$

The direct connection between the reduced evolution of $\mathcal{B}$ given by (2.3) and the cocycle $U_t$ is one of the important results of quantum stochastic calculus [23] which is the object of the next section. For the moment we only mention that in the weak coupling limit [1], $U_t$ is the solution of the stochastic differential equation [23], [34], [33]

$$dU_t = \{V_f dA_f^*(t) - V_f^* dA_f(t) + V_s dA_s^*(t) - V_s^* dA_s(t) - \frac{1}{2} V^* V dt\} U_t, \quad U_0 = \mathbf{1}. \tag{2.4}$$

We will see in the next section that if $U_t$ satisfies (2.4) the following *dilation diagram* [26], [27] commutes:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\ \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1} \Big\downarrow & & \Big\uparrow \mathrm{Id}\otimes\phi\otimes\phi \\ \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s & \xrightarrow{\hat{T}_t = \mathrm{Ad}[\tilde{U}_t]} & \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s \end{array} \tag{2.5}$$

i.e. for all $X \in \mathcal{B}$ : $T_t(X) = \big(\mathrm{Id}\otimes\phi\otimes\phi\big)\big(\hat{T}_t(X\otimes\mathbf{1}\otimes\mathbf{1})\big)$, where $\phi$ is the vacuum state on $\mathcal{W}$, and $\mathbf{1}$ is the identity operator in $\mathcal{W}$. Any dilation of the semigroup $T_t$ with Bose fields is unitarily equivalent with the above one under certain minimality requirements.

The dilation diagram can also be read in the Schrödinger picture if we reverse the arrows: start with a state $\rho$ of the system $\mathcal{B}$ in the upper right hand corner, then this state undergoes the following sequence of maps

$$\rho \mapsto \rho\otimes\phi\otimes\phi \mapsto \rho\otimes\phi\otimes\phi\circ\hat{T}_t = \hat{T}_{t*}(\rho\otimes\phi\otimes\phi) \mapsto \mathrm{Tr}_{\mathcal{F}^f\otimes\mathcal{F}^s}(\hat{T}_{t*}(\rho\otimes\phi\otimes\phi)).$$

This means that at $t = 0$, the atom in the state $\rho$ is coupled to the electromagnetic field in the vacuum state, after $t$ seconds of unitary evolution the partial trace over the field is taken.

# 3 Quantum stochastic calculus

Here, we briefly discuss the quantum stochastic calculus developed by Hudson and Parthasarathy [23]. For a detailed treatment of the subject we refer to [34] and [33]. The exposition here is a bit broader than strictly necessary for the construction of the cocycle of the previous section. However, the general description [34] presented here is needed in section 7.

Let $\mathcal{H}$ be a Hilbert space. We define the *bosonic* or *symmetric Fock space* over $\mathcal{H}$ by

$$\mathcal{F}(\mathcal{H}) := \mathbb{C} \oplus \bigoplus_{k=1}^{\infty} \mathcal{H}^{\otimes_s k}.$$

In the previous section we had $\mathcal{H} = L^2(\mathbb{R})$. For every $f \in \mathcal{H}$ we define the *exponential vector* $e(f) \in \mathcal{F}(\mathcal{H})$ in the following way

$$e(f) := 1 \oplus \bigoplus_{k=1}^{\infty} \frac{1}{\sqrt{k!}} f^{\otimes k}.$$

The inner product of two exponential vectors $e(f)$ and $e(g)$ is $\langle e(f), e(g)\rangle = \exp(\langle f, g\rangle)$. We denote the *vacuum vector* $e(0) = 1 \oplus 0 \oplus 0 \oplus \ldots$ also by $\Phi$. The span of all exponential vectors, denoted $\mathcal{D}$, forms a dense subspace of $\mathcal{F}(\mathcal{H})$.

Let $\xi$ be a projection (on the Hilbert space $H$) valued measure on $\mathbb{R}$ with no jump points, i.e. $\xi(\{t\}) = 0$ for all $t \in \mathbb{R}$. Denote by $\mathcal{H}_{t]}, \mathcal{H}_{[s,t]}$ and $\mathcal{H}_{[t}$ the ranges of the projections $\xi((-\infty, t]), \xi([s,t])$ and $\xi([t,\infty))$, respectively. For a vector $f \in \mathcal{H}$ we denote $f_{t]} := \xi((-\infty,t])f$, $f_{[s,t]} := \xi([s,t])f$ and $f_{[t} := \xi([t,\infty))f$. Let us write $\mathcal{H}$ as the direct sum $\mathcal{H}_{t]} \oplus \mathcal{H}_{[t}$, then $\mathcal{F}(\mathcal{H})$ is unitarily equivalent with $\mathcal{F}(\mathcal{H}_{t]}) \otimes \mathcal{F}(\mathcal{H}_{[t})$ through the identification $e(f) \cong e(f_{t]}) \otimes e(f_{[t})$. For notational convenience the tensor product signs between exponential vectors are often omitted. The algebra $\mathcal{W} := \mathcal{B}\big(\mathcal{F}(\mathcal{H})\big)$ also splits as a tensor product $\mathcal{W}_{t]} \otimes \mathcal{W}_{[t}$ where $\mathcal{W}_{t]} := \mathcal{B}\big(\mathcal{F}(\mathcal{H}_{t]})\big)$ and $\mathcal{W}_{[t} := \mathcal{B}\big(\mathcal{F}(\mathcal{H}_{[t})\big)$.

A map $m : \mathbb{R}_+ \to \mathcal{H} : t \mapsto m_t$ is called a *$\xi$-martingale* if $m_t \in \mathcal{H}_{t]}$ for all $t$ and $\xi([0,s])m_t = m_s$ for all $s < t$. For $m$ and $m'$ $\xi$-martingales, there exists a complex valued measure (of finite variation on every bounded interval), denoted $\langle\langle m, m'\rangle\rangle$ on $\mathbb{R}_+$, satisfying

$$\langle\langle m, m'\rangle\rangle\big([0,t]\big) = \langle m_t, m'_t\rangle, \tag{3.1}$$

for all $t \geq 0$. Let $m$ be a $\xi$-martingale. The annihilation operator $A(m_t)$ and creation operator $A^*(m_t)$ are defined on the domain $\mathcal{D}$ by

$$\begin{aligned} &A(m_t)e(g) = \langle m_t, g\rangle e(g), \qquad g \in \mathcal{H}, \\ &\big\langle e(h), A^*(m_t)e(g)\big\rangle_{\mathcal{F}(\mathcal{H})} = \langle h, m_t\rangle \big\langle e(h), e(g)\big\rangle_{\mathcal{F}(\mathcal{H})}, \quad h, g \in \mathcal{H}. \end{aligned} \tag{3.2}$$

Let $M_t$ be one of the processes $A(m_t)$ or $A^*(m_t)$ for some $\xi$-martingale $m$. The following factorisation property [23], [34] makes the definition of stochastic integration against $M_t$ possible

$$(M_t - M_s)e(f) = e(f_{s]})\big\{(M_t - M_s)e(f_{[s,t]})\big\}e(f_{[t}),$$

with $(M_t - M_s)e(f_{[s,t]}) \in \mathcal{F}(\mathcal{H}_{[s,t]})$. We first define the stochastic integral for the so-called *simple* operator processes with values in the atom and noise algebra $\mathcal{B} \otimes \mathcal{W}$ where $\mathcal{B} := M_n$ and $\mathcal{W}$ is the algebra of all bounded operators on the Fock space $\mathcal{F}(\mathcal{H})$.

**Definition 3.1:** Let $\{L_s\}_{0 \leq s \leq t}$ be an adapted (i.e. $L_s \in \mathcal{B} \otimes \mathcal{B}(\mathcal{F}(H_{s]}))$ for all $0 \leq s \leq t$) simple process with respect to the partition $\{s_0 = 0, s_1, \ldots, s_p = t\}$ in the sense that $L_s = L_{s_j}$ whenever $s_j \leq s < s_{j+1}$. Then the stochastic integral of $L$ with respect to $M$ on $\mathbb{C}^n \otimes \mathcal{D}$ is given by [23], [34]:

$$\int_0^t L_s dM_s\ fe(u) := \sum_{j=0}^{p-1} \big(L_{s_j} fe(u_{s_j]})\big)\big((M_{s_{j+1}} - M_{s_j})e(u_{[s_j,s_{j+1}]})\big)e(u_{[s_{j+1}}).$$

By the usual approximation by simple processes we can extend the definition of the stochastic integral to a large class of stochastically integrable processes [23], [34].

We simplify our notation by writing $dX_t = L_t dM_t$ for $X_t = X_0 + \int_0^t L_s dM_s$. Note that the definition of the stochastic integral implies that the increments $dM_s$ lie in the future, i.e. $dM_s \in \mathcal{W}_{[s}$. Another consequence of the definition of the stochastic integral is that its expectation with respect to the vacuum state $\langle \Phi, \cdot \Phi \rangle$ is always 0 due to the fact that the increments $dA(m_t)$ and $dA^*(m_t)$ have zero expectation values in the vacuum. This will often simplify calculations of expectations, our strategy being that of trying to bring these increments to act on the vacuum state thus eliminating a large number of differentials.

The following theorem of Hudson and Parthasarathy extends the Itô rule of classical probability theory.

**Theorem 3.2:** ***(Quantum Itô rule [23], [34])*** *Let $M_1$ and $M_2$ each be one of the processes $A(m_t)$ or $A^*(m'_t)$. Then $M_1 M_2$ is an adapted process satisfying the relation:*

$$d(M_1 M_2) = M_1 dM_2 + M_2 dM_1 + dM_1 dM_2,$$

*where $dM_1 dM_2$ is given by the quantum Itô table:*

| $dM_1 \backslash dM_2$ | $dA^*(m'_t)$ | $dA(m'_t)$ |
|---|---|---|
| $dA^*(m_t)$ | 0 | 0 |
| $dA(m_t)$ | $d\langle\langle m, m' \rangle\rangle$ | 0 |

**Notation.** The quantum Itô rule will be used for calculating differentials of products of Itô integrals. Let $\{Z_i\}_{i=1,\ldots,p}$ be Itô integrals, then

$$d(Z_1 Z_2 \ldots Z_p) = \sum_{\substack{\nu \subset \{1,\ldots,p\} \\ \nu \neq \emptyset}} [\nu]$$

where the sum runs over all *non-empty* subsets of $\{1, \ldots, p\}$ and for any $\nu = \{i_1, \ldots i_k\}$, the term $[\nu]$ is the contribution to $d(Z_1 Z_2 \ldots Z_p)$ coming from differentiating only the terms with indices in the set $\{i_1, \ldots i_k\}$ and preserving the order of the factors in the product. For example the differential $d(Z_1 Z_2 Z_3)$ contains terms of the type $[2] = Z_1 (dZ_2) Z_3$, $[13] = (dZ_1) Z_2 (dZ_3)$, and $[123] = (dZ_1)(dZ_2)(dZ_3)$.

