Quasi-translations and counterexamples to the Homogeneous Dependence Problem

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Abstract

In this article, the author gives counterexamples to the Linear Dependence Problem for Homogeneous Nilpotent Jacobians for dimension 5 and up. This problem has been formulated as a conjecture/problem by several authors ([6], [7], [10], [11], [12]) in connection to the Jacobian conjecture. In dimension 10 and up, cubic counterexamples are given.

In the construction of these counterexamples, the author makes use of so-called quasi-translations, a special type of invertible polynomial maps. Quasi-translations can also be seen as a special type of locally nilpotent derivations.

Key words: Jacobian (Conjecture), Quasi-translation, Linear Dependence Problem

MSC: 14R15; 14R20; 14R99

Introduction

Write $\mathcal{J}H$ for the Jacobian of H. The Linear Dependence Problem for Homogeneous Nilpotent Jacobians asserts that, if $H = (H_1, H_2, \ldots, H_n)$ is a homogeneous polynomial map such that $\mathcal{J}H$ is nilpotent, then the rows of $\mathcal{J}H$ are dependent over $\mathbb{C}$, i.e. there are $\lambda_i \in \mathbb{C}$, not all zero, such that

$$\lambda_1 \mathcal{J}H_1 + \lambda_2 \mathcal{J}H_2 + \ldots + \lambda_n \mathcal{J}H_n = 0$$

or equivalently, the components of H are linearly dependent over $\mathbb{C}$, i.e. there are $\lambda_i \in \mathbb{C}$, not all zero, such that

$$\lambda_1 H_1 + \lambda_2 H_2 + \ldots + \lambda_n H_n = 0$$

Let d be the degree of H. Positive answers to the Linear Dependence Problem for Homogeneous Nilpotent Jacobians are known in the following cases

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- $\operatorname{rk} \mathcal{J}H \le 1$ (including the case $n \le 2$, also if H is not homogeneous, see [1] or [7]),
- $n = 3$, d arbitrary (by van den Essen and the author, see [4]),
- $n = 4$, $d \le 3$ (by Hubbers, see [10] or [7]),

I will give counterexamples to the Linear Dependence Problem for Homogeneous Nilpotent Jacobians in dimension 5 and up, including cubic counterexamples in dimension 10 and up. On the Cubic Linear Dependence Problem for Homogeneous Nilpotent Jacobians, C. Olech put a bottle of polish vodka for the one who first solved the problem by way of either a proof or a counterexample, see [12].

Write $x = (x_1, x_2, \ldots, x_n)$. We call the polynomial map $x+H$ a *quasi-translation* if its polynomial inverse is $x - H$. In section 1, I will show that for quasi-translations $x + H$, $\mathcal{J}H$ is nilpotent. I will construct counterexamples of the quasi-translation type to the Linear Dependence Problem for Homogeneous Nilpotent Jacobians in dimension 6 and up. For these counterexamples $H = (H_1, H_2, \ldots, H_n)$, the associated derivation

$$D = \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$$

is locally nilpotent, such that $\ker D$ does not contain coordinates. In particular, it gives a new class of locally nilpotent derivations of maximal rank. The first examples of such derivations were constructed by Freudenburg in [8].

1 Quasi-translations

Let $H = (H_1, H_2, \ldots, H_n)$ be a polynomial map and $D = \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$ be the derivation associated with it. H does not need to be homogeneous yet.

The following proposition makes clear that quasi-translations can be seen as a special kind of both nilpotent Jacobians and locally nilpotent derivations.

Proposition 1.1 *$x + H$ is a quasi-translation, if and only if $D^2 x_i = 0$ for all i. Furthermore, $\mathcal{J}H$ is nilpotent in case $x + H$ is a quasi-translation.*

Proof:

i) Assume $x + H$ is a quasi-translation. Then $(x - H) \circ (x + H) = x$, whence

$$H(x + H) = (x - (x - H)) \circ (x + H) = x + H - x = H$$

and

$$H(x + (m + 1)H) = H(x + H + mH) = H(x + H + mH(x + H))$$

Assume that $H(x + mH) = H(x)$. Substituting $x = x + H$ in it gives

$$H(x + (m + 1)H) = H(x + H + mH(x + H)) = H(x + H) = H$$

whence

$$H(x + mH) = H$$

follows for all $m \in \mathbb{N}$ by induction. Consequently,

$$H(x + tH) = H \tag{1}$$

is an equality of polynomial maps, where t is a new indeterminate. If we first differentiate (1) to t and then substitute $t = 0$, then we get

$$\mathcal{J}H \cdot H = 0 \tag{2}$$

which is equivalent to $D^2x_i = DH_i = 0$ for all i.

