A NEW CHARACTERIZATION OF THE UNIT BALL OF $H^2$

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(Communicated by Juha M. Heinonen)

ABSTRACT. We derive a new expression for the norm of $H^2$ functions; we present some well-known results in a different setting.

INTRODUCTION

In 1915, Pick [3] proved the following result.

**Theorem 1.** Let $g$ be an analytic function on the unit disc $\Delta$ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in $\Delta$ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have

\[ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k) \overline{g(z_l)}}{1 - z_k \overline{z_l}} \lambda_k \overline{\lambda_l} \geq 0. \]  

Ahlfors [1], page 3, gives an elegant proof of this characterization of the unit ball of $H^\infty$.

In this note we shall present a characterization of the unit ball of $H^2$. Our main tool will be an explicit solution of the “minimal interpolation problem” for $H^2$ (see [2], page 141). As a byproduct we obtain a new proof of Pick’s theorem.

DESCRIPTION OF THE MAIN RESULT

Let $z_1, z_2, \ldots, z_n$ be a sequence in $\Delta$, and let $b$ be the Blaschke product generated by the sequence

\[ b(z) = \prod_{j=1}^{n} \frac{z - z_j}{1 - \overline{z_j} z}. \]

We shall prove that the following conditions are equivalent for continuous functions $f$ on $\Delta$:

1) $f$ lies in the unit ball of $H^2$.

2) For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in $\Delta$ we have

\[ \sum_{k=1}^{n} \sum_{l=1}^{n} f(z_k) \overline{f(z_l)} \cdot \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k) \overline{b'(z_l)}} \leq 1. \]
Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in $\Delta$ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{ f \in H^2 : f(z_j) = w_j, \ j = 1, 2, \ldots, n \}.$$  

$\Lambda$ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)l'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of $H^p$ spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k)b'(z_k)},$$

$\varphi \in \Lambda$, and $\varphi$ is analytic on some neighbourhood of $\overline{\Delta}$. $\Lambda$ is a hyperplane in $H^2$.

With $\varphi$ and $b$ defined as in (4) and (2) we have

$$\Lambda = \{ \varphi + bg ; g \in H^2 \}.$$

**Theorem 2.** $\varphi$ is the unique solution of the “minimal interpolation problem”, i.e., for every $f \in \Lambda \setminus \{ \varphi \}$ we have $\|f\|_2 > \|\varphi\|_2$.

**Proof.** It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + bg$ we have

$$\langle f - \varphi, \varphi \rangle = \langle bg, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} b(e^{it})g(e^{it})\overline{\varphi(e^{it})} dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} b(e^{it})g(e^{it})\overline{b(e^{it})b'(e^{it})} \sum_{k=1}^{n} \overline{w_k} \int_{e^{-it} = z_k} \frac{g(z)}{1 - \overline{z_k}z} dz dt.$$

Note that $\|b(e^{it})\|^2 = 1$. Thus,

$$\langle f - \varphi, \varphi \rangle = \sum_{k=1}^{n} \frac{\overline{w_k}}{2\pi b'(z_k)} \int_0^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it}z_k} dt$$

$$= \sum_{k=1}^{n} \frac{\overline{w_k}}{b'(z_k)} \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z}_k z} dz = 0,$$

because the integrand is analytic on $\Delta$.  

It will be convenient to have an explicit expression for $||\varphi||_2^2$:

$$
||\varphi||_2^2 = \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = \frac{1}{2\pi} \sum_{k=1}^n \sum_{t=1}^n w_k \overline{w_l} \frac{2\pi}{b'(z_k)b'(z_l)} \int_0^{2\pi} (e^{it} - z_k)(e^{-it} - \overline{z_l}) dt
$$

$$
= \frac{1}{2\pi} \sum_{k=1}^n \sum_{t=1}^n w_k \overline{w_l} \int_0^{2\pi} \frac{dz}{b'(z_k)b'(z_l)}
$$

$$
= \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

There are, of course, many other expressions for $||\varphi||_2$.

**Theorem 3.**

$$
||\varphi||_2 = \max \left\{ \left\{ \sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} \right\} : f \in H^2, ||f||_2 \leq 1 \right\}.
$$

**Proof.**

$$
\sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)\varphi(z)}{b(z)} dz;
$$

hence, by Schwarz’s inequality we have

$$
\left| \sum_{k=1}^n \frac{w_k f(z_k)}{b'(z_k)} \right| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(e^{it})| \cdot |\varphi(e^{it})| dt \leq ||f||_2 \cdot ||\varphi||_2 \leq ||\varphi||_2.
$$

Equality holds for the function $f : z \rightarrow \frac{1}{||\varphi||_2} \sum_{k=1}^n \frac{w_k}{(1 - z_k \overline{z_l})b'(z_k)}$.

