Recent progress on the Jacobian Conjecture

Michiel de Bondt and Arno van den Essen

In memory of S. Łojasiewicz

Abstract

In this paper we describe some recent developments concerning the Jacobian Conjecture(JC). First we describe Drużkowski's result in [6] which asserts that it suffices to study the JC for Drużkowski mappings of the form $x + (Ax)^{*3}$ with $A^2 = 0$. Then we describe the authors' result of [2] which asserts that it suffices to study the JC for so-called gradient mappings i.e. mappings of the form $x - \nabla f$, with $f \in k^{[n]}$ homogeneous of degree 4. Using this result we explain Zhao's reformulation of the JC which asserts the following: for every homogeneous polynomial $f \in k^{[n]}$ (of degree 4) the hypothesis $\Delta^m(f^m) = 0$ for all $m \geq 1$ implies that $\Delta^{m-1}(f^m) = 0$ for all large m (Δ is the Laplace operator). In the last section we descibe Kumar's formulation of the JC in terms of smoothness of a certain family of hypersurfaces. [1]

Introduction

Since the first appearance of the JC in [12] various papers have been published concerning this conjecture. One of the milestones is undoubtedly the classical paper [1] of Bass, Connell and Wright from 1982. This paper gave an impuls to the field of polynomial automorphisms, which is now flourishing as never before. To mention a few highlights: the counterexample to the real Jacobian Conjecture by Pinchuk in [14], 1994, proofs of the 2-dimensional Markus-Yamabe Conjecture by Glutsuk, Fessler and Gutierrez in [9],[8] and [10], the polynomial counterexamples to the Markus-Yamabe Conjecture in all dimensions ≥ 3 by Cima, van den Essen, Gasull, Hubbers and Mañosas in [4], 1995, the proof of the linearization conjecture for $\mathbb{C}^*$-actions on $\mathbb{C}^3$ by Kaliman, Koras, Makar-Limanov and Russell in [11] and recently the negative solution of the tame generators conjecture by Shestakov and Umirbaev in [15]. However, since the famous reduction theorems of Bass, Connell, Wright/Yagzhev [16] and Drużkowski [5], not much progress has been made towards the Jacobian Conjecture. The aim of this paper is to report on some surprising new reduction theorems, which go far beyond the classical reductions mentioned before. The two most important papers in this respect are [2] and [17]. In the authors' paper [3] a survey is given of various results related to the paper [2]. Therefore in this paper we will focus our

[1] 14R15, Jacobian Conjecture, Hessian Conjecture, Laplace Operator

attention on Zhao's paper [17] (see section 3). First we recall in section 2 the main result of [2], on which Zhao's result is based. Finally in the last section we describe one more consequence of the main theorem of [2]: namely a reformulation of the JC, due to Mohan Kumar, in terms of smoothness of a family of hypersurfaces.

1 The classical reduction theorems and Drużkowski's recent reduction

Throughout this paper k denotes an algebraically closed field of characteristic zero and by $k^{[n]}$ or $k[x]$ we denote the n-variable polynomial ring $k[x_1, \ldots, x_n]$. Recall that the Jacobian Conjecture asserts that a polynomial map $F : k^n \to k^n$ is invertible if $\det JF \in k^*$, where $JF = (\frac{\partial F_i}{\partial x_j})$ denotes the Jacobian matrix of F.
In [1] Bass, Connell and Wright and in [16] Yagzhev showed that it suffices to investigate the JC for all $n \geq 1$ and all polynomial maps of the form $F = x + H$, where $H = (H_1, \ldots, H_n)$ is homogeneous (of degree 3) and JH nilpotent (in fact they show that for such homogeneous maps H the condition $\det JF \in k^*$ is equivalent to JH being nilpotent). A little later Drużkowski in [5] showed that one may even assume that each H_i is of the form L_i^3, where L_i is a linear form. In other words it suffices to study the JC for polynomial maps of the form $x + (Ax)^{*3}$, where $A \in M_n(k)$ and $(v_1, \ldots, v_n)^{*3}$ denotes the vector $(v_1^3, \ldots, v_n^3)$. More recently Drużkowski in [6] obtained the following improvement of his reduction theorem.

