Stochastic Schrödinger equations

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Abstract

A derivation of stochastic Schrödinger equations is given using quantum filtering theory. We study an open system in contact with its environment, the electromagnetic field. Continuous observation of the field yields information on the system: it is possible to keep track in real time of the best estimate of the system's quantum state given the observations made. This estimate satisfies a stochastic Schrödinger equation, which can be derived from the quantum stochastic differential equation for the interaction picture evolution of system and field together. Throughout the paper we focus on the basic example of resonance fluorescence.

1 Introduction

It has long been recognized that continuous time measurements can not be described by the standard projection postulate of quantum mechanics. In the late 60's, beginning 70's, Davies developed a theory for continuous time measurement [13] culminating in his book [14]. His mathematical work became known to the quantum optics community through the paper with Srinivas on photon counting [31].

The late 80's brought renewed interest to the theory of continuous time measurement. For instance the waiting time distribution of fluorescence photons of a two-level atom driven by a laser was obtained by associating a continuous evolution to the atom in between photon detections and jumps at the moments a photon is detected [11]. In this way every record of photon detection times determines a trajectory in the state space of the atom. Averaging over all possible detection records leads to the well-known description of the dissipative evolution of the atom by a master equation. Advantage of the trajectory approach is the fact that an initially pure state will remain pure along the whole trajectory. This allows for the use of state vectors instead of density matrices, significantly speeding up computer simulations [28], [12], [16], [9].

Infinitesimally, the quantum trajectories are solutions of a stochastic differential equation with the measurement process as the noise term. The change in the state is given by the sum of two terms: a deterministic one proportional with dt and a stochastic one proportional to the number of detected photons dN_t in the interval dt. For other schemes such as homodyne detection the corresponding stochastic differential equation is obtained as the diffusive limit of photon counting where the jumps in the state space decrease in size but become increasingly frequent [2], [9], [35]. In this limit the stochastic term in the differential equation is replaced by a process with continuous paths.

The stochastic Schrödinger equations obtained in this way had been postulated before by Gisin [17], [18], [15], in an attempt to generalize the customary unitary evolution in quantum mechanics. The stochastic terms are seen as randomness originating from the measurement process. However, in this approach the correspondence between the different quantum state diffusion equations and the measurements that can be performed is not emphasized.

Another approach originated from the development of quantum stochastic calculus [20], [29], generalizing the classical Itô table to quantum noises represented by creation and annihilation operators (see Section 6). Barchielli saw the relevance of this new calculus for quantum optics [1]. Indeed, in the Markovian approximation the interaction between a quantum system and the electromagnetic field is governed by a unitary solution of a quantum stochastic differential equation in the sense of [20].

Belavkin was the first to see the connection between quantum measurement theory and classical filtering theory [22], in which one estimates a signal or system process when observing a function of the signal in the presence of noise. This is done by deriving the filtering equation which is a stochastic differential equation for the expectation value of the system process conditioned on outcomes of the observation process. Belavkin extended the filtering theory [6], [5] to allow for the quantum noises of [20]. Stochastic Schrödinger equations turn out to be examples of the quantum filtering or Belavkin equation [4], [7].

Aim of this paper is to give an elementary presentation of quantum filtering theory. We construct the expectation of an observable conditioned on outcomes of a given measurement process. The differential form of this conditional expectation is the stochastic Schrödinger equation associated with the given measurement. At the heart of the derivation lies the Itô table of quantum stochastic calculus enabling a fast computation of the equation. The procedure is summarized in a small recipe in Section 7.

To illustrate the theory we consequently focus on the basic example of resonance fluorescence of a two-level atom for which we consider photon counting and homodyne detection measurement schemes. The stochastic Schrödinger equations for these examples are derived in two ways, once via the usual approach using quantum trajectories and a diffusive limit, and once using quantum filtering theory. In this way we hope to emphasize how conceptually different both methods are.

This paper is organised as follows. Sections 2 and 3 serve as an introduction to the guiding example of this paper: resonance fluorescence of a two-level atom driven by a laser. In Section 2 we put the photon counting description of resonance fluorescence by Davies [8], [11], [10] into the form of a stochastic differential equation driven by the counting process. In Section 3 we discuss the homodyne detection scheme as a diffusive limit of the photon counting measurement, arriving at a stochastic differential equation driven by a diffusion process. The equations of Sections 2 and 3 will be rederived later in a more general way using quantum filtering theory.

In Section 4 we introduce the concept of conditional expectation in quantum mechanics by first illustrating it in some simple, motivating examples. Section 5 describes the dissipative evolution of the open system within the Markov approximation. The joint evolution of the system and its environment, the quantized electromagnetic field, is given by unitaries satisfying a quantum stochastic differential equation. Given a measurement of some field observables it is shown how to condition the state of the system on outcomes of the measurement using the construction of Section 4. Section 6 is a short review of quantum stochastic calculus and its applications to open systems. Sections 5 and 6 describe dilation theory and quantum stochastic calculus in a nutshell.

Section 7 contains the derivation of the quantum filtering equation, the stochastic differential equation for the conditional expectation. This equation is the stochastic Schrödinger equation for the given measurement. This part ends with a recipe for computing stochastic Schrödinger equations for a large class of quantum systems and measurements. The end of the article connects to Sections 2 and 3 by showing how the recipe works in our main example.

2 The Davies process

We consider a two-level atom in interaction with the quantized electromagnetic field. The state of the atom is described by a 2×2-density matrix ρ, i.e. $\rho \geq 0$, and $\mathrm{Tr}\rho = 1$. Atom and field together perform a unitary, thus reversible evolution, but by taking a partial trace over the electromagnetic field we are left with an irreversible, dissipative evolution of the atom alone. In the so called Markov limit it is given by a norm continuous semigroup $\{T_t\}_{t\geq 0}$ of completely positive maps. A central example discussed in this paper is resonance fluorescence. Here the atom is driven by a laser on the *forward* channel, while in the *side* channel a photon counting measurement is performed. For the time being we will suppress the oscillations of the laser for reasons of simplicity. In this case the Lindblad generator of T_t, or Liouvillian L is given by (cf. [9]):

$$\frac{d}{dt}\Big|_{t=0} T_t(\rho) = L(\rho) = -i[H,\rho] + i\frac{\Omega}{2}[V+V^*,\rho] - \frac{1}{2}\{V^*V,\rho\} + V\rho V^*, \text{ where } V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad (2.1)$$

$H := \frac{\omega_0}{2}\sigma_z$ is the Hamiltonian of the atom, and Ω is the *Rabi frequency*.
The master equation (2.1) can be *unravelled* in many ways depending on what photon detection measurement is performed. By unravelling the master equation we mean writing L as the sum $\mathcal{L}+\mathcal{J}$, where $\mathcal{J}$ represents the instantaneous state change taking place when detecting a photon, and $\mathcal{L}$ describes the smooth state variation in between these instants. The unravelling for photon counting in the side channel is given by [9]

$$\mathcal{L}(\rho) = -i[H,\rho] + i\frac{\Omega}{2}[V+V^*,\rho] - \frac{1}{2}\{V^*V,\rho\} + (1-|\kappa_s|^2)V\rho V^* \text{ and } \mathcal{J}(\rho) = |\kappa_s|^2 V\rho V^*,$$

with $|\kappa_s|^2$ the decay rate into the side channel.
An outcome of the measurement over an arbitrary finite time interval $[0,t)$ is the set of times $\{t_1, t_2, \ldots, t_k\}$ at which photons are detected in the side channel of the field. The number of detected photons can be arbitrary, thus the space of outcomes is

$$\Omega\left([0,t)\right) := \bigcup_{n=0}^{\infty} \Omega_n\left([0,t)\right) = \bigcup_{n=0}^{\infty} \{\sigma \subset [0,t);\ |\sigma| = n\}$$

also called the *Guichardet space* [19]. In order to describe the probability distribution of the outcomes we need to make $\Omega\left([0,t)\right)$ into a measure space. Let us consider the space of n-tuples $[0,t)^n$ with its Borel σ-algebra and the measure $\frac{1}{n!}\lambda_n$ where λ_n is the Lebesgue measure. Then the map

$$j_n : [0,t)^n \ni (t_1, \ldots, t_n) \to \{t_1, \ldots, t_n\} \in \Omega_n\left([0,t)\right)$$

induces the σ-algebra $\Sigma_n([0,t))$ and the measure μ_n on $\Omega_n([0,t))$. We define now the measure μ on $\Omega([0,t))$ such that $\mu(\{\emptyset\}) = 1$ and $\mu = \mu_n$ on $\Omega_n([0,t))$. We will abbreviate $\Omega\big([0,t)\big)$ and $\Sigma\big([0,t)\big)$ to Ω^t and Σ^t, respectively.
Davies was the first to show [14] (see also [9], [8]) that the unnormalized state of the 2-level atom at time t with initial state ρ, and conditioned on the outcome of the experiment being in a set $E \in \Sigma^t$ is given by:

$$\mathcal{M}^t[E](\rho) = \int_E W_t(\omega)(\rho)d\mu(\omega),$$

where for $\omega = \{t_1, \ldots, t_k\} \in \Omega^t$ with $0 \leq t_1 \leq \ldots \leq t_k < t$ we have

$$W_t(\omega)(\rho) := \exp\big((t-t_k)\mathcal{L}\big)\mathcal{J} \ldots \mathcal{J} \exp\big((t_2 - t_1)\mathcal{L}\big)\mathcal{J}\exp\big(t_1\mathcal{L}\big)(\rho).$$

Furthermore, $\mathbb{P}^t_\rho[E] := \mathrm{Tr}(\mathcal{M}^t[E](\rho))$ is the probability that the event E occurs if the initial state is ρ. The family of prababilty measures $\{\mathbb{P}^t_\rho\}_{t\geq 0}$ is consistent, i.e. $\mathbb{P}^{t+s}_\rho[E] = \mathbb{P}^t_\rho[E]$ for all $E \in \Sigma^t$, $s \geq 0$, see [8], hence by Kolmogorov's extension theorem it extends to a single probability measure $\mathbb{P}_\rho$ on the σ-algebra Σ^∞, of the set Ω^∞.
On the measure space $(\Omega^\infty, \Sigma^\infty, \mathbb{P}_\rho)$ we define the following random variables:

$$N_t : \ \Omega^\infty \to \mathbb{N} : \ \omega \mapsto |\omega \cap [0,t)|,$$

counting the number of photons detected in the side channel up to time t. The counting process $\{N_t\}_{t\geq 0}$ has differential $dN_t := N_{t+dt} - N_t$ satisfying $dN_t(\omega) = 1$ if $t \in \omega$ and $dN_t(\omega) = 0$ otherwise. Therefore we have the following Itô rules: $dN_t dN_t = dN_t$ and $dN_t dt = 0$, (cf. [2]).
To emphasise the fact that the evolution of the 2-level atom is stochastic, we will regard the normalized density matrix as a random variable $\{\rho^t_\bullet\}_{t\geq 0}$ with values in the 2×2-density matrices defined as follows:

$$\rho^t_\bullet : \ \Omega^\infty \to M_2 : \ \omega \mapsto \rho^t_\omega := \frac{W_t\big(\omega \cap [0,t)\big)(\rho)}{\mathrm{Tr}\Big(W_t\big(\omega \cap [0,t)\big)(\rho)\Big)}. \tag{2.2}$$

The processes N_t and $\rho^t_\bullet$ are related through the stochastic differential equation $d\rho^t_\bullet = \alpha_t dt + \beta_t dN_t$. Following [2] we will now determine the processes α_t and β_t by differentiating (2.2). If $t \in \omega$ then $dN_t(\omega) = 1$, i.e. the differential dt is negligible compared to $dN_t = 1$, therefore:

$$\beta_t(\omega) = \rho^{t+dt}_\omega - \rho^t_\omega = \frac{\mathcal{J}(\rho^t_\omega)}{\mathrm{Tr}\big(\mathcal{J}(\rho^t_\omega)\big)} - \rho^t_\omega. \tag{2.3}$$

On the other hand, if $t \notin \omega$ then $dN_t(\omega) = 0$, i.e. dN_t is negligible compared to dt. Therefore it is only the dt term that contributes:

$$\begin{aligned}\alpha_t(\omega) &= \frac{d}{ds}\Big|_{s=t} \frac{\exp\big((s-t)\mathcal{L}\big)(\rho^t_\omega)}{\mathrm{Tr}\Big(\exp\big((s-t)\mathcal{L}\big)(\rho^t_\omega)\Big)} = \\ \mathcal{L}(\rho^t_\omega) &- \frac{\rho^t_\omega}{\mathrm{Tr}(\rho^t_\omega)^2}\mathrm{Tr}\big(\mathcal{L}(\rho^t_\omega)\big) = \mathcal{L}(\rho^t_\omega) + \mathrm{Tr}\big(\mathcal{J}(\rho^t_\omega)\big)\rho^t_\omega,\end{aligned} \tag{2.4}$$

where we used that $\mathrm{Tr}\big(\mathcal{L}(\rho^t_\omega)\big) = -\mathrm{Tr}\big(\mathcal{J}(\rho^t_\omega)\big)$, as a consequence of the fact that $\mathrm{Tr}\big(L(\sigma)\big) = 0$ for all density matrices σ. Substituting (2.3) and (2.4) into $d\rho^t_\bullet = \alpha_t dt + \beta_t dN_t$ we get the following *stochastic Schrödinger equation* for the state evolution of the 2-level atom if we are counting photons in the side channel (cf. [2], [10]):

$$d\rho^t_\bullet = L(\rho^t_\bullet)dt + \Big(\frac{\mathcal{J}(\rho^t_\bullet)}{\mathrm{Tr}\big(\mathcal{J}(\rho^t_\bullet)\big)} - \rho^t_\bullet\Big)\Big(dN_t - \mathrm{Tr}\big(\mathcal{J}(\rho^t_\bullet)\big)dt\Big). \tag{2.5}$$

The differential $dM_t := dN_t - \mathrm{Tr}\big(\mathcal{J}(\rho_\bullet^t)\big)dt$ and the initial condition $M_0 = 0$ define an important process M_t called the *innovating martingale*, discussed in more detail in Section 7.

