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Quantum Mechanics and Representation Theory: The New Synthesis

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Abstract. Quantum mechanics and representation theory, in the sense of unitary representations of groups on Hilbert spaces, were practically born together between 1925–1927, and have continued to enrich each other till the present day. Following a brief historical introduction, we focus on a relatively new aspect of the interaction between quantum mechanics and representation theory, based on the use of K-theory of C^*-algebras. In particular, the study of the K-theory of the reduced C^*-algebra of a locally compact group (which for a compact group is just its representation ring) has culminated in two fundamental conjectures, which are closely related to quantum theory and index theory, namely the Baum–Connes conjecture and the Guillemin–Sternberg conjecture. Although these conjectures were both formulated in 1982, and turn out to be closely related, so far there has been no interplay between them whatsoever, either mathematically or sociologically. This is presumably because the Baum–Connes conjecture is nontrivial only for noncompact groups, with current emphasis entirely on discrete groups, whereas the Guillemin–Sternberg conjecture has so far only been stated for compact Lie groups. As an elementary introduction to both conjectures in one go, indicating how the latter can be generalized to the noncompact case, this paper is a modest attempt to change this state of affairs.

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1. The Old Synthesis

Recall that a unitary representation of a (topological) group G on a Hilbert space H is a (strongly continuous) homomorphism of G into the group of unitary operators on H. A brief look at some key dates in the history of physics and mathematics shows that quantum mechanics and unitary representation theory were born together:

1925 Finite-dimensional representation theory of semi-simple Lie groups (Weyl).
1925 Matrix mechanics (Heisenberg).
1926 Wave mechanics (Schrödinger).
1926 Quantum mechanics (Dirac).
1927 Hilbert space formulation of quantum mechanics (von Neumann).
1927 Unitary group representations on Hilbert space (Peter–Weyl).

In fact, this is no coincidence, as the concept of a unitary group representation on a possibly infinite-dimensional Hilbert space was directly inspired by quantum mechanics. Hermann Weyl asked himself which type of infinite-dimensional topological vector space would give him the best possible generalization of the decomposition theory of the regular representation of a finite group to the compact case, and found the answer in John von Neumann's brand new concept of abstract Hilbert space. This concept, in turn, was directly inspired by the recent development of quantum mechanics, with which Weyl was thoroughly familiar. To quote George Mackey [31]:

> The Lebesgue integral and Hilbert spaces of square-integrable functions were still strange and unfamiliar objects to most mathematicians in the 1920s, however, and a systematic theory of group representations on an infinite-dimensional Hilbert space was slow to develop. When it did, this development was directly inspired by the discovery of quantum mechanics in the period between 1924 and 1927, especially by von Neumann's success in putting this theory on a rigorous, mathematically coherent form based on the theory of operators in Hilbert space.

It remains a remarkable example of the 'unreasonable effectiveness of physics in mathematics' that among all possible functional-analytic settings Hilbert spaces turned out to present such a good context for representation theory. In mathematics, unitary representation theory generalises Fourier analysis, turns out to be relevant to practically all fields of mathematics ranging from probability theory to number theory, and in addition has created a field of its own [31]. In quantum physics, unitary representations come in whenever symmetry plays a role, from solid state physics to elementary particles and quantum field theory.

The point of this paper is to explain that a second notion of representation, closely related to the unitary one, still plays a role in quantum mechanics. To put this in perspective, let us mention a few highlights of the 'old synthesis' of unitary representation theory and quantum mechanics, which paved the way for the 'new synthesis' discussed shortly.

- *Uniqueness of the Schrödinger representation* (von Neumann, 1931). As listed above, quantum mechanics was originally discovered in two seemingly totally different forms, matrix mechanics and wave mechanics. Following heuristic arguments by physicists such as Pauli and Schrödinger, von Neumann used the unitary representation theory of what is now called the Heisenberg group to show that these versions of quantum mechanics were not merely unitarily equivalent, but even unique up to such equivalence [47]. See [34] for a detailed historical discussion of von Neumann's decisive intervention.
- *Representation theory of the Poincaré group* (Wigner, 1939). Wigner's classification of all unitary representations of the Poincaré group provided the second example (following von Neumann's study of the Heisenberg group just mentioned) where the complete representation theory of a Lie group that

was neither compact nor Abelian was found [50]. Here Wigner was directly motivated by a problem in quantum physics, namely the classification of relativistic particles. Moreover, this was the first full-fledged application of the induction technique later developed by Mackey [31].

- *Geometric quantization* (Kostant, Souriau, 1966). This is a technique purporting to construct unitary representations of Lie groups from suitable actions on symplectic manifolds, which in classical mechanics play the role of phase spaces [25, 43]. It encompasses both Kirillov's method of finding representations of nilpotent groups from coadjoint orbits [23, 24] and the Borel–Weil construction of irreducible representations of compact Lie groups on spaces of holomorphic sections of holomorphic vector bundles (cf. [9]). The two highlights just mentioned both fall within the scope of geometric quantization, and in fact provided motivating examples (the first for Kostant and the second for Souriau). Geometric quantization in some sense culminated in the Guillemin–Sternberg conjecture, which will play a crucial role in what follows; cf. [17] for a modern treatment. Also cf. Section 2 below.
- *Berezin quantization* (Berezin, 1973). This is in some sense a competitor to geometric quantization, which is well adapted to discuss the relationship between quantization on symmetric domains and representation theory. See [44] and [26] for complementary treatments, also cf. the remainder of this volume.

Although the interaction between quantum mechanics and representation theory in the above tradition remains fascinating and continues unabated, we now turn to a relatively new technique in this interaction. This is the K-theory of C^*-algebras, which tool in our opinion has given rise to a 'new synthesis' of these fields.

2. Group C^*-Algebras

First recall that a C^*-algebra A is a Banach algebra with involution such that $\|a^*a\| = \|a\|^2$ for all $a \in A$; equivalently, A is isomorphic to an involutive norm-closed subalgebra of the algebra of all bounded operators on some Hilbert space. A representation of a C^*-algebra on a Hilbert space H is a linear map $\pi\colon A \to B(H)$ (where $B(H)$ is the algebra of all bounded operators on H) such that $\pi(ab) = \pi(a)\pi(b)$ and $\pi(a)^* = \pi(a^*)$. Such maps are automatically continuous. The basic equivalence relation between C^*-algebras is not isomorphism but Morita equivalence: provided that the C^*-algebras in question have a so-called countable approximate identity, two C^*-algebras A and B are said to be Morita equivalent, written $A \overset{M}{\sim} B$, when they are *stably* isomorphic. This, in turn, means that A is isomorphic to $B \otimes B_0(H)$, or vice versa, where $B_0(H)$ is the C^*-algebra of all compact operators on some Hilbert space H. This concept is due to Marc Rieffel; see [39] for a modern treatment.

Our basic examples will be group C^*-algebras. Let G be a locally compact group with Haar measure dx, assumed unimodular for simplicity. Then $L^1(G)$ is an involutive Banach algebra with (convolution) product $f*g(x) = \int_G dy f(xy^{-1})g(y)$ and involution $f^*(x) = f(x^{-1})$. The L^1-norm is not a C^*-norm, but $L^1(G)$ can be equipped with a new norm $\|f\| = \sup_\pi\{\|\pi(f)\|\}$, where the supremum is taken over all continuous representations π of $L^1(G)$ on a Hilbert space. This new norm is indeed a C^*-norm by construction; the completion of $L^1(G)$ in this norm is the group C^*-algebra $C^*(G)$. The basic feature of $C^*(G)$ is that its representation theory in the above sense coincides with the unitary representation theory of G: there is a bijective correspondence between nondegenerate representations $\pi(C^*(G))$ (i.e., representations that are not identically zero on a proper closed subspace of the representation space) and unitary representations $U(G)$ on a Hilbert space H, given, for $f \in L^1(G)$, by $\pi(f) = \int_G dx f(x)U(x)$. This correspondence preserves (topological) irreducibility.