Let us return to the setup of section 2. We now make sense of equation (2.4). Note that the Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The forward and side channel both have their own copy of $L^2(\mathbb{R})$. The projection valued measure $\xi$ is given by

$$\xi(I)(f_f \oplus f_s) = (\chi_I f_f) \oplus (\chi_I f_s), \qquad f_f, f_s \in L^2(\mathbb{R}),$$

for all Borel subsets $I$ of $\mathbb{R}$. Here $\chi_I$ denotes the *indicator function* of $I$, i.e. the function that takes the value 1 on $I$ and is 0 elsewhere.

The maps $m^f : \mathbb{R}_+ \to \mathcal{H} : t \mapsto \chi_{[0,t]} \oplus 0$ and $m^s : \mathbb{R}_+ \to \mathcal{H} : t \mapsto 0 \oplus \chi_{[0,t]}$ are $\xi$-martingales. We denote the annihilation $A(m_t^f)$, $A(m_t^s)$ and creation operators $A^*(m_t^f)$, $A^*(m_t^s)$, defined on $\mathcal{D}$ by (3.2), more compactly by $A_f(t)$, $A_s(t)$, $A_f^*(t)$ and $A_s^*(t)$, respectively. The calculus for stochastic integrals with respect to $A_\sigma(t)$ and $A_\nu^*(t)$, $\sigma, \nu \in \{f, s\}$ is then given by the *Hudson-Parthasarathy Itô table* [23], [34]:

| $dM_1\backslash dM_2$ | $dA_\nu^*(t)$ | $dA_\nu(t)$ |
|---|---|---|
| $dA_\sigma^*(t)$ | 0 | 0 |
| $dA_\sigma(t)$ | $\delta_{\sigma\nu}dt$ | 0 |

Let us introduce the selfadjoint quantum noise $\beta_t$ describing the interaction between the quantum system $\mathcal{B} = M_n(\mathbb{C})$ and the electromagnetic field

$$d\beta_t := -i\big(V_f dA_f^*(t) - V_f^* dA_f(t) + V_s dA_s^*(t) - V_s^* dA_s(t)\big), \qquad \beta_0 = 0. \tag{3.3}$$

It is clear in our example of spontaneous decay of a two-level system that this noise represents an interaction consisting of creations of excitations of the two-level system accompanied by annihilations of photons in the decay channels and vice versa. It describes the interaction of the electromagnetic field, in which we distinguished two decay channels, and the two-level system in the weak coupling limit [1]. We let the cocycle $U_t$ of section 2, providing the evolution in the weak coupling limit of the two-level system and field together, be given by the quantum stochastic differential equation

$$\begin{aligned}
&dU_t = \{id\beta_t - \frac{1}{2}(d\beta_t)^2\}U_t = \\
&\{V_f dA_f^*(t) - V_f^* dA_f(t) + V_s dA_s^*(t) - V_s^* dA_s(t) - \frac{1}{2}V^*V dt\}U_t, \\
&U_0 = \mathbf{1}.
\end{aligned}$$

We can now check that the dilation diagram (2.5) commutes. Using the continuous tensor product structure of the Fock space $\mathcal{F}(\mathcal{H})$, it is easy to see that following the lower part of diagram (2.5) defines a semigroup on $\mathcal{B}$, i.e. we only have to show that it is generated by the Lindblad operator $L$ of equation (2.3). For all $X \in \mathcal{B}$

$$d\mathrm{Id} \otimes \phi \otimes \phi\big(\hat{T}_t(X \otimes \mathbf{1} \otimes \mathbf{1})\big) = \mathrm{Id} \otimes \phi \otimes \phi\big(d(U_t^*(X \otimes \mathbf{1} \otimes \mathbf{1})U_t)\big).$$

Using the notation below Theorem 3.2 with $Z_1 = U_t^*$ and $Z_2 = (X \otimes \mathbf{1} \otimes \mathbf{1})U_t$, we find

$$d\mathrm{Id} \otimes \phi \otimes \phi\big(\hat{T}_t(X \otimes \mathbf{1} \otimes \mathbf{1})\big) = \mathrm{Id} \otimes \phi \otimes \phi\big([1] + [2] + [12]\big).$$

With the aid of the Hudson-Parthasarathy Itô table we can evaluate these terms. We are only interested in the $dt$-terms since the expectation with respect to the vacuum kills the other terms. The terms [1] and [2] provide the anticommutators $-\frac{1}{2}\{V_f^*V_f, X\}dt$ and $-\frac{1}{2}\{V_s^*V_s, X\}dt$ and [12] provides the terms $V_f^*XV_f dt$ and $V_s^*XV_s dt$, proving our claim.

We now change the situation in diagram (2.5) by introducing a laser on the forward channel, i.e. the forward channel is now in a *coherent state* $\gamma_h := \langle\psi(h), \cdot\, \psi(h)\rangle$ where $\psi(h) := \exp(-\frac{1}{2}\|h\|^2)e(h)$, the exponential vector $e(h)$ for some $h \in L^2(\mathbb{R}_+)$ normalised to unity. The laser will be used to send control-pulses to the system $\mathcal{B}$.

This leads to the following dilation diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t^h} & \mathcal{B} \\ {\scriptstyle \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1}}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathrm{Id}\otimes\gamma_h\otimes\phi} \\ \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s & \xrightarrow{\hat{T}_t=\mathrm{Ad}[\tilde{U}_t]} & \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s \end{array} \tag{3.4}$$

i.e. the evolution on $\mathcal{B}$ has changed and it is in general *not* a semigroup. Denote by $W(h)$ the unitary *Weyl* or *displacement operator* defined on $\mathcal{D}$ by: $W(h)\psi(g) = \exp(-2i\mathrm{Im}\langle h, g\rangle)\psi(g+h)$. Note that $W(h)\Phi = W(h)\psi(0) = \psi(h)$, so that we can write for all $X \in \mathcal{B}$

$$\begin{aligned} &T_t^h(X) = \mathrm{Id}\otimes\gamma_h\otimes\phi(U_t^* X\otimes\mathbf{1}\otimes\mathbf{1}U_t) = \\ &\mathrm{Id}\otimes\phi\otimes\phi\big(W_f(h)^* U_t^* X\otimes\mathbf{1}\otimes\mathbf{1}U_t W_f(h)\big) = \\ &\mathrm{Id}\otimes\phi\otimes\phi\big(W_f(h_{t]})^* U_t^* X\otimes\mathbf{1}\otimes\mathbf{1}U_t W_f(h_{t]})\big), \end{aligned}$$

where $h_{t]} = h\chi_{(0,t]}$ and $W_f(h) := \mathbf{1}\otimes W(h)\otimes\mathbf{1}$. Defining $U_t^h := U_t W_f(h_{t]})$, together with the quantum stochastic differential equation for $W_f(h_{t]})$ [34]

$$dW_f(h_{t]}) = \{h(t)dA_f^*(t) - \overline{h}(t)dA_f(t) - \frac{1}{2}|h(t)|^2 dt\}W_f(h_{t]}), \quad W_f(h_{0]}) = \mathbf{1},$$

and the Itô rules leads to the following quantum stochastic differential equation for $U_t^h$

$$\begin{aligned} dU_t^h = \Big\{&\big(V_f + h(t)\big)dA_f^*(t) - \big(V_f^* + \overline{h}(t)\big)dA_f(t)\big) + V_s dA_s^*(t) - V_s^* dA_s(t) - \\ &\frac{1}{2}\big(|h(t)|^2 + V^*V + 2h(t)V_f^*\big)dt\Big\}U_t^h, \quad U_0^h = \mathbf{1}. \end{aligned} \tag{3.5}$$

Therefore, the dilation diagram (3.4) is equivalent to

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t^h} & \mathcal{B} \\ {\scriptstyle \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1}}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathrm{Id}\otimes\phi\otimes\phi} \\ \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s & \xrightarrow{\hat{T}_t^h=\mathrm{Ad}[\tilde{U}_t^h]} & \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s \end{array} \tag{3.6}$$

In the following, we will often omit the superscript $h$ to simplify the notation. Define a Hamiltonian by $H := i\big(\overline{h}(t)V_f - h(t)V_f^*\big)$, then following the lower part of diagram (3.6) and using Itô's rules, see Theorem 3.2, shows that the time dependent generator of the dissipative evolution $T_t^h$ in the presence of the laser on the control channel is given by

$$L(X) = i[H, X] + V^*XV - \frac{1}{2}\{V^*V, X\}. \tag{3.7}$$

Later on we will choose $h$ in a suitable way in order to exert control on the system $\mathcal{B}$.

# 4 The Belavkin equation

Let us now turn our attention to the side channel. In this channel an observable is measured continuously in time. Goal is to briefly show how to derive a stochastic differential equation for the stochastic state evolution of the system $\mathcal{B}$ conditioned on the outcome of the measurement process. The method described below is known as quantum filtering, see [10] and [12] for a more detailed treatment.