3 Homodyne detection

We change the experimental setup described in the previous section by introducing a *local oscillator*, i.e. a one mode oscillator in a coherent state given by the normalised vector in $l^2(\mathbb{N})$

$$\psi(\alpha_t) := \exp\big(\frac{-|\alpha_t|^2}{2}\big)(1, \alpha_t, \frac{\alpha_t^2}{\sqrt{2}}, \frac{\alpha_t^3}{\sqrt{6}}, \ldots), \tag{3.1}$$

for a certain $\alpha_t \in \mathbb{C}$. We take $\alpha_t = \frac{w_t}{\varepsilon}$, where w_t is a complex number with modulus $|w_t| = 1$, and $\varepsilon > 0$. The number ε is inversely proportional to the intensity of the oscillator. Later on we will let the intensity go to infinity, i.e. $\varepsilon \to 0$. The phase ϕ_t of the oscillator is represented by $w_t = \exp(i\phi_t)$, with $\phi_t = \phi_0 + \omega_{lo}t$, where ω_{lo} is the frequency of the oscillator.
The local oscillator is coupled to a channel in the electromagnetic field, the local oscillator beam. The field is initially in the vacuum state. The local oscillator and the field are coupled in such a way that every time a photon is detected in the beam, a jump on the local oscillator occurs, given by the operation

$$\mathcal{J}_{lo}(\rho) = A_{lo}\rho A_{lo}^*, \tag{3.2}$$

where A_{lo} is the annihilation operator corresponding to the mode of the local oscillator. The coherent state $\psi(\alpha_t)$ is an eigenstate of the jump operator A_{lo} at eigenvalue α_t.
Now we are ready to discuss the homodyne detection scheme. Instead of directly counting photons in the side channel we first mix the side channel with the local oscillator beam with the help of a fifty-fifty beam splitter. In one of the emerging beams a photon counting measurement is performed. A detected photon can come from the atom through the side channel or from the local oscillator via the local oscillator beam. Therefore the jump operator on states σ of the atom and the oscillator together, is the sum of the respective jump operators:

$$\mathcal{J}_{a\otimes lo}(\sigma) = (\kappa_s V \otimes I + I \otimes A_{lo})\sigma(\overline{\kappa}_s V^* \otimes I + I \otimes A_{lo}^*).$$

An initial product state $\rho \otimes |\psi(\alpha_t)\rangle\langle\psi(\alpha_t)|$ of the 2-level atom and the local oscillator will remain a product after the jump since $\psi(\alpha_t)$ is an eigenvector of the annihilation operator. Tracing out the local oscillator yields the following jump operation for the atom in the homodyne setup:

$$\mathcal{J}_a(\rho) = \mathrm{Tr}_{lo}\Big(\mathcal{J}_{a\otimes lo}\big(\rho \otimes |\psi(\alpha_t)\rangle\langle\psi(\alpha_t)|\big)\Big) = \big(\kappa_s V + \frac{w_t}{\varepsilon}\big)\rho\big(\overline{\kappa}_s V^* + \frac{\overline{w}_t}{\varepsilon}\big).$$

In the same way as in Section 2, we can derive the following stochastic Schrödinger equation for the state evolution of the two-level atom when counting photons after mixing the side channel and the local oscillator beam [2] [10]:

$$d\rho_\bullet^t = L(\rho_\bullet^t)dt + \frac{1}{\varepsilon}\Big(\frac{\mathcal{J}_a(\rho_\bullet^t)}{\mathrm{Tr}\big(\mathcal{J}_a(\rho_\bullet^t)\big)} - \rho_\bullet^t\Big)\varepsilon\Big(dN_t - \mathrm{Tr}\big(\mathcal{J}_a(\rho_\bullet^t)\big)dt\Big), \tag{3.3}$$

where the extra ε's are introduced for future convenience. We will again use the abbreviation: $dM_t^a = dN_t - \mathrm{Tr}\big(\mathcal{J}_a(\rho_\bullet^t)\big)dt$ for the innovating martingale (see Section 7). In the homodyne detection scheme the intensity of the local oscillator beam is taken extremely large, i.e. we are interested in the limit $\varepsilon \to 0$ [2], [9], [35]. Then the number of detected photons becomes very large and it makes sense to scale and center N_t, obtaining in this way the process with differential $dW_t^\varepsilon := \varepsilon dN_t - dt/\varepsilon$ and $W_0^\varepsilon = 0$. We find the following Itô rules for dW_t^ε:

$$\begin{aligned} dW_t^\varepsilon dW_t^\varepsilon &= \big(\varepsilon dN_t - \frac{1}{\varepsilon}dt\big)\big(\varepsilon dN_t - \frac{1}{\varepsilon}dt\big) = \varepsilon^2 dN_t = \varepsilon dW_t^\varepsilon + dt, \\ dW_t^\varepsilon dt &= 0. \end{aligned}$$

In the limit $\varepsilon \to 0$ this becomes $dW_t dW_t = dt$ and $dW_t dt = 0$, i.e. the process $W_t := \lim_{\varepsilon\to 0} W_t^\varepsilon$ is a diffusion. It is actually this scaled and centered process that is being observed and not the individual photon counts N_t, see [9]. We pass now to the evaluation of the limit of (3.3):

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\frac{\mathcal{J}_a(\rho_\bullet^t)}{\mathrm{Tr}(\mathcal{J}_a(\rho_\bullet^t))} - \rho_\bullet^t\Big) = w_t\overline{\kappa}_s\rho_\bullet^t V^* + \overline{w}_t\kappa_s V\rho_\bullet^t - \mathrm{Tr}(w_t\overline{\kappa}_s\rho_\bullet^t V^* + \overline{w}_t\kappa_s V\rho_\bullet^t)\rho_\bullet^t.$$

This leads to the following stochastic Schrödinger equation for the homodyne detection scheme [2], [10], [35]

$$d\rho_\bullet^t = L(\rho_\bullet^t)dt + \big(w_t\overline{\kappa}_s\rho_\bullet^t V^* + \overline{w}_t\kappa_s V\rho_\bullet^t - \mathrm{Tr}(w_t\overline{\kappa}_s\rho_\bullet^t V^* + \overline{w}_t\kappa_s V\rho_\bullet^t)\rho_\bullet^t\ \big)\, dM_t^{hd}, \tag{3.4}$$

for all states $\rho \in M_2$, where

$$dM_t^{hd} := dW_t - \mathrm{Tr}(w_t\overline{\kappa}_s\rho_\bullet^t V^* + \overline{w}_t\kappa_s V\rho_\bullet^t)dt. \tag{3.5}$$

Let $a_s(t)$ and $a_b(t)$ denote the annihilation operators for the side channel and the local oscillator beam, respectively. They satisfy the canonical commutation relations

$$[a_i(t), a_j^*(r)] = \delta_{i,j}\delta(t-r), \quad i,j \in \{s,b\}.$$

Smearing with a quadratically integrable function f gives

$$A_i(f) = \int f(t)a_i(t)dt, \quad i \in \{s,b\}.$$

By definition, the stochastic process $\{N_t\}_{t\geq 0}$ counting the number of detected photons has the same law as the the number operator $\Lambda(t)$ up to time t for the beam on which the measurement is performed. Formally we can write

$$\Lambda(t) = \int_0^t \big(a_s^*(r)\otimes I + I\otimes a_b^*(r)\big)\big(a_s(r)\otimes I + I\otimes a_b(r)\big)dr.$$

The oscillator beam is at time t in the coherent state $\psi\left(\frac{f_t}{\varepsilon}\right)$, where $f_t \in L^2(\mathbb{R})$ is the function $r \mapsto w_r\chi_{[0,t]}(r)$. Since the state of the local oscillator beam is an eigenvector of the annihilation operator $a_b(r)$

$$a_b(r)\psi\left(\frac{f_t}{\varepsilon}\right) = \frac{w_r}{\varepsilon}\psi\left(\frac{f_t}{\varepsilon}\right),$$

we find

$$\begin{aligned} \varepsilon\Lambda(t) - \frac{t}{\varepsilon} &= \varepsilon\Lambda_s(t)\otimes I + \varepsilon\int_0^t \big(\frac{w_r}{\varepsilon}a_s^*(r) + \frac{\overline{w}_r}{\varepsilon}a_s(r)\big)\otimes I + \frac{|w_r|^2}{\varepsilon^2}dr - \frac{t}{\varepsilon} \\ &= \varepsilon\Lambda_s(t)\otimes I + \big(A_s^*(f_t) + A_s(f_t)\big)\otimes I. \end{aligned}$$

The operator $X_\phi(t) := A_s^*(f_t) + A_s(f_t)$ is called a *field quadrature*. We conclude that in the limit $\varepsilon \to 0$ the homodyne detection is a setup for continuous time measurement of the field quadratures $X_\phi(t)$ of the side channel. (cf. [9]).

4 Conditional expectations

In the remainder of this article we will derive the equations (2.5) and (3.4) in a different way. We will develop a general way to derive Belavkin equations (or stochastic Schrödinger equations). The counting experiment and the homodyne detection experiment, described in the previous sections, serve as examples in this general framework. The method we describe here closely follows Belavkin's original paper on quantum filtering theory [6]. Our approach differs in its construction of the conditional expectation, which is the topic of this section.
Let us remind the concept of conditional expectation from probability theory. Let $(\Omega, \Sigma, \mathbb{P})$ be a probability space describing the "world" and $\Sigma' \subset \Sigma$ a σ-algebra of events to which "we have access". A random variable f on $(\Omega, \Sigma, \mathbb{P})$ with $\mathbb{E}(|f|) < \infty$ can be projected to its conditional expectation $\mathcal{E}(f)$ which is measurable with respect to Σ' and satisfies

$$\int_E f \mathrm{d}\mathbb{P} = \int_E \mathcal{E}(f) \mathrm{d}\mathbb{P}$$

for all events E in Σ'. Our information about the state of that part of the world to which we have access, can be summarized in a probability distribution $\mathbb{Q}$ on Σ'. Then the predicted expectation of f given this information is $\int_\Omega \mathcal{E}(f)\mathrm{d}\mathbb{Q}$. We will extend this now to quantum systems and measurements.