In what follows, the so-called reduced group C^*-algebra $C_r^*(G)$ plays an important role. This is the image of $C^*(G)$ in the representation π_L defined by the (left) regular representation $U_L(G)$ on $L^2(G)$ under the above correspondence; for unimodular groups one just has $\pi_L(f)\psi = f * \psi$. Hence $C_r^*(G) \cong C^*(G)/\ker(\pi_L)$. We say that a unitary representation $U(G)$ is weakly contained in $L^2(G)$ when the corresponding representation $\pi(C^*(G))$ satisfies the following condition: $\pi_L(f) = 0$ for some $f \in C^*(G)$ implies $\pi(f) = 0$. One then has a bijective correspondence between nondegenerate representations of $C_r^*(G)$ and unitary representations $U(G)$ that are weakly contained in $L^2(G)$. The irreducibles among the latter form, up to equivalence, the reduced unitary dual $\hat{G}_r$. This is a closed subspace of $\hat{G}$.

Here are some basic examples [6, 41].

(1) G Abelian: $C^*(G) = C_r^*(G) \cong C_0(\hat{G})$, where $\hat{G}$ is the unitary dual of G and $C_0(X)$ is the space of all continuous functions on a locally compact Hausdorff space X that vanish at infinity. This easily follows from the theory of Fourier analysis on Abelian groups.
(2) G compact: $C^*(G) = C_r^*(G) \overset{M}{\sim} C_0(\hat{G})$, which is immediate from the Peter–Weyl theory.
(3) G connected complex semi-simple Lie: $C_r^*(G) \overset{M}{\sim} C_0(\hat{G}_r)$, where the reduced unitary dual is given in standard notation as $\hat{G}_r = \widehat{MA}/W$, in terms of the minimal parabolic subgroup $P = MAN$ and Weyl group W. In this case, $\hat{G}_r$ happens to be Hausdorff. See [38].
(4) G connected real reductive Lie: this time $C_r^*(G)$ is in general no longer Morita equivalent to a commutative C^*-algebra; see [48]. This is closely related to the fact that, in general, $\hat{G}_r$ fails to be Hausdorff.

These examples illustrate the philosophy of noncommutative geometry (see [11], esp. §II.4), in which $C_r^*(G)$ is seen as the correct description of $\hat{G}_r$ whenever the latter is non-Hausdorff and $C_0(\hat{G}_r)$ therefore fails to characterize or even describe it. When $\hat{G}_r$ is Hausdorff, $C_r^*(G)$ is expected to be Morita equivalent to $C_0(\hat{G}_r)$ (although the author is unaware of a general theorem in this direction), so that the two C^*-algebras are interchangeable when analyzed with the typical tools of noncommutative geometry, all of which are Morita invariant. In this ideology, the inclusion $\hat{G}_r \hookrightarrow \hat{G}$ is dual to the projection $C^*(G) \to C^*_r(G)$.

In addition, group C^*-algebras emerge quite naturally in quantization theory. Recall that a Poisson algebra is a commutative algebra $\tilde{A}$ over $\mathbb{C}$ equipped with a Lie bracket $\{\, ,\, \}$, such that for each $f \in \tilde{A}$ the map $g \mapsto \{f, g\}$ is a derivation of $\tilde{A}$ as a commutative algebra, and that a Poisson manifold P is a manifold equipped with a Lie bracket on $\tilde{A} = C^\infty(P)$ (called a Poisson bracket), such that it becomes a Poisson algebra with respect to pointwise multiplication. Poisson manifolds (or rather the Poisson algebras they define) are the classical analogues of C^*-algebras [26]. The most fundamental properties of a Poisson manifold P are firstly that each $f \in C^\infty(P)$ defines a vector field ξ_f by $\xi_f(g) = \{f, g\}$ (called the Hamiltonian vector field of f), and secondly that P is foliated by its symplectic leaves. Here a symplectic manifold is simply regarded as a Poisson manifold with the property that the Hamiltonian vector fields span its tangent bundle, and the symplectic leaves of P are simply the integral manifolds of the distribution defined by the Hamiltonian vector fields. See [49].

The idea of deformation quantization is to 'deform' a Poisson algebra into a certain type of noncommutative associative algebra; different schools use different algebras here, but following Rieffel [40] we prefer to work with C^*-algebras. The case in point is the so-called Lie–Poisson manifold $\mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of a Lie group G, whose dual vector space $\mathfrak{g}^*$ is equipped with the unique Poisson structure that on linear functions (i.e., elements of $\mathfrak{g}^{**} = \mathfrak{g}$) is the Lie bracket. Now, both $C^*(G)$ and $C^*_r(G)$ turn out to be deformations of the Lie–Poisson algebra $C^\infty(\mathfrak{g}^*)$ in the strongest possible sense [26]. The connectedness properties of G seem too subtle for $\mathfrak{g}^*$ to catch; the claim holds for any Lie group G with Lie algebra $\mathfrak{g}$. See below for the difference between $C^*(G)$ and $C^*_r(G)$ in the light of $\mathfrak{g}^*$.

This result may be seen in the light of the 'orbit philosophy' of geometric quantization, as follows. In this philosophy, one attempts to associate a unitary irreducible representation of a Lie group G to a coadjoint orbit in $\mathfrak{g}^*$, seen as a symplectic manifold (in fact, the symplectic structure of a coadjoint orbit used in geometric quantization is precisely the one it inherits as a symplectic leaf of $\mathfrak{g}^*$). This is sometimes completely successful, as in the case of exponential nilpotent Lie groups, where one obtains a bijective correspondence between coadjoint orbits and unitary irreducible representations [23]. In other cases one has partial success, as for compact Lie groups, where any irreducible representation corresponds to some orbit, but not all orbits can be quantized into a representation. Sometimes the alleged correspondence breaks down altogether. See [24] for a status report. However, the orbit philosophy may be turned into a theorem if one changes its meaning somewhat [26].

To do so, we recall the notion of a strongly Hamiltonian action of a Lie group G on a symplectic manifold M (cf. [1] or [26]). This means firstly that the action is smooth, secondly, that the symplectic form ω is G invariant, and thirdly that a so-called equivariant momentum map $J\colon M \to \mathfrak{g}^*$ exists. Defining functions J_X for each $X \in \mathfrak{g}$ by $J_X(m) = \langle J(m), X\rangle$, by definition of J one then has $\xi_{J_X} = X^M$, where X^M is the fundamental vector field of X defined by the G action. Such actions are the classical analogue of unitary representations of G.

The representation theory of $C^*(G)$ has a classical counterpart as well. Following Alan Weinstein [49], we define a representation of a Poisson manifold P as a complete symplectic realization, that is, as a complete Poisson map $M \xrightarrow{\rho} P$, where M is a symplectic manifold; the completeness property means that complete Hamiltonian vector fields on P pull back to complete vector fields on M. When G is connected and simply connected, it then follows (cf. Thm. III.1.2.6 in [26]) that there is a bijective correspondence between representations of the Lie–Poisson manifold $\mathfrak{g}^*$ and strongly Hamiltonian actions of G with complete momentum map. This is evidently the classical analogue of the correspondence between unitary representations of G and nondegenerate representations of $C^*(G)$.

Returning to the orbit philosophy, we note that there is a satisfactory notion of irreducibility of a representation of a Poisson manifold P, namely the condition that the Hamiltonian vector fields $\xi_{\rho^* f}$, $f \in C^\infty(P)$, span the tangent bundle of M. In then follows (cf. Thm. I.2.6.7 in [26]) that the irreducible representations of P are given by its symplectic leaves, with ρ given by the inclusion map (or by covering spaces thereof). Now the symplectic leaves of $\mathfrak{g}^*$ are precisely its coadjoint orbits, so that the irreducible representations of $\mathfrak{g}^*$ are just its coadjoint orbits (or their coverings). In other words, the Lie–Poisson manifold $\mathfrak{g}^*$ is just the assembly of its irreducible representations (modulo possible covering spaces).

Instead of deforming or quantizing single such irreducible representations, namely coadjoint orbits, one should better quantize all of them simultaneously, which amounts to quantizing $\mathfrak{g}^*$. As mentioned, this leads to $C^*(G)$ (or $C^*_r(G)$), whose irreducible representations, in turn, bijectively correspond the to the unitary irreducible representations of G (that are weakly contained in $L^2(G)$). Therefore, in some sense $\mathfrak{g}^*/\mathrm{Ad}^*(G)$ is the classical counterpart of $\hat{G}_r$ (perhaps $\hat{G}$ should be compared with the space of all irreducible representations of $\mathfrak{g}^*$ up to symplectomorphism). In conclusion, the successful quantization of $\mathfrak{g}^*$ as a whole by $C^*(G)$ replaces the somewhat flawed programme of quantizing individual coadjoint orbits in $\mathfrak{g}^*$ by unitary irreducible representations of G.