ii) Assume $D^2x_i = 0$ for all i. Then $H_i \in \ker D$ for all i, whence

$$H_i = (\exp D)H_i = H_i((\exp D)x_1, \dots, (\exp D)x_n) = H_i(x + H)$$

and

$$(x - H) \circ (x + H) = x + H - H(x + H) = x + H - H = x$$

iii) Since $x - tH$ is an invertible polynomial map, the entries of

$$(\mathcal{J}(x - tH))^{-1} = (I_n - t\mathcal{J}H)^{-1} = I_n + t\mathcal{J}H + t^2(\mathcal{J}H)^2 + \cdots$$

are contained in $\mathbb{C}[t]$. It follows that $\mathcal{J}H$ is nilpotent.

This completes the proof of proposition 1.1. $\square$

The following proposition characterizes the homogeneous relations between the components of H:

Proposition 1.2 *Let g be homogeneous of degree e. Then $D^{e+1}g = 0$. Furthermore, $D^e g = 0$, if and only if $g(H) = 0$.*

Proof: Looking at the coefficient of t^e in

$$(\exp tD)g = g((\exp tD)x_1, \dots, (\exp tD)x_n) = g(x_1 + tH_1, ..., x_n + tH_n)$$

we get

$$D^e g = e!g(H) \tag{3}$$

and the assertions of proposition 1.2 follow. $\square$

Corollary 1.3 *A linear form R is a linear relation between the components of H, if and only if $R \in \ker D$.*

The above corollary shows that quasi-translations $x+H$ without linear relations between the components of H are a special type of locally nilpotent derivations without linear kernel elements. We will construct such quasi-translations.

2 Homogeneous quasi-translations with linearly independent components

Put

$$H_{A,B}^{[n]} := \begin{pmatrix} B(Ax_1 - Bx_2) \\ A(Ax_1 - Bx_2) \\ B(Ax_3 - Bx_4) \\ A(Ax_3 - Bx_4) \\ \vdots \\ B(Ax_{n-1} - Bx_n) \\ A(Ax_{n-1} - Bx_n) \end{pmatrix} \qquad (n \text{ even})$$

where A and B are indeterminates so that

$$D_{A,B}^{[n]} := \sum_{i=1}^{n} (H_{A,B}^{[n]})_i \frac{\partial}{\partial x_i}$$

is a derivation on the ring $\mathbb{C}[A,B,x]$. Then $(\mathcal{J}H_{A,B}^{[n]})^2 = 0$, for

$$\mathcal{J}H_{A,B}^{[n]} = \begin{pmatrix} AB & -B^2 & & & \emptyset \\ A^2 & -AB & & & \\ & & \ddots & & \\ & & & AB & -B^2 \\ \emptyset & & & A^2 & -AB \end{pmatrix}$$

Since $H_{A,B}^{[n]}$ is homogeneous of degree 1, it follows from Eulers formula that

$$\mathcal{J}H_{A,B}^{[n]} \cdot H_{A,B}^{[n]} = \mathcal{J}H_{A,B}^{[n]} \cdot \mathcal{J}H_{A,B}^{[n]} \cdot x = 0$$

So $D_{A,B}^{[n]}$ satisfies

$$(D_{A,B}^{[n]})^2 x_i = 0 \tag{4}$$

for all i.

If $i, j \le n/2$, then

$$\begin{aligned} D_{A,B}^{[n]}(x_{2i-1}x_{2j}) &= (H_{A,B}^{[n]})_{2i-1}x_{2j} + x_{2i-1}(H_{A,B}^{[n]})_{2j} \\ &= B(Ax_{2i-1} - Bx_{2i})x_{2j} + x_{2i-1}A(Ax_{2j-1} - Bx_{2j}) \\ &= -B^2 x_{2i}x_{2j} + A^2 x_{2i-1}x_{2j-1} \end{aligned}$$

The right hand side is symmetric in i, j, whence

$$x_{2i-1}x_{2j} - x_{2i}x_{2j-1} \in \ker D_{A,B}^{[n]} \quad (i, j \le n/2) \tag{5}$$

Now assume that $n \ge 6$ even and take

$$a := x_1x_4 - x_2x_3$$

and

$$b := x_3x_6 - x_4x_5$$

Since $a, A, b, B \in \ker D^{[n]}_{A,B}$, $D^{[n]}_{A,B}$ induces a derivation $D^{[n]}_{a,b}$ on $\mathbb{C}[A,B,x]/(A - a, B - b) = \mathbb{C}[x]$, for which $a, b \in \ker D^{[n]}_{a,b}$. From (4), it follows that

$$(D^{[n]}_{a,b})^2 x_i = 0 \tag{6}$$

for all i.