The guiding example is that of an n level atom described by the algebra $\mathcal{B} := M_n$ undergoing a transformation given by a completely positive unit preserving map $T : \mathcal{B} \to \mathcal{B}$ with the following Kraus decomposition $T(X) = \sum_{i\in\Omega} V_i^* X V_i$. The elements of Ω can be seen as the possible measurement outcomes. For any initial state ρ of $\mathcal{B}$ and measurement result $i \in \Omega$, the state after the measurement is given by

$$\rho_i = V_i \rho V_i^* / \mathrm{Tr}(V_i \rho V_i^*),$$

and the probability distribution of the outcomes is $p = \sum_{i\in\Omega} p_i \delta_i$ where δ_i is the atomic measure at i, and $p_i = \mathrm{Tr}(V_i\rho V_i^*)$, which without loss of generality can be assumed to be strictly positive. We represent the measurement by an instrument, that is the completely positive map with the following action on states

$$\mathcal{M} : \ M_n^* \to M_n^* \otimes \ell^1(\Omega) : \ \rho \mapsto \sum_{i\in\Omega} \rho_i \otimes p_i \delta_i. \tag{4.1}$$

Let $X \in \mathcal{B}$ be an observable of the system. Its expectation after the measurement, given that the result $i \in \Omega$ has been obtained is $\mathrm{Tr}(\rho_i X)$. The function

$$\mathcal{E}(X) : \ \Omega \to \mathbb{C} : \ i \mapsto \mathrm{Tr}(\rho_i X)$$

is the *conditional expectation* of X onto $\ell^\infty(\Omega)$. If $q = \sum q_i \delta_i$ is a probability distribution on Ω then $\sum q_i \mathcal{E}(X)(i)$ represents the expectation of X on a statistical ensemble for which the distribution of the measurement outcomes is q. We extend the conditional expectation to the linear map

$$\mathcal{E} : \ \mathcal{B} \otimes \ell^\infty(\Omega) \to \ell^\infty(\Omega) \subset \mathcal{B} \otimes \ell^\infty(\Omega)$$

such that for any element $A : i \mapsto A_i$ in $\mathcal{B} \otimes \ell^\infty(\Omega) \cong \ell^\infty(\Omega \to \mathcal{B})$ we have

$$\mathcal{E}(A) : \ i \mapsto \mathrm{Tr}(\rho_i A_i).$$

This map has the following obvious properties: it is idempotent and has norm one. Moreover, it is the unique linear map with these properties preserving the state $\mathcal{M}(\rho)$ on $\mathcal{B} \otimes \ell^\infty(\Omega)$. For this reason we will call $\mathcal{E}$, the conditional expectation with respect to $\mathcal{M}(\rho)$. Its dual can be seen as an extension of probability distribitions $q \in \ell^1(\Omega)$ to states on $\mathcal{B} \otimes \ell^\infty(\Omega)$

$$\mathcal{E}^* : \ q \mapsto \sum_{i\in\Omega} \rho_i \otimes q_i \delta_i.$$

Thus while the measurement (4.1) provides a state $\mathcal{M}(\rho)$ on $\mathcal{B} \otimes \ell^\infty(\Omega)$, the conditional expectation with respect to $\mathcal{M}(\rho)$ extends probability distributions $q \in \ell^1(\Omega)$ of outcomes, to states on $\mathcal{B} \otimes \ell^\infty(\Omega)$, and in particular on $\mathcal{B}$ which represents the state after the measurement given the outcomes distribution q.

With this example in mind we pass to a more general setup which will be needed in deriving the stochastic Schrödinger equations. Let $\mathcal{A}$ be a unital *-algebra of bounded operators on a Hilbert space $\mathbb{H}$ whose selfadjoint elements represent the observables of a quantum system. It is natural from the physical point of view to assume that $\mathcal{A}$ is strongly closed, i.e. if $\{A_n\}_{n\geq 0}$ is a sequence of operators in $\mathcal{A}$ such that $\|A_n\psi\| \to \|A\psi\|$ for any vector ψ in $\mathbb{H}$ and a fixed bounded operator A, then $A \in \mathcal{A}$. From the mathematical point of view this leads to the rich theory of von Neumann algebras inspired initially by quantum mechanics, but can as well be seen as the generalization of probability theory to the non-commutative world of quantum mechanics. Indeed, the building blocks of quantum systems are matrix algebras, while probability spaces can be encoded into their *commutative* algebra of bounded random variables $L^\infty(\Omega, \Sigma, \mathbb{P})$ which appeared already in the example above. A state is described by a density matrix in the first case or a probability distribution in the second, in general it is a positive normalized linear functional $\psi : \mathcal{A} \to \mathbb{C}$ which is continuous with respect to the weak*-topology, the natural topolgy on a von Neumann algebra seen as the dual of a Banach space [21].

Definition 4.1: Let $\mathcal{B}$ be a von Neumann subalgebra of a von Neumann algebra $\mathcal{A}$ of operators on a (separable) Hilbert space $\mathbb{H}$. A *conditional expectation* of $\mathcal{A}$ onto $\mathcal{B}$ is a linear surjective map $\mathcal{E} : \ \mathcal{A} \to \mathcal{B}$, such that:

1. $\mathcal{E}^2 = \mathcal{E}$ ($\mathcal{E}$ is idempotent),
2. $\forall_{A\in\mathcal{A}} : \ \|\mathcal{E}(A)\| \leq \|A\|$ ($\mathcal{E}$ is normcontractive).

In [33] it has been shown that the conditions 1 and 2 are equivalent to $\mathcal{E}$ being an identity preserving, completely positive map, and satisfying the *module property*

$$\mathcal{E}(B_1AB_2) = B_1\mathcal{E}(A)B_2, \qquad \text{for all } B_1, B_2 \in \mathcal{B}, \text{ and } A \in \mathcal{A}, \tag{4.2}$$

generalizing a similar property of conditional expectations in classical probability theory (cf. [34]).

In analogy to the classical case we are particularly interested in the conditional expectation which leaves a given state ρ on $\mathcal{A}$ invariant, i.e. $\rho \circ \mathcal{E} = \rho$. However such a map does not always exist, but if it exists then it is unique [32] and will be denoted $\mathcal{E}_\rho$. Using $\mathcal{E}_\rho$ we can extend states σ on $\mathcal{B}$ to states $\sigma \circ \mathcal{E}_\rho$ of $\mathcal{A}$ which should be interpreted as the updated state of $\mathcal{A}$ after receiving the information (for instance through a measurement) that the subsystem $\mathcal{B}$ is in the state σ (cf. [25]).
In the remainder of this section we will construct the conditional expectation $\mathcal{E}_\rho$ from a von Neumann algebra $\mathcal{A}$ onto its *center* $\mathcal{C} := \{C \in \mathcal{A}; \ AC = CA \text{ for all } A \in \mathcal{A}\}$ leaving a given state ρ on $\mathcal{A}$

invariant. The center $\mathcal{C}$ is a commutative von Neumann algebra and is therefore isomorphic to some $L^\infty(\Omega, \Sigma, \mathbb{P})$. In our guiding example the center of $\mathcal{B} \otimes \ell^\infty(\Omega)$ is $\ell^\infty(\Omega)$. Later on (see section 6) this role will be played by the commutative algebra of the observed process with Ω the space of all paths of measurement records.

Theorem 4.2: *There exists a unique conditional expectation $\mathcal{E}_\rho : \mathcal{A} \to \mathcal{C}$ which leaves the state ρ on $\mathcal{A}$ invariant.*

Proof. The proof is based on the central decomposition of $\mathcal{A}$ [21]. In our guiding example, $\mathcal{B} \otimes \ell^\infty(\Omega)$ is isomorphic to $\oplus_{i\in\Omega}\mathcal{B}_i$ where the $\mathcal{B}_i$'s are copies of $\mathcal{B}$. In general we can identify the center $\mathcal{C}$ with some $L^\infty(\Omega, \Sigma, \mathbb{P})$ where $\mathbb{P}$ corresponds to the restriction of ρ to $\mathcal{C}$. We will ignore for simplicity all issues related with measurability in the following constructions. The Hilbert space $\mathbb{H}$ has a direct integral representation $\mathbb{H} = \int_\Omega^\oplus \mathbb{H}_\omega \mathbb{P}(d\omega)$ in the sense that there exists a family of Hilbert spaces $\{\mathbb{H}_\omega\}_{\omega\in\Omega}$ and for any $\psi \in \mathbb{H}$ there exists a map $\omega \mapsto \psi_\omega \in \mathbb{H}_\omega$ such that

$$\langle \psi, \phi \rangle = \int_\Omega \langle \psi_\omega, \phi_\omega \rangle \mathbb{P}(d\omega).$$

The von Neumann algebra $\mathcal{A}$ has a *central decomposition* $\mathcal{A} = \int_\Omega^\oplus \mathcal{A}_\omega \mathbb{P}(d\omega)$ in the sense that there exists a family $\{\mathcal{A}_\omega\}_{\omega\in\Omega}$ of von Neumann algebras with trivial center, or factors, and for any $A \in \mathcal{A}$ there is a map $\omega \mapsto A_\omega \in \mathcal{A}_\omega$ such that $(A\psi)_\omega = A_\omega \psi_\omega$ for all $\psi \in \mathbb{H}$ and $\mathbb{P}$-almost all $\omega \in \Omega$. The state ρ on $\mathcal{A}$ has a decomposition in states ρ_ω on $\mathcal{A}_\omega$ such that for any $A \in \mathcal{A}$ its expectation is obtained by integrating with respect to $\mathbb{P}$ the expectations of its components A_ω:

$$\rho(A) = \int_\Omega \rho_\omega(A_\omega)\mathbb{P}(d\omega). \tag{4.3}$$

The map $\mathcal{E}_\rho : \ \mathcal{A} \to \mathcal{C}$ defined by

$$\mathcal{E}_\rho(A) : \ \omega \mapsto \rho_\omega(A_\omega)$$

for all $A \in \mathcal{A}$ is the desired conditional expectation. One can easily verify that this map is linear, identity preserving, completely positive (as a positive map onto a commutative von Neumann algebra), and has the module property. Thus, $\mathcal{E}_\rho$ is a conditional expectation and leaves the state ρ invariant by 4.3. Uniqueness follows from [32]. □

It is helpful to think of the state ρ and an arbitrary operator A as maps $\rho_\bullet : \ \omega \mapsto \rho_\omega$, and respectively $A_\bullet : \ \omega \mapsto A_\omega$. The conditional expectation $\mathcal{E}_\rho(A)$ is the function $\rho_\bullet(A_\bullet) : \ \omega \mapsto \rho_\omega(A_\omega)$.

5 The dilation

Let $\mathcal{B}$ be the observable algebra of a given quantum system on the Hilbert space $\mathbb{H}$. In the case of resonance fluorescence $\mathcal{B}$ will be all 2×2 matrices M_2, the algebra of observables for the 2-level atom. The irreversible evolution of the system in the Heisenberg picture is given by the norm continuous semigroup $\{T_t\}_{t\geq 0}$ of completely positive maps $T_t: \ \mathcal{B} \to \mathcal{B}$. By Lindblad's theorem [26] we have $T_t = \exp(tL)$ where the generator $L: \mathcal{B} \to \mathcal{B}$ has the following action

$$L(X) = i[H, X] + \sum_{j=1}^{k} V_j^* X V_j - \frac{1}{2}\{V_j^* V_j, X\}, \tag{5.1}$$

where H and the V_j's are fixed elements of $\mathcal{B}$, H being selfadjoint.
We can see the irreversible evolution as stemming from a *reversible* evolution of the system $\mathcal{B}$ coupled to an environment, which will be the electromagnetic field. We model a channel in the field by the bosonic or symmetric Fock space over the Hilbert space $L^2(\mathbb{R})$ of square integrable wave functions on the real line, i.e.

$$\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathbb{R})^{\otimes_s n}.$$

The algebra generated by the field observables on $\mathcal{F}$ contains all bounded operators and we denote it by $\mathcal{W}$. For the dilation we will need k independent copies of this algebra $\mathcal{W}^{\otimes k}$.
The free evolution of the field is given by the unitary group S_t, the second quantization of the *left* shift $s(t)$ on $L^2(\mathbb{R})$, i.e. $s(t): f \mapsto f(\cdot + t)$. In the Heisenberg picture the evolution on $\mathcal{W}$ is

$$W \mapsto S_t^* W S_t := \mathrm{Ad}[S_t](W).$$

The atom and field together form a closed quantum system, thus their joint evolution is given by a one-parameter group $\{\hat{T}_t\}_{t\in\mathbb{R}}$ of $*$-automorphisms on $\mathcal{B} \otimes \mathcal{W}^{\otimes k}$:

$$X \mapsto \hat{U}_t^* X \hat{U}_t := \mathrm{Ad}[\hat{U}_t](X).$$

The group $\hat{U}_t$ is a perturbation of the free evolution without interaction. We describe this perturbation by the family of unitaries $U_t := S_{-t}^{\otimes k} \hat{U}_t$ for all $t \in \mathbb{R}$ satisfying the *cocycle* identity

$$U_{t+s} = S_{-s}^{\otimes k} U_t S_s^{\otimes k} U_s, \qquad \text{for all } t, s \in \mathbb{R}.$$