3. K-Theory of C^*-Algebras

Referring the reader to textbooks such as [7] for a detailed discussion, we now briefly review the basic definitions in the K-theory of C^*-algebras. First, when A has a unit, $K_0(A)$ is the Abelian group generated by homotopy classes $[p]$ of projections p (i.e., $p^* = p^2 = p$) in $M_\infty(A)$ and relations $[p] + [q] = [p \oplus q]$. Here the involutive algebra $M_\infty(A)$ is the direct (algebraic) limit in n of the $n \times n$ matrices over A, where n is arbitrary but finite; the notion of homotopy is given by path-connectedness with respect to the norm-topology. Equivalently, $K_0(A)$ is the Abelian group generated by isomorphism classes of finitely generated projective modules E over A, with relations $[E] + [F] = [E \oplus F]$, so that the C^*-algebraic definition of K_0 happens to coincide with the purely algebraic one. As a special case, for $A = C(X)$ and X compact Hausdorff one recovers the topological K-theory of Atiyah and Hirzebruch, which is defined in terms of vector bundles over X, through the natural isomorphsim $K_0(C(X)) \cong K^0(X)$.

This definition is functorial, in that an involutive homomorphism ρ: $A \to B$ induces a homomorphism ρ_*: $K_0(A) \to K_0(B)$ in the obvious way, sending $[p]$ to $[\rho(p)]$. Functoriality is then used to define $K_0(A)$ also when A has no unit: in that case, $K_0(A)$ is defined as the kernel of the map π_*: $K_0(\tilde{A}) \to K_0(\tilde{A}/A) \cong \mathbb{Z}$, where $\tilde{A}$ is the unitization of A, so that $\tilde{A}/A \cong \mathbb{C}$, with $K_0(\mathbb{C}) \cong K^0(\text{point}) \cong \mathbb{Z}$. The group C^*-algebras $C^*(G)$ and $C^*_r(G)$ have a unit iff G is discrete; hence for nondiscrete G one always has to work with finitely generated projective modules over $\widetilde{C^*(G)}$. In addition, one has $K_0(C_0(X)) \cong K^0(X)$, where K^0 stands for topological K-theory with compact support.

C^*-algebraic K-theory starts to differ from algebraic K-theory for higher K-groups: for a C^*-algebra A, with or without unit, one defines $K_i(A) = K_0(S^i A)$, where $A \mapsto SA$ is the suspension map, given by $SA = C_0(\mathbb{R}, A)$, and S^i is its ith iteration. The basic (and in some sense the only) theorem in C^*-algebraic K-theory is Bott periodicity, stating that $K_{i+2}(A) \cong K_i(A)$, natural in A. Consequently, the long exact sequence of K-theory collapses into a cyclic six-term sequence: given a short exact sequence $0 \to J \to A \to B \to 0$ of C^*-algebras, one has the periodic exact sequence

$$K_0(B) \cong K_2(B) \to K_1(J) \to K_1(A) \to K_1(B) \to K_0(J) \to K_0(A) \to K_0(B).$$

Neither the definiton of the higher K-groups, nor Bott periodicity are shared by algebraic K-theory. However, a fundamental property of both theories lies in the fact that K-theory is invariant under Morita equivalence.

In what follows, for reasons to emerge we shall concentrate on $K_\bullet(C^*_r(G))$ (where $\bullet = 0, 1$). On the one hand, this object is related to the finitely generated projective modules over $C^*_r(G)$ (or its unitization), which are generally different from its Hilbert space representations as a C^*-algebra. As we shall see, this presents a genuinely new starting point for the interaction between quantum mechanics and representation theory. But on the other hand, since $K(A)$ is interpreted as a noncommutative topological invariant of a C^*-algebra A [11], $K_\bullet(C^*_r(G))$ gives a rough or topological picture of the reduced unitary representation theory of G. For example, returning to the list above, we have $K_\bullet(C^*_r(G)) \cong K^\bullet(\hat{G}_r)$ in the first three cases (for the fourth cf. [6, 48]).

However, already the first two cases illustrate how diverse the K-theoretic picture of the representation theory of G can be. For compact G, one obviously has

$$K_0(C_r^*(G)) = F_{ab}(\hat{G}) = \bigoplus_{\hat{G}} \mathbb{Z};$$

$$K_1(C_r^*(G)) = 0.$$

Here $F_{ab}(\hat{G})$ is the free Abelian group on the unitary dual $\hat{G} = \hat{G}_r$. This means that each element of the unitary dual of G provides a generator of K_0, without relations between the generators. On the other hand, for the simplest noncompact group, $G = \mathbb{Z}$, one has

$$K_0(C_r^*(\mathbb{Z})) = \mathbb{Z};$$

$$K_1(C_r^*(\mathbb{Z})) = \mathbb{Z}.$$

Since $\hat{\mathbb{Z}} = \mathbb{T}$, we conclude that the unitary dual as a whole only provides one generator of K_0, whereas, in contrast to the compact case, this time K_1 is nonvanishing.

It should be clear already from these very simple examples that $K_\bullet(C_r^*(G))$ is an interesting object of study, which for compact G is entirely understood. In that case, it is useful to note that the representation ring $R(G)$ of a compact group (which is the Abelian group generated by equivalence classes $[V]$ of finite-dimensional representations V of G with relations $[V] + [W] = [V \oplus W]$) is isomorphic to $K_0(C_r^*(G))$. If V is irreducible, the pertinent isomorphism maps the generator $[V] \in R(G)$ to the homotopy class of any projection p in $C_r^*(G) \equiv \pi_L(C^*(G))$ for which $\pi_R(G) \restriction pL^2(G) \cong V$, where π_R is the right-regular representation.

A slight refinement of this isomorphism $R(G) \cong K_0(C_r^*(G))$ for compact G shows that, for general almost connected Lie groups (i.e., having finitely many connected components), each irreducible representation $U(G)$ in the discrete series (that is, $U(G)$ is isomorphic to a direct summand of $L^2(G)$) provides a generator of $K_0(C_r^*(G))$ without relations. Hence one has

$$\bigoplus_{\hat{G}_d} \mathbb{Z} \subset K_0(C_r^*(G)).$$

Indeed, if $U(G)$ acts on H, then for any $\psi, \varphi \in H$ the function $f_{\varphi,\psi}\colon x \mapsto (\varphi, U(x)\psi)$ lies in $C_r^*(G)$, and $p_{\varphi,\psi} = d_U f_{\varphi,\psi}$ is a projection for suitable $d_U > 0$ (i.e., the formal dimension of U). One then has $H \cong \pi_L(p_{\varphi,\psi})L^2(G)$ as unitary G modules, and the homotopy class $[p_{\varphi,\psi}]$, which is independent of the coice of ψ and φ, is the generator $[U]$ in $K_0(C_r^*(G))$ corresponding to the discrete series representation U [29].

The modern point of view is that $K_\bullet(C_r^*(G))$ is the appropriate generalization of the representation ring $R(G)$ to the noncompact case. As we shall see, the Baum–Connes conjecture describes the structure of $K_\bullet(C_r^*(G))$. To understand the philosophy behind both this conjecture and the one due to Guillemin–Sternberg discussed afterwards, we recall two key ideas from quantum theory, in fact dating from 1928–1930, the days of the 'old synthesis' of quantum mechanics and representation theory. It is these two ideas that make the 'new synthesis' effective.

4. The Dirac Operator

We first recall the concept of a Dirac operator, originally defined by Dirac in 1928 in his relativistic description of electrons. First, the real Clifford algebra Cl $\mathbb{R}^n$) is generated by 'Dirac matrices' $\gamma_1, \ldots, \gamma_n$ satisfying $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta^{\mu\nu}$. When n is even, which is the important case in what follows, the complexification $\mathbb{C}l(\mathbb{R}^n)$ is isomorphic to a simple matrix algebra with an obviously unique irreducible representation on a space of 'spinors' S_n. Under the even part of $\mathbb{C}l(\mathbb{R}^n)$ (i.e., the part generated by an even number of Dirac matrices), one has a decomposition $S_n = S_n^+ \oplus S_n^-$. (When n is odd, S_n carries an irreducible representation of $\mathbb{C}l(\mathbb{R}^n)$.)