Theorem 1.1 *(Drużkowski, 2000). It suffices to investigate the JC for all $n \geq 1$ and all polynomial maps of the form $x + (Ax)^{*3}$ with the additional property that $A^2 = 0$.*

Proof. Let $F := x + (Ax)^{*3} : k^n \to k^n$ and $i \in k$ satisfy $i^2 = -1$. Put $F_* := x + 2i(Ax)^{*3}$. Observe that $F_* = zF(z^{-1}x)$, where $z^2 = \frac{1}{2i}$. So F is invertible iff F_* is invertible iff $\hat{F} := (F_*, y) = (x + 2i(Ax)^{*3}, y) : k^{2n} \to k^{2n}$ is invertible. Now put

$$Q := (x + iy, y + (A(x + iy))^{*3}) \text{ and } S := (x - iy, y).$$

Then $G := S \circ \hat{F} \circ Q$ is invertible iff $\hat{F}$ is invertible. Furthermore, one readily verifies that $G = (x, y) + (N(x, y))^{*3}$, where

$$N := \begin{pmatrix} -iA & A \\ A & iA \end{pmatrix}$$

which satisfies $N^2 = 0$.

2 Reduction to the symmetric case

Let JH be a Jacobian matrix. Then one easily verifies that JH is symmetric iff H is a gradient mapping i.e. $H = \nabla f (= (f_{x_1}, \ldots, f_{x_n}))$ for some $f \in k[x]$. The main result of [2] asserts that it suffices to investigate the JC for all $n \geq 2$ and all $F : k^n \to k^n$ of the form $F = x + \nabla f$ (with $J(\nabla f)$ nilpotent). More precisely we have

Theorem 2.1 *(de Bondt, van den Essen, 2003). If the JC is true for all polynomial maps $F : k^{2n} \to k^{2n}$ of the form $x + \nabla f$, with $J(\nabla f)$ nilpotent (and homogeneous), then the JC is true for all polynomial maps of the form $x + H : k^n \to k^n$ with JH nilpotent (and homogeneous).*

The proof of this result is based on the next lemma. Recall that

$$J(\nabla f) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right) =: \mathcal{H}(f)$$

the Hessian of f. The standard bilinear form on k^m we denote by $<,>$.

Lemma 2.2 *Let $H = (H_1(x), \ldots, H_n(x)) \in k[x]^n$ and $y_1, \ldots, y_n$ new variables. Put $f := f_H = (-i) < H(x + iy), y >$. Then JH is nilpotent iff $\mathcal{H}(f)$ is nilpotent.*

Proof. (1) $\mathcal{H}(f)$ is nilpotent iff det $(TI_{2n} - \mathcal{H}(f)) = T^{2n}$. Put $S := (x - iy, y)$ and S_0 the corresponding matrix in $M_{2n}(k)$. Then
$g := f \circ S = (-i) < H(x), y >$ and

(2) $\mathcal{H}(g) = \begin{pmatrix} * & (-i)(JH)^t \\ (-i)JH & 0 \end{pmatrix}$.

Furthermore

(3) $\mathcal{H}(g) = S_0^t \mathcal{H}(f)_{|S(x,y)}.S_0$.

Since det $S_0 = 1$ we get from (1) and (3) $\mathcal{H}(f)$ is nilpotent iff

$$\det S_0^t(TI_{2n} - \mathcal{H}(f))_{|S(x,y)} S_0 = T^{2n} \text{ iff } \det\,(TS_0^t S_0 - \mathcal{H}(g)) = T^{2n}.$$

Since $S_0^t S_0 = \begin{pmatrix} I_n & -iI_n \\ -iI_n & 0 \end{pmatrix}$ we get from (2)

$$\mathcal{H}(f) \text{ is nilpotent iff } \det \begin{pmatrix} * & -iTI_n + i(JH)^t \\ -iTI_n + iJH & 0 \end{pmatrix} = T^{2n}.$$