In this paper the observable $Y_t^s$ of the field that is measured continuously in time will always be a field quadrature, i.e.

$$Y_t^s := \mathbf{1}_{\mathcal{B}} \otimes \mathbf{1}_{\mathcal{W}^f} \otimes \Big( \big(e^{-i\phi} A_s(t) + e^{i\phi} A_s^*(t)\big) \otimes \mathbf{1}_{\mathcal{W}^s_{[t}} \Big) \; \in \; \mathcal{B} \otimes \mathcal{W}^f \otimes \Big(\mathcal{W}^s_{t]} \otimes \mathcal{W}^s_{[t}\Big), \quad (4.1)$$

for some phase $\phi \in [0, 2\pi)$. Such a field quadrature measurement can be performed by a homodyne detection experiment. See [10], [12] for measurement of other observables. Let $\rho$ be the initial state of the quantum system $\mathcal{B}$. We describe the measurement process in the interaction picture, i.e. the shift part of $\breve{U}_t := (S_t \otimes S_t)U_t$ acts on the observables while the cocycle part $U_t$, given by equation (3.5) with the superscript $h$ suppressed, acts on the states

$$\rho^t(X) := \rho \otimes \phi(U_t^* X U_t), \qquad X \in \mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s.$$

Let $\mathcal{C}_t$ be the von Neumann algebra generated by the family of observables $\{Y_r^s;\ 0 \leq r \leq t\}$. Since $Y_r^s$ and $Y_t^s$ commute for all $r, t \geq 0$ the algebra $\mathcal{C}_t$ is commutative. The algebras $\{\mathcal{C}_t\}_{t\geq 0}$ form a growing family, that is $\mathcal{C}_s \subset \mathcal{C}_t$ for all $s \leq t$. Thus we can define the inductive limit $\mathcal{C}_\infty := \lim_{t\to\infty} \mathcal{C}_t$, which is the smallest von Neumann algebra containing all $\mathcal{C}_t$. It follows via Kolmogorov's extension theorem, see [12] Theorem 5.1, that there exists a unique state $\rho^\infty$ on $\mathcal{C}_\infty$ which coincides with $\rho^t$ when restricted to $\mathcal{C}_t \subset \mathcal{C}_\infty$ for all $t \geq 0$. From spectral theory it follows that there exists a measure space $(\Omega, \Sigma, \mathbb{P}_\rho)$ and a growing family $\{\Sigma_t\}_{t\geq 0}$ of $\sigma$-subalgebras of $\Sigma$, such that $(\mathcal{C}_\infty, \rho^\infty)$ and $(\mathcal{C}_t, \rho^t)$ are isomorphic to $L^\infty(\Omega, \Sigma, \mathbb{P}_\rho)$ and $L^\infty(\Omega, \Sigma_t, \mathbb{P}_\rho)$, respectively. The space $\Omega$ should be interpreted as the paths of the observed process $Y_r^s$ when the measurement is continued infinitely long. The $\sigma$-algebras $\Sigma_t$ contain the events up to time $t$.

In the Heisenberg picture, when a measurement of an observable $Y$ with discrete spectrum $Sp(Y)$ has been performed, all observables in $\mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s$ have to be sandwiched with the projection corresponding to the observed measurement result. If the result of the measurement is unknown, but the measurement has taken place, an observable takes the form of a direct sum over all possible outcomes of the original observable sandwiched with the projections corresponding to the outcomes, i.e.

$$X_{\text{after meas.}} = \bigoplus_{y \in Sp(Y)} P_y X P_y \qquad X \in \mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s.$$

Note that this procedure destroys all coherences between different measurement results. Moreover, it maps all observables in $\mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s$ to the commutant of the algebra generated by the measured observable.

Therefore, in analogy with the above, when a process $\{Y_r^s\}_{0\le r\le t}$ has been measured continuosly in time, we can restrict to the algebra $\mathcal{A}_t \subset \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s$ which is the *commutant* of the observed process

$$\mathcal{A}_t := \mathcal{C}_t' := \{X\in\mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s;\ XC = CX,\ \forall C\in\mathcal{C}_t\}.$$

We call $\mathcal{A}_t$ the algebra of observables that are *not demolished* [10] by observing the process $\{Y_r^s\}_{0\le r\le t}$. Note that from the double commutant theorem it follows that $\mathcal{C}_t$ is the *center* of $\mathcal{A}_t$, i.e. $\mathcal{C}_t = \{C\in\mathcal{A}_t;\ AC = CA,\ \forall A\in\mathcal{A}_t\}$.

We investigate the situation of the previous paragraph more abstractly for a moment, i.e. let $\mathcal{A}$ be a von Neumann algebra of operators on some Hilbert space $\mathbb{H}$ and let $\mathcal{C}$ be its center. Let $\rho$ denote a state on the algebra $\mathcal{A}$. We will now explain the decomposition of $\mathcal{A}$ over its center $\mathcal{C}$, see [24] for all details and proofs. We can identify the center $\mathcal{C}$ with some $L^\infty(\Omega,\Sigma,\mathbb{P})$ where $\mathbb{P}$ corresponds to the restriction of $\rho$ to $\mathcal{C}$. The Hilbert space $\mathbb{H}$ has a direct integral representation $\mathbb{H} = \int^\oplus \mathbb{H}_\omega\mathbb{P}(d\omega)$ in the sense that there exists a family of Hilbert spaces $\{\mathbb{H}_\omega\}_{\omega\in\Omega}$ and for any $\psi\in\mathbb{H}$ there exists a map $\omega\mapsto\psi_\omega\in\mathbb{H}_\omega$ such that

$$\langle\psi,\phi\rangle = \int_\Omega \langle\psi_\omega,\phi_\omega\rangle\mathbb{P}(d\omega) \qquad \psi,\phi\in\mathbb{H}.$$

The von Neumann algebra $\mathcal{A}$ has a *central decomposition* $\mathcal{A} = \int^\oplus \mathcal{A}_\omega\mathbb{P}(d\omega)$ in the sense that there exists a family $\{\mathcal{A}_\omega\}_{\omega\in\Omega}$ of von Neumann algebras with trivial center, or factors, and for any $A\in\mathcal{A}$ there is a map $\omega\mapsto A_\omega\in\mathcal{A}_\omega$ such that $(A\psi)_\omega = A_\omega\psi_\omega$ for all $\psi\in\mathbb{H}$ and $\mathbb{P}$-almost all $\omega\in\Omega$. The state $\rho$ on $\mathcal{A}$ has a decomposition in states $\rho_\omega$ on $\mathcal{A}_\omega$ such that for any $A\in\mathcal{A}$ its expectation is obtained by integrating with respect to $\mathbb{P}$ the expectations of its components $A_\omega$:

$$\rho(A) = \int_\Omega \rho_\omega(A_\omega)\mathbb{P}(d\omega).$$

Loosely speaking the component $A_\omega\in\mathcal{A}_\omega$ is the operator $A\in\mathcal{A}$ sandwiched with the projection corresponding to a measurement result $\omega$. Moreover, the state $\rho_\omega$ is the state $\rho$ conditioned on the measurement result $\omega$. For all $X\in\mathcal{A}$ we denote by $\rho_\bullet(X_\bullet)$ the function $\omega\mapsto\rho_\omega(X_\omega)$. The complex number $\rho_\omega(X_\omega)$ is the expectation of the observable $X$ in the state $\rho$ conditioned on measurement result $\omega$.

Define a map $\mathcal{E}_\rho:\ \mathcal{A}\to\mathcal{C}\cong L^\infty(\Omega,\Sigma,\mathbb{P})$ by $\mathcal{E}_\rho(X) := \rho_\bullet(X_\bullet)$ for all $X\in\mathcal{A}$. It is easily verified, see also [12], that this map is linear, surjective, identity preserving, completely positive, it satisfies the *module property*

$$\mathcal{E}_\rho(C_1XC_2) = C_1\mathcal{E}_\rho(X)C_2, \qquad C_1,C_2\in\mathcal{C},\ X\in A,$$

and it leaves the state $\rho$ invariant, i.e. $\rho\big(\mathcal{E}_\rho(X)\big) = \rho(X)$ for all $X\in\mathcal{A}$. These properties uniquely determine the map $\mathcal{E}_\rho$, see [40]. It is called the *conditional expectation* of $\mathcal{A}$ onto $\mathcal{C}$ with respect to $\rho$. Returning to the original problem, i.e. a whole family of algebras $\mathcal{A}_t$ with center $\mathcal{C}_t$, we get a family of conditional expectations $\mathcal{E}_{\rho^t}:\ \mathcal{A}_t\to\mathcal{C}_t$. We denote $\mathcal{E}_{\rho^t}$ more compactly by $\mathcal{E}^t$.

Apart from the family of quantum mechanical conditional expectations $\mathcal{E}^t$, there is also a family of conditional expectations in the classical sense that plays an important role in the following. Denote by $\mathbb{E}^t_\rho$ the unique classical conditional expectation from $\mathcal{C}_\infty \cong L^\infty(\Omega, \Sigma, \mathbb{P}_\rho)$ onto $\mathcal{C}_t \cong L^\infty(\Omega, \Sigma_t, \mathbb{P}_\rho)$ that leaves the state $\rho^\infty$, or equivalently, the expectation with respect to $\mathbb{P}_\rho$ invariant, i.e. $\rho^\infty \circ \mathbb{E}^t = \rho^\infty$. These conditional expectations satisfy the *tower property*, that is $\mathbb{E}^s_\rho\big(\mathbb{E}^t_\rho(C)\big) = \mathbb{E}^s_\rho(C)$ for all $C \in \mathcal{C}_\infty$ and $t \geq s \geq 0$. $\mathbb{E}^0_\rho$ is the expectation with respect to $\mathbb{P}_\rho$ and will simply be denoted $\mathbb{E}_\rho$. Note that the tower property for $s = 0$ is just the invariance of the state $\rho^\infty (= \mathbb{E}_\rho)$.

For all $t \geq 0$ and $X \in \mathcal{B}$ the operator $X \otimes \mathbf{1} \otimes \mathbf{1} \in \mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s$ commutes with the observed process $\{Y^s_r\}_{0 \leq r \leq t}$ up to time $t$, i.e. $\mathcal{B} \subset \mathcal{A}_t$. Therefore we can define for all $X \in \mathcal{B}$ a process $\{M^X_t\}_{t\geq 0}$ in the algebra $\mathcal{C}_\infty \cong L^\infty(\Omega, \Sigma, \mathbb{P}_\sigma)$ by

$$M^X_t := \mathcal{E}^t(X) - \mathcal{E}^0(X) - \int_0^r \mathcal{E}^r\big(L(X)\big)dr, \tag{4.2}$$

where $L: \mathcal{B} \to \mathcal{B}$ is the Liouvillian of equation (3.7). From the definition it is clear that $M^X_t$ is an element of $\mathcal{C}_t$ for all $t \geq 0$. The process $\{M^X_t\}_{t\geq 0}$ is a martingale, i.e. for all $0 \leq s \leq t$ we have $\mathbb{E}^s_\rho(M^X_t) = M^X_s$, see [10], [12] for details and a proof. In differential form equation (4.2) reads

$$d\rho^t_\bullet(X) = \rho^t_\bullet\big(L(X)\big)dt + dM^X_t,$$

where we have used that $X_\bullet$ is the constant function $\omega \mapsto X$. This equation is the *Belavkin equation* [9], [10], [12].