The direct connection between the reduced evolution of the atom given by (5.1) and the cocycle U_t is one of the important results of quantum stochastic calculus [20] which makes the object of Section 6. For the moment we only mention that in the Markov limit, U_t is the solution of the stochastic differential equation [20], [29], [27]

$$dU_t = \{V_j dA_j^*(t) - V_j^* dA_j(t) - (iH + \frac{1}{2} V_j^* V_j)dt\}U_t, \qquad U_0 = \mathbf{1}, \tag{5.2}$$

where the repeated index j is meant to be summed over. The quantum Markov dilation can be summarized by the following diagram (see [23], [24]):

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t} & \mathcal{B} \\ {\scriptstyle \mathrm{Id}\otimes\mathbf{1}^{\otimes k}}\Big\downarrow & & \Big\uparrow{\scriptstyle \mathrm{Id}\otimes\phi^{\otimes k}} \\ \mathcal{B}\otimes\mathcal{W}^{\otimes k} & \xrightarrow{\hat{T}_t} & \mathcal{B}\otimes\mathcal{W}^{\otimes k} \end{array} \tag{5.3}$$

i.e. for all $X \in \mathcal{B}: \ T_t(X) = \big(\mathrm{Id} \otimes \phi^{\otimes k}\big)\big(\hat{T}_t(X \otimes \mathbf{1}^{\otimes k})\big)$, where ϕ is the vacuum state on $\mathcal{W}$, and $\mathbf{1}$ is the identity operator in $\mathcal{W}$. Any dilation of the semigroup T_t with Bose fields is unitarily equivalent with the above one under certain minimality requirements. The diagram can also be read in the

Schrödinger picture if we reverse the arrows: start with a state ρ of the system $\mathcal{B}$ in the upper right hand corner, then this state undergoes the following sequence of maps:

$$\rho \mapsto \rho \otimes \phi^{\otimes k} \mapsto (\rho \otimes \phi^{\otimes k}) \circ \hat{T}_t = \hat{T}_{t*}(\rho \otimes \phi^{\otimes k}) \mapsto \mathrm{Tr}_{\mathcal{F}^{\otimes k}}\big(\hat{T}_{t*}(\rho \otimes \phi^{\otimes k})\big).$$

This means that at $t = 0$, the atom in state ρ is coupled to the k channels in the vacuum state, and after t seconds of unitary evolution we take the partial trace taken over the k channels.

We would now like to introduce the measurement process. It turns out that this can be best described in the *interaction picture*, where we let the shift part of $\hat{U}_t = S_t^{\otimes k} U_t$ act on the observables while the cocycle part acts on the states:

$$\rho^t(X) := \rho \otimes \phi^{\otimes k}(U_t^* X U_t) \tag{5.4}$$

for all $X \in \mathcal{B} \otimes \mathcal{W}^{\otimes k}$. It is well known that for the Bose field for arbitrary time t we can split the noise algebra as a tensor product

$$\mathcal{W} = \mathcal{W}_{0)} \otimes \mathcal{W}_{[0,t)} \otimes \mathcal{W}_{[t}$$

with each term being the algebra generated by those fields over test functions with support in the corresponding subspace of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = L^2\big((-\infty, 0)\big) \oplus L^2\big([0,t)\big) \oplus L^2\big([t, \infty)\big).$$

Such a continous tensor product structure is called a *filtration* and it is essential in the development of quantum stochastic calculus reviewed in Section 6. The observables which we measure in an arbitrary time interval $[0,t)$ form a commuting family of selfadjoint operators $\{Y_s\}_{0 \le s \le t}$ whose spectral projections belong to the middle part of the tensor product $\mathcal{W}_{[0,t)}$. In the Davies process $Y_s = \Lambda(s)$, i.e. the number operator up to time s, while in the homodyne case $Y_s = X_\phi(s)$. Notice that the part $\mathcal{W}_{0)}$ will not play any significant role as it corresponds to "what happened before we started our experiment".

Let $\mathcal{C}_t$ be the commutative von Neumann generated by the observed process up to time t, $\{Y_s\}_{0 \le s \le t}$ $(t \ge 0)$, seen as a subalgebra of $\mathcal{B} \otimes \mathcal{W}^{\otimes k}$. By a theorem on von Neumann algebras, $\mathcal{C}_t$ is equal to the double commutant of the observed process up to time t: $\mathcal{C}_t = \{Y_s;\ 0 \le s \le t\}''$, with the *commutant* $\mathcal{S}'$ of a subset $\mathcal{S}$ of $\mathcal{B} \otimes \mathcal{W}^{\otimes k}$ being defined by $\mathcal{S}' := \{X \in \mathcal{B} \otimes \mathcal{W}^{\otimes k};\ XS = SX\ \forall S \in \mathcal{S}\}$. The algebras $\{\mathcal{C}_t\}_{t \ge 0}$ form a growing family, that is $\mathcal{C}_s \subset \mathcal{C}_t$ for all $s \le t$. Thus we can define the inductive limit $\mathcal{C}_\infty := \lim_{t \to \infty} \mathcal{C}_t$, which is the smallest von Neumann algebra containing all $\mathcal{C}_t$. On the other hand for each $t \ge 0$ we have a state on $\mathcal{C}_t$ given by the restriction of the state ρ^t of the whole system defined by (5.4). We will show now that that the states ρ^t for different times "agree with each other".

Theorem 5.1: *On the commutative algebra $\mathcal{C}_\infty$ there exists a unique state ρ^∞ which coincides with ρ^t when restricted to $\mathcal{C}_t \subset \mathcal{C}_\infty$, for all $t \ge 0$. In particular there exists a measure space $(\Omega, \Sigma, \mathbb{P}_\rho)$ such that $(\mathcal{C}_\infty, \rho^\infty)$ is isomorphic with $L^\infty(\Omega, \Sigma, \mathbb{P}_\rho)$ and a growing family $\{\Sigma_t\}_{t \ge 0}$ of σ-subalgebras of Σ such that $(\mathcal{C}_t, \rho^t) \cong L^\infty(\Omega, \Sigma_t, \mathbb{P}_\rho)$.*

Proof. In the following we will drop the extensive notation of tensoring identity operators when representing operators in $\mathcal{W}_{[s,t)}$ for all $s, t \in \mathbb{R}$. Let $X \in \mathcal{C}_s$, in particular $X \in \mathcal{W}_{[0,s)}^{\otimes k}$. By (5.2), $U_t \in \mathcal{B} \otimes \mathcal{W}_{[0,t)}^{\otimes k}$, because the coefficients of the stochastic differential equation lie in $\mathcal{B} \otimes \mathcal{W}_{[0,t)}^{\otimes k}$. This implies that $S_{-s}^{\otimes k} U_t S_s^{\otimes k} \in \mathcal{B} \otimes \mathcal{W}_{[s,t+s)}^{\otimes k}$. Using the tensor product structure of $\mathcal{W}^{\otimes k}$, we see that $\mathcal{W}_{[0,s)}^{\otimes k}$

and $\mathcal{B} \otimes \mathcal{W}^{\otimes k}_{[s,t+s)}$ commute, and in particular X commutes with $S^{\otimes k}_{-s} U_t S^{\otimes k}_s$. Then

$$\begin{aligned} \rho^{t+s}(X) &= \rho^0(U^*_{t+s} X U_{t+s}) = \rho^0(U^*_s (S^{\otimes k}_{-s} U_t S^{\otimes k}_s)^* X S^{\otimes k}_{-s} U_t S^{\otimes k}_s U_s) \\ &= \rho^0(U^*_s X U_s) = \rho^s(X). \end{aligned} \tag{5.5}$$

This implies that the limit state ρ^∞ on $\mathcal{C}_\infty$ with the desired properties exists, in analogy to the Kolmogorov extension theorem for probability measures. As seen in the previous section, $(\mathcal{C}_\infty, \rho^\infty)$ is isomorphic to $L^\infty(\Omega, \Sigma, \mathbb{P}_\rho)$ for some probability space $(\Omega, \Sigma, \mathbb{P}_\rho)$. The subalgebras $(\mathcal{C}_t, \rho^t)$ are isomorphic to $L^\infty(\Omega, \Sigma_t, \mathbb{P}_\rho)$ for some growing family $\{\Sigma_t\}_{t\geq 0}$ of σ-subalgebras of Σ. □

Remark. From spectral theory it follows that the measure space (Ω^t, Σ_t) coincides with the joint spectrum of $\{Y_s\}_{s\leq t}$, i.e. Ω^t is the set of all paths of the process up to time t. For the example of the counting process this means that Ω^t is the Guichardet space of the interval $[0, t)$, which is the set of all sets of instants representing a "click" of the photon counter, i.e. it is the set of all paths of the counting process.

We define now $\mathcal{A}_t := \mathcal{C}_t'$ for all $t \geq 0$, i.e. $\mathcal{A}_t$ is the commutant of $\mathcal{C}_t$, then $\mathcal{C}_t$ is the center of the von Neumann algebra $\mathcal{A}_t$. Notice that the observable algebra of the atom $\mathcal{B}$ is contained in $\mathcal{A}_t$. By Theorem 4.2 we can construct a family of conditional expectations $\{\mathcal{E}^t_{\rho^t} : \mathcal{A}_t \to \mathcal{C}_t\}_{t\geq 0}$. For each t, $\mathcal{E}^t_{\rho^t}$ depends on the state of the "world" at that moment ρ^t, keeping this in mind we will simply denote it by $\mathcal{E}^t$. An important property of $\mathcal{E}^t$ is that $\rho^\infty \circ \mathcal{E}^t = \rho^t \circ \mathcal{E}^t = \rho^t$, since the range of $\mathcal{E}^t$ is $\mathcal{C}_t$ and $\mathcal{E}^t$ leaves ρ^t invariant.

For an element $X \in \mathcal{A}_t$, $\mathcal{E}^t(X)$ is an element in $\mathcal{C}_t$, i.e. a function on Ω_t. Its value in a point $\omega \in \Omega_t$, i.e. an outcomes record up to time t, is the expectation value of X given the observed path ω after t time units. We will use the notation $\mathcal{E}^t(X) := \rho^t_\bullet(X_\bullet)$ defined in the end of Section 4 to emphasise the fact that this is a function on Ω_t. When restricted to $\mathcal{B} \otimes \mathcal{C}_t$ the conditional expectation is precisely of the type discussed in our guiding example in Section 4.
There exists no conditional expectation from $\mathcal{B} \otimes \mathcal{W}$ onto $\mathcal{C}_t$ since performing the measurement has *demolished* the information about observables that do not commute with the observed process [6]. We call $\mathcal{A}_t$ the algebra of observables that are *not demolished* [6] by observing the process $\{Y_s\}_{0\leq s\leq t}$. This means that performing the experiment and ignoring the outcomes gives the same time evolution on $\mathcal{A}_t$ as when no measurement was done.

From classical probability it follows that for all $t \geq 0$ there exists a unique conditional expectation $\mathbb{E}^t_\rho : \mathcal{C}_\infty \to \mathcal{C}_t$ that leaves the state ρ^∞ invariant, i.e. $\rho^\infty \circ \mathbb{E}^t_\rho = \rho^\infty$. These conditional expectations have the *tower property*, i.e. $\mathbb{E}^s_\rho \circ \mathbb{E}^t_\rho = \mathbb{E}^s_\rho$ for all $t \geq s \geq 0$, which is often very useful in calculations. $\mathbb{E}^0_\rho$ is the expectation with respect to $\mathbb{P}_\rho$, and will simply be denoted $\mathbb{E}_\rho$. Note that the tower property for $s = 0$ is exactly the invariance of the state $\rho^\infty (= \mathbb{E}_\rho)$.