Since for $n \times n$ matrices A and B we have that

$$\det \begin{pmatrix} * & -iA \\ -iB & 0 \end{pmatrix} = \det A \det B$$

we get $\mathcal{H}(f)$ is nilpotent iff det $(TI_n - (JH)^t)$det $(TI_n - JH) = T^{2n}$ iff

$$\det\,(TI_n - JH) = T^n \text{ iff } JH \text{ is nilpotent.}$$

Proof of theorem 2.1. Let $H = (H_1(x), \ldots, H_n(x))$ with JH nilpotent (and H homogeneous). Let $f = f_H$ be as in lemma 2.2. Then $\mathcal{H}(f)$ is nilpotent (and f is homogeneous). So by our hypothesis

$$G := (x_1 + f_{x_1}, \ldots, x_n + f_{x_n}, y_1 + f_{y_1}, \ldots, y_n + f_{y_n})$$

is invertible. Consequently, with S as in the proof of lemma 2.2, $S^{-1} \circ G \circ S$ is invertible too. An easy calculation shows that

$$S^{-1} \circ G \circ S = (x_1 + H_1(x), \ldots, x_n + H_n(x), *, \ldots, *).$$

Since this last map is invertible, the desired result follows from the next lemma.

Lemma 2.3 *If $\tilde{F} := (F_1(x), \ldots, F_n(x), *, \ldots, *) : k^{2n} \to k^{2n}$ is invertible, then $F := (F_1(x), \ldots, F_n(x)) : k^n \to k^n$ is invertible.*

Proof. Let $(G_1(x,y), \ldots, G_n(x,y), *, \ldots, *)$ be the inverse of $\tilde{F}$. Then in particular

$$F_i(G_1(x,y), \ldots, G_n(x,y)) = x_i \text{ for all } i.$$

So $F_i(G_1(x,0), \ldots, G_n(x,0)) = x_i$ for all i, which means that F is invertible with inverse $(G_1(x,0), \ldots, G_n(x,0))$.

Combining theorem 2.1 with the classical Bass, Connell, Wright/ Yagzhev reduction theorem we get

Corollary 2.4 . *The following statements are equivalent*
i) The Jacobian Conjecture.
ii) The Jacobian Conjecture for polynomial maps of the form $x + \nabla f$ with $\mathcal{H}(f)$ nilpotent and f homgeneous of degree 4.

3 Zhao's Laplace operator formulation of the Jacobian Conjecture

In the previous section we saw that it suffices to investigate the JC for polynomial maps of the form $x + \nabla f$ with $\mathcal{H}(f)(= J(\nabla f))$ nilpotent (and we may even assume that f is homogeneous of degree 4).
In [17] Zhao uses this result to obtain a remarkable reformulation of the JC. Recall that the Laplace operator, denoted Δ, is equal to $\partial_1^2 + \ldots + \partial_n^2$ ($\partial_i := \frac{\partial}{\partial x_i}$).

Theorem 3.1 *(Zhao, 2004). The JC is equivalent to each of the following statements.*
i) If f is a homogeneous polynomial of degree ≥ 3 such that $\Delta^m(f^m) = 0$ for all $m \geq 1$, then $\Delta^{m-1}(f^m) = 0$ for all large m.
ii) If f is a homogeneous polynomial of degree 4 such that $\Delta^m(f^m) = 0$ for all $m \geq 1$, then $\Delta^{m-1}(f^m) = 0$ for all large m.