Denote by $\tilde{Y}^s_t$ the process given by the following stochastic differential equation

$$d\tilde{Y}^s_t = dY^s_t - \mathcal{E}^t(e^{i\phi}V^*_s + e^{-i\phi}V_s)dt, \quad \tilde{Y}^s_0 = 0.$$

The process $\tilde{Y}^s_t$ is a martingale, i.e. for all $0 \leq r \leq t$ we have $\mathbb{E}^r_\rho(\tilde{Y}^s_t) = \tilde{Y}^s_r$, see [10], [12] for details and a proof. We call $\tilde{Y}^s_t$ the *innovating martingale* of the observed process $Y^s_t$. The link between the martingale $M^X_t$ and the observed process $Y^s_t$ is provided by the martingale representation theorem which states that there exists a stochastically integrable process $\eta^X_t$ such that

$$dM^X_t = \eta^X_t d\tilde{Y}^s_t = \eta^X_t\big(dY^s_t - \mathcal{E}^t(e^{i\phi}V^*_s + e^{-i\phi}V_s)dt\big).$$

The process $\eta^X_t$ can be calculated by using that $\mathcal{E}^t$ leaves $\rho^t$ invariant [10]. We refer to [12] for the details, the result is

$$\eta^X_t = \mathcal{E}^t(e^{i\phi}V^*_s X + e^{-i\phi}XV_s) - \mathcal{E}^t(e^{i\phi}V^*_s + e^{-i\phi}V_s)\mathcal{E}^t(X).$$

This leads to the Belavkin equation [10], [12]

$$\begin{aligned} d\rho^t_\bullet(X) = \rho^t_\bullet\big(L(X)\big)dt + \Big(\rho^t_\bullet(e^{i\phi}V^*_s X + e^{-i\phi}XV_s) - \rho^t_\bullet(e^{i\phi}V^*_s + e^{-i\phi}V_s)\rho^t_\bullet(X)\Big) \times \\ \times \Big(dY^s_t - \rho^t_\bullet(e^{i\phi}V^*_s + e^{-i\phi_t}V_s)dt\Big) \quad X \in \mathcal{B}. \end{aligned} \tag{4.3}$$

This equation tells us how the state of the system $\mathcal{B}$ evolves over an infinitesimal time $dt$ depending on what we observe for the measurement process $dY_t^s$. Since $Y_t^s$ is a martingale with variance $t$ on the space of the Wiener process, it must be the Wiener process itself.

# 5 Control: the essentially commutative case

In this section we focus on dilations that are essentially commutative [28]. We will use the results of the measurement of $Y_t^s$ to control the time evolution $T_t$ of the system $\mathcal{B}$ in order to bring it as close to the identity map as possible. For essentially commutative dilations this can be done (nearly) perfectly. This section serves as a guiding example for the more realistic situations described in sections 6 and 8.

Let $V$ be selfadjoint, i.e. $V = V^*$. The discussion below can easily be adapted to fit the situation where $V = -V^*$. Define for $\sigma = f, s$ field observables $Y_t^\sigma := i(A_\sigma^*(t) - A_\sigma(t)) \in \mathcal{W}_{t]}^\sigma$. Using $V = V^*$, equation (2.4), i.e. the laser on the forward channel is off, simplifies to

$$dU_t = \Big\{ -iV_f dY_t^f - iV_s dY_t^s - \frac{1}{2}V^2 dt \Big\} U_t, \qquad U_0 = \mathbf{1}. \tag{5.1}$$

This means that for $t \geq 0$ the solution $U_t$ is an element of $\mathcal{B} \otimes \mathcal{C}_t$, with $\mathcal{C}_t$ the commutative von Neumann algebra generated by the process $\{Y_r^f \otimes Y_r^s\}_{0 \leq r \leq t}$. (We have dropped the extensive notation with the identities tensored to the $Y_r$'s.) This means that we can restrict the dilation of diagram (3.6) to $\mathcal{B} \otimes \mathcal{C}_\infty$, i.e.

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\ {\scriptstyle \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1}}\downarrow & & \uparrow{\scriptstyle \mathrm{Id}\otimes\phi\otimes\phi} \\ \mathcal{B}\otimes\mathcal{C}_\infty & \xrightarrow{\hat{T}_t} & \mathcal{B}\otimes\mathcal{C}_\infty \end{array} \tag{5.2}$$

A dilation for which the relative commutant of the embedding of the algebra $\mathcal{B}$ into the subalgebra of $\mathcal{B} \otimes \mathcal{W}^f \otimes \mathcal{W}^s$ generated by $\{U_t^* X \otimes \mathbf{1} \otimes \mathbf{1} U_t;\ X \in \mathcal{B},\ t \geq 0\}$ is commutative, is called *essentially commutative* [28]. Although we restrict the discussion to the essentially commutative dilation determined by equation (5.1), the results of this section can be extended to all essentially commutative dilations [28].

If the dilation is essentially commutative the derivation of the Belavkin equation is extremely simple. Since $U_t$ is not demolished by observing $\{Y_r^s\}_{0 \leq r \leq t}$, i.e. it is an element of the commutant of $\mathcal{C}_t$, we can just calculate $d(U_t^* X \otimes \mathbf{1} \otimes \mathbf{1} U_t)$ using the quantum Itô rules and decompose it over the paths of the measurement process. It is clear that this leads exactly to the Belavkin equation of the previous section (4.3) with $\phi = \frac{\pi}{2}$ and $V = V^*$

$$d\rho_\bullet^t(X) = \rho_\bullet^t\big(L(X)\big)dt + i\rho_\bullet^t\big([V_s, X]\big)dY_t^s, \tag{5.3}$$

where $L$ is as in equation (3.7) with $H$ is 0, i.e. there has been no control yet. In general, however, we do not have the decomposition of $U_t$ over the center and we have to resort to the methods of the previous section. Note that for $\phi = \frac{\pi}{2}$ and $V = V^*$ we have $d\tilde{Y}_t^s = dY_t^s$, i.e. $Y_t^s$ is the Wiener process. This means that the measurement process is *non-informative* [21], i.e. since here there is no state dependent drift term, we do not gain information about the state $\rho_\bullet^t$ by observing $Y_t^s$.

Let $\rho^0$ be the density matrix of the initial state of the system $\mathcal{B}$. We observe $Y_t^s$ from time 0 to time $\tau$. Suppose that the laser is off in that time interval, i.e. $h(t) = 0$ for $0 \le t < \tau$. Then the stochastic density matrix at time $\tau$ is given by (5.3)

$$\tilde{\rho}_\bullet^\tau = \rho^0 + \int_0^\tau V\tilde{\rho}_\bullet^t V^* dt - \frac{1}{2}\{V^*V, \tilde{\rho}_\bullet^t\}dt + i[\tilde{\rho}_\bullet^t, V_s]dY_t^s,$$

where the tilde has been introduced to remind us that this is the state *before* control has taken place. In the time interval from 0 to $\tau$ we have observed $Y_t^s$ and therefore we can determine the difference $\Delta(\tau) := Y_\tau^s - Y_0^s$ at time $\tau$. Then we want to control the state $\tilde{\rho}_\bullet^\tau$ with a unitary

$$U_c^\tau := \exp\big(i\Delta(\tau)V_s\big),$$

i.e. the density matrix after control is given by $\rho_\bullet^\tau = U_c^\tau\tilde{\rho}_\bullet^\tau U_c^{\tau *}$. This can be done by supplying a very sharply peaked laser pulse to the system, i.e. take

$$h(t) = -i\frac{\kappa_s\Delta(\tau)}{2\kappa_f}\delta_\tau(t), \qquad 0 \le t < 2\tau,$$

where $\delta_\tau$ is the delta function at time $\tau$. Then $H = -\Delta(\tau)\delta_\tau V_s$ in equation (3.7), i.e. at time $\tau$ all terms in equation (3.7) are negligable with respect to the commutator with $H$. At time $\tau$ this commutator performs a Rabi oscillation exactly of size $U_c^\tau$. After having applied the control unitaries the state $\rho_\bullet^\tau$ is taken as the new initial state $\rho^0$ and the control scheme is repeated after every $\tau$ time units.

Note that the control unitary $U_c^\tau$ satisfies the following stochastic differential equation

$$dU_c^\tau = \{iV_s dY_\tau^s - \frac{1}{2}V_s^2 d\tau\}U_c^\tau = U_c^\tau\{iV_s dY_\tau^s - \frac{1}{2}V_s^2 d\tau\}, \qquad U_c^0 = \mathbf{1}.$$

Recall that we have the Itô rules $dY_t^s dY_t^s = dt$, $dY_t^s dt = dt dY_t^s = 0$, $dY_t^f dt = dt dY_t^f = 0$ and $dY_t^s dY_t^f = dY_t^f dY_t^s = 0$. Using the notation below Theorem 3.2 with $Z_1 = U_c^\tau$, $Z_2 = \tilde{\rho}_\bullet^\tau$ and $Z_3 = U_c^{\tau *}$ we find infinitesimally at $\tau = 0$, i.e. $\tau$ should be very small or equivalently we should correct with extremely high frequency

$$d\rho_\bullet^\tau\Big|_{\tau=0} = \big([1] + [2] + [3] + [12] + [13] + [23] + [123]\big)\Big|_{\tau=0}. \tag{5.4}$$

Note that it immediately follows from Itô's rules that $[123] = 0$.