6 Quantum stochastic calculus

In this section we briefly discuss the quantum stochastic calculus developed by Hudson and Parthasarathy [20]. For a detailed treatment of the subject we refer to [29] and [27]. Let $\mathcal{F}(\mathcal{H})$ denote the symmetric (or bosonic) Fock space over the one particle space $\mathcal{H} := \mathbb{C}^k \otimes L^2(\mathbb{R}_+) = L^2(\{1, 2, \dots k\} \times \mathbb{R}_+)$. The

space $\mathbb{C}^k$ describes the k channels we identified in the electromagnetic field. As in the previous section we denote the algebra of bounded operators on the one channel Fock space $\mathcal{F}(\mathbb{R}_+)$ by $\mathcal{W}$, and on the k channels $\mathcal{F}(\mathcal{H})$ by $\mathcal{W}^{\otimes k}$.
For every $f \in \mathcal{H}$ we define the *exponential vector* $e(f) \in \mathcal{F}(\mathcal{H})$ in the following way:

$$e(f) := 1 \oplus \bigoplus_{n=1}^{\infty} \frac{1}{\sqrt{n!}} f^{\otimes n},$$

which differs from the coherent vector by a normalization factor. The inner products of two exponential vectors $e(f)$ and $e(g)$ is $\langle e(f), e(g)\rangle = \exp(\langle f, g\rangle)$. Note that the span of all exponential vectors, denoted $\mathcal{D}$, forms a dense subspace of $\mathcal{F}(\mathcal{H})$. Let f_j be the j'th component of $f \in \mathcal{H}$ for $j = 1, 2, \ldots, k$. The annihilation operator $A_j(t)$, creation operator $A_j^*(t)$ and number operator $\Lambda_{ij}(t)$ are defined on the domain $\mathcal{D}$ by

$$\begin{aligned}
A_j(t)e(f) &= \langle \chi_{[0,t]},\, f_j\rangle e(f) = \int_0^t f_j(s)ds\; e(f) \\
\big\langle e(g),\, A_j^*(t)e(f)\big\rangle &= \langle g_j,\, \chi_{[0,t]}\rangle \big\langle e(g),\, e(f)\big\rangle = \int_0^t \overline{g}_j(s)ds\; \exp(\langle f, g\rangle) \\
\big\langle e(g),\, \Lambda_{ij}(t)e(f)\big\rangle &= \langle g_i,\, \chi_{[0,t]} f_j\rangle \big\langle e(g),\, e(f)\big\rangle = \int_0^t \overline{g}_i(s) f_j(s)ds\; \exp(\langle f, g\rangle).
\end{aligned}$$

The operator $\Lambda_{ii}(t)$ is the usual counting operator for the i'th channel. Let us write $L^2(\mathbb{R}^+)$ as direct sum $L^2([0,t]) \oplus L^2([t,\infty])$, then $\mathcal{F}(L^2(\mathbb{R}_+))$ is unitarily equivalent with $\mathcal{F}(L^2([0,t]) \otimes \mathcal{F}(L^2[t,\infty))$ through the identification $e(f) \cong e(f_{t]}) \otimes e(f_{[t})$, with $f_{t]} = f\chi_{[0,t]}$ and $f_{[t} = f\chi_{[t,\infty)}$. We will also use the notation $f_{[s,t]}$ for $f\chi_{[s,t]}$ and omit the tensor product signs between exponential vectors. The same procedure can be carried out for all the k channels.

Let M_t be one of the processes $A_j(t), A_j^*(t)$ or $\Lambda_{ij}(t)$. The following factorisability property [20], [29] makes the definition of stochastic integration against M_t possible

$$(M_t - M_s)e(f) = e(f_{s]})\big\{(M_t - M_s)e(f_{[s,t]})\big\}e(f_{[t}),$$

with $(M_t - M_s)e(f_{[s,t]}) \in \mathcal{F}\big(\mathbb{C}^k \otimes L^2([s,t])\big)$. We firstly define the stochastic integral for the so called *simple* operator processes with values in the atom and noise algebra $\mathcal{B} \otimes \mathcal{W}^{\otimes k}$ where $\mathcal{B} := M_n$

Definition 6.1: Let $\{L_s\}_{0 \le s \le t}$ be an adapted (i.e. $L_s \in \mathcal{B} \otimes \mathcal{W}_{s]}$ for all $0 \le s \le t$) simple process with respect to the partition $\{s_0 = 0, s_1, \ldots, s_p = t\}$ in the sense that $L_s = L_{s_j}$ whenever $s_j \le s < s_{j+1}$. Then the stochastic integral of L with respect to M on $\mathbb{C}^n \otimes \mathcal{D}$ is given by [20], [29]:

$$\int_0^t L_s dM_s\; fe(u) := \sum_{j=0}^{p-1} \big(L_{s_j} fe(u_{s_j]})\big)\big((M_{s_{j+1}} - M_{s_j})e(u_{[s_j,s_{j+1}]})\big)e(u_{[s_{j+1}}).$$

By the usual approximation by simple processes we can extend the definition of the stochastic integral to a large class of stochastically integrable processes [20], [29]. We simplify our notation by writing $dX_t = L_t dM_t$ for $X_t = X_0 + \int_0^t L_s dM_s$. Note that the definition of the stochastic integral implies that the increments dM_s lie in the future, i.e. $dM_s \in \mathcal{W}_{[s}$. Another consequence of the definition of the stochastic integral is that its expectation with respect to the vacuum state ϕ is always 0 due to the fact that the increments $dA_j, dA_j^*, d\Lambda_{ij}$ have zero expectation values in the vacuum. This will often simplify calculations of expectations, our strategy being that of trying to bring these increments to act on the vacuum state thus eliminating a large number of differentials.

The following theorem of Hudson and Parthasarathy extends the Itô rule of classical probability theory.

Theorem 6.2: ***(Quantum Itô rule [20], [29])*** *Let M_1 and M_2 be one of the processes A_j, A_j^* or Λ_{ij}. Then M_1M_2 is an adapted process satisfying the relation:*

$$dM_1M_2 = M_1dM_2 + M_2dM_1 + dM_1dM_2,$$

where dM_1dM_2 is given by the quantum Itô table:

$dM_1 \backslash dM_2$	dA_i^*	$d\Lambda_{ij}$	dA_i
dA_k^*	0	0	0
$d\Lambda_{kl}$	$\delta_{li}dA_k^*$	$\delta_{li}d\Lambda_{kj}$	0
dA_k	$\delta_{ki}dt$	$\delta_{ki}dA_j$	0

Notation. The quantum Itô rule will be used for calculating differentials of products of Itô integrals. Let $\{Z_i\}_{i=1,\ldots,p}$ be Itô integrals, then

$$d(Z_1Z_2\ldots Z_p) = \sum_{\substack{\nu\subset\{1,\ldots,p\}\\ \nu\neq\emptyset}} [\nu]$$

where the sum runs over all *non-empty* subsets of $\{1,\ldots,p\}$ and for any $\nu = \{i_1,\ldots i_k\}$, the term $[\nu]$ is the contribution to $d(Z_1Z_2\ldots Z_p)$ coming from differentiating only the terms with indices in the set $\{i_1,\ldots i_k\}$ and preserving the order of the factors in the product. For example the differential $d(Z_1Z_2Z_3)$ contains terms of the type $[2] = Z_1(dZ_2)Z_3$, $[13] = (dZ_1)Z_2(dZ_3)$, and $[123] = (dZ_1)(dZ_2)(dZ_3)$.

Let V_j for $j = 1,2,\ldots,k$, and H be operators in $\mathcal{B}$ with H is selfadjoint. Let S be a unitary operator on $\mathbb{C}^n \otimes l^2(\{1,2,\ldots,k\})$ with $S_{ij} = \langle i, Sj\rangle \in \mathcal{B}$ the "matrix elements" in the basis $\{|i>: i = 1,\ldots,k\}$ of $\mathbb{C}^k$. Then there exists a unique unitary solution for the following quantum stochastic differential equation [20], [29]

$$dU_t = \big\{V_jdA_j^*(t) + (S_{ij} - \delta_{ij})d\Lambda_{ij}(t) - V_i^*S_{ij}dA_j(t) - (iH + \frac{1}{2}V_j^*V_j)dt\big\}U_t, \qquad (6.1)$$

with initial condition $U_0 = \mathbf{1}$, where again repeated indices have been summed. Equation (5.2), providing the cocycle of unitaries perturbing the free evolution of the electromagnetic field is an example of such an equation. The terms $d\Lambda_{ij}$ in equation (6.1) describe direct scattering between the channels in the electromagnetic field [3]. We have omitted this effect for the sake of simplicity, i.e. we always take $S_{ij} = \delta_{ij}$.

We can now check the claim made in Section 5 that the dilation diagram 5.3 commutes. It is easy to see that following the lower part of the diagram defines a semigroup on $\mathcal{B}$. We have to show it is generated by L. For all $X \in \mathcal{B}$ we have

$$d\ \mathrm{Id} \otimes \phi^k\big(\hat{T}_t(X \otimes \mathbf{1}^{\otimes k})\big) = \mathrm{Id} \otimes \phi^k(d\ U_t^*X \otimes \mathbf{1}^{\otimes k}U_t).$$

Using the Itô rules we obtain

$$d\, U_t^* X \otimes \mathbf{1}^{\otimes k} U_t = (dU_t^*) X \otimes \mathbf{1}^{\otimes k} U_t + U_t^* X \otimes \mathbf{1}^{\otimes k} dU_t + (dU_t^*) X \otimes \mathbf{1}^{\otimes k} dU_t.$$

With the aid of the Itô table we can evaluate these terms. We are only interested in the dt-terms since the expectation with respect to the vacuum kills the other terms. Then we obtain: $d\,\mathrm{Id} \otimes \phi^k (U_t^* X \otimes \mathbf{1}^{\otimes k} U_t) = \mathrm{Id} \otimes \phi^k (U_t^* L(X) \otimes \mathbf{1}^{\otimes k} U_t) dt$, proving the claim.

Now we return to the example of resonance fluorescence. Suppose the laser is off, then we have spontaneous decay of the 2-level atom into the field which is in the vacuum state. For future convenience we already distinguish a *forward* and a *side* channel in the field, the Liouvillian is then given by

$$L(X) = i[H, X] + \sum_{\sigma = f,s} V_\sigma^* X V_\sigma - \frac{1}{2}\{V_\sigma^* V_\sigma, X\},$$

where

$$V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad V_f = \kappa_f V, \quad V_s = \kappa_s V, \quad |\kappa_f|^2 + |\kappa_s|^2 = 1,$$

with $|\kappa_f|^2$ and $|\kappa_s|^2$ the decay rates into the forward and side channel respectively.
The dilation of the quantum dynamical system $(M_2, \{T_t = \exp(tL)\}_{t \geq 0})$, is now given by the closed system $(M_2 \otimes \mathcal{W}_f \otimes \mathcal{W}_s, \{\hat{T}_t\}_{t \in \mathbb{R}})$ with unitary cocycle given by

$$dU_t^{sd} = \{V_f dA_f^*(t) - V_f^* dA_f(t) + V_s dA_s^*(t) - V_s^* dA_s(t) - (iH + \frac{1}{2} V^* V) dt\} U_t^{sd}, \quad U_0^{sd} = \mathbf{1},$$

where the superscript sd reminds us of the fact that the laser is off, i.e. we are considering spontaneous decay. We can summarize this in the following dilation diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t = \exp(tL)} & \mathcal{B} \\ {\scriptstyle \mathrm{Id} \otimes \mathbf{1} \otimes \mathbf{1}} \big\downarrow & & \big\uparrow {\scriptstyle \mathrm{Id} \otimes \phi \otimes \phi} \\ \mathcal{B} \otimes \mathcal{W}_f \otimes \mathcal{W}_s & \xrightarrow{\hat{T}_t^{sd} = \mathrm{Ad}[\hat{U}_t^{sd}]} & \mathcal{B} \otimes \mathcal{W}_f \otimes \mathcal{W}_s \end{array}$$

where $\hat{U}_t^{sd}$ is given by $S_t \otimes S_t U_t^{sd}$ for $t \geq 0$.
We change this setting by introducing a laser on the forward channel, i.e. the forward channel is now in a coherent state (see 3.1) $\gamma_h := \langle \psi(h), \cdot\, \psi(h) \rangle$ for some $h \in L^2(\mathbb{R}_+)$. This leads to the following dilation diagram