The Dirac operator on $\mathbb{R}^n$ is now defined as

$$\not{D} = \sum_{\mu=1}^{n} \gamma^\mu \frac{\partial}{\partial x^\mu}\colon\ C_c^\infty(\mathbb{R}^n, S_n^+) \to C_c^\infty(\mathbb{R}^n, S_n^-).$$

(When n is odd, one has $\not{D}\colon\ C_c^\infty(\mathbb{R}^n, S_n) \to C_c^\infty(\mathbb{R}^n, S_n)$.)

On a Riemannian manifold M one may perform this construction at least locally. Globally, the Dirac operator may be defined when M admits a so-called spin structure. Firstly, let the compact Lie group Spin(n) be the well-known nontrivial connected double cover of SO(n), with covering projection $\tau\colon$ Spin$(n) \to$ SO(n). Now a spin structure on M is a principal Spin(n)-bundle $\mathfrak{S}$ over M together with an isomorphism $\mathfrak{S} \times_{\mathrm{Spin}(n)} \mathbb{R}^n \cong TM$ of vector bundles; the bundle on the left-hand side is the bundle associated to $\mathfrak{S}$ by τ composed with the defining representation of SO(n) on $\mathbb{R}^n$. For even n, the $S^\pm$ carry irreducible unitary representations of Spin(n), in terms of which the spinor bundles $\mathcal{S}_n^\pm = \mathfrak{S} \times_{\mathrm{Spin}(n)} S_n^\pm$ are defined.

The Dirac operator may then locally be expressed as follows. One first constructs local 'vielbeins' e_μ, $\mu = 1, \ldots, n = \dim(M)$, which are vector fields that are orthonormal with respect to the Riemannian metric g (i.e., $g_x(e_\mu, e_\nu) = \delta_{\mu\nu}$ at each point x). The Riemannian connection ∇ on TM then defines the Levi-Civita symbols Γ by $\nabla_{e_\mu} e_\nu = \Gamma^\rho_{\mu\nu} e_\rho$, in terms of which the Dirac operator is [16]

$$\not{D} = \sum_{\mu=1}^{n} \gamma^\mu (e_\mu - \tfrac{1}{4}\Gamma^\rho_{\mu\nu}\gamma^\rho\gamma^\nu)\colon\ C_c^\infty(M, \mathcal{S}_n^+) \to C_c^\infty(M, \mathcal{S}_n^-).$$

(When n is odd, one of course has $\not{D}\colon\ C_c^\infty(M, \mathcal{S}_n) \to C_c^\infty(M, \mathcal{S}_n)$, where $\mathcal{S}_n = \mathfrak{S} \times_{\mathrm{Spin}(n)} S_n$.)

As on $\mathbb{R}^n$, this procedure defines a first-order linear partial differential operator, probably first written down in this generality by Atiyah and Singer. It may be interesting to quote Michael Atiyah at this point [3]:

> The geometrical significance of spinors is still very mysterious. They appear out of some slick algebra, but the geometrical meaning is obscure (...) When Singer and I were investigating these questions we 'rediscovered' for ourselves the Dirac operator. Had we been better educated in physics, or had there been

the kind of dialogue with physicists that is now common, we would have got there much sooner.

In the case of interest to representation theory, one can explicitly write down the spinor bundles [37]. Let G be an almost connected Lie group with maximal compact subgroup K. Then $T_e(G/K) \cong \mathbb{R}^n$ carries a representation $\pi(K)$ that is defined in the obvious geometric way (i.e., given by the derivative of the left K action on G/K). We assume that $n = \dim(G/K)$ is even. The assumption that G/K has a G invariant spin structure turns out to be equivalent to the assumption that $\pi\colon K \to \mathrm{SO}(n)$ lifts to $\tilde{\pi}\colon K \to \mathrm{Spin}(n)$ (that is, $\pi = \tau \circ \tilde{\pi}$). On this assumption, the spinor bundles on G/K are simply given as the associated vector bundles $\mathcal{S}^{\pm} = G \times_K S_n^{\pm}$, the vector bundles associated to the standard principal K bundle $G \to G/K$ by the representations of K on $S_n^{\pm}$ obtained by composing the pertinent representations of $\mathrm{Spin}(n)$ with $\tilde{\pi}$. We have now dropped the index n. Also, the Dirac operator may be written purely group-theoretically [37].

We will also need the more general concept of a $\mathrm{Spin}^{\mathbb{C}}$ structure on a manifold M and its associated Dirac operator, which we briefly recall [15–17]. Here the group $\mathrm{Spin}(n)$ in the above discussion in replaced by $\mathrm{Spin}^{\mathbb{C}}(n)$, which is a nontrivial central extension of $\mathrm{SO}(n)$ by $U(1)$, defined as $\mathrm{Spin}^{\mathbb{C}}(n) = \mathrm{Spin}(n) \times_{Z_2} U(1)$, where Z_2 is seen as the subgroup $\{(1, 1), (-1, -1)\}$ of $\mathrm{spin}(n) \times U(1)$. Thus one has the obvious homomorphisms $\pi\colon \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \cong \mathrm{Spin}(n)/Z_2$, given by projection on the first factor, and det: $\mathrm{Spin}^{\mathbb{C}}(n) \to U(1)$, defined by $[x, z] \mapsto z^2$.

A $\mathrm{Spin}^{\mathbb{C}}$ structure on M is a principal $\mathrm{Spin}^{\mathbb{C}}(n)$-bundle $\mathfrak{P}$ over M with an isomorphism $\mathfrak{P} \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{R}^n \cong TM$ of vector bundles, where this time the bundle on the left-hand side is the vector bundle associated to $\mathfrak{P}$ by the defining representation of $\mathrm{SO}(n)$ seen as a representation of $\mathrm{Spin}^{\mathbb{C}}(n)$ through π. The spinor bundles are defined in the same way as above, i.e., as $\mathcal{S}_n^{\pm} = \mathfrak{P} \times_{\mathrm{Spin}^{\mathbb{C}}(n)} S_n^{\pm}$ (for even n), where we note that the representations of $\mathrm{Spin}(n)$ on $S_n^{\pm}$ extend to $\mathrm{Spin}^{\mathbb{C}}(n)$ in the obvious way (by letting $U(1)$ act in its defining representation). As a new feature compared to the $\mathrm{Spin}(n)$ case, one now in addition has a line bundle $L = \mathfrak{P} \times_{\mathrm{Spin}^{\mathbb{C}}(n)} \mathbb{C}$ associated to $\mathfrak{P}$ by the defining representation of $U(1)$ on $\mathbb{C}$, seen as a representation of $\mathrm{Spin}^{\mathbb{C}}(n)$ through det. This line bundle will be crucial in relating the $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator to quantization later on.

To define the $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator, the Riemannian connection no longer suffices; one now needs a connection on $\mathfrak{P}$ that induces the given Riemannian connection on its associated bundle TM. The connection on $\mathfrak{P}$ then induces a certain connection on the associated line bundle L (with 1-form A), as well as some connection on $\mathcal{S}_n = \mathfrak{P} \times_{\mathrm{Spin}^{\mathbb{C}}(n)} S_n$. This eventually defines $\not{D}\colon C_c^{\infty}(M, \mathcal{S}_n^{+}) \to C_c^{\infty}(M, \mathcal{S}_n^{-})$ by restriction of $\not{D}\colon C_c^{\infty}(M, \mathcal{S}_n) \to C_c^{\infty}(M, \mathcal{S}_n)$, since $S_n = S_n^{+} \oplus S_n^{-}$. A local expression for $\not{D}$ may be written down by replacing e_{μ} by $e_{\mu} + \frac{1}{2} i A_{\mu}$ in the above formula for the Spin Dirac operator [16]. Physically, this means that one has added a coupling to an Abelian gauge field, usually associated to the theory of electromagnetism (whereas the Riemannian connection is related to gravity). See [16, 17] for a global formula.

In both the Spin and the $\mathrm{Spin}^{\mathbb{C}}$ case, one may twist the Dirac operator by taking the tensor product of $\mathcal{S}^{\pm}$ with a vector bundle E over M; choosing a connection on E, this yields the twisted Dirac operator $\not{D}_E$: $C_c^{\infty}(M, \mathcal{S}_n \otimes E) \to C_c^{\infty}(M, \mathcal{S}_n \otimes E)$.