For $W \in \mathcal{B}$ we denote by $L_W$ the Lindblad operator corresponding to $W$ acting on density matrices $\rho$, i.e.

$$L_W(\rho) := W\rho W^* - \frac{1}{2}\{W^*W, \rho\}.$$

Then we can write $([1] + [3] + [13])|_{\tau=0} = L_{V_s}(\rho^0)d\tau - i[\rho^0, V_s]dY_0^s$ and $([2] + [12] + [23])|_{\tau=0} = L_V(\rho^0)d\tau + i[\rho^0, V_s]dY_0^s - 2L_{V_s}(\rho^0)d\tau$. Therefore we get $d\rho_\bullet^\tau|_{\tau=0} =$

$L_{V_f}(\rho^0)d\tau$ and since we repeat the control every $\tau$ time units with $\tau$ very small, i.e. we take $\tau$ infinitesimal, this leads to the following deterministic state evolution

$$d\rho^t = L_{V_f}(\rho^t)dt.$$

This means we only have dissipation into the forward channel. We can take $\kappa_f$ arbitrarily small which means we have succeeded in freezing the state evolution nearly perfectly, i.e. the control scheme *restores quantum information* in the sense of [21].

# 6 Control without squeezing

We now return to the more realistic situation of spontaneous decay of a two-level atom to its ground state. We are again interested in controlling the state of a system in order to get as close as possible to freezing its state evolution. However, in trying to do this, we encounter problems that motivate the investigation put forward in the sections to come.

Guided by the previous section we write $V$ of equation (2.2) as the sum $V = V_R + iV_I$ with $V_R$ and $V_I$ selfadjoint, i.e.

$$V_R := \frac{V+V^*}{2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V_I := \frac{V-V^*}{2i} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{6.1}$$

Denote for $\sigma = f, s$ : $V_R^\sigma := \kappa_\sigma V_R$, $V_I^\sigma := \kappa_\sigma V_I$, $Y_R^\sigma(t) := i\big(A_\sigma^*(t) - A_\sigma(t)\big)$ and $Y_I^\sigma(t) := A_\sigma^*(t) + A_\sigma(t)$. Then equation (2.4), i.e. the laser is off, can be written as

$$dU_t := \Big\{\Big(\sum_{\sigma=f,s} iV_I^\sigma dY_I^\sigma(t) - iV_R^\sigma dY_R^\sigma(t)\Big) - \frac{1}{2}V^*V dt\Big\}U_t, \qquad U_0 = \mathbf{1}. \tag{6.2}$$

Since the noises $Y_R^s(t)$ and $Y_I^s(t)$ in the side channel do not commute we can not observe them both simultaneously.

In the following we choose to observe $Y_R^s(t)$ and to keep notation simple we denote it by $Y_t$. The Belavkin equation for observation of $Y_t$ follows from equation (4.3)

$$\begin{aligned} d\rho_\bullet^t = L(\rho_\bullet^t)dt + i\Big(\rho_\bullet^t V_s^* - V_s\rho_\bullet^t - \mathrm{Tr}(\rho_\bullet^t V_s^* - V_s\rho_\bullet^t)\rho_\bullet^t\Big)\times \\ \times\Big(dY_t - i\mathrm{Tr}(\rho_\bullet^t V_s^* - V_s\rho_\bullet^t)dt\Big). \end{aligned}$$

where $L$ is given by equation (3.7) with $H = 0$. Using the relation $\rho_\bullet^t V_s^* - V_s\rho_\bullet^t = [\rho_\bullet^t, V_R^s] - i\{\rho_\bullet^t, V_I^s\}$, this equation simplifies to

$$d\rho_\bullet^t = L(\rho_\bullet^t)dt + \Big(i[\rho_\bullet^t, V_R^s] + \{\rho_\bullet^t, V_I^s\} - 2\mathrm{Tr}(\rho_\bullet^t V_I^s)\rho_\bullet^t\Big)\Big(dY_t - 2\mathrm{Tr}(\rho_\bullet^t V_I^s)dt\Big). \tag{6.3}$$

Note that $Y_t$ is a Wiener process plus a stochastic drift term that depends on the state of the two-level atom. By observing $Y_t$ we can estimate this drift term and in this way obtain information about the state $\rho_\bullet^t$.

We run a control scheme similar to the one in the previous section, i.e. we choose $h(t) := -i\frac{\kappa_s\Delta(\tau)}{2\kappa_f}\delta_\tau(t)$ for $0 \le t < 2\tau$. Then we get a control unitary $U_c^\tau = \exp\big(i\Delta(\tau)V_R^s\big)$, satisfying the stochastic differential equation

$$dU_c^\tau = \{iV_R^s dY_\tau - \frac{1}{2}{V_R^s}^2 d\tau\}U_c^\tau = U_c^\tau\{iV_R^s dY_\tau - \frac{1}{2}{V_R^s}^2 d\tau\}, \qquad U_c^0 = \mathbf{1}.$$

The state after control is again given by $\rho_\bullet^\tau := U_c^\tau \tilde{\rho}_\bullet^\tau U_c^{\tau *}$ where $\tilde{\rho}_\bullet^\tau$ is given by the Belavkin equation (6.3). We use the notation below Theorem 3.2 with $Z_1 = U_c^\tau$, $Z_2 = \tilde{\rho}_\bullet^\tau$ and $Z_3 = U_c^{\tau *}$. For infinitesimal $\tau$ evaluated at $\tau = 0$, this leads to equation (5.4), i.e.

$$d\rho_\bullet^\tau\Big|_{\tau=0} = \big([1] + [2] + [3] + [12] + [13] + [23] + [123]\big)\Big|_{\tau=0}.$$

Again $[123] = 0$ and further $([1] + [3] + [13])|_{\tau=0} = L_{V_R^s}(\rho^0)d\tau - i[\rho^0, V_R^s]dY_0$. Furthermore we have

$$[2]\Big|_{\tau=0} = L_V(\rho^0)d\tau + \Big(i[\rho^0, V_R^s] + \{\rho^0, V_I^s\} - 2\mathrm{Tr}(\rho^0 V_I^s)\rho^0\Big)\Big(dY_0 - 2\mathrm{Tr}(\rho^0 V_I^s)d\tau\Big),$$

$$\big([12] + [23]\big)\Big|_{\tau=0} = -2L_{V_R^s}(\rho^0)d\tau + i[V_R^s, \{\rho^0, V_I^s\}]d\tau - 2\mathrm{Tr}(\rho^0 V_I^s)i[V_R^s, \rho^0]d\tau.$$

A calculation shows that

$$L_V(\rho^0) - L_{V_R^s}(\rho^0) + i[V_R^s, \{\rho^0, V_I^s\}] = L_{V_f}(\rho^0) + L_{V_I^s}(\rho^0) + \frac{i}{2}[V_I^s V_R^s + V_R^s V_I^s, \rho^0].$$

Since for spontaneous decay $V_I^s V_R^s + V_R^s V_I^s = 0$, we get

$$d\rho_\bullet^\tau\Big|_{\tau=0} = L_{V_f}(\rho^0)d\tau + L_{V_I^s}(\rho^0)d\tau + \Big(\{\rho^0, V_I^s\} - 2\mathrm{Tr}(\rho^0 V_I^s)\rho^0\Big)\Big(dY_0 - 2\mathrm{Tr}(\rho^0 V_I^s)d\tau\Big).$$

Since we repeat the control every $\tau$ time units with $\tau$ very small, i.e. we take $\tau$ infinitesimal, this leads to the following stochastic time evolution for the density matrix of the two-level atom

$$d\rho_\bullet^t = L_{V_f}(\rho_\bullet^t)dt + L_{V_I^s}(\rho_\bullet^t)dt + \Big(\{\rho_\bullet^t, V_I^s\} - 2\mathrm{Tr}(\rho_\bullet^t V_I^s)\rho_\bullet^t\Big)\Big(dY_t - 2\mathrm{Tr}(\rho_\bullet^t V_I^s)dt\Big).$$

The first term on the right hand side is harmless, we already encountered it in the previous section, by taking $\kappa_f$ small enough we can make it as small as we want. The third term is also harmless. Since $Y_t - \int_0^t 2\mathrm{Tr}(\rho_\bullet^r V_I^s)dr$ is a martingale it vanishes when we average over all possible outcomes for $Y_t$. However, the second term reflects the fact that we can not observe $Y_I^s$ and correct it simultaneously with $Y_R^s$. The next sections are devoted to finding a way around this problem.

# 7 Squeezed states and their calculus

In this section we drop the assumption that the side channel of the field is initially in the vacuum state. We take a step back and rethink our model for (a channel in) the field. For the vacuum state we are going to end up with the description we have already used this far. Goal of the description below is to incorporate the situation where the initial state of the side channel is a so-called *squeezed* state. In a sqeezed state the variance of one of the quadratures $Y_R^s$ and $Y_I^s$ decreases while the other one increases as a result of Heisenberg's uncertainty relation. In the next section we will observe the increased quadrature and correct it. The disturbing effect of the other quadrature has decreased as a result of the squeezing.

Let $H$ be the real space of quadratically integrable $\mathbb{R}^2$-valued functions on $\mathbb{R}$. On $H$ we define a symplectic form $\sigma: H \times H \to \mathbb{R}$ by

$$\sigma(f,g) := -\int_{\mathbb{R}} (f_1 f_2) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d\lambda, \qquad f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\ g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in H,$$

where $\lambda$ denotes the Lebesgue measure on $\mathbb{R}$. For notational convenience we define

$$J_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We will describe (a channel in) the electromagnetic field by the $C^*$-algebra of *canonical commutation relations* $CCR(H,\sigma)$ over the symplectic space $(H,\sigma)$.

The algebra $CCR(H,\sigma)$ is defined as the $C^*$-algebra generated by abstract elements $\{W(f);\ f \in H\}$ satisfying relations

$$\begin{aligned} &1. \quad W(f)^* = W(-f), \qquad f \in H, \\ &2. \quad W(f)W(g) = \exp\big(-i\sigma(f,g)\big)W(f+g), \qquad f,g \in H. \end{aligned} \tag{7.1}$$

The second relation is called the *Weyl relation.* It follows from [37] that the $C^*$-algebra $CCR(H,\sigma)$ exists and moreover that it is unique up to isomorphism. Furthermore it immediately follows from (7.1) that $W(f)$ is unitary for all $f \in H$.

Let $\alpha: H \times H \to \mathbb{R}$ be a symmetric positive bilinear form satisfying

$$\sigma(f,g)^2 \leq \alpha(f,f)\alpha(g,g), \qquad f,g \in H. \tag{7.2}$$

It is well known (cf. [35]) that if $\alpha$ satisfies (7.2) then there exists a unique state $\gamma$ on the $C^*$-algebra $CCR(H,\sigma)$ satisfying

$$\gamma\big(W(f)\big) = \exp\big(-\frac{1}{2}\alpha(f,f)\big), \qquad f \in H. \tag{7.3}$$

Such a state $\gamma$ on $CCR(H,\sigma)$ is called a *quasifree* state.