An immediate result from Theorem 2 is

**Corollary.** For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of $\Delta$ we have

$$
\sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
$$

**Proof.** Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

$$
||1||_2 = 1,
$$

we have

$$
1 \geq ||\varphi||_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

$$
1 = \varphi(0)^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\varphi(e^{it})|^2 dt = ||\varphi||_2^2 = \sum_{k=1}^n \sum_{l=1}^n \frac{1}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)}.
$$

If $0 \notin \{z_1, z_2, \ldots, z_n\}$, there is strict inequality.

Because of the uniqueness of $\varphi$ there can be equality only if

$$
b(z) \sum_{k=1}^n \frac{1}{(z - z_k)b'(z_k)} = 1.
$$
In this identity for rational functions we let \( z \to \infty \). Since \( z_j \neq 0 \), \( \lim_{z \to \infty} b(z) \) has a finite value. Therefore, the left-hand side has limit zero.

**Remark.** The corollary shows that a function satisfying (1) also satisfies (3).

The fact that \( \varphi \in \Lambda \) has an interesting reformulation. We start with a lemma.

**Lemma 1.** The partial fraction decomposition of \( \varphi \) is

\[
\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{z}_l z_j)(1 - \bar{z}_l z_k)b'(z_k)b'(z_l)}.
\]

**Proof.** An elegant way to prove this is to compute both sides of the following identity.

For \( z \in \Delta \) we have

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(\zeta)}{1 - \zeta z} \frac{d\zeta}{\zeta}.
\]

The left-hand side is equal to

\[
\frac{1}{2\pi i} \int_{\Gamma} \varphi(\zeta) d\zeta = \varphi(z),
\]

while the right-hand side is equal to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{\zeta - z} \sum_{k=1}^{n} \frac{\bar{w}_k}{(\zeta - z_k)b'(z_k)} \frac{1}{1 - \zeta z} \frac{d\zeta}{\zeta},
\]

i.e., to the complex conjugate of

\[
\frac{1}{2\pi i} \int_{\Gamma} \frac{b(\zeta)}{\zeta - z} \sum_{k=1}^{n} \frac{\bar{w}_k}{(1 - \bar{z}_k \zeta)b'(z_k)} \frac{1}{1 - \bar{z}_k \zeta} \frac{d\zeta}{\zeta}.
\]

Calculation of the residues at the points \( z_1, z_2, \ldots, z_n \) lead to (5).

The condition \( \varphi \in \Lambda \) implies that \( \varphi(z_j) = w_j, \ j = 1, \ldots, n \), i.e.,

\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1 - \bar{z}_l z_j)(1 - \bar{z}_l z_k)b'(z_k)b'(z_l)} = w_j.
\]

This is equivalent to the assertion that the matrices

\[
B = (\beta_{jk})
\]

and its conjugate \( \overline{B} = (\overline{\beta}_{jk}) \) where

\[
\beta_{jk} = \frac{1}{(1 - \bar{z}_j z_k)b'(z_k)}
\]

are inverses of each other, i.e., \( B \) and \( \overline{B} \) are unitary.
Proof of the main result

Theorem 4. Let \( f \) be a continuous function on the unit disc in the complex plane. Then the following conditions are equivalent:

1. \( f \) is analytic and \( f \) lies in the unit ball of \( H^2 \).
2. For every \( n \in \mathbb{N} \) and for every sequence \( z_1, z_2, \ldots, z_n \) of mutually distinct points in \( \Delta \) we have
   \[
   \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
   \]

Proof. We split up the proof into two lemmas.

Lemma 2. Let \( f \) belong to the unit ball of \( H^2 \), and let a sequence of mutually distinct points \( z_1, z_2, \ldots, z_n \) in \( \Delta \) be given. Then (3) holds.

Proof. Define \( w_j = f(z_j) \). \( f \) lies in the hyperplane \( \Lambda \) and the element \( \varphi \) of \( \Lambda \) with minimal norm satisfies
\[
\|\varphi\|_2 \leq \|f\|_2 \leq 1.
\]
Use of the explicit expression for \( \|\varphi\|_2 \) leads to (3).

Lemma 3. Let \( f \) be continuous and assume that \( f \) satisfies (3). Then \( f \) is analytic and \( f \) lies in the unit ball of \( H^2 \).

Proof. We apply (3) for the case \( n = 1 \); an easy computation shows that
\[
|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}}
\]
for every choice of \( z \in \Delta \).