Put

$$c := x_1x_6 - x_2x_5$$

and

$$f := c(ax_{n-1} - bx_n)$$

Since $D^{[n]}_{a,b}$ is locally nilpotent, $\ker D^{[n]}_{a,b}$ is factorially closed. From $c \in \ker D$, $(H^{[n]}_{a,b})_n \in \ker D^{[n]}_{a,b}$ and $(ax_{n-1} - bx_n) \mid (H^{[n]}_{a,b})_n$, $f \in \ker D^{[n]}_{a,b}$ follows.

Theorem 2.1 *Assume $n \geq 6$ even. Then $x + H^{[n]}_{a,b}$ and $(x, x_{n+1}) + (H^{[n]}_{a,b}, f)$ are quasi-translations, for which there are no linear relations between the components of $H^{[n]}_{a,b}$ resp. $(H^{[n]}_{a,b}, f)$. Furthermore, $H^{[n]}_{a,b}$ resp. $(H^{[n]}_{a,b}, f)$ is homogeneous of degree 5.*

Proof: We already saw that $x + H^{[n]}_{a,b}$ is a quasi-translation. Since $f \in \ker D^{[n]}_{a,b}$ and the components of $(H^{[n]}_{a,b}, f)$ are contained in $\mathbb{C}[x]$ (no x_{n+1}), $\mathcal{J}_{n+1}(H^{[n]}_{a,b}, f)^2 \cdot (x, x_{n+1}) = 0$, i.e. $(x, x_{n+1}) + (H^{[n]}_{a,b}, f)$ is a quasi-translation.

So it remains to show the linear independence of the components of $H^{[n]}_{a,b}$ resp. $(H^{[n]}_{a,b}, f)$. For that purpose, define $y = (t, t^2 - 1, t^3, t^4, t^5 - 1, t^6, t^7, \ldots, t^n)$. Then $a(y) = t^3$, $b(y) = t^4$ and $c(y) = t^5 + t^2 - 1$. It follows that for all $i \leq n/2$,

$$(H^{[n]}_{a,b})_{2i-1}(y) = -t^{8+2i} + O(t^{7+2i})$$

$$(H^{[n]}_{a,b})_{2i}(y) = -t^{7+2i} + O(t^{6+2i})$$

and

$$f = -t^{9+n} + O(t^{8+n})$$

So all components of $H^{[n]}_{a,b}(y)$ resp. $(H^{[n]}_{a,b}, f)(y)$ have different order in t, and the linear independence of the components of $H^{[n]}_{a,b}$ resp. $(H^{[n]}_{a,b}, f)$ follows. $\square$

Corollary 2.2 *The Linear Dependence Problem for Homogeneous Jacobians has a negative answer in dimension 6 and up.*

In 1876 already, P. Gordan and M. Nöther proved that in dimension $n \leq 4$, no counterexamples of the quasi-translation type to the Linear Dependence Problem for Homogeneous Jacobians exist, see [9]. More generally, they showed that for a quasi-translation $x+H$ in any dimension $n \geq 3$, with H homogeneous and $\operatorname{rk} \mathcal{J}H \leq 2$, there are even two independent linear combinations between the components of H. In proposition 1.2 of [4], it is shown that $\operatorname{rk} \mathcal{J}H \leq n-2$ if $x+H$ is a quasi-translation with H homogeneous and $n \geq 3$, whence $\operatorname{rk} \mathcal{J}H \leq 2$ in case $n \leq 4$.

P. Gordan and M. Nöther use geometric methods to get this result. In [5], the author gives an algebraic proof of a slightly more general result. P. Gordan and M. Nöther did some research in dimension 5 as well, the only dimension for which it is not known yet whether the components of H need to be linearly dependent. In [5], it is shown that one only need to consider quasi-translations $x+H$, where H is of the form

$$H = (h_1(p,q), h_2(p,q), h_3(p,q), h_4(p,q), r)$$

with $h = (h_1, h_2, h_3, h_4)$ homogeneous and p, q homogeneous of the same degree. In the spirit of C. Olech, I promise a bottle of Joustra Beerenburg (Frisian spirit) for the one who first solves the problem whether for quasi-translations $x+H$ in dimension 5 with H homogeneous, the components of H need to be linearly dependent.