In this paper we focus on a particular class of quasifree states $\gamma_{nc}$ indexed by a parameter $n \in \mathbb{R}$ and a complex parameter $c = a + ib, a, b \in \mathbb{R}$. These states will turn out to be the *squeezed white noise* states of the field as they are encountered in quantum optics after a Markov approximation is made (cf. [20]). They are defined through equation (7.3) with a symmetric positive bilinear form $\alpha_{nc}$ given by [22]

$$\alpha_{nc}(f,g) = \int_{\mathbb{R}} (f_1 f_2) \begin{pmatrix} 2n+1+2a & 2b \\ 2b & 2n+1-2a \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} d\lambda, \qquad f,g \in H.$$

For notational convenience we define

$$Q_{nc} := \begin{pmatrix} 2n+1+2a & 2b \\ 2b & 2n+1-2a \end{pmatrix}.$$

Condition (7.2) leads to the restrictions $n(n+1) \geq |c|^2$ and $n \geq 0$. For $n = c = 0$ we get the usual *vacuum state* and for $c = 0$ we end up with a chaotic temperature state. More details on the interpretation of this class of states will follow below.

A real linear map $J : H \to H$ is called *multiplication by $i$* if it satisfies $J^2 = -\text{id}$. Then $H$ is a complex vector space with the usual addition and the scalar multiplication given by $(x+iy)f = xf + yJf$ for all $x, y \in \mathbb{R}$.

**Lemma 7.1:** *Let $n \geq 0$ and $n(n+1) \geq |c|^2$. $H$ can be considered as a complex vector space equipped with an inner product given by*

$$\langle f, g\rangle_{nc} = \alpha_{nc}(f,g) + i\sigma(f,g), \qquad f, g \in H, \tag{7.4}$$

*if and only if $n(n+1) = |c|^2$. In this case multiplication by $i$ is given by $J_{nc} = J_0 Q_{nc}$.*

*Proof.* Since the inner product (7.4) is linear in its second argument $J_{nc}$ has to satisfy $\sigma(f, J_{nc}g) = \alpha_{nc}(f,g)$ for all $f, g \in H$. It easily follows from $n \geq 0$ and $n(n+1) \geq |c|^2$ that $Q_{nc}$ is non degenerate. Therefore $J_{nc}$ has to satisfy $-J_0 J_{nc} = Q_{nc}$ which is equivalent to $J_{nc} = J_0 Q_{nc}$. $J_{nc}$ is multiplication by $i$ if and only if $J_{nc}^2 = -\text{id}$, which is equivalent to

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2n+1+2a & 2b \\ 2b & 2n+1-2a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2n+1+2a & 2b \\ 2b & 2n+1-2a \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Longleftrightarrow$$

$$\begin{pmatrix} 4b^2 - (2n+1+2a)(2n+1-2a) & 0 \\ 0 & 4b^2 - (2n+1+2a)(2n+1-2a) \end{pmatrix} =$$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Longleftrightarrow \qquad |c|^2 = n^2 + n.$$

□

In the following we will always be in the situation of Lemma 7.1, i.e. $n \geq 0$ and $n(n+1) = |c|^2$. The states of Lemma 7.1, i.e. states that allow for the definition of an inner product on $H$ through (7.4), are called *Fock states* (name will become apparent in a minute). We denote the complex Hilbert space given by the pair $(H, J_{nc})$ equipped with the inner product of (7.4) by $H_{nc}$. Note that $H_{00}$ is just the space $L^2(\mathbb{R})$ of all quadratically integrable functions on the real line $\mathbb{R}$. The representation of $CCR(H, \sigma)$ discussed below is actually the GNS-representation with respect to a Fock state $\gamma_{nc}$, see [35] for the details.

Fix $n \geq 0$ and $c \in \mathbb{C}$ such that $n(n+1) = |c|^2$. Recall that the bosonic Fock space over $H_{nc}$ was defined as

$$\mathcal{F}_{nc} := \mathbb{C} \oplus \bigoplus_{k=1}^{\infty} H_{nc}^{\otimes_s k},$$

and that for all $f$ in $H_{nc}$ the *exponential vector* is given by $e(f) := 1 \oplus \bigoplus_{k=1}^{\infty} \frac{1}{\sqrt{k!}} f^{\otimes k}$. The span of all exponential vectors was denoted $\mathcal{D}$ and the *vacuum vector* $e(0) = 1 \oplus 0 \oplus 0 \oplus \ldots$ was also written as $\Phi$. On the dense domain $\mathcal{D}$ we define for all $f \in H_{nc}$ operators $W_{nc}(f)$ by

$$W_{nc}(f)e(g) := \exp\big( - \langle f, g\rangle_{nc} - \frac{1}{2}\alpha_{nc}(f,f)\big)e(f+g), \qquad f, g \in H_{nc}.$$

They are isometric and therefore uniquely extend to unitary operators on $\mathcal{F}_{nc}$. The mapping $\Pi_{nc} : W(f) \mapsto W_{nc}(f)$ uniquely defines a linear map $\Pi_{nc}$ from $CCR(H,\sigma)$ into the bounded operators on the bosonic Fock space. The map $\Pi_{nc}$ preserves the relations 1. and 2. of (7.1) defining $CCR(H,\sigma)$, i.e. it is a representation of the canonical commutation relations on $\mathcal{F}_{nc}$. The state $\gamma_{nc}$ is now given by the vector $\Phi \in \mathcal{F}_{nc}$, i.e.

$$\gamma_{nc}(X) = \langle \Phi, \Pi_{nc}(X)\Phi\rangle_{nc}, \qquad X \in CCR(H,\sigma).$$

The triple $(\mathcal{F}_{nc}, \Pi_{nc}, \Phi)$ is the GNS-triple corresponding to the state $\gamma_{nc}$, cf. [35]. The algebra of observables for the electromagnetic field in the Fock state $\gamma_{nc}$ is modelled by the von Neumann algebra $\mathcal{W}_{nc}$ generated by $\{W_{nc}(f);\ f \in H_{nc}\}$, which is just all bounded operators on $\mathcal{F}_{nc}$.

**Remark.** We can reduce the case of a non Fock quasifree state to a Fock state by doubling the space $H$ to $H \oplus H$. We can embed the algebra of canonical commutation relations over $H$ into the algebra of canonical commutation relations over $H \oplus H$ and view the state on $CCR(H,\sigma)$ as the restriction of a Fock state on this bigger algebra (cf. [35]). In this way we get representations on a doubled up Fock space. Then the algebra of observables is not the whole algebra of bounded operators but a true subalgebra.

The dilation of the semigroup $T_t$ of diagram (2.5) serves as our starting point. We change it by replacing the vacuum state $\phi = \gamma_{00}$ on the side channel by the Fock state $\gamma_{nc}$ described above. The dilation diagram then changes to

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t^{nc}} & \mathcal{B} \\ \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1} \big\downarrow & & \big\uparrow \mathrm{Id}\otimes\phi\otimes\gamma_{nc} \\ \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s_{nc} & \xrightarrow{\hat{T}_t^{nc}} & \mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s_{nc} \end{array} \qquad (7.5)$$

Coupling the quantum system to a field in another state than the vacuum has changed its reduced dynamics to $T_t^{nc}$. Changing the representation space of the algebra of canonical commutation relations from $\mathcal{F} = \mathcal{F}_{00}$ to $\mathcal{F}_{nc}$ also means that we have to describe the joint evolution of the system and (the two channels in) the field in this representation. Making sense of the group $T_t^{nc}$ will be our main concern for the remainder of this section.

For all $f \in H$ the family of operators $\{W_{nc}(tf)\}_{t\in\mathbb{R}}$ forms a one-parameter group, continuous in the strong operator topology. Therefore it follows from Stone's theorem that for all $f \in H$ there exists a selfadjoint $B_{nc}(f)$ such that

$$W_{nc}(tf) = \exp\big(itB_{nc}(f)\big).$$

The operators $B_{nc}(f)$ are called *field operators*. The domain of the operator $B_{nc}(f_k)\ldots B_{nc}(f_1)$ contains $\mathcal{D}$ for every $f_1, \ldots f_k \in H$ and $k \in \mathbb{N}$ (cf. [35]). For $f, g \in H$ and $t \in \mathbb{R}$ it follows from the Weyl relation that on the domain $\mathcal{D}$

$$\begin{aligned} &1. \quad B_{nc}(tf) = tB_{nc}(f),\\ &2. \quad B_{nc}(f+g) = B_{nc}(f) + B_{nc}(g),\\ &3. \quad [B_{nc}(f), B_{nc}(g)] = 2i\sigma(f,g). \end{aligned} \tag{7.6}$$

Let $H_0$ be the real Hilbert space $\{f \in H;\ f = (f_1, 0)\}$. From (7.6.3) it immediately follows that the family of operators $\{B_{nc}(f); f \in H_0\}$ is commutative. Using spectral theory, they can be realised as random variables on a single measure space. If the field described by the algebra $CCR(H,\sigma)$ is in the Fock state $\gamma_{nc}$, then the joint characteristic function of the random variables $B_{nc}(f_1), B_{nc}(f_2), \ldots, B_{nc}(f_k)$ is for $t_1, \ldots, t_k \in \mathbb{R}$ given by

$$\Big\langle \Phi, \exp\big(it_1B_{nc}(f_1)\big)\exp\big(it_2B_{nc}(f_2)\big)\ldots\exp\big(it_kB_{nc}(f_k)\big)\Phi\Big\rangle =$$

$$\Big\langle \Phi, \exp\big(i\sum_{i=1}^k t_iB_{nc}(f_i)\big)\Phi\Big\rangle = \gamma_{nc}\Big(W\big(\sum_{i=1}^k t_if_i\big)\Big) = \exp\Big(-\frac{1}{2}\sum_{i,j=1}^k t_it_j\alpha_{nc}(f_i,f_j)\Big),$$

i.e. their joint distribution is Gaussian with covariance matrix $\alpha_{nc}(f_i, f_j)$. In a similar way it can be shown that the family $\{B_{nc}(J_0f);\ f \in H_0\}$ is commutative and the joint distribution of the random variables $B_{nc}(J_0f_1), \ldots B_{nc}(J_0f_k)$ is Gaussian with covariance matrix $\alpha_{nc}(J_0f_i, J_0f_j)$. The Gaussianity of these fields, the covariance matrix and the condition $|c|^2 = n^2+n$ are exactly the defining properties of a squeezed vacuum state in the quantum optics literature, cf. [20].

