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A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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ABSTRACT. We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

INTRODUCTION

In 1915, Pick [3] proved the following result.

Theorem 1. Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k) \overline{g(z_l)}}{1 - z_k \overline{z_l}} \lambda_k \lambda_l \geq 0.
$$

Ahlfors [1], page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “minimal interpolation problem” for $H^2$ (see [2], page 141). As a byproduct we obtain a new proof of Pick’s theorem.

DESCRIPTION OF THE MAIN RESULT

Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by the sequence

$$
b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}.
$$

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$:

1) $f$ lies in the unit ball of $H^2$.
2) For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k) f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k) b'(z_l)} \leq 1.
$$

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For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define
\[
\Lambda = \{ f \in H^2 : f(z_j) = w_j, \ j = 1, 2, \ldots, n \}.
\]
$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial
\[
\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)l'(z_k)},
\]
where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function
\[
\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \frac{w_k}{b'(z_k)(1 - |z_k|^2)},
\]
with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with
\[
\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)},
\]
$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\overline{\Delta}$. $\Lambda$ is a hyperplane in $H^2$.

With $\varphi$ and $b$ defined as in (4) and (2) we have
\[
\Lambda = \{ \varphi + gb; \ g \in H^2 \}.
\]

**Theorem 2.** $\varphi$ is the unique solution of the “minimal interpolation problem”, i.e., for every $f \in \Lambda \backslash \{ \varphi \}$ we have $\|f\|_2 > \|\varphi\|_2$.

**Proof.** It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|_2 = \|\varphi\|_2 + \|f - \varphi\|_2$).

From the decomposition $f = \varphi + bg$ we have
\[
(f - \varphi, \varphi) = (bg, \varphi) = \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it})g(e^{it})\varphi(e^{it})dt
\]
\[
= \frac{1}{2\pi} \int_{0}^{2\pi} b(e^{it})g(e^{it})b(e^{it}) \sum_{k=1}^{n} \frac{\overline{w}_k}{(e^{-it} - \overline{z}_k)b'(\overline{z}_k)}dt.
\]
Note that $|b(e^{it})|^2 = 1$. Thus,
\[
(f - \varphi, \varphi) = \sum_{k=1}^{n} \frac{\overline{w}_k}{2\pi b'(z_k)} \int_{0}^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it}\overline{z}_k}dt
\]
\[
= \sum_{k=1}^{n} \frac{\overline{w}_k}{b'(z_k)} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z}_k z}dz = 0,
\]
because the integrand is analytic on $\Delta$.  

Preliminaries
It will be convenient to have an explicit expression for $\|\varphi\|_2^2$:

$$
\|\varphi\|_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^n \sum_{l=1}^n w_k \overline{w_l} \int_0^{2\pi} \frac{dt}{(e^{it} - z_k)(e^{-it} - \overline{z_l})} = \frac{1}{2\pi} \sum_{k=1}^n \sum_{l=1}^n w_k \overline{w_l} \int_0^{2\pi} \frac{dz}{b'(z_k)b'(z_l)(1 - z_k \overline{z_l})} = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l} \frac{b'(z_k)}{b'(z_l)}}.
$$

There are, of course, many other expressions for $\|\varphi\|_2^2$.

**Theorem 3.**

$$
\|\varphi\|_2 = \max \left\{ \left| \sum_{k=1}^n w_k f(z_k) \right| : f \in H^2, \|f\|_2 \leq 1 \right\}.
$$

**Proof.**

$$
\sum_{k=1}^n w_k f(z_k) = \frac{1}{2\pi i} \int_{\Gamma} f(z) \varphi(z) b(z) \frac{dz}{b'(z_k)};
$$

hence, by Schwarz’s inequality we have

$$
\left| \sum_{k=1}^n w_k f(z_k) \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \leq \|f\|_2 \cdot \|\varphi\|_2 \leq \|\varphi\|_2.
$$

Equality holds for the function $f : z \rightarrow -\frac{1}{\|\varphi\|_2} \sum_{k=1}^n \frac{w_k}{(1 - z_k \overline{z_l})b'(z_k)}$.

An immediate result from Theorem 2 is

**Corollary.** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

**Proof.** Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

$$
\|1\|_2 = 1,
$$

we have

$$
1 \geq \|\varphi\|_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \frac{1}{b'(z_k)b'(z_l)}.
$$

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

$$
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \frac{1}{b'(z_k)b'(z_l)}.
$$

If $0 \notin \{z_1, z_2, \ldots, z_n\}$, there is strict inequality.

Because of the uniqueness of $\varphi$ there can be equality only if

$$
b(z) \sum_{k=1}^n \frac{1}{(z - z_k)b'(z_k)} = 1.
$$
In this identity for rational functions we let $z \to \infty$. Since $z_j \neq 0$, $\lim_{z \to \infty} b(z)$ has a finite value. Therefore, the left-hand side has limit zero.

**Remark.** The corollary shows that a function satisfying (1) also satisfies (3).

The fact that $\varphi \in \Lambda$ has an interesting reformulation. We start with a lemma.