In the remainder of this section we give a somewhat simplified proof of this result. We start with some notations and generalities.
If R is a commutative ring, then $R[[x]]$ denotes the ring $R[[x_1, \ldots, x_n]]$ of formal power series in $x_1, \ldots, x_n$ over R. The order of an element g of $R[[x]]$, denoted $o(g)$, is by definition the smallest degree of a monomial appearing in g if $g \neq 0$ and $o(g) = \infty$ if $g = 0$. More generally, if $H = (H_1, \ldots, H_n) \in R[[x]]^n$ then $o(H)$ denotes the minimum of the $o(H_i)$.
Now let $H \in k[[x]]^n$ with $o(H) \geq 2$. Then the formal map $F = x - H$ satisfies det $JF(0) = 1$. So it has a formal inverse. To study this inverse the crucial idea in [17] is to embed F in a family of such maps. More precisely, let t be a new variable and let $A := k[t]$. Then define

$$F_t := x - tH(x) \in A[[x]]^n.$$

Since det $(J_xF_t)(0) = 1$ it follows from the formal inverse function theorem ([7], 1.1.2) that F_t has a unique formal inverse, say G_t in $A[[x]]^n$, which is of the form $x + U_t(x)$ with $o(U_t) \geq 2$. Setting $t = 0$ in $F_t(G_t(x)) = x$ we get $G_0(x) = x$. So $U_t(x) = tN_t(x)$ for some $N_t(x) \in A[[x]]^n$. Hence

$$G_t(x) = x + N_t(x).$$

Consequently, the equation $G_t(F_t(x)) = x$ implies that $x - tH(x) + tN_t(F_t(x)) = x$, whence

(4) $N_t(F_t(x)) = H(x)$.

By the chainrule we get $JN_t(F_t).JF_t = JH$. Using $JF_t = I - tJH$ this gives

$$JN_t(F_t) = JH.(I - JH)^{-1} = \sum_{k=1}^{\infty}(JH)^k(x)t^{k-1}.$$

Writing ∂_t for $\frac{\partial}{\partial t}$ we get

Proposition 3.2 *$N_t(x)$ is the unique formal solution of the Cauchy problem*

(5) $\partial_t(N_t) = JN_t.N_t$, $N_{t=0}(x) = H(x)$.

Proof. The initial condition follows directly from (4). Furthermore, differentiating (4) with respect to t gives $\partial_t(N_t)(F_t) - (JN_t)H = 0$. Composing from the right with G_t and using (4) gives the desired result.

From now on we assume that JH is **symmetric**. So $H = \nabla f$ for some unique $f \in k[[x]]$ with $o(f) \geq 3$. It follows from (5) that $JN_t(F_t)$ is symmetric and hence so is $JN_t(x)$. Consequently there exists a unique $Q_t \in A[[x]]$ with $o(Q_t) \geq 3$ such that $N_t(x) = \nabla Q_t$. So $G_t(x) = x + t\nabla Q_t$. Writing $<,>$ for the standard bilinear form we have

Proposition 3.3 *Q_t is the unique solution of the Cauchy problem*

(6) $\partial_t(Q_t) = \frac{1}{2} < \nabla Q_t, \nabla Q_t >$, $Q_{t=0} = f$.

Proof. Using $N_t = \nabla Q_t$ and 3.2 we get $\nabla(\partial_t(Q_t)) = \partial_t(\nabla Q_t) = JN_t.\nabla Q_t$. Also one easily verifies that

$$\nabla(\tfrac{1}{2} < \nabla Q_t, \nabla Q_t >) = \mathcal{H}(Q_t).\nabla Q_t = JN_t.\nabla Q_t.$$

So $\nabla(\partial_t(Q_t)) = \nabla(\frac{1}{2} < \nabla Q_t, \nabla Q_t >)$. This implies the first equality in (6), since the polynomials in this equation have no constant term. Finally, using (4) we get that $\nabla Q_{t=0} = N_0 = H = \nabla f$, which gives $Q_{t=0} = f$.

In order to investigate JC one should, according 3.4. study polynomial maps $x - \nabla f$ with $\mathcal{H}(f)$ nilpotent. Therefore we call an element $f \in k[[x]]$ which matrix $\mathcal{H}(f)$ is nilpotent, **Hesse Nilpotent**, HN for short.