**Definition 7.2:** Fix $n \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $|c|^2 = n^2 + n$. On the domain $\mathcal{D} \subset \mathcal{F}_{nc}$ we define *creation* and *annihilation* operators by

$$A^*_{nc}(f) := \frac{1}{2}\big(B_{nc}(f) - iB_{nc}(J_{nc}f)\big), \quad A_{nc}(f) := \frac{1}{2}\big(B_{nc}(f) + iB_{nc}(J_{nc}f)\big), \quad f \in H,$$
$$A^*_0(f) := \frac{1}{2}\big(B_{nc}(f) - iB_{nc}(J_0f)\big), \quad A_0(f) := \frac{1}{2}\big(B_{nc}(f) + iB_{nc}(J_0f)\big), \quad f \in H.$$

It immediately follows from equation (7.6.3) that these operators satisfy the following commutation relations $[A_0(f), A_0(g)] = [A^*_0(f), A^*_0(g)] = [A_{nc}(f), A_{nc}(g)] = [A^*_{nc}(f), A^*_{nc}(g)] = 0$, $[A_{nc}(f), A^*_{nc}(g)] = \langle f, g\rangle_{nc}$ and $[A_0(f), A^*_0(g)] = \langle f, g\rangle_{00}$ for all

$f, g \in H$. Moreover, it is a standard result (cf. [35]) that for Fock states $A_{nc}(f)\Phi = 0,\ f \in H$. Furthermore, we can build up the symmetric Fock space by acting with creation operators on the vacuum. From all these properties it easily follows that for all $h, f, g \in H$

$$A_{nc}(f)e(g) = \langle f, g\rangle_{nc} e(g), \quad \text{and} \quad \big\langle e(h), A^*_{nc}(f)e(g)\big\rangle_{\mathcal{F}_{nc}} = \langle h, f\rangle_{nc}\big\langle e(h), e(g)\big\rangle_{\mathcal{F}_{nc}},$$

i.e. $A_{nc}(f)$ and $A^*_{nc}(f)$ satisfy the relations of section 3. This means we can define stochastic integrals with respect to $A_{nc}$ and $A^*_{nc}$.

Define the following (non-atomic) projection valued measure $\xi$ on the direct sum Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \oplus H_{nc}$ consisting of a copy of $L^2(\mathbb{R})$ for the forward channel and a copy of $H_{nc}$ for the side channel by

$$\xi(I):\ L^2(\mathbb{R}) \oplus H_{nc} \to L^2(\mathbb{R}) \oplus H_{nc}:\ g \oplus \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \mapsto g\chi_I \oplus \begin{pmatrix} f_1\chi_I \\ f_2\chi_I \end{pmatrix},$$

for all Borel subsets $I$ of $\mathbb{R}$. Here $\chi_I$ denotes the indicator function of the set $I$. Define $\xi$-martingales by

$$m^f:\ \mathbb{R}_+ \to \mathcal{H}:\ t \mapsto m^f_t := \chi_{[0,t]} \oplus \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$
$$m^s:\ \mathbb{R}_+ \to \mathcal{H}:\ t \mapsto m^s_t := 0 \oplus \begin{pmatrix} \chi_{[0,t]} \\ 0 \end{pmatrix}.$$

The measure $\langle\langle m^s, m^s\rangle\rangle$ is then given by $\langle\langle m^s, m^s\rangle\rangle\big([0,t]\big) = \langle m^s_t, m^s_t\rangle_{nc} = (2n + 1 + 2a)t$. For $A_{nc}\big(\frac{m^s_t}{\sqrt{2n+1+2a}}\big)$ and $A^*_{nc}\big(\frac{m^s_t}{\sqrt{2n+1+2a}}\big)$ we introduce the shorthand notation $A_s(t)$ and $A^*_s(t)$, respectively. Note that for stochastic integrals with respect to $A_s(t)$ and $A^*_s(t)$ we find the Hudson-Parthasarathy Itô table. We denote $A_0(m^s_t)$ and $A^*_0(m^s_t)$ more compactly by $A_0(t)$ and $A^*_0(t)$. The following lemma enables the definition of stochastic integrals with respect to $A_0(t)$ and $A^*_0(t)$.

**Lemma 7.3:** *Let $n \in \mathbb{R}$ and $c \in \mathbb{C}$ such that $n(n+1) = |c|^2$. Then for all $t \geq 0$ we can write $A_0(t)$ and $A^*_0(t)$ as the following linear combinations of $A_s(t)$ and $A^*_s(t)$*

$$A_0(t) = \frac{n + c}{\sqrt{2n+1+2a}}A^*_s(t) + \frac{n+1+c}{\sqrt{2n+1+2a}}A_s(t),$$
$$A^*_0(t) = \frac{n + \overline{c}}{\sqrt{2n+1+2a}}A_s(t) + \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}}A^*_s(t),$$

*where $a$ is the real part of $c$.*

*Proof.* From Defintion 7.2 and $J_{nc} = J_0 Q_{nc}$ it follows that for all $f \in H$

$$A_0(f) = \frac{1}{2}\big(B_{nc}(f) + iB_{nc}(J_0 f)\big) = \frac{1}{2}\big(B_{nc}(f) + iB_{nc}(J_{nc}Q^{-1}_{nc} f)\big)$$
$$= \frac{1}{2}\big(A^*_{nc}(f) + A_{nc}(f) - A^*_{nc}(Q^{-1}_{nc} f) + A_{nc}(Q^{-1}_{nc} f)\big).$$

Using $J_{nc} = J_0 Q_{nc}$ and

$$Q_{nc}^{-1} = \begin{pmatrix} \frac{1+4b^2}{2n+1+2a} & -2b \\ -2b & 2n+1+2a \end{pmatrix},$$

we find for the $\xi$-martingale $m_t$

$$Q_{nc}^{-1} m_t = \begin{pmatrix} \frac{1+4b^2}{2n+1+2a}\chi_{[0,t]} \\ -2b\chi_{[0,t]} \end{pmatrix} = \frac{1}{2n+1+2a}\begin{pmatrix} \chi_{[0,t]} \\ 0 \end{pmatrix} - \frac{2b}{2n+1+2a} J_{nc} \begin{pmatrix} \chi_{[0,t]} \\ 0 \end{pmatrix}$$
$$= \Big(\frac{1}{2n+1+2a} - \frac{2b}{2n+1+2a} J_{nc}\Big) m_t.$$

From Definition 7.2 and equation (7.6) we see that $A_{nc}(J_{nc}f) = -iA_{nc}(f)$ and $A_{nc}^*(J_{nc}f) = iA_{nc}^*(f)$ for all $f \in H$. Therefore it follows that

$$A_0(t) = \frac{1}{2}\Bigg(A_{nc}^*(m_t) + A_{nc}(m_t) - \Big(\frac{1}{2n+1+2a} - \frac{2bi}{2n+1+2a}\Big)A_{nc}^*(m_t) +$$
$$\Big(\frac{1}{2n+1+2a} + \frac{2bi}{2n+1+2a}\Big)A_{nc}(m_t)\Bigg) =$$
$$\frac{n+c}{2n+1+2a}A_{nc}^*(m_t) + \frac{n+1+c}{2n+1+2a}A_{nc}(m_t) =$$
$$\frac{n+c}{\sqrt{2n+1+2a}}A_s^*(t) + \frac{n+1+c}{\sqrt{2n+1+2a}}A_s(t).$$

□

Clearly, we now define for all stochastically integrable processes $L_t$ stochastic integrals $L_t dA_0(t)$ and $L_t dA_0^*(t)$ by $L_t \frac{n+c}{\sqrt{2n+1+2a}} dA_s^*(t) + L_t \frac{n+1+c}{\sqrt{2n+1+2a}} dA_s(t)$ and $L_t \frac{n+\overline{c}}{\sqrt{2n+1+2a}} dA_s(t) + L_t \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}} dA_s^*(t)$, respectively. Using the Hudson-Parthasarathy Itô table it easily follows that the calculus of these stochastic integrals is given by the *squeezed noise Itô table* [20], [22]:

| $dM_1 \backslash dM_2$ | $dA_0^*(t)$ | $dA_0(t)$ |
|---|---|---|
| $dA_0^*(t)$ | $\overline{c}dt$ | $ndt$ |
| $dA_0(t)$ | $(n+1)dt$ | $cdt$ |

We are now in a position to explain the construction of $\breve{T}_t^{nc}$ in the dilation diagram (7.5). The free evolution of the side channel is again given by the unitary group $S_t$, the second quantization of the left shift $s(t)$ on $H_{nc}$, i.e.

$$s(t)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1(\cdot + t) \\ f_2(\cdot + t) \end{pmatrix}, \qquad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H_{nc}.$$

In the Heisenberg picture the free evolution on $\mathcal{W}_{nc}$ is then given by $X \mapsto S_t^* X S_t$. The system $\mathcal{B}$ and field together form a closed system, thus their joint evolution is given by a one-parameter group $\hat{U}_t$ of unitaries, leading to a Heisenberg picture evolution $\hat{T}_t^{nc} := \mathrm{Ad}[\hat{U}_t]$ on $\mathcal{B}\otimes\mathcal{W}^f\otimes\mathcal{W}^s_{nc}$. The group $\hat{U}_t$ is a perturbation of the free evolution. As in the vacuum case of section 2, we let this perturbation be given by the cocycle of unitaries $U_t := (S_{-t}\otimes S_{-t})\hat{U}_t$. The stochastic differential equation (2.4) that was satisfied by the cocycle $U_t$ when the side channel was still in the vacuum state is now changed. The quantum noise of equation (3.3) takes the form

$$d\beta_t = -i\big(V_f dA_f(t) - V_f^* dA_f(t) + V_s dA_0^*(t) - V_s^* dA_0(t)\big), \qquad \beta_0 = 0,$$

in the squeezed noise representation. If the field is in the vacuum state the operators $A_0(t)$ and $A_s(t)$ coincide. $A_0(t)$ should be interpreted as the annihilation operator of a photon in the side channel and $A_s(t)$ should be interpreted as the annihilation operator of a squeezed excitation of in the side channel, i.e. a quasiparticle consisting out of many photons. Using Lemma 7.3 we find

$$id\beta_t = V_f dA_f^*(t) - V_f^* dA_f(t) + \Big(\frac{n+1+\overline{c}}{\sqrt{2n+1+2a}}V_s - \frac{n+c}{\sqrt{2n+1+2a}}V_s^*\Big)dA_s^*(t) - \Big(\frac{n+1+c}{\sqrt{2n+1+2a}}V_s^* - \frac{n+\overline{c}}{\sqrt{2n+1+2a}}V_s\Big)dA_s(t).$$