5. Restriction to Zero Eigenspace

The second key idea from quantum mechanics announced at the end of Section 3 goes back to Wigner; in its original form it seems almost too trivial to mention, though it forms the basis of almost all of quantum physics. Let $U(G)$ be a representation of a group G on a Hilbert space H, and let T: $H \to H$ be a bounded operator such that $TU(x) = U(x)T$ for all $x \in G$. Then $\ker(T) \subset H$ is stable under G, so that it carries a representation of G.

An important refinement of this idea is to start from a (bounded) operator T: $H_1 \to H_2$, where both Hilbert spaces H_i carry representations $U_i(G)$, and $TU_1(x) = U_2(x)T$ for all x. The idea is then to take the formal difference as 'virtual representation' of G:

$$G\text{-index}(T) = [\ker(T)] - [\ker^*(T)],$$

where $\ker^*(T) \equiv \ker(T^*) = \mathrm{coker}(T)$. Here the square brackets denote the unitary isomorphism class of the representation space in question. This expression makes immediate sense when G is compact and T is Fredholm: in that case G-index(T) is an element of the representation ring $R(G)$. When G is trivial, one simply has the usual Fredholm index of T. However, when G fails to be compact or T is not Fredholm, G-index seems to lack an a priori meaning.

In their remarkable work on the geometric construction of the discrete series for semisimple Lie groups, Atiyah and Schmid [4] had to make sense of the G-index in the following situation. They assumed that G is a connected real semisimple Lie group G with finite centre, with maximal compact subgroup K such that $\mathrm{rank}(G) = \mathrm{rank}(K)$, so that G has a discrete series. Using the notation of the previous section, following Parthasarathy [37], they took an irreducible representation V_{μ} of K, $\mu \in \hat{K}$, and defined the twisted spinor bundles $\mathcal{S}_{\mu}^{\pm} = G \times_K S_n^{\pm} \otimes V_{\mu}$, with associated twisted Dirac operator $\not{D}_{\mu}$: $C_c^{\infty}(G/K, \mathcal{S}_{\mu}^{+}) \to C_c^{\infty}(G/K, \mathcal{S}_{\mu}^{-})$. (This extends to an unbounded operator between the respective Hilbert spaces of L^2-sections, but since $\not{D}_{\mu}$ is elliptic is it not really necessary to perform this extension. Alternatively, one may work with the bounded operator $\not{D}_{\mu}/\sqrt{1 + \not{D}_{\mu}^* \not{D}_{\mu}}$.)

In this case, the L^2-kernel of $\not{D}_{\mu}$ is infinite-dimensional, and under certain conditions on μ defines a discrete series representation of G. In fact, writing $\mathcal{H}_{\mu}^{+} = \ker(\not{D}_{\mu})$ and $\mathcal{H}_{\mu}^{-} = \ker^*(\not{D}_{\mu})$, $\mathcal{H}_{\mu}^{-}$ always vanishes, whereas $\mathcal{H}_{\mu}^{+}$ is zero when $\mu + \rho_c$ is singular (in standard terminology), and carries an irreducible discrete series representation of G for any nonsingular value of $\mu + \rho_c$. Moreover, when G is linear its entire discrete series arises in this way [4].

In order to prove this result, Atiyah and Schmid introduced a discrete subgroup $\Gamma \subset G$, which acts freely and properly on G/K such that $\Gamma\backslash G/K$ is compact. They then replaced G-index$(\not{D}_\mu)$ by

$$\text{index}_\Gamma(\not{D}_\mu) = \dim_\Gamma(\ker(\not{D}_\mu)) - \dim_\Gamma(\ker^*(\not{D}_\mu)),$$

where $\dim_\Gamma(-)$ measures the Murray–von Neumann dimension of $-$ as a module of the von Neumann algebra $W^*(\Gamma)$ (which is the weak completion of $C_r^*(\Gamma)$ acting on $L^2(\Gamma)$). They then controlled $\mathcal{H}_\mu^\pm$ by an appeal to Atiyah's L^2-index theorem [2], which in this case states that $\text{index}_\Gamma(\not{D}_\mu) = \text{index}(\hat{\not{D}}_\mu)$. Here $\hat{\not{D}}_\mu$ is the operator on $\Gamma\backslash G/K$ whose lift to G/K is $\not{D}_\mu$; recall that the latter is invariant under G, hence under Γ.

With due respect, it should be stated that the introduction of Γ is a completely artificial trick. In the spirit of Section 3, it would be much more satisfactory to look at G-index$(\not{D}_\mu)$ as an element of $K_0(C_r^*(G))$. In that case, the Dirac induction procedure of Parthasarathy used above would simply define a map from $R(K)$ to $K_0(C_r^*(G))$. (When n is odd, G-index$(\not{D}_\mu)$ would be an element of $K_1(C_r^*(G))$, and in general Dirac induction defines a map $R(K) \to K_\bullet(C_r^*(G))$.) This was indeed achieved by Kasparov [20] and Connes and Moscovici [12] (also cf. [11]). We first explain their procedure in general, and then return to the situation of Atiyah and Schmid.

6. The Analytic Assembly Map

The Fredholm index of an operator T measures the size of $\ker(T)$ and $\ker^*(T)$ as $\mathbb{C}$-modules. The basic idea in regarding G-index(T) as an element of $K_0(C_r^*(G))$ is to measure the size of these spaces as $C_r^*(G)$-modules; if they happen to be finitely generated and projective, one obtains a well-defined element of $K_0(C_r^*(G))$ even when they are infinite-dimensional as vector spaces over $\mathbb{C}$. The best way to do this in the type of examples relevant to quantum mechanics and representation theory is to regard T as an element of the so-called equivariant K-homology group $K_\bullet^G(M)$ defined by Kasparov [20, 22], and to subsequently define its G-index as the image of its class in $K_\bullet^G(M)$ under the so-called analytic assembly map

$$G\text{-index:}\ K_\bullet^G(M) \to K_\bullet(C_r^*(G)).$$

We now explain this procedure, following [6] (cf. [45, 46] for additional details as well as an alternative treatment; see also [11, 19]).

It turns out that for unimodular groups G the analytic assembly map factors through $K_\bullet(C^*(G))$, that is, one may start from G-index construed as a map from $K_\bullet^G(M)$ to $K_\bullet(C^*(G))$, and subsequently compose this map with the push-forward in K-theory of the canonical projection $C^*(G) \to C_r^*(G)$. This was shown for discrete G in [46], but the arguments apparently work for unimodular groups in general. This is of little importance for the Baum–Connes conjecture, but it is crucial for the generalized Guillemin–Sternberg conjecture discussed later on.

Let M be a locally compact space (usually a finite-dimensional manifold, equipped with a proper G action, where G is a locally compact group, in our setting usually a Lie group) for which M/G is compact. An even equivariant K-homology cycle over M consists of a pair of Hilbert spaces $H^{\pm}$, each carrying a representation $\pi^{\pm}$ of the C^*-algebra $C_0(M)$, as well as a unitary representation $U^{\pm}(G)$, with a bounded operator F: $H^+ \to H^-$ such that:

(1) $U^{\pm}(x)\pi^{\pm}(f)U^{\pm}(x)^* = \pi^{\pm}(f_x)$, where $f_x(m) = f(x^{-1}m)$;
(2) $F\pi^+(f) - \pi^-(f)F$ is a compact operator for each $f \in C_0(M)$;
(3) $FU^+(x) - U^-(x)F = 0$ for each $x \in G$;
(4) F admits an 'almost' parametrix Q: $H^- \to H^+$, i.e., $\pi^-(f)(FQ-1)$ and $(QF-1)\pi^+(f)$ are compact.

When M is compact and G is compact, this simply implies that F is Fredholm. In that case, the basic examples of such cycles come from G equivariant elliptic pseudodifferential operators P: $L^2(M, E^+) \to L^2(M, E^-)$, where $E^{\pm}$ are vector bundles over M equipped with a G action. (Similarly, odd cycles consist of a single Hilbert space H with F: $H \to H$.)

Kasparov defined an equivalence relation on such cycles, basically given by a suitable version of homotopy, whose equivalence classes turn out to form an Abelian group, the equivariant K-homology group $K_0^G(M)$. In practice, one can do all computations at the level of the cycles; the precise definition of the equivalence relation only occurs in proofs. We will therefore not state it explicitly here; cf. [7] for a detailed treatment. (Similarly, one has a group $K_1^G(M)$.)