Although in dimension 4, no counterexamples of the quasi-translation type exist to the Linear Dependence Problem for Homogeneous Nilpotent Jacobians, this problem is still open in dimension 4 for degree 4 and up.

3 Other homogeneous Jacobians without linear dependences

The main theorem of this section is the following:

Theorem 3.1 *Let $H = (H_1, H_2, \ldots, H_s, \ldots, H_n)$ and*

$$D = \sum_{i=1}^{s} H_i \frac{\partial}{\partial x_i}$$

Assume that $H_i \in \ker D$ for all i and $H_i \in \mathbb{C}[x_1, x_2, \ldots, x_{i-1}]$ for all $i > s$. Then $x+H$ is an invertible polynomial map. In particular, $\mathcal{J}H$ is nilpotent if H is homogeneous of degree 2 at least.

Proof: Since $D^2 x_i = 0$ for all i, $H_{s+1} \in \ker D$ and $H_{s+1} \in \mathbb{C}[x_1, x_2, \ldots, x_s]$, we have

$$\begin{aligned} H_{s+1} &= (\exp D) H_{s+1} \\ &= H_{s+1}((\exp D)x_1, (\exp D)x_2, \ldots, (\exp D)x_s) \\ &= H_{s+1}(x_1 + H_1, x_2 + H_2, \ldots, x_s + H_s) \end{aligned}$$

So if we define the polynomial map G_i as

$$G_i = (x_1, x_2, \ldots, x_{i-1}, x_i - H_i, x_{i+1}, \ldots, x_n)$$

for all $i > s$, then

$$G_{s+1} \circ (x+H) = (x_1 + H_1, x_2 + H_2, \ldots, x_s + H_s, x_{s+1}, x_{s+2} + H_{s+2}, \ldots, x_n + H_n)$$

By induction on s,

$$G_n \circ G_{n-1} \circ \cdots \circ G_{s+1} \circ (x+H) = (x_1 + H_1, x_2 + H_2, \ldots, x_s + H_s, x_{s+1}, x_{s+2}, \ldots, x_n)$$

follows, which is the quasi-translation corresponding to D and hence invertible. Consequently, $x+H$ is invertible, since each G_i with $i > s$ is. So $\det(I_n + \mathcal{J}H) = \det \mathcal{J}(x+H) \in \mathbb{C}^*$. This implies that $\mathcal{J}H$ is nilpotent in case H is homogeneous of degree at least 2 (see [7, 6.2.11]). □

Again, take $a := x_1 x_4 - x_2 x_3$ and $b := x_3 x_6 - x_4 x_5$.

Corollary 3.2 *Put*

$$H = (x_5 H^{[4]}_{a, x_5^2}, a^3)$$

Then H is a counterexample in dimension 5 to the Linear Dependence Problem for Homogeneous Jacobians. Observe that H has degree 6.

Proof: We apply theorem 3.1 with $s = 4$, so let

$$D = \sum_{i=1}^{4} H_i \frac{\partial}{\partial x_i}$$

Then D and $D^{[4]}_{a, x_5^2}$ have the same kernel, which contains x_5. It follows that $D^2 x_i = 0$ for all i. Furthermore, $a^3 \in \ker D$ and $a^3 \in \mathbb{C}[x_1, x_2, x_3, x_4]$, so $x + H$ is invertible. Since H is homogeneous of degree 6, the nilpotency of $\mathcal{J}H$ follows.

To show the linear independence of the components of H, put $y = (t+1, t^2, t^3, t^4, 1)$. Then $a(y) = t^4$ and the leading terms to t in the components of $H(y)$ are

$$t^5, t^9, t^7, t^{11}, t^{12}$$

whence the components of H are linearly independent. □

Corollary 3.3 *Let $n \geq 6$ and put*

$$H = (x_5 H^{[4]}_{x_5,x_6}, a^2, x_5^4, \ldots, x_{n-1}^4)$$

Then H is a counterexample of degree 4 to the Linear Dependence Problem for Homogeneous Jacobians.

Proof: Again we apply theorem 3.1 with $s = 4$. The nilpotency of $\mathcal{J}H$ follows in a similar matter as in the previous corollary.