Lemma 3.4 *Let $f \in k[[x]]$ with $o(f) \geq 3$. Then f is HN iff $\Delta Q_t = 0$.*

Proof. Observe that $JN_t = J(\nabla Q_t) = \mathcal{H}(Q_t)$, whence Tr JN_t=Tr $\mathcal{H}(Q_t) = \Delta Q_t$. Since $H = \nabla f$ we also have $JH = \mathcal{H}(f)$. Then it follows from (5) by taking traces that

(7) $(\Delta Q_t)(F_t) = \sum_{k=1}^{\infty}$ Tr $\mathcal{H}(f)^k t^{k-1}$.

Finally, f is HN iff Tr $\mathcal{H}(f)^k = 0$ for all $k \geq 1$ iff $(\Delta Q_t)(F_t) = 0$ iff $\Delta Q_t = 0$.

Now we are able to give Zhao's main theorem, which gives a beautiful formula for Q_t (and hence for the formal inverse $G_t = x + \nabla Q_t$) in case f is HN. In fact his theorem gives the following more general result.

Theorem 3.5 *(Zhao, 2004). Let $f \in k[[x]]^n$ with $o(f) \geq 3$ and HN. Then*

(8) $Q_t^k = k! \sum_{m=0}^{\infty} \frac{t^m}{2^m m!(m+k)!} \Delta^m (f^{m+k})$ for all $k \geq 1$.

Proof. Introduce a new variable s and consider the generating function of the sequence $\{Q_t^k/k!\}$ i.e. $U := exp(sQ_t)$.
Claim: U is the unique solution of the Cauchy problem

(9) $\partial_t(U) = \frac{1}{2s}\Delta U$, $U(t=0) = exp(sf)$.

To prove this claim observe that, using (6), we get

(10) $\partial_t(U) = \frac{s}{2} < \nabla Q_t, \nabla Q_t > U$ and $U(t=0) = exp(sf)$.

Furthermore, $\Delta U = s \sum_i \partial_i(\partial_i(Q_t)U) = s\Delta(Q_t)U + s^2 \sum_i \partial_i(Q_t)^2 U$

$= s^2 < \nabla Q_t, \nabla Q_t > U$, since $\Delta Q_t = 0$ by 3.4. So

(11) $\Delta U = s^2 < \nabla Q_t, \nabla Q_t > U$.

From (10) and (11) we get (9). However, also the formal series

$$\sum_{k=0}^{\infty} \frac{t^k}{(2s)^k k!} \Delta^k (exp(sf))$$

is a solution of the Cauchy problem (9), as one easily verifies. So by the uniqueness we obtain that this series is equal to $exp(sQ_t)$. Comparing the coefficients of s^k for all $k \geq 1$ in this equation we obtain (8).

As an immediate consequence of (8) we get

Corollary 3.6 *Let $f \in k[[x]]$ with $o(f) \geq 3$ and f HN. Then $\Delta^m(f^m) = 0$ for all $m \geq 1$*

Proof. By (3.4) $\Delta Q_t = 0$. Then use (8) with $k = 1$.

Now we show that the converse holds as well i.e.

Theorem 3.7 *Let $f \in k[[x]]$ with $o(f) \geq 3$. Then f is HN iff $\Delta^m(f^m) = 0$ for all $m \geq 1$ iff $\Delta^m(f^m) = 0$ for all $1 \leq k \leq n$.*

The proof of this result follows directly from the next result with $k = n$, using the fact that an $n \times n$ matrx A over a domain is nilpotent iff Tr $A^k = 0$ for all $1 \leq k \leq n$.

Proposition 3.8 *Let $v_m(f) := \Delta^m(f^m)$ and $u_m(f) := Tr\,\mathcal{H}(f)^m$ for all $m \geq 1$. Let $k \geq 1$. If $v_1(f) = \ldots = v_k(f) = 0$, then $u_1(f) = \ldots = u_k(f) = 0$.*

The proof of this result is based on the following lemma in which we use the symbol "*" to denote a non-zero constant in k.