To define the analytic assembly map, one forms the subspaces $H_c^{\pm} = \pi^{\pm}(C_c(M))H^{\pm} \subset H^{\pm}$. By the covariance property 1 above and the assumed properness of the G action on M, the function $\langle\psi,\varphi\rangle$: $x \mapsto (\psi, U^{\pm}(x)\varphi)$ lies in $C_c(G)$ for every $\psi, \varphi \in H_c^{\pm}$. The key observation is now that this function is positive as an element of the C^*-algebra $C_r^*(G)$, of which $C_c(G)$ is here regarded as a (dense) subspace. See [26], Thm. IV.2.5.4, or [45], p. 49. In general, $\langle\psi,\varphi\rangle$ will fail to define a positive element of $C^*(G)$, which is the reason why the analytic assembly map takes values in $K_\bullet(C_r^*(G))$ rather than $K_\bullet(C^*(G))$. However, when G is unimodular, $\langle\psi,\varphi\rangle$ may be shown to be positive even in $C^*(G)$; see [46] (extending the given proof from discrete to unimodular groups in the obvious way).

By this positivity property, one may then renorm $H_c^{\pm}$ by means of

$$\|\psi\|^2 = \|\langle\psi,\psi\rangle\|_{C_r^*(G)},$$

and complete to Banach spaces $\mathfrak{H}^{\pm}$. It is not difficult to show that $C_c(G)$ (as a convolution algebra) acts on $H_c^{\pm}$ from the right by means of $\psi f = \int_G dx f(x^{-1})U^{\pm}(x)$ (in this formula we assume that G is unimodular), and that this extends to a right action of $C_r^*(G)$ on $\mathfrak{H}^{\pm}$. (Technically, $\mathfrak{H}^{\pm}$ are Hilbert C^*-modules over $C_r^*(G)$, cf. [11, 26, 45].) When G is unimodular, this applies, *mutatis mutandis*, to $C^*(G)$ as well.

To prove that F extends to a bounded operator $F'\colon \mathfrak{H}^+ \to \mathfrak{H}^-$ one has to assume that F is properly supported, i.e., that for each $f \in C_c(M)$ there is a $g \in C_c(M)$ such that $F\pi^+(f) = \pi^-(g)F\pi^+(f)$. (When $H^\pm$ consist of L^2-sections of vector bundles, this just means that F maps sections of compact support into each other.) This condition can always be met by changing representatives within a given equivalence class (cf. [45, 46]).

Under favourable circumstances, $\ker(F')$ and $\ker^*(F')$ are finitely generated projective modules over $C^*_r(G)$ (when G is not discrete one should use the unitization $\widetilde{C^*_r(G)}$ of $C^*_r(G)$ here and in some of the following expressions, noting that the G-index always lies in $K_0(C^*_r(G))$ itself), in which case

$$G\text{-index}(F) = [\ker(F')] - [\ker^*(F')] \in K_0(C^*_r(G)).$$

Here the square brackets stand either for the isomorphism class of the pertinent $C^*_r(G)$ module, or the homotopy class of the associated projection in $M_\infty(C^*_r(G))$. In general, one has to find a representative of the given cycle within its equivalence class such that $\ker(F')$ and $\ker^*(F')$ are finitely generated projective; this can always be done.

When G is compact and F Fredholm, the symbol G-index used here essentially coincides with its earlier use in Section 5; even in that case $\mathfrak{H}^\pm$ differs from $H^\pm$, but for Fredholm operators T, upon the identification $R(G) \cong K_0(C^*_r(G))$ the object $G\text{-index}(T) \in R(G)$ as defined in Section 5 coincides with $G\text{-index}(T) \in K_0(C^*_r(G))$ as defined in the present section.

In any case, the map G-index passes to equivalence classes of K-homology cycles, and with slight abuse of notation defines the analytic assembly map

$$G\text{-index}\colon\ K^G_\bullet(M) \to K_\bullet(C^*_r(G)).$$

When G is unimodular, one may also replace $C^*_r(G)$ by $C^*(G)$ here.

7. The Baum–Connes Conjecture

Let G be an almost connected unimodular Lie group. The procedure in Section 5 is valid also in this general situation, and combined with the analytic assembly map of the preceding section one obtains a 'Dirac induction' map

$$\mathcal{DI}\colon\ R(K) \to K_0(C^*_r(G)),$$

defined on generators $[V_\mu] \in R(G)$ $(\mu \in \hat{H})$ by

$$\mathcal{DI}([V_\mu]) = G\text{-index}(\not{D}_\mu).$$

(When $\dim(G/K)$ is odd, one obtains a map $\mathcal{DI}\colon\ R(K) \to K_1(C^*_r(G))$.)

Assuming that G/K admits a G invariant Spin structure, and that its dimension is even, the Connes–Kasparov conjecture [11, 21] (which is a special case of the Baum–Connes conjecture, see below) states that the Dirac induction map

$\mathcal{DI}$ is an isomorphism, so that $K_0(C_r^*(G)) \cong R(K)$. (In addition, it states that $K_1(C_r^*(G)) = 0$. When $\dim(G/K)$ is odd, the conjecture is that $K_0(C_r^*(G)) = 0$ and $K_1(C_r^*(G)) \cong R(K)$ through Dirac induction. See Section 6 in [10] for the situation where G/K has no G invariant Spin structure.) When G in addition is assumed to be semisimple with finite centre, the Connes–Kasparov conjecture would imply exhaustion of the discrete series by Dirac induction, reproving the result of Atiyah and Schmid (this implication was recognized by Connes around 1980).

Namely, as we have seen in Section 3, each member U of the discrete series provides a generator $[U]$ of $K_0(C_r^*(G))$. If $\mathcal{DI}$ is an isomorphism, one may take $\mathcal{DI}^{-1}([U]) \in R(K)$. The latter is a priori of the form $\mathcal{DI}^{-1}([U]) = \sum_\mu n_\mu [V_\mu]$, with $n_\mu \in \mathbb{Z}$, but since (like Mackey induction) Dirac induction commutes with taking direct sums, by assumption one has $[U] = \sum_\mu n_\mu \mathcal{DI}([V_\mu])$. It follows that $\mathcal{DI}^{-1}([U])$ consist of a single term with coefficient 1. One finally adds the vanishing theorems for $\mathcal{H}_\mu^-$ in [4] to replace G-index$(\not{D}_\mu)$ by $\ker(\not{D}_\mu)$; this is where the assumptions of semisimplicity and finite centre are used (cf. [29] for the complete argument). The existence of the discrete series whenever $\mathrm{rank}(G) = \mathrm{rank}(K)$ does not appear to have been proved by such techniques; however, the implication that G has no discrete series when $\mathrm{rank}(G) \neq \mathrm{rank}(K)$ follows quite easily [29].

The Connes–Kasparov conjecture was proved by Wassermann [48] using explicit knowlegde of the representation theory of G, and by V. Lafforgue without such input [28].

The Connes–Kasparov conjecture is a special case of the Baum–Connes conjecture [5, 6]. The starting point of the latter is the classifying space $\underline{E}G$ for proper G actions (where G is a locally compact group) [6, 45]. This is a final object in the homotopy category of coparacompact metrizable proper G spaces and continuous equivariant maps; in other words, for any proper G space P (for which P/G is paracompact) there is a continuous G equivariant map $P \to \underline{E}G$ that is unique up to G equivariant homotopy. This notion should be compared with the usual classifying space EG for principal (i.e, free and proper) actions. The basic example is that G/K is a model for $\underline{E}G$ whenever G is almost connected.

As we have seen, a locally compact proper G space M for which M/G is compact has an associated equivariant K-homology group $K_\bullet^G(M)$, with analytic assembly map G-index: $K_\bullet^G(M) \to K_\bullet(C_r^*(G))$. The basic idea of the Baum–Connes conjecture is that this map is an isomorphism for $M = \underline{E}G$, that is,

$$G\text{-index}: K_\bullet^G(\underline{E}G) \stackrel{\cong}{\to} K_\bullet(C_r^*(G)).$$

However, since $\underline{E}G/G$ is not necessarily compact, one has to define the left-hand side as the direct limit of $K_\bullet^G(M)$ over all subspaces $M \subset \underline{E}G$ that are stable and cocompact for the given G action. In this formulation, the Baum–Connes conjecture simply states that every element of $K_\bullet(C_r^*(G))$ is a G-index in a suitably injective way.