**Lemma 1.** The partial fraction decomposition of $\varphi$ is

$$\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1-z_l)(1-z_l z_k) b'(z_k) b'(z_l)}.$$  

**Proof.** An elegant way to prove this is to compute both sides of the following identity.

For $z \in \Delta$ we have

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1-\zeta z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1-\zeta z} \frac{d\zeta}{\zeta}.$$  

The left-hand side is equal to

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1-\zeta z} \frac{d\zeta}{\zeta} = \varphi(z),$$  

while the right-hand side is equal to the complex conjugate of

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{b(\zeta)} \sum_{k=1}^{n} \frac{\bar{w}_k}{(1-\zeta z_k) b'(z_k) 1-\zeta z_k} \cdot \frac{1}{1-\zeta z_k} \frac{d\zeta}{\zeta},$$  

i.e., to the complex conjugate of

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{b(\zeta)} \sum_{k=1}^{n} \frac{\bar{w}_k}{(1-\zeta z_k) b'(z_k) 1-\zeta z_k} \cdot \frac{d\zeta}{\zeta}.$$  

Calculation of the residues at the points $z_1, z_2, \ldots, z_n$ lead to (5).

The condition $\varphi \in \Lambda$ implies that $\varphi(z_j) = w_j$, $j = 1, \ldots, n$, i.e.,

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1-z_l)(1-z_l z_k) b'(z_k) b'(z_l)} = w_j.$$  

This is equivalent to the assertion that the matrices

$$B = (\beta_{ik})$$  

and its conjugate $\overline{B} = (\overline{\beta}_{ik})$ where

$$\beta_{ik} = \frac{1}{(1-z_i z_k) b'(z_k)}$$  

are inverses of each other, i.e., $B$ and $\overline{B}$ are unitary.
Theorem 4. Let $f$ be a continuous function on the unit disc in the complex plane. Then the following conditions are equivalent:

1. $f$ is analytic and $f$ lies in the unit ball of $H^2$.
2. For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_kz_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$ 

Proof. We split up the proof into two lemmas.

Lemma 2. Let $f$ belong to the unit ball of $H^2$, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ be given. Then (3) holds.

Proof. Define $w_j = f(z_j)$. $f$ lies in the hyperplane $\Lambda$ and the element $\varphi$ of $\Lambda$ with minimal norm satisfies

$$\|\varphi\|_2 \leq \|f\|_2 \leq 1.$$ 

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

Lemma 3. Let $f$ be continuous and assume that $f$ satisfies (3). Then $f$ is analytic and $f$ lies in the unit ball of $H^2$.

Proof. We apply (3) for the case $n = 1$; an easy computation shows that

$$|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}}$$ 

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\Delta_\rho$. For every $n$ there is a function $\varphi_n$ with

$$\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n,$$

and

$$\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_kz_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$ 

Thus, $\varphi_n$ lies in the unit ball of $H^2$, and so by Lemma 2, we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in $\Delta$

$$\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\varphi_n(\zeta_l)}{1 - \zeta_k\zeta_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.$$ 

It follows from (6) that

$$|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}},$$

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\Delta_\rho$. Therefore, it contains a locally uniformly convergent subsequence $\varphi_{n_j}$. At the points $z_1, z_2, \ldots$ the subsequence converges to $f$. By the continuity of $f$ and the fact that $\{z_1, z_2, \ldots\}$ is dense in $\Delta_\rho$, we see that

$$\lim_{n_j \to \infty} \varphi_{n_j} = f.$$
This shows that \( f \) is analytic on \( \Delta_p \) for all \( p < 1 \). Because of uniform convergence on \( \Gamma_r \), we have

\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.
\]

Thus, \( f \in H^2 \) and \( \|f\|_2 \leq 1 \).

Lemma 2 and Lemma 3 together constitute a proof of the theorem.

**Corollary.** For \( f \in H^2 \) we define

\[
\nu(f) = \sup \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} : z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
\]

Then \( \nu(f) = \|f\|_2^2 \).

**Proof.** Assume that \( \nu(f) = 1 \). Then by Lemma 3 \( \|f\|_2^2 \leq 1 \). If \( \|f\|_2^2 < \lambda^2 < 1 \) for some \( \lambda \), then we have \( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt < \lambda^2 < 1 \) which is impossible by Lemma 2.

In a similar way we can show that \( \|f\|_2^2 = 1 \) implies that \( \nu(f) = 1 \). By the homogeneity of \( \nu \) and \( \| \cdot \|_2 \) it follows that for all \( f \in H^2 \): \( \nu(f) = \|f\|_2^2 \).

**Pick’s Theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let \( g \) belong to the unit ball of \( H^\infty \), and let \( z_1, z_2, \ldots, z_n \) be a sequence of mutually distinct points in \( \Delta \). Let \( w_1, w_2, \ldots, w_n \) be an arbitrary sequence of complex numbers. We consider the hyperplanes \( A \) and \( A_g \) where

\[
A_g = \{ f \in H^2 : f(z_j) = w_jg(z_j), j = 1, 2, \ldots, n \}.
\]

Of course, if \( f \in A \), then \( g \cdot f \in A_g \), and by Theorem 2 applied to \( A_g \) we have

\[
\|gf\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} w_kg(z_k)w_lg(z_l) \cdot \frac{1}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}.
\]

Let \( \varphi \) be, as before, the element of \( A \) with smallest norm. From \( \|g\|_\infty \leq 1 \) we obtain

\[
\|g\varphi\|_2 \leq \|\varphi\|_2.
\]

Combining these steps leads to

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} w_k \bar{w}_l \cdot \frac{1}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} \geq \|\varphi\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \bar{w}_l g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)},
\]

i.e.,

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{w_k \bar{w}_l}{b'(z_k)b'(z_l)} \geq 0.
\]
and since the sequence $w_1, w_2, \ldots, w_n$ is arbitrary, we have for all choices of $\lambda_1, \lambda_2, \ldots, \lambda_n$,

$$
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \overline{z_l}} \cdot \lambda_k \overline{\lambda_l} \geq 0.
$$

By the choice $n = 1$, $\lambda_1 = 1$ we see that the converse is trivial.

References


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