To show the linear independence of the components of H, put $y = (t, t^2, t^3, t^4+1, 1, t^6, t^7, \ldots, t^n)$. Then $a(y) = t$ and the leading terms to t of $H(y)$ are

$$-t^{14}, -t^8, -t^{16}, -t^{10}, t^8, 1, t^{24}, \ldots, t^{4(n-1)}$$

whence the components of H are linearly indepenent. $\square$

Corollary 3.4 *Let $n \geq 10$ and put*

$$H = (H^{[6]}_{x_9,x_{10}}, x_9 a, x_9 b, x_8 a - x_7 b, x_9^3, x_{10}^3, \ldots, x_{n-1}^3)$$

Then H is a cubic counterexample to the Linear Dependence Problem for Homogeneous Jacobians.

Proof: We apply theorem 3.1 with $s = 8$, so let

$$D = \sum_{i=1}^{8} H_i \frac{\partial}{\partial x_i}$$

From $a, b \in \ker D$ and the construction of H_7 and H_8, $H_9 \in \ker D$ follows. So the conditions of theorem 3.1 are fulfilled and $\mathcal{J}H$ is nilpotent.

Put $y = (t, t^2, t^3 + 1, t^4, t^5, t^6, t^7, t^8, 1, t^{10}, t^{11}, \ldots, t^n)$. Then $a(y) = -t^2$ and $b(y) = t^6$. So the leading terms to t of $H(y)$ are

$$-t^{22}, -t^{12}, -t^{24}, -t^{14}, -t^{26}, -t^{16}, -t^2, t^6, -t^{13}, 1, t^{30}, \ldots, t^{3(n-1)}$$

So the components of H are linearly independent. $\square$

4 Final remarks

In [2], techniques are given to make Jacobians symmetric by way of stabilization, such that the rows of the Jacobian remain linearly independent over $\mathbb{C}$. The Jacobian becomes a Hessian, i.e. the Jacobian of a gradient.

Assume we have a nilpotent Jacobian $\mathcal{J}H$ with $H = (H_1, H_2, \ldots, H_n)$, which rows are linearly independent over $\mathbb{C}$. Then the columns do not need to be linearly independent. But if $\mathcal{J}H \cdot \lambda = 0$ for some nonzero $\lambda \in \mathbb{C}^n$, then each H_i is contained in the kernel of

$$D = \sum_{i=1}^{n} \lambda_i \frac{\partial}{\partial x_i}$$

Now there is a linear coordinate system $y = y_1, y_2, \ldots, y_n$ such that $y_i \in \ker D$ for all $i \le n-1$, whence H can be expressed as a polynomial map in $y_1, y_2, \ldots, y_{n-1}$. So the last column of $\mathcal{J}_y H$ is zero, where $\mathcal{J}_y$ is the Jacobian with respect to y. Proceeding this way, we can ensure that the first s columns of $\mathcal{J}_z(H)$ are linearly independent over $\mathbb{C}$ and the last $n-s$ columns of $\mathcal{J}_z(H)$ are zero for some linear coordinates system z, i.e. $H_i \in \mathbb{C}[z_1, z_2, \ldots, z_s]$ for all i.

Theorem 4.1 *Assume $H \in \mathbb{C}[x_1, x_2, \ldots, x_s]^n$ has a nilpotent Jacobian. Assume further that both the rows of $\mathcal{J}H$ and the first s columns of $\mathcal{J}H$ are linearly independent over $\mathbb{C}$. Then*

$$h = \sum_{i=1}^{n} x_i H_i(x_1 + \mathrm{i}x_{n+1}, x_2 + \mathrm{i}x_{n+2}, \ldots, x_s + \mathrm{i}x_{n+s}) + \sum_{i=s+1}^{m} (x_i + \mathrm{i}x_{n+i})^{d+1}$$

has a nilpotent Hessian if $s \le m \le n$. Furthermore, the rows of $\mathcal{H}h = \mathcal{J}(\nabla h)$ are linearly independent. Moreover, h is homogeneous in case H is homogeneous of degree d.

This theorem can be proved with techniques presented in [2].

Talking about Hessians, you might wonder why P. Gordan and and M. Nöther were interested in (homogeneous) quasi-translations. The reason is that they were studying (homogeneous) Hessians with determinant zero. For such Hessians $\mathcal{H}h$, there exists a (homogeneous) relation R between the components of ∇h. Now if $h \in \mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_n]$, then $x + H$ with

$$H = \nabla R \circ \nabla h$$

is a quasi-translation, see [3] or just the original paper [9]. It is remarkable that P. Gordan and M. Nöther already juggled with nilpotent derivations before derivations were invented.

For quadratic homogeneous maps, in particular quadratic linear maps, the linear dependence problem was first stated as Conjecture 11.3 by K. Rusek in [13]. This conjecture is still open.

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