The Baum–Connes conjecture has now been proved for a large number of (locally compact) groups; for example, Chabert *et al.* established its truth for almost connected groups [10], and V. Lafforgue proved the conjecture for a certain class of groups that spectacularly included groups with Kazhdan's property T [28]. However, the conjecture is now widely believed to be false in general. See [45] for a recent introduction, concentrating on discrete groups.

When G is an almost connected Lie group, the Baum–Connes conjecture may be reformulated in terms of K-theory alone (i.e., the K-homology group on the left-hand side may be replaced by a K-theory group) [11]. This reformulation, which Connes intended as a generalization of Bott periodicity, turns out to be closely related to deformation quantization [27]. The basic idea of 'Connes's version of the Baum–Connes conjecture' is as follows. For any manifold M, one has the K-theoretic Poincaré duality property $K_\bullet(M) \cong K^\bullet(T^*M)$ [11, 19]. When M carries a proper G action, this extends to an isomorphism

$$K_\bullet^G(M) \cong K_\bullet(C_0(T^*M) \rtimes_r G),$$

where the G action on T^*M is the pullback of its action on G, defining the reduced crossed product C^*-algebra $C_0(T^*M) \rtimes_r G$ (cf. [11]). In addition, G evidently acts on $B_0(L^2(M))$, and it turns out that the C^*-algebras $B_0(L^2(M)) \rtimes_r G$ and $C_r^*(G)$ are Morita equivalent. Thus the analytic assembly map G-index: $K_\bullet^G(M) \to K_\bullet(C_r^*(G)$ can be translated into an equivalent map

$$G\text{-quant: } K_\bullet(C_0(T^*M) \rtimes_r G) \to K_\bullet(B_0(L^2(M)) \rtimes_r G).$$

As our notation already indicates, this map turns out to be a G equivariant version of the standard Weyl quantization of the cotangent bundle T^*M, translated into K-theory as suggested by the Connes–Higsom formalism of E-theory (cf. [11]). In general, without a group action, the K-theory of T^*M will not be rigid under deformation quantization, except in very special cases such as $M = \mathbb{R}^n$ (in which case the rigidity property $K^\bullet(\mathbb{R}^{2n}) \cong K_\bullet(B_0(L^2(M)) = K^\bullet(\text{point})$ is a special case of Bott periodicity). However, the Baum–Connes conjecture assures this property in the very special case where $M = G/K$ *and* Weyl quantization of its cotangent bundle is twisted by G. By [10], this property indeed holds.

Either way, when G is almost connected one can use Bott periodicity to show that

$$K_0^G(G/K) \cong K_\bullet(C_0(T^*(G/K)) \rtimes_r G) \cong R(K),$$

under the assumptions that G admits a G invariant Spin structure, and that its dimension is even (cf. [10] for the general case). Thus the Connes–Kasparov conjecture is a special case of the Baum–Connes conjecture.

8. Geometric Quantization

As noted in Section 1, geometric quantization was originally a far-reaching generalization of among other things the Borel–Weil construction of irreducible representations of compact Lie groups by holomorphic induction. However, the same may be said of the Baum–Connes conjecture, in which Dirac induction plays such a crucial role. In fact, in its current version, which apparently goes back to unpublished remarks by Raoul Bott, geometric quantization is nothing but the G-index of a suitable Dirac operator. Since it fits into the K-theoretic discussion so far, we now review Bott's version of geometric quantization (cf. [17, 36] for more details).

The first step is to canonically associate a $\mathrm{Spin}^{\mathbb{C}}$ structure to a given symplectic manifold (M, ω). First, one picks an almost complex structure $\mathcal{J}$ on M that is compatible with ω (in that $\omega(-, \mathcal{J}-)$ is positive definite and symmetric, i.e., a metric). This $\mathcal{J}$ canonically induces a $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{P}_{\mathcal{J}}$ on TM [15, 17], but this is not the right one to use here.

To proceed, one needs to assume that (M, ω) is called prequantizable, in that the cohomology class $[\omega]/2\pi$ in $H^2(M, \mathbb{R})$ is integral. This means that it lies in the image of $H^2(M, \mathbb{Z})$ under the natural homomorphism $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$. In that case, there exists a line bundle L_ω over M whose first Chern class $c_1(L_\omega)$ maps to $[\omega]/2\pi$ under this homomorphism; L_ω is called a prequantization line bundle over M. The $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{P}$ needed to quantize M is the one obtained by twisting $\mathfrak{P}_{\mathcal{J}}$ with L_ω. This means (cf. [17], App. D.2.7) that

$$\mathfrak{P} = \mathfrak{P}_{\mathcal{J}} \times_{\ker(\pi)} U(L_\omega),$$

where $\pi\colon \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$ was defined in Section 4 (note that $\ker(\pi) \cong U(1)$), and $U(L_\omega) \subset L_\omega$ is the unit circle bundle.

In a slight variation on this definition [17, 36], one assumes that $(M, 2\omega)$ rather than (M, ω) is prequantizable; this turns out to solve a number of technical problems in quantization theory. On this assumption, there exists a $\mathrm{Spin}^{\mathbb{C}}$ structure $\tilde{\mathfrak{P}}$ on M whose associated line bundle is $L_{2\omega}$. This is explicitly given by

$$\tilde{\mathfrak{P}} = \mathfrak{P}_{\mathcal{J}} \times_{\ker(\pi)} U(\tilde{L}_\omega),$$

where $\tilde{L}_\omega$ is a line bundle over M for which $\tilde{L}_\omega^2 = L_{2\omega} \otimes \kappa$ (here $\kappa = \det(T^*_{\mathbb{C}}M)$). In case that L_ω as well as $\sqrt{\kappa}$ exist, one has $\tilde{L}_\omega = L_\omega \otimes \sqrt{\kappa}$ (so that $\tilde{\mathfrak{P}}$ gives the so-called half-form correction to geometric quantization), but it often happens that neither L_ω nor $\sqrt{\kappa}$ exists, in which case one may still be able to make sense of $\tilde{\mathfrak{P}}$. See [36] for an interesting class of examples of this phenomenon.

When M is compact, the coarsest notion of quantization is to say that (M, ω) is quantized by the Fredholm index of the Dirac operator $\not{D}$ defined by either one of the above $\mathrm{Spin}^{\mathbb{C}}$ structures ($\not{D}$ is elliptic, hence Fredholm). This is just a number, which may even be negative! Now suppose M carries a strongly Hamiltonian action of a Lie group G. When both M and G are compact in the situation of the preceding paragraph, the $\mathrm{Spin}^{\mathbb{C}}$ structure is G invariant, and the geometric quantization of (M, ω) with the given G action $G \circlearrowright M$ is defined as

$$Q(G \circlearrowright M, \omega) = G\text{-index}(\not{D}) \in R(G).$$

This quantization does not depend on the choice of $\mathcal{J}$ made in the construction of $\not{D}$, though it does depend on L_ω or $L_{2\omega}$ (up to isomorphism) [17].

This may seem to be an outrageous abstraction of any notion of quantization, but in fact this definition is rather closely related to the original goal of geometric quantization, which was that a symplectic manifold with strongly Hamiltonian group action should be mapped into a Hilbert space carrying a unitary representation of the group. Technically, this representation was constructed as the kernel of some first-order partial differential operator defined by a (possibly complex) polarization on M [17]. This procedure has a long history, going back to the irreducibility of the Schrödinger representation of basic quantum mechanics [47]. It often led to certain difficulties, which are obviated if one uses the appropriate $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator. Hence in the new approach one has firstly replaced polarizations by $\mathrm{Spin}^{\mathbb{C}}$ Dirac operators, secondly replaced the kernel by the index, and thirdly passed to the representation ring. Now, when the symplectic form ω is sufficiently 'large,' $\ker^*(\not{D})$ tends to vanish [8], so that $Q(G \circlearrowright M, \omega)$ is really the image in $R(G)$ of a unitary representation of G. It turns out that this new approach to geometric quantization is much more flexible than its predecessors.