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t^h} & \mathcal{B} \\ {\scriptstyle \mathrm{Id} \otimes \mathbf{1} \otimes \mathbf{1}} \big\downarrow & & \big\uparrow {\scriptstyle \mathrm{Id} \otimes \gamma_h \otimes \phi} \\ \mathcal{B} \otimes \mathcal{W}_f \otimes \mathcal{W}_s & \xrightarrow{\hat{T}_t^{sd} = \mathrm{Ad}[\hat{U}_t^{sd}]} & \mathcal{B} \otimes \mathcal{W}_f \otimes \mathcal{W}_s \end{array} \tag{6.2}$$

i.e. the evolution on $\mathcal{B}$ has changed and it is in general *not* a semigroup. Denote by $W(h)$ the unitary *Weyl* or *displacement operator* defined on $\mathcal{D}$ by: $W(h)\psi(f) = \exp(-2i\mathrm{Im}\langle h, f \rangle)\psi(f + h)$. Note that $W(h)\phi = W(h)\psi(0) = \psi(h)$, so that we can write

$$T_t^h(X) = \mathrm{Id} \otimes \gamma_h \otimes \phi(U_t^{sd*} X \otimes \mathbf{1} \otimes \mathbf{1} U_t^{sd}) = \mathrm{Id} \otimes \phi \otimes \phi\big(W_f(h)^* U_t^{sd*} X \otimes \mathbf{1} \otimes \mathbf{1} U_t^{sd} W_f(h)\big) =$$
$$\mathrm{Id} \otimes \phi \otimes \phi\big(W_f(h_{t]})^* U_t^{sd*} X \otimes \mathbf{1} \otimes \mathbf{1} U_t^{sd} W_f(h_{t]})\big),$$

where $h_{t]} := h\chi_{[0,t]}$ and $W_f(h) := \mathbf{1} \otimes W(h) \otimes \mathbf{1}$. Defining $U_t := U_t^{sd} W_f(h_{t]})$, together with the stochastic differential equation for $W_f(h_{t]})$ [29]

$$dW_f(h_{t]}) = \{h(t)dA_f^*(t) - \overline{h}(t)dA_f(t) - \frac{1}{2}|h(t)|^2 dt\}W_f(h_{t]}), \qquad W_f(h_0) = \mathbf{1},$$

and the Itô rules leads to the following quantum stochastic differential equation for U_t:

$$\begin{aligned} dU_t = \ & \big\{(V_f + h(t))dA_f^*(t) - (V_f^* + \overline{h}(t))dA_f(t) \ + V_s dA_s^*(t) - V_s^* dA_s(t) \ - \\ & -\big(iH + \tfrac{1}{2}(|h(t)|^2 + V^*V + 2h(t)V_f^*)\big)dt \ \big\}U_t, \quad U_0 = \mathbf{1}. \end{aligned}$$

Define $\tilde{V}_f := V_f + h(t)$, $\tilde{V}_s := V_s$ and $\tilde{H} := H + i\frac{1}{2}(\overline{h}(t)V_f - h(t)V_f^*)$ then this reads

$$dU_t = \sum_{\sigma=f,s} \{\tilde{V}_\sigma dA_\sigma^*(t) - \tilde{V}_\sigma^* dA_\sigma - \frac{1}{2}(i\tilde{H} + \tilde{V}_\sigma^*\tilde{V}_\sigma)dt\}U_t, \quad U_0 = \mathbf{1}. \tag{6.3}$$

The time dependent generator of the dissipative evolution in the presence of the laser on the forward channel is

$$L(X) = i[\tilde{H}, X] + \sum_{\sigma=f,s} \tilde{V}_\sigma^* X \tilde{V}_\sigma - \frac{1}{2}\{\tilde{V}_\sigma^*\tilde{V}_\sigma, X\}. \tag{6.4}$$

Therefore the diagram for resonance fluorescence (6.2) is equivalent to

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{T_t^h} & \mathcal{B} \\ {\scriptstyle \mathrm{Id}\otimes\mathbf{1}\otimes\mathbf{1}}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{Id}\otimes\phi\otimes\phi} \\ \mathcal{B}\otimes\mathcal{W}_f\otimes\mathcal{W}_s & \xrightarrow{\hat{T}_t = \mathrm{Ad}[\hat{U}_t]} & \mathcal{B}\otimes W_f\otimes\mathcal{W}_s \end{array}$$

where $\hat{U}_t$ is given by $S_t \otimes S_t U_t$ for $t \geq 0$. For $h(t) = -i\Omega/\kappa_f$, we find the master equation for resonance fluorescence (2.1). From now on we will no longer suppress the oscillations of the laser, i.e. we take $h(t) = -i\exp(i\omega t)\Omega/\kappa_f$. Then we find

$$L(X) = i[H, X] - i\frac{\Omega}{2}[e^{-i\omega t}V + e^{i\omega t}V^*, X] - \frac{1}{2}\{V^*V, X\} + V^*XV,$$

note that the laser is resonant when $\omega = \omega_0$.

7 Belavkin's stochastic Schrödinger equations

Now we are ready to derive a stochastic differential equation for the process $\mathcal{E}^t(X)$. In the next section we will see that this equation leads to the stochastic Schrödinger equations (2.5) and (3.4), that we already encountered in Sections 2 and 3.

Definition 7.1: Let X be an element of $\mathcal{B} := M_n$. Define the process $\{M_t^X\}_{t\geq 0}$ in the algebra $\mathcal{C}_\infty \cong L^\infty(\Omega, \Sigma, \mathbb{P}_\rho)$, generated by the observed process $\{Y_t\}_{t\geq 0}$ (see Section 5) by

$$M_t^X := \mathcal{E}^t(X) - \mathcal{E}^0(X) - \int_0^t \mathcal{E}^r\big(L(X)\big)dr,$$

where $L: \mathcal{B} \to \mathcal{B}$ is the Liouvillian. In the following we suppress the superscript X in M_t^X to simplify our notation.

Note that from the above definition it is clear that M_t is an element of $\mathcal{C}_t$ for all $t \geq 0$. The following theorem first appeared (in a more general form and with a different proof) in [6] and is at the heart of quantum filtering theory. We prove it using the properties of conditional expectations. For simplicity we have restricted to observing a process in the field $\mathcal{W}^{\otimes k}$. The theory can be extended to processes that are in $\mathcal{B} \otimes \mathcal{W}^{\otimes k}$, transforming it into a more interesting filtering theory. For the stochastic Schrödinger equations arising in quantum optics our approach is general enough.

Theorem 7.2: *The process $\{M_t\}_{t\geq 0}$ of definition 7.1 is a martingale with respect to the filtration $\{\Sigma_t\}_{t\geq 0}$ of Ω and the measure $\mathbb{P}_\rho$, i.e. for all $t \geq s \geq 0$ we have: $\mathbb{E}_\rho^s(M_t) = M_s$.*

Proof. From the module property of the conditional expectation it follows that $\mathbb{E}_\rho^s(M_t) = M_s$ for $t \geq s \geq 0$ is equivalent to $\mathbb{E}_\rho^s(M_t - M_s) = 0$ for $t \geq s \geq 0$. This means we have to prove for all $t \geq s \geq 0$ and $E \in \Sigma_s$:

$$\int_E \mathbb{E}_\rho^s(M_t - M_s)(\omega)\mathbb{P}_\rho(d\omega) = 0,$$

which, by the tower property, is equivalent to

$$\int_E (M_t - M_s)(\omega)\mathbb{P}_\rho(d\omega) = 0, \tag{7.1}$$

i.e. $\mathbb{E}_\rho\big(\chi_E(M_t - M_s)\big) = 0$. Now using Definition 7.1 and again the module property of the conditional expectation we find, writing E also for the projection corresponding to χ_E

$$\begin{aligned}
\mathbb{E}_\rho\big(\chi_E(M_t - M_s)\big) &= \rho^\infty\Big(\mathcal{E}^t(X \otimes E) - \mathcal{E}^s(X \otimes E) - \int_s^t \mathcal{E}^r\big(L(X) \otimes E\big)dr\Big) \\
&= \rho^t(X \otimes E) - \rho^s(X \otimes E) - \int_s^t \rho^r\big(L(X) \otimes E\big)dr.
\end{aligned}$$

This means we have to prove: $d\rho^t(X \otimes E) - \rho^t\big(L(X) \otimes E\big)dt = 0$, for all $t \geq s$. Note that $\rho^t(X \otimes E) = \rho^0(U_t^* X \otimes EU_t) = \rho \otimes \phi^{\otimes k}(U_t^* X \otimes EU_t)$. Therefore $d\rho^t(X \otimes E) = \rho \otimes \phi^{\otimes k}\big(d(U_t^* X \otimes EU_t)\big)$. We will use the notation below Theorem 6.2 with $Z_1 = U_t^*$ and $Z_2 = X \otimes EU_t$. Using the quantum Itô table and the fact that only the dt terms survive after taking a vacuum expectation, we find:

$$\begin{aligned}
d\rho^0(U_t^* X \otimes EU_t) &= \rho^0\big([1]\big) + \rho^0\big([2]\big) + \rho^0\big([12]\big), \quad \text{where} \\
\rho^0\big([1]\big) + \rho^0\big([2]\big) &= \rho^0\big(U_t^*(i[H, X] \otimes E - \frac{1}{2}\{V_j^* V_j, X\} \otimes E)U_t\big)dt \\
\rho^0\big([12]\big) &= \rho^0\big(U_t^*(V_j^* X V_j) \otimes EU_t\big)dt.
\end{aligned}$$

This means $d\rho^t(X \otimes E) = \rho^t\big(L(X) \otimes E\big)dt$, for all $t \geq s$, proving the theorem. □

Note that in the proof of the above theorem we have used that the projection $E \in \mathcal{C}_s$ commutes with the increments $dA_j(s)$, $dA_j^*(s)$, ds and with the processes in front of the increments in equation (5.2), i.e. V_j, V_j^*, $V_j^* V_j$ and H. If the theory is extended to a more general filtering theory [6], then

these requirements become real restrictions on the process $\{Y_t\}_{t\geq 0}$. If they are satisfied the observed process $\{Y_t\}_{t\geq 0}$ is said to be *self non demolition* [6].
Definition 7.1 implies the following stochastic differential equation for the process $\mathcal{E}^t(X)$

$$d\mathcal{E}^t(X) = \mathcal{E}^t\big(L(X)\big)dt + dM_t, \tag{7.2}$$

called the *Belavkin equation*. The only thing that remains to be done is linking the increment dM_t to the increment of the observed process dY_t.
Let us assume that the observed process $\{Y_t\}_{t\geq 0}$ satisfies a quantum stochastic differential equation

$$dY_t = \alpha_j(t)dA_j^*(t) + \beta_{ij}(t)d\Lambda_{ij}(t) + \alpha_j^*(t)dA_j(t) + \delta(t)dt,$$

for some adapted stochastically integrable processes α_j, β_{ij}, and δ, such that $\alpha_j(t), \beta_{ij}(t), \delta(t) \in \mathcal{W}_{t]}^{\otimes k}$ for all $t \geq 0$, and $\beta_{ij}^* = \beta_{ji}$, $\delta = \delta^*$ since Y_t is selfadjoint. Furthermore, since the observed process $\{Y_t\}_{t\geq 0}$ is commutative, we have $[dY_t, Y_s] = 0$ for all $s \leq t$, which leads to

$$\begin{aligned}
&[\alpha_j(t), Y_s]dA_j^*(t) + [\beta_{ij}(t), Y_s]d\Lambda_{ij}(t) + [\alpha_j^*(t), Y_s]dA_j(t) + [\delta(t), Y_s]dt = 0 \quad \Rightarrow \\
&[\alpha_j(t), Y_s] = 0, \quad [\beta_{ij}(t), Y_s] = 0, \quad [\alpha_j^*(t), Y_s] = 0, \quad [\delta(t), Y_s] = 0,
\end{aligned}$$

i.e. $\alpha_j(t), \beta_{ij}(t), \alpha_j^*(t), \delta(t) \in \mathcal{A}_t$. This enables us to define a process $\tilde{Y}_t$ by

$$\begin{aligned}
d\tilde{Y}_t = &\Big(\alpha_j(t)dA_j^*(t) - \mathcal{E}^t\big(V_j^*\alpha_j(t)\big)dt\Big) + \Big(\beta_{ij}(t)d\Lambda_{ij}(t) - \mathcal{E}^t\big(V_i^*\beta_{ij}(t)V_j\big)dt\Big) + \\
&\Big(\alpha_j^*(t)dA_j(t) - \mathcal{E}^t\big(\alpha_j^*(t)V_j\big)dt\Big), \qquad \tilde{Y}_0 = 0,
\end{aligned} \tag{7.3}$$

i.e. we have the following splitting of Y_t:

$$Y_t = Y_0 + \tilde{Y}_t + \int_0^t \Big(\mathcal{E}^s\big(V_j^*\alpha_j(s)\big) + \mathcal{E}^s\big(V_i^*\beta_{ij}(s)V_j\big) + \mathcal{E}^s\big(\alpha_j^*(s)V_j\big) + \delta(s)\Big)ds, \tag{7.4}$$

which in view of the following theorem is the semi-martingale splitting of Y_t. The process $\tilde{Y}_t$ is called the *innovating martingale* of the observed process Y_t.