Define

$$V_{nc} := \frac{n+1+\overline{c}}{\sqrt{2n+1+2a}}V_s - \frac{n+c}{\sqrt{2n+1+2a}}V_s^*, \tag{7.7}$$

then the quantum stochastic differential equation for the cocycle $U_t$ is given by

$$dU_t = \{V_f dA_f^*(t) - V_f^* dA_f(t) + V_{nc} dA_s^*(t) - V_{nc}^* dA_s(t) - \frac{1}{2}(V_{nc}^*V_{nc} + V_f^*V_f)dt\}U_t,$$
$$U_0 = \mathbf{1}. \tag{7.8}$$

In a similar way as in section 3 this leads to the Lindblad operator for the semigroup $T_t^{nc} = \exp(tL_{nc})$:

$$L_{nc}(X) = V_f^* X V_f - \frac{1}{2}\{V_f^*V_f, X\} + V_{nc}^* X V_{nc} - \frac{1}{2}\{V_{nc}^*V_{nc}, X\}, \qquad X \in \mathcal{B}.$$

# 8 Control with squeezing

Note that the operator $V_{nc}$ of equation (7.7) for strongly squeezed fields, i.e. $n$ and $c$ are big, is very close to being skew-selfadjoint. Therefore for strongly squeezed fields the dilation is very close to being essentially commutative. In this section we exploit this idea and control the skew-selfadjoint part of $V_{nc}$.

Write again $V = V_R + iV_I$ with $V_R$ and $V_I$ the selfadjoint operators of equation (6.1). We will again use for $X \in \{R, I\}$ and $\sigma \in \{f, s\}$ the notation $V_X^\sigma := \kappa_\sigma V_X$. Furthermore we introduce:

$$W_R := \frac{V_R^s}{\sqrt{2n+1+2a}} \quad \text{and} \quad W_I := \frac{V_I^s + i(n+c)V_s^* - i(n+\overline{c})V_s}{\sqrt{2n+1+2a}},$$

i.e. $V_{nc} = W_R + iW_I$ with $W_R$ and $W_I$ selfadjoint. Defining $Y_R^\sigma(t) := i\big(A_\sigma^*(t) - A_\sigma(t)\big)$ and $Y_I^\sigma(t) := A_\sigma^*(t) + A_\sigma(t)$, $\sigma \in \{f, s\}$ equation (7.8), i.e. the laser is off, becomes

$$dU_t = \Big\{iV_I^f dY_I^f - iV_R^f dY_R^f + iW_I dY_I^s(t) - iW_R dY_R^s(t) - \frac{1}{2}(V_f^* V_f + V_{nc}^* V_{nc})dt\Big\}U_t,$$
$$U_0 = \mathbf{1}.$$

Using a homodyne detection scheme we can observe the quadratures $X_\phi(t) := e^{-i\phi}A_0(t) + e^{i\phi}A_0(t)$ for $\phi \in [0, 2\pi)$. With the help of Lemma 7.3 this can be written as

$$X_\phi(t) = \frac{e^{-i\phi}(n + c) + e^{i\phi}(n + 1 + \overline{c})}{\sqrt{2n + 1 + 2a}}A_s^*(t) + \frac{e^{-i\phi}(n + 1 + c) + e^{i\phi}(n + \overline{c})}{\sqrt{2n + 1 + 2a}}A_s(t).$$

For simplicity we assume that $c$ is real, i.e. $c = a$. Note that the variance of $X_0$ has increased due to the squeezing, while the variance of $X_{\frac{\pi}{2}}$ has decreased. Therefore we choose to observe $Y_t := X_0(t) = \sqrt{2n + 1 + 2a}Y_I^s(t)$.

The Belavkin equation for observing $Y_t$ when the laser is still off, follows from equation (4.3)

$$d\rho_\bullet^t = L_{nc}(\rho_\bullet^t)dt + \frac{i[W_I, \rho_\bullet^t] + \{W_R, \rho_\bullet^t\} - 2\mathrm{Tr}\big(\rho_\bullet^t W_R\big)\rho_\bullet^t}{\sqrt{2n + 1 + 2a}}\Big(dY_t - 2\mathrm{Tr}\big(\rho_\bullet^t V_R^s\big)dt\Big). \quad (8.1)$$

Note that the observed process $Y_t$ is a drift, represented by the term $2\mathrm{Tr}\big(\rho_\bullet^t V_R^s\big)dt$ plus an amplified Wiener process, i.e. amplified up to a variance of $(2n + 1 + 2a)t$. Through the drift term we gain information on the state of the two-level system. However, for strong squeezing, i.e. $n$ and $a$ big, this information gets lost in the noise of the amplified Wiener process. In the limit for squeezing to infinity, the measurement scheme is again non-informative, just as in the essentially commutative case.

We run a control scheme as in section 6 only now based on the observation of $Y_t$. We correct with the control unitary given by

$$U_c^\tau = \exp\Big(-i\frac{\Delta(\tau)W_I}{\sqrt{2n + 1 + 2a}}\Big),$$

where $\Delta(\tau) := Y_\tau - Y_0$. Note that for $c$ real, i.e. $c = a$ we have

$$W_I = \frac{i\kappa_s\sqrt{2n + 1 + 2a}}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \sqrt{2n + 1 + 2a}V_I^s,$$

i.e. we can realise this control unitary by applying a laser pulse determined by $h(t) = \frac{\kappa_s\Delta(\tau)}{2\kappa_f}\delta_\tau(t)$ for $0 \leq t < 2\tau$. The control unitary satisfies the following quantum stochastic differential equation

$$dU_c^\tau = \{-iV_I^s dY_\tau - \frac{2n + 1 + 2a}{2}{V_I^s}^2 d\tau\}U_c^\tau = U_c^\tau\{-iV_I^s dY_\tau - \frac{2n + 1 + 2a}{2}{V_I^s}^2 d\tau\},$$
$$U_c^0 = \mathbf{1}.$$

The state after control is again given by $\rho^\tau_\bullet := U^\tau_c \tilde{\rho}^\tau_\bullet U^{\tau *}_c$ where $\tilde{\rho}^\tau_\bullet$ is given by the Belavkin equation (8.1). We use the notation below Theorem 3.2 with $Z_1 = U^\tau_c$, $Z_2 = \tilde{\rho}^\tau_\bullet$ and $Z_3 = U^{\tau *}_c$. For infinitesimal $\tau$ evaluated at $\tau = 0$, this leads to equation (5.4), i.e.

$$d\rho^\tau_\bullet\Big|_{\tau=0} = \big([1] + [2] + [3] + [12] + [13] + [23] + [123]\big)\Big|_{\tau=0}.$$

Again $[123] = 0$ and further $([1] + [3] + [13])|_{\tau=0} = L_{W_I}(\rho^0)d\tau + i[\rho^0, V^s_I]dY_0$. Furthermore we have

$$[2]\Big|_{\tau=0} = L_{nc}(\rho^0)d\tau + \Big(i[V^s_I, \rho^0] + \frac{\{V^s_R, \rho^0\} - 2\mathrm{Tr}\big(\rho^0 V^s_R\big)\rho^0}{2n+1+2a}\Big)\Big(dY_0 - 2\mathrm{Tr}(\rho^0 V^s_R)d\tau\Big),$$

$$\big([12] + [23]\big)\Big|_{\tau=0} = -2L_{W_I}(\rho^0)d\tau - i\big[W_I, \{W_R, \rho^0\}\big]d\tau + 2\mathrm{Tr}\big(\rho^0 W_R\big)i[W_I, \rho^0]d\tau.$$

A calculation shows that

$$L_{nc}(\rho^0) - L_{W_I}(\rho^0) - i\big[W_I, \{W_R, \rho^0\}\big] = L_{V_f}(\rho^0) + L_{W_R}(\rho^0) - \frac{i}{2}[W_I W_R + W_R W_I, \rho^0].$$

Since $W_I W_R + W_R W_I = 0$ for real $c$ and $\mathrm{Tr}\big(\rho^0 W_R\big)[W_I, \rho^0] = \mathrm{Tr}(\rho^0 V^s_R)[V^s_I, \rho^0]$ , we get

$$d\rho^\tau_\bullet\Big|_{\tau=0} = L_{V_f}(\rho^0)d\tau + L_{W_R}(\rho^0)d\tau + \Big(\frac{\{V^s_R, \rho^0\} - 2\mathrm{Tr}\big(\rho^0 V^s_R\big)\rho^0}{2n+1+2a}\Big)\Big(dY_0 - 2\mathrm{Tr}(\rho^0 V^s_R)d\tau\Big).$$

Since we repeat the control every $\tau$ time units with $\tau$ very small, i.e. we take $\tau$ infinitesimal, this leads to the following stochastic time evolution for the density matrix of the two-level atom

$$d\rho^t_\bullet = L_{V_f}(\rho^t_\bullet)dt + \frac{L_{V^s_R}(\rho^t_\bullet)}{2n+1+2a}dt + \Big(\frac{\{V^s_R, \rho^t_\bullet\} - 2\mathrm{Tr}\big(\rho^t_\bullet V^s_R\big)\rho^t_\bullet}{2n+1+2a}\Big)\Big(dY_t - 2\mathrm{Tr}(\rho^t_\bullet V^s_R)dt\Big).$$

The first term on the right hand side is again due to the fact that we did not measure and correct the forward channel. It is harmless, since we can take $\kappa_f$ arbitrarily small. The other two terms converge to 0 when squeezing goes to infinity. Therefore, in the limit, this control scheme restores quantum information.

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