So far, we have assumed that both G and M are compact. However, it will be clear from the preceding part of this paper that this quantization prescription applies much more generally. Indeed, both G and M may be noncompact, as long as the given strongly Hamiltonian G action on M is proper and cocompact, and the pertinent $\mathrm{Spin}^{\mathbb{C}}$ structure may be chosen so as to be G invariant. In fact, one may soften the latter requirement, since in the definition of equivariant K-homology it is possible to replace condition (3) in Section 6 by the requirement that $FU^+(x) - U^-(x)F$ be compact rather than zero; cf. [46]. Under these conditions, one may put

$$Q(G \circlearrowright M, \omega) = G\text{-index}(\not{D}) \in K_0(C^*(G)).$$

Here we have assumed G to be unimodular; cf. Section 6. This is the case in practically all applications in quantization theory, but as we have seen, in general one may always defined $Q(G \circlearrowright M, \omega)$ as an element of $K_0(C_r^*(G))$.

9. The Guillemin–Sternberg Conjecture

The Guillemin–Sternberg conjecture is a mathematical implementation of the idea that 'quantization commutes with reduction.' This idea goes back to Dirac in physics [14], who was the first to study constrained Hamiltonian systems and their quantization in a systematic way. His ideas were later brought into a more decent mathematical shape. For example, his construction of the physical phase space of a classical system supposed to be invariant under the action of a Lie group was formalized as Marsden–Weinstein reduction [1, 26]. Namely, the symplectic quotient or Marsden–Weinstein reduced space defined by a given strongly Hamiltonian G action on a symplectic manifold M, as above, is $M^0 = J^{-1}(0)/G$ [1]. In case that 0 is a regular value of J and the G action is proper and free on $J^{-1}(0)$, M^0 is a manifold, which moreover carries a unique symplectic form ω^0 with the property $i^*\omega = \pi^*\omega^0$. Here $i\colon\ J^{-1}(0) \hookrightarrow M$ is the inclusion and $\pi\colon\ J^{-1}(0) \to M^0$ is the projection map. Thus Marsden–Weinstein reduction produces a new symplectic manifold (M^0, ω^0) from a given symplectic manifold (M, ω) equipped with a strongly Hamiltonian G action. If the stated assumptions are not met, singularities may arise in the reduced space (cf. [42]).

Guillemin and Sternberg gave the first mathematical version of the 'quantization commutes with reduction' idea [18]. They considered the case in which the symplectic manifold M is compact, prequantizable, and equipped with a positive-definite complex polarization $\mathcal{J}$. Under these circumstances, the original scenario of geometric quantization applies in its optimal form: one picks a connection ∇ on the prequantization line bunlde L_ω whose curvature is ω, and defines the Hilbert space $H(M)$ allegedly quantizing M as the space $H(M) = H^0(M, L_\omega)$ of polarized sections of L_ω (i.e., of sections annihilated by all ∇_X, $X \in \mathcal{J}$).

The Hilbert space $H(M)$ then carries a natural unitary representation of G determined by the classical data, as polarized sections of L_ω are mapped into each other by the pullback of the G action. Moreover, it turns out that the reduced space M^0 inherits all relevant structures on M (except, of course, the G action), so that it is quantizable as well, in the same fashion. This leads to the original version of the Guillemin–Sternberg conjecture, namely

$$\dim(H^0(M, L_\omega)^G) = \dim(H^0(M^0, L_{\omega^0})),$$

which Guillemin and Sternberg indeed managed to prove. The idea of the proof is to define a map from $H^0(M, L_\omega)^G$ to $H^0(M^0, L_{\omega^0})$ by simply restricting a G invariant polarized section of L_ω to $J^{-1}(0)$; this map is then shown to be a linear isomorphism [18].

We now turn to the modern version of the conjecture [17, 32], which in part arose from the need to adapt the preceding discussion to less favourable situations.

For compact G and M, both $\mathrm{Spin}^{\mathbb{C}}$ structures on M used in Bott's definition of quantization explained in the preceding section turn out to descend to the reduced space M^0 [17, 32, 36]. We denote the associated $\mathrm{Spin}^{\mathbb{C}}$ Dirac operator on M^0 by $\not{D}_0$. Furthermore, for compact G one has a map $R(G) \to \mathbb{Z}$, given on generators of $R(G)$ by $[V] \mapsto \dim(V_0)$, where V_0 is the G invariant part of V. We write α_0 for the image of $\alpha \in R(G)$ under this map.

For compact G and M the modern Guillemin–Sternberg conjecture then reads

$$G\text{-index}(\not{D})_0 = \text{index}(\not{D}_0).$$

This conjecture was proved in [32] for the Dirac operator determined by the $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{P}$ of the preceding section; it even holds when 0 fails to be a regular value of J [33]. Also see [17, 35] for other proofs and further references. For the Spin$^{\mathbb{C}}$ structure $\tilde{\mathfrak{P}}$ the conjecture is not true in the form stated; instead, one needs to perform Marsden–Weinstein reduction at ρ_c rather than 0 [36]. This 'ρ_c-shift' is a well-known phenomenon in representation theory, cf. also [4] and Section 5 above.

It is remarkable that there have been no attempts to generalize the Guillemin–Sternberg conjecture (in any of its versions) to the case where G and M are non-compact. From the point of view of this paper it is obvious how this should be done at least for the modern formulation. We have already noted that G-index($\not{D}$) as a definition of geometric quantization can be generalized. We now need to assume that G is unimodular. The analogue of the above map $R(G) \to \mathbb{Z}$ is then the map $K_0(C^*(G)) \to \mathbb{Z}$ functorially induced by the involutive homomorphism $C^*(G) \to \mathbb{C}$, given by extension (from $L^1(G)$ to $C^*(G)$) of the map $f \mapsto \int_G dx\, f(x)$. This map is indeed continuous; in fact, it is just the representation of $C^*(G)$ on $\mathbb{C}$ corresponding to the trivial representation of G (one cannot work with $C^*_r(G)$ here, since this representation may not lie in $\hat{G}_r$; in fact, it does if and only if G is amenable). These observations immediately lead to a far-reaching generalization of the Guillemin–Sternberg conjecture, which is formally the same statement as in the compact case, the symbols now being reinterpreted as appropriate.

As a first example, consider the case where $G = \Gamma$ is discrete and infinite. One then simply has $M^0 = M/\Gamma$, and $\not{D}_0$ is just the operator on M/Γ whose lift is $\not{D}$. To compare this situation with Atiyah's L^2-index theorem [2] already mentioned (cf. Section 5), note that $\text{index}_\Gamma(\not{D})$ may be obtained from $\not{D}$ in a K-theoretic approach [11, 13]: first, one takes Γ-index($\not{D}$) $\in K_0(C^*_r(\Gamma))$, and secondly one takes the trace $f \mapsto f(e)$, which defines a cyclic cocycle on $C^*_r(\Gamma)$. The pairing of this cocycle with Γ-index($\not{D}$) is precisely $\text{index}_\Gamma(\not{D})$ (also cf. [30] for a slightly different approach). Hence, roughly speaking, Atiyah's index theorem comes from the map $f \mapsto f(e)$ on $C^*_r(\Gamma)$, whereras the generalized Guillemin–Sternberg conjecture emerges from the map $f \mapsto \sum_{\gamma\in\Gamma} f(\gamma)$ on $C^*(\Gamma)$. Since the right-hand side of the L^2-index theorem is the same as the right-hand side of the Guillemin–Sternberg conjecture, on use of the L^2-index theorem the present generalization of this conjecture is equivalent to

$$\Gamma\text{-index}(\not{D})_0 = \text{index}_\Gamma(\not{D}),$$

where, as just explained, the left-hand side is the image of Γ-index($\not{D}$) $\in K_0(C^*(\Gamma))$ under the map $K_0(C^*(\Gamma)) \to \mathbb{Z}$ induced in K-theory by the map $f \mapsto \sum_\gamma f(\gamma)$. Note that this equality is only required for the (first) Spin$^{\mathbb{C}}$ Dirac operator defined by the symplectic structure, whereas the L^2-index theorem holds for any elliptic operator.

To close, we remark that there seems to be quite some technical overlap between the Baum–Connes conjecture and the generalized Guillemin–Sternberg conjecture, which perhaps was hidden so far because the former is only nontrivial for noncompact groups, whereas all literature on the latter was concerned with compact groups.

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