Theorem 7.3: *The process $\{\tilde{Y}_t\}_{t\geq 0}$ is a martingale with respect to the filtration $\{\Sigma_t\}_{t\geq 0}$ of Ω and the measure $\mathbb{P}_\rho$, i.e. for all $t \geq s \geq 0$ we have: $\mathbb{E}_\rho^s(\tilde{Y}_t) = \tilde{Y}_s$.*

Proof. We need to prove that for all $t \geq s \geq 0$: $\mathbb{E}_\rho^s(\tilde{Y}_t - \tilde{Y}_s) = 0$. This means we have to prove for all $t \geq s \geq 0$ and $E \in \Sigma_s$:

$$\begin{aligned}
&\int_E \mathbb{E}_\rho^s(\tilde{Y}_t - \tilde{Y}_s)(\omega)\mathbb{P}_\rho(d\omega) = 0 \iff \int_E (\tilde{Y}_t - \tilde{Y}_s)(\omega)\mathbb{P}_\rho(d\omega) = 0 \iff \\
&\mathbb{E}_\rho\left(Y_tE - Y_sE - \int_s^t \Big(\mathcal{E}^r\big(V_j^*\alpha_j(r)\big)E + \mathcal{E}^r\big(V_i^*\beta_{ij}(r)V_j\big)E + \mathcal{E}^r\big(\alpha_j^*(r)V_j\big)E + \delta(r)E\Big)dr\right) = 0 \iff \\
&\rho^t(Y_tE) - \rho^s(Y_sE) = \int_s^t \rho^r\Big(\mathcal{E}^r\big(V_j^*\alpha_j(r)\big)E + \mathcal{E}^r\big(V_i^*\beta_{ij}(r)V_j\big)E + \mathcal{E}^r\big(\alpha_j^*(r)V_j\big)E + \delta(r)E\Big)dr.
\end{aligned}$$

For $t = s$ this is okay, so it remains to be shown that for all $t \geq s \geq 0$ and $E \in \Sigma_s$:

$$\begin{aligned}
&d\rho^t(Y_tE) = \rho^t\Big(\mathcal{E}^t\big(V_j^*\alpha_j(t)\big)E + \mathcal{E}^t\big(V_i^*\beta_{ij}(t)V_j\big)E + \mathcal{E}^t\big(\alpha_j^*(t)V_j\big)E + \delta(t)E\Big)dt \iff \\
&d\rho^0(U_t^*Y_tEU_t) = \rho^t\Big(\mathcal{E}^t\big(V_j^*\alpha_j(t)\big)E + \mathcal{E}^t\big(V_i^*\beta_{ij}(t)V_j\big)E + \mathcal{E}^t\big(\alpha_j^*(t)V_j\big)E + \delta(t)E\Big)dt.
\end{aligned}$$

We define: $Z_1(t) := U_t^*$, $Z_2(t) := Y_t E$ and $Z_3(t) := U_t$ then we find, using the notation below Theorem 6.2: $d\rho^0(U_t^* Y_t E U_t) = \rho^0([1]+[2]+[3]+[12]+[13]+[23]+[123])$. Remember $\rho^0 = \rho \otimes \phi^{\otimes k}$, i.e. we are only interested in the dt terms, since the vacuum kills all other terms. The terms [1], [3] and [13] together make up the usual Lindblad term and since $L(\mathbf{1}) = 0$ we do not have to consider them. Furthermore, term [2] contributes $U_t^*\delta(t)EU_t dt$, term [12] contributes $U_t^* V_j^* \alpha_j(t) EU_t dt$, term [23] contributes $U_t^* \alpha_j^*(t) V_j E U_t dt$ and term [123] contributes $U_t^* V_i^* \beta_{ij}(t) V_j U_t dt$, therefore we get

$$
\begin{aligned}
&d\rho^0(U_t^* Y_t E U_t) = \rho^0\big(U_t^*\alpha_j^*(t)V_j E U_t + U_t^* V_i^* \beta_{ij}(t) V_j U_t + U_t^* V_j^* \alpha_j(t) E U_t + U_t^* \delta(t) E U_t\big)dt = \\
&\rho^t\big(\alpha_j^*(t) V_j E + V_i^* \beta_{ij}(t) V_j + V_j^* \alpha_j(t) E + \delta(t) E\big)dt = \\
&\rho^t\Big(\mathcal{E}^t\big(V_j^*\alpha_j(t)\big)E + \mathcal{E}^t\big(V_i^*\beta_{ij}(t)V_j\big)E + \mathcal{E}^t\big(\alpha_j^*(t)V_j\big)E + \delta(t)E\Big)dt,
\end{aligned}
$$

proving the theorem. □

Remark. In the probability literature an adapted process which can be written as the sum of a martingale and a finite variation process is called a semimartingale [30]. The Theorems 7.2 and 7.3 show that M_t and Y_t are semimartingales.

We now represent the martingale M_t from Definition 7.1 as an integral over the innovating martingale (cf. [22]) by

$$dM_t = \eta_t d\tilde{Y}_t \tag{7.5}$$

for some stochastically integrable process η_t, which together with equation (7.4) provides the link between dM_t and dY_t. We are left with the problem of determining η_t, which we will carry out in the next section for the examples of Section 2 and 3. Here we just give the recipe for finding η_t.

Recipe. Define for all integrable adapted processes b_t and c_t a process B_t in $\mathcal{C}_\infty$ by

$$dB_t = b_t d\tilde{Y}_t + c_t dt. \tag{7.6}$$

These processes form a dense subalgebra of $\mathcal{C}_\infty$. Now determine η_t from the fact that $\mathcal{E}^t$ leaves ρ^t invariant [6], i.e. for all B_t

$$\rho^t\big(\mathcal{E}^t(B_t X)\big) = \rho^t(B_t X).$$

From this it follows that for all B_t

$$d\rho^0\big(U_t^* B_t(\mathcal{E}^t(X) - X)U_t\big) = 0. \tag{7.7}$$

We evaluate the differential $d\big(U_t^* B_t(\mathcal{E}^t(X) - X)U_t\big)$ using the quantum Itô rules. Since $\rho^0 = \rho\otimes\phi^{\otimes k}$ we can restrict to the dt terms, since the others die on the vacuum. We will use the notation below Theorem 6.2 with $Z_1(t) = U_t^*$, $Z_2(t) = B_t, Z_3(t) = \mathcal{E}^t(X) - X$ and $Z_4(t) = U_t$. The following lemma simplifies the calculation considerably.

Lemma 7.4: *The sum of all terms in which Z_2 is not differentiated has zero expectation: $\rho^0([1] + [3] + [4] + [13] + [14] + [34] + [134]) = 0$.*

Proof. The dt terms of [3] are $U_t^* B_t \mathcal{E}^t\big(L(X)\big)U_t dt$ and $-U_t^* B_t \eta_t\big(\mathcal{E}^t(V_j^*\alpha_j) + \mathcal{E}^t(V_i^*\beta_{ij}V_j) + \mathcal{E}^t(\alpha_j^* V_j)\big)U_t dt$. Using the fact that $\mathcal{E}^t$ leaves ρ^t invariant we see that the term $U_t^* B_t \mathcal{E}^t\big(L(X)\big)U_t dt$ cancels against the

dt terms of [1], [4] and [14], which make up the Lindblad generator L with a minus sign. The other term of [3] is cancelled in expectation against the dt terms of [13], [34] and [134], since

$$\rho^0([13]) = \rho^t(B_t\eta_t V_j^*\alpha_j)dt = \rho^t\big(\mathcal{E}^t(B_t\eta_t V_j^*\alpha_j)\big)dt = \rho^t\big(B_t\eta_t\mathcal{E}^t(V_j^*\alpha_j)\big)dt$$
$$\rho^0([34]) = \rho^t(B_t\eta_t\alpha_j^* V_j)dt = \rho^t\big(\mathcal{E}^t(B_t\eta_t\alpha_j^* V_j)\big)dt = \rho^t\big(B_t\eta_t\mathcal{E}^t(\alpha_j^* V_j)\big)dt$$
$$\rho^0([134]) = \rho^t(B_t\eta_t V_i^*\beta_{ij}V_j)dt = \rho^t\big(\mathcal{E}^t(B_t\eta_t V_i^*\beta_{ij}V_j)\big)dt = \rho^t\big(B_t\eta_t\mathcal{E}^t(V_i^*\beta_{ij}V_j)\big)dt.$$

□

Using equation (7.3), the fact that $\mathcal{E}^t$ leaves ρ^t invariant and the module property, we find that the term [2] has expectation zero as well

$$\rho^0([2]) = \rho^t\left(b_t d\tilde{Y}_t(\mathcal{E}^t(X) - X)\right) = -\rho^t\left(b_t\mathcal{E}^t(V_j^*\alpha_j(t) + \alpha_j^*(t)V_j + V_i^*\beta_{ij}V_j)(\mathcal{E}^t(X) - X)\right)dt =$$
$$-\rho^t\left(b_t\mathcal{E}^t(V_j^*\alpha_j(t) + \alpha_j^*(t)V_j + V_i^*\beta_{ij}V_j)\mathcal{E}^t(\mathcal{E}^t(X) - X)\right)dt = 0.$$

Thus, only the terms containing no B_t nor c_t can contribute non-trivially. This leads to an equation allowing us to obtain an expression for η_t by solving

$$\rho^0([12] + [23] + [24] + [123] + [124] + [234] + [1234]) = 0. \tag{7.8}$$

Although this can be carried out in full generality, we will provide the solution only for our main examples, the photon counting and homodyne detection experiments for a resonance fluorescence setup, in the next section.

8 Examples

We now return to the example considered in Section 2. We were considering a 2-level atom in interaction with the electromagnetic field. The interaction was given by a cocycle U_t satisfying equation (6.3). The observed process is the number operator in the side channel, i.e. $Y_t = \Lambda_{ss}(t)$. Therefore $d\tilde{Y}_t = d\Lambda_{ss}(t) - \mathcal{E}^t(V_s^*V_s)dt$. Recall now the notation $Z_1(t) = U_t^*$, $Z_2(t) = B_t$, $Z_3(t) = \mathcal{E}^t(X) - X$ and $Z_4(t) = U_t$, their differentials are given by

$$\begin{aligned}
dU_t^* &= U_t^*\sum_{\sigma=f,s}\{\tilde{V}_\sigma^* dA_\sigma(t) - \tilde{V}_\sigma dA_\sigma^*(t) - \frac{1}{2}(-i\tilde{H} + \tilde{V}_\sigma^*\tilde{V}_\sigma)dt\}\\
dB_t &= b_t d\Lambda_{ss}(t) + \big(c_t - b_t\mathcal{E}^t(V_s^*V_s)\big)dt\\
d(\mathcal{E}^t(X) - X) &= \eta_t d\Lambda_{ss}(t) + \Big(\mathcal{E}^t\big(L(X)\big) - \eta_t\mathcal{E}^t(V_s^*V_s)\Big)dt\\
dU_t &= \sum_{\sigma=f,s}\{\tilde{V}_\sigma dA_\sigma^*(t) - \tilde{V}_\sigma^* dA_\sigma(t) - \frac{1}{2}(i\tilde{H} + \tilde{V}_\sigma^*\tilde{V}_\sigma)dt\}U_t.
\end{aligned}$$

Following the recipe of the previous section we now only have to determine the dt terms of [12], [23], [24], [124], [123], [124] and [1234]. All of these terms are zero in expectation with respect to ρ^0, except for [124] and [1234]

$$\begin{aligned}
\rho^0\big([124]\big) &= \rho^0\Big(U_t^* b_t V_s^*\big(\mathcal{E}^t(X) - X\big)V_sU_t\Big)dt\\
\rho^0\big([1234]\big) &= \rho^0\big(U_t^* b_t\eta_t V_s^*V_sU_t\big)dt.
\end{aligned}$$

For all b_t the sum of these terms has to be 0 in expectation, i.e.

$$\forall b_t:\ \rho^t\left(b_t\Big(V_s^*\big(\mathcal{E}^t(X)-X\big)V_s+\eta_t V_s^*V_s\Big)\right)dt=0 \iff$$
$$\forall b_t:\ \rho^t\left(\mathcal{E}^t\left(b_t\Big(V_s^*\big(\mathcal{E}^t(X)-X\big)V_s+\eta_t V_s^*V_s\Big)\right)\right)dt=0 \iff$$
$$\forall b_t:\ \rho^t\left(b_t\Big(\mathcal{E}^t(X)\mathcal{E}^t(V_s^*V_s)-\mathcal{E}^t(V_s^*XV_s)+\eta_t\mathcal{E}^t(V_s^*V_s)\Big)\right)dt=0 \iff$$
$$\eta_t=\frac{\mathcal{E}^t(V_s^*XV_s)}{\mathcal{E}^t(V_s^*V_s)}-\mathcal{E}^t(X).$$