Lemma 3.9 *Let $k \geq 1$ and $u_1(f) = \ldots = u_k(f) = 0$. Then for all $m \geq 1$*

(12) $\partial_t^l Q_t^m \equiv *\Delta^l Q_t^{m+l} (mod\, t^{(k+1)-l})$, for all $1 \leq l \leq k$.

Proof. i) By induction on l. First the case $l = 1$. Observe

$$\partial_t Q_t^m = m Q_t^{m-1} \partial_t(Q_t) = *Q_t^{m-1} < \nabla Q_t, \nabla Q_t > \text{ (by (6))}.$$

So we need to show that $Q_t^{m-1} < \nabla Q_t, \nabla Q_t > \equiv \Delta(Q_t^{m+1})(mod\, t^k)$. Therefore observe that

$$\Delta(Q_t^{m+1}) = *Q_t^m \Delta Q_t + *Q_t^{m-1} < \nabla Q_t, \nabla Q_t >.$$

Since by (7) and the hypothesis $\Delta Q_t \equiv 0(mod\, t^k)$, the case $l = 1$ follows.
ii) Now assume (12) for $1 \leq l \leq k$. Applying ∂_t to (12) gives

(13) $\partial_t^{l+1} Q_t^m \equiv *\Delta^l \partial_t(Q_t^{m+l})(mod\, t^{(k+1)-(l+1)})$.

From the case $l = 1$ with $m + l$ instead of m we get

$$\partial_t(Q_t^{m+l}) \equiv *\Delta Q_t^{m+l+1}(mod\, t^k).$$

Combining this with (13) gives the desired result for $l + 1$.

Proof of 3.8. By induction on k. The case $k = 1$ is obvious since $v_1(f) = u_1(f)$. So assume 3.8 for $k \geq 1$ and lets prove it for $k + 1$. So we assume that $v_1(f) = \ldots = v_{k+1}(f) = 0$. In particular the induction hypothesis implies that $u_1(f) = \ldots = u_k(f) = 0$. So by (7) $\Delta Q_t \equiv u_{k+1}(f) t^k (mod\, t^{k+1})$. Consequently

$$u_{k+1}(f) = \tfrac{1}{k!} \partial_t^k (\Delta Q_t)_{t=0}.$$

Furthermore, applying Δ to (12) with $l = k$ and $m = 1$ we get

$$\partial_t^k(\Delta Q_t) \equiv *\Delta^{k+1} Q_t^{k+1}(mod\, t).$$

So, using $Q_0 = f$ (by (6)), we get $u_{k+1}(f) = *\Delta^{k+1} Q_0^{k+1} = *\Delta^{k+1} f^{k+1} = v_{k+1} = 0$, as desired.

Now we are finally able to give

Proof of theorem 3.1. Let f be homogeneous of degree 4. Substituting $t = 1$ in (8) with $k = 1$ we get that the formal inverse of $x - \nabla f$ is of the form $x + \nabla Q$, where

$$Q = \sum_{m=0}^{\infty} \tfrac{1}{2^m m!(m+1)!} \Delta^m (f^{m+1}).$$

Since the condition $\mathcal{H}(f)$ is nilpotent is equivalent to the conditions described in 3.7, the desired result follows readily from 2.4.

4 Kumar's formulation of the Jacobian Conjecture

We conclude this paper with an observation of Mohan Kumar ([13]) which describes the Jacobian Conjecture as a problem concerning the smoothness of some hypersurfaces.

Theorem 4.1 *(Kumar, 2004) The Jacobian Conjecture is equivalent to the following statement:*

(S) For every homogeneous HN polynomial f of degree 4, every $1 \le i \le n$ and every $t \in k^*$, the hypersurface

$$S(t,i) := x_i + f_{x_i} + \tfrac{t}{2} f_{x_i x_i} + \tfrac{t^2}{6} f_{x_i x_i x_i}$$

has no singularities.