Let \( 0 < r < \rho < 1 \), and let \( z_1, z_2, z_3, \ldots \) be an enumeration of the rational points of \( \overline{\Delta}_\rho \). For every \( n \) there is a function \( \varphi_n \) with
\[
\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n,
\]
and
\[
\|\varphi_n\|_2^2 = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \overline{z_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
\]
Thus, \( \varphi_n \) lies in the unit ball of \( H^2 \), and so by Lemma 2, we have for every sequence \( \zeta_1, \zeta_2, \ldots, \zeta_n \) in \( \Delta \)
\[
\sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k \overline{\zeta_l}} \cdot \frac{1}{b'(z_k)b'(z_l)} \leq 1.
\]
It follows from (6) that
\[
|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}},
\]
hence the sequence \( \varphi_1, \varphi_2, \ldots \) is uniformly bounded on \( \overline{\Delta}_\rho \). Therefore, it contains a locally uniformly convergent subsequence \( \varphi_{n_j} \). At the points \( z_1, z_2, \ldots \) the subsequence converges to \( f \). By the continuity of \( f \) and the fact that \( \{z_1, z_2, \ldots\} \) is dense in \( \Delta_\rho \), we see that
\[
\lim_{n_j \to \infty} \varphi_{n_j} = f.
\]
This shows that \( f \) is analytic on \( \Delta_p \) for all \( p < 1 \). Because of uniform convergence on \( \Gamma_r \), we have
\[
\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j \to \infty} \frac{1}{2\pi} \int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \leq 1.
\]
Thus, \( f \in H^2 \) and \( \|f\|_2 \leq 1 \).

Lemma 2 and Lemma 3 together constitute a proof of the theorem.

**Corollary.** For \( f \in H^2 \) we define
\[
\nu(f) = \sup \left\{ \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{f(z_k)f(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} ; z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta \right\}.
\]

Then \( \nu(f) = \|f\|_2^2 \).

**Proof.** Assume that \( \nu(f) = 1 \). Then by Lemma 3 \( \|f\|_2^2 \leq 1 \). If \( \|f\|_2^2 < \lambda^2 < 1 \) for some \( \lambda \), then we have \( \|\frac{1}{2} f\| < 1 \) but \( \nu\left(\frac{1}{2} f\right) > 1 \) which is impossible by Lemma 2.

In a similar way we can show that \( \|f\|_2 = 1 \) implies that \( \nu(f) = 1 \). By the homogeneity of \( \nu \) and \( \| \cdot \|_2 \) it follows that for all \( f \in H^2 \) : \( \nu(f) = \|f\|_2^2 \).

**Pick’s Theorem**

As an application of our results we shall give a proof of Pick’s theorem.

Let \( g \) belong to the unit ball of \( H^\infty \), and let \( z_1, z_2, \ldots, z_n \) be a sequence of mutually distinct points in \( \Delta \). Let \( w_1, w_2, \ldots, w_n \) be an arbitrary sequence of complex numbers. We consider the hyperplanes \( \Lambda \) and \( \Lambda_g \) where
\[
\Lambda_g = \{ f \in H^2 : f(z_j) = w_jg(z_j), j = 1, 2, \ldots, n \}.
\]

Of course, if \( f \in \Delta \), then \( g \cdot f \in \Delta_g \), and by Theorem 2 applied to \( \Lambda_g \) we have
\[
\|g \cdot f\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k g(z_k) w_l g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)}.
\]

Let \( \varphi \) be, as before, the element of \( \Lambda \) with smallest \( L^2 \)-norm. From \( \|g\|_\infty \leq 1 \) we obtain
\[
\|g \varphi\|_2 \leq \|\varphi\|_2.
\]

Combining these steps leads to
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \bar{w}_l}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)} = \|\varphi\|_2^2 \geq \|g \varphi\|_2^2 \geq \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k \bar{w}_l g(z_k) g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{1}{b'(z_k)b'(z_l)},
\]
i.e., to
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)g(z_l)}{1 - z_k \bar{z}_l} \cdot \frac{w_k \bar{w}_l}{b'(z_k)b'(z_l)} \geq 0,
\]
and since the sequence \(w_1, w_2, \ldots, w_n\) is arbitrary, we have for all choices of \(\lambda_1, \lambda_2, \ldots, \lambda_n\),
\[
\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k\overline{z_l}} \cdot \lambda_k \overline{\lambda_l} \geq 0.
\]
By the choice \(n = 1, \lambda_1 = 1\) we see that the converse is trivial.

References


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