Substituting the expressions for η_t and $\tilde{Y}_t$ into equation (7.2) we obtain the Belavkin equation for photon counting in the side channel

$$d\mathcal{E}^t(X)=\mathcal{E}^t\big(L(X)\big)dt+\Big(\frac{\mathcal{E}^t(V_s^*XV_s)}{\mathcal{E}^t(V_s^*V_s)}-\mathcal{E}^t(X)\Big)\big(d\Lambda_{ss}(t)-\mathcal{E}^t(V_s^*V_s)dt\big). \tag{8.1}$$

Now recall that $\mathcal{E}^t(X)=\rho^t_\bullet(X_\bullet)$, i.e. it is the function $\Omega_t\to\mathbb{C}:\ \omega\mapsto\rho^t_\omega(X_\omega)$. For all $X\in\mathcal{B}=M_2$, the M_2 valued function $X_\bullet$ is the constant function $\omega\mapsto X$. Therefore for all X in $\mathcal{B}$, the Belavkin equation (8.1) is equivalent to

$$d\rho^t_\bullet(X)=\rho^t_\bullet\big(L(X)\big)dt+\Big(\frac{\rho^t_\bullet(V_s^*XV_s)}{\rho^t_\bullet(V_s^*V_s)}-\rho^t_\bullet(X)\Big)\big(d\Lambda_{ss}(t)-\rho^t_\bullet(V_s^*V_s)dt\big),$$

which is equivalent to the Belavkin equation of Section 2, equation (2.5). In simulating the above equation we can take for $Y_t=\Lambda_{ss}(t)$ the unique jump process with independent jumps and rate $\rho^t_\bullet(V_s^*V_s)$, since $\Lambda_{ss}(t)-\int_0^t\rho^r_\bullet(V_s^*V_s)dr$ has to be a martingale.

Let us now turn to the homodyne detection scheme which we already discussed in Section 3. The observed process is now $Y_t=X_\phi(t)=A_s^*(f_t)+A_s(f_t)$ (see Section 3 for the definition of f_t). This means the innovating martingale $\tilde{Y}_t$ satisfies $d\tilde{Y}_t=e^{i\phi_t}dA_s^*(t)+e^{-i\phi_t}dA_s(t)-\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)dt$, where $\phi_t=\phi_0+\omega_{lo}t$ with ω_{lo} the frequency of the local oscillator. Therefore we find different differentials for B_t and $\mathcal{E}^t(X)-X$ than we had in the photon counting case

$$dB_t=b_t\big(e^{i\phi_t}dA_s^*(t)+e^{-i\phi_t}dA_s(t)\big)+\big(c_t-b_t\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\big)dt$$
$$d(\mathcal{E}^t(X)-X)=\eta_t\big(e^{i\phi_t}dA_s^*(t)+e^{-i\phi_t}dA_s(t)\big)+\Big(\mathcal{E}^t\big(L(X)\big)-\eta_t\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\Big)dt$$

Following the recipe of the previous section we now only have to determine the dt terms of [12], [23], [24], [124], [123], [124] and [1234]. All of these terms are zero in expectation with respect to ρ^0, except for [12], [23] and [24]

$$\rho^0\big([12]\big)=\rho^0\Big(U_t^*e^{i\phi_t}V_s^*b_t\big(\mathcal{E}^t(X)-X\big)U_t\Big)dt$$
$$\rho^0\big([23]\big)=\rho^0(U_t^*b_t\eta_tU_t)dt$$
$$\rho^0\big([24]\big)=\rho^0\Big(U_t^*b_t\big(\mathcal{E}^t(X)-X\big)e^{-i\phi_t}V_sU_t\Big)dt.$$

For all b_t the sum of these terms has to be 0 in expectation, i.e.

$$\begin{aligned}
&\forall b_t :\ \rho^t\left(b_t\Big(e^{i\phi_t}V_s^*\big(\mathcal{E}^t(X)-X\big)+\big(\mathcal{E}^t(X)-X\big)e^{-i\phi_t}V_s+\eta_t\Big)\right)dt=0 \iff \\
&\forall b_t :\ \rho^t\left(\mathcal{E}^t\left(b_t\Big(e^{i\phi_t}V_s^*\big(\mathcal{E}^t(X)-X\big)+\big(\mathcal{E}^t(X)-X\big)e^{-i\phi_t}V_s+\eta_t\Big)\right)\right)dt=0 \iff \\
&\forall b_t :\ \rho^t\Big(b_t\big(-\mathcal{E}^t(e^{i\phi_t}V_s^*X+e^{-i\phi_t}XV_s)+\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\mathcal{E}^t(X)+\eta_t\big)\Big)dt=0 \iff \\
&\eta_t=\mathcal{E}^t(e^{i\phi_t}V_s^*X+e^{-i\phi_t}XV_s)-\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\mathcal{E}^t(X).
\end{aligned}$$

Substituting the expressions for η_t and $\tilde{Y}_t$ into equation (7.2) we obtain the Belavkin equation for the homodyne detection scheme

$$\begin{aligned}
&d\mathcal{E}^t(X)=\mathcal{E}^t\big(L(X)\big)dt+\big(\mathcal{E}^t(e^{i\phi_t}V_s^*X+e^{-i\phi_t}XV_s)-\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\mathcal{E}^t(X)\big)\times \\
&\times\big(e^{i\phi_t}dA_s^*(t)+e^{-i\phi_t}dA_s(t)-\mathcal{E}^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)dt\big).
\end{aligned}\tag{8.2}$$

Now recall that $\mathcal{E}^t(X)=\rho_\bullet^t(X_\bullet)$, i.e. it is the function $\Omega_t\to\mathbb{C}:\ \omega\mapsto\rho_\omega^t(X_\omega)$. For all $X\in\mathcal{B}=M_2$, the M_2 valued function $X_\bullet$ is the constant function $\omega\mapsto X$. Therefore for all X in $\mathcal{B}$, the Belavkin equation (8.2) is equivalent to

$$\begin{aligned}
&d\rho_\bullet^t(X)=\rho_\bullet^t\big(L(X)\big)dt+\big(\rho_\bullet^t(e^{i\phi_t}V_s^*X+e^{-i\phi_t}XV_s)-\rho_\bullet^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)\rho_\bullet^t(X)\big)\times \\
&\times\big(e^{i\phi_t}dA_s^*(t)+e^{-i\phi_t}dA_s(t)-\rho_\bullet^t(e^{i\phi_t}V_s^*+e^{-i\phi_t}V_s)dt\big),
\end{aligned}$$

which is equivalent to the Belavkin equation of Section 3, equation (3.4). Since $A_s^*(f_t)+A_s(f_t)-\int_0^t\rho_\bullet^r(e^{i\phi_r}V_s^*+e^{-i\phi_r}V_s)dr$ is a martingale with variance t on the space of the Wiener process, it must be the Wiener process itself.

References

[1] Barchielli A., Measurement theory and stochastic differential equations in quantum mechanics, *Phys. Rev. A*, **34**, (1986), 1642–1649.

[2] Barchielli, A., Belavkin, V.P., Measurements continuous in time and a posteriori states in quantum mechanics, *J. Phys. A*, **24**, (1991), 1495–1514.

[3] Barchielli, A, Lupieri, G., Quantum stochastic models of two-level atoms and electromagnetic cross sections, *J. Math. Phys.*, **41**, (2000), 7181–7205.

[4] Belavkin V., A new wave equation for a continuous nondemolition measurement, *Physics letters A*, **140**, (1989), 355–358.

[5] Belavkin V., Quantum Continual Measurements and a Posteriori Collapse on CCR, *Commun. Math. Phys*, **146**, (1992), 611–635.

[6] Belavkin V., Quantum stochastic calculus and quantum nonlinear filtering, *Journal of Multivariate Analysis*, **42**, (1992), 171–201.

[7] Belavkin, V.P., Staszewski, P., A quantum particle undergoing continuous observation, *Physics letters A*, **140**, (1989), 359–362.

[8] Bouten, L.M., Maassen, J.D.M., Kümmerer, B., Constructing the Davies process of resonance fluorescence with quantum stochastic calculus, *Optics and Spectroscopy*, **94**, (2003), 911–919.

[9] Carmichael H., *An Open Systems Approach to Quantum Optics*, Springer-Verlag, Berlin Heidelberg New-York (1993).

[10] Carmichael H., Stochastic Schrödinger equations: what they mean and what they can do, in Eberly, Mandel, Wolf, editors, *Coherence and Quantum Optics*, volume VII, Plenum Press, New York (1996).

[11] Carmichael, H.J., Singh, S., Vyas, R., Rice, P.R., Photoelectron waiting times and atomic state reduction in resonance fluorescence, *Phys. Rev. A*, **39**, (1989), 1200–1218.

[12] Dalibard, J., Castin, Y., Mølmer, K., Wave-function approach to dissipative processes in quantum optics, *Phys. Rev. Lett.*, **68**, (1992), 580–583.

[13] Davies E., Quantum stochastic processes, *Commun. Math. Phys.*, **15**, (1969), 277–304.

[14] Davies E., *The Quantum Theory of Open Systems*, Academic Press, London New-York San Francisco (1976).

[15] Diósi, L., Gisin, N., Halliwell, J., Percival, I.C., Decoherent histories and quantum state diffusion, *Phys. Rev. Lett.*, **74**, (1995), 203–207.

[16] Gardiner, C.W., Parkins, A.S., Zoller, P., Wave-function quantum stochastic differential equations and quantum-jump simulation methods, *Phys. Rev. A*, **46**, (1992), 4363–4381.

[17] Gisin N., Quantum measurements and stochastic processes, *Phys. Rev. Lett.*, **52**, (1984), 1657–1660.

[18] Gisin N., Stochastic quantum dynamics and relativity, *Helv. Phys. Acta*, **62**, (1989), 363–371.

[19] Guichardet A., *Symmetric Hilbert Spaces and Related Topics*, volume 261 of *Lect. Notes Math.*, Springer (1972).

[20] Hudson, R.L., Parthasarathy, K.R., Quantum Itô's formula and stochastic evolutions, *Commun. Math. Phys.*, **93**, (1984), 301–323.

[21] Kadison, R.V., Ringrose, J.R., *Fundamentals of the Theory of Operator Algebras*, volume II, Academic Press, San Diego (1986).

[22] Kallianpur G., *Stochastic Filtering Theory*, Springer, Berlin (1980).

[23] Kümmerer B., *A dilation theory for completely positive operators*, Ph.D. thesis, Eberhard-Karls-Universität, Tübingen (1982).

[24] Kümmerer B., Markov Dilations on W^*-algebras, *J. Funct. Anal.*, **63**, (1985), 139–177.

[25] Kümmerer B., Quantum Markov processes, in Buchleitner, A., Hornberger, K., editors, *Coherent evolution in noisy environments*, volume 611 of *Lecture Notes in Physics*, Springer-Verlag, Berlin Heidelberg (2002).

[26] Lindblad G., On the generators of quantum dynamical semigroups, *Commun. Math. Phys.*, **48**, (1976), 119–130.

[27] Meyer P., *Quantum Probability for Probabilists*, Springer, Berlin (1993).

[28] Mølmer, K., Castin, Y., Dalibard, J., Monte Carlo wave-function method in quantum optics, *J. Opt. Soc. Am. B*, **10**, (1993), 524–538.

[29] Parthasarathy K., *An Introduction to Quantum Stochastic Calculus*, Birkhäuser, Basel (1992).

[30] Rogers, L.C.G., Williams, D., *Diffusions, Markov Processes, and Martingales*, Wiley Series in Probability and Mathematical Statistics (1987).

[31] Srinivas, M.D., Davies, E.B., Photon counting probabilities in quantum optics, *Optica Acta*, **28**, (1981), 981–996.

[32] Takesaki M., Conditional expectations in von Neumann algebras, *J. Funct. Anal.*, **9**, (1971), 306–321.

[33] Tomiyama J., On the projections of norm one in W^*-algebras, *Proc. Japan Acad.*, **33**, (1957), 608–612.

[34] Williams D., *Probability with Martingales*, Cambridge University Press (1991).

[35] Wiseman, H.M., Milburn, G.J., Quantum theory of field-quadrature measurements, *Phys. Rev. A*, **47**, (1993), 642–662.