Proof. i) First assume (S). Let f be a homogeneous HN polynomial of degree 4. According (2.4) and [7], 4.2.1 it suffices to show that $F := x + \nabla f$ is injective. Therefore suppose that $F(a) = F(a+b)$ for some $a, b \in k^n$ with $b \neq 0$. Choose an orthogonal T such that $T^{-1}b = (t, 0, \ldots, 0)$, for some $t \in k^*$. Put $g := f \circ T$. Then $G := x + \nabla g = x + T^t \circ \nabla f \circ T = T^{-1} \circ F \circ T$ and $G(T^{-1}a) = G(T^{-1}a + T^{-1}b)$. So replacing F by G and f by g we may assume that $b = (t, 0, \ldots, 0)$ for some $t \in k^*$.
ii) Now consider the assumption

(14) $(x + \nabla f)(a + (t, 0, \ldots, 0)) = (x + \nabla f)(a)$.

Put $a_* := (a_2, \ldots, a_n)$. Then looking at the first component of (14) we get $a_1 + t + f_{x_1}(a_1 + t, a_*) = a_1 + f_{x_1}(a)$. Expanding $f_{x_1}(a_1 + t, a_*)$ in its Taylor series we deduce that

(15) $t + t f_{x_1 x_1}(a) + \frac{t^2}{2} f_{x_1 x_1 x_1}(a) + \frac{t^3}{6} f_{x_1 x_1 x_1 x_1}(a) = 0$.

For $2 \le i \le n$, looking at the i-th component of (14) gives

(16) $t f_{x_i x_1}(a) + \frac{t^2}{2} f_{x_i x_1 x_1}(a) + \frac{t^3}{6} f_{x_i x_1 x_1 x_1}(a) = 0$

Dividing by $t \in k^*$ we deduce from (15) and (16) that the hypersurface $S(t,1)$ has a singularity at a, contradiction.
ii) Conversely assume that the JC holds. If for some homogeneous HN polynomial f of degree 4, some $1 \le i \le n$ and some $t \in k^*$ the hypersurface $S(t,i)$ has a singularity, say at $a \in k^n$, then reading backwards the argument in ii) we find from (14) that the map $x + \nabla f$ is not injective. In particular $x + \nabla f$ is not invertible, contradicting the JC since f is HN i.e. $J(\nabla f)$ is nilpotent.

To conclude this paper we give the following interesting observation, also due to Kumar.

Proposition 4.2 *Let f be a homogeneous HN polynomial of degree 4. Then for every $i \le i \le n$ and every $t \in k$ the hypersurface*

$$R(t,i) := x_i + f_{x_i} + t f_{x_i x_i} + \tfrac{t^2}{2} f_{x_i x_i x_i}$$

has no singularities.

Proof. We may assume that $i = 1$. Let $b := (t, 0, \ldots, 0)$ and $x_* := (x_2, \ldots, x_n)$. Since $\mathcal{H}(f)$ is nilpotent, so is $\mathcal{H}(f)(x_1 + t, x_*) = \mathcal{H}(f(x_1 + t, x_*))$. Using Taylor's expansion we get

$$f(x_1 + t, x_*) = f(x) + tf_{x_1}(x) + \tfrac{t^2}{2} f_{x_1 x_1}(x) + \ldots$$

Since "taking the Hessian" of a polynomial is additive, we see that

$$\mathcal{H}(f(x_1 + t, x_*)) = \mathcal{H}(f) + t\mathcal{H}(f_{x_1}) + \tfrac{t^2}{2} \mathcal{H}(f_{x_1 x_1}).$$

The first row of this matrix is $\nabla(R(t, 1) - x_1)$ and thus $\nabla(R(t, 1))$ is the first row of the invertible matrix $I_n - \mathcal{H}(f(x_1 + t, x_*))$, which implies that the hypersurfac $R(t, 1)$ has no singularities.

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Authors' address:
University of Nijmegen
Department of Mathematics
Postbus 9010
6500 GL Nijmegen
THE NETHERLANDS

Email: debondt@math.ru.nl, essen@math.ru.nl