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A NEW CHARACTERIZATION OF THE UNIT BALL OF H^2

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ABSTRACT. We derive a new expression for the norm of H^2 functions; we present some well-known results in a different setting.

INTRODUCTION

In 1915, Pick [3] proved the following result.

Theorem 1. *Let g be an analytic function on the unit disc Δ in the complex plane. Then $|g(z)| \leq 1$ for all $z \in \Delta$ if and only if for all $n \in \mathbb{N}$, for all sequences $z_1, z_2, \ldots, z_n$ in Δ and for all sequences $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have*

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1-g(z_k)\overline{g(z_l)}}{1-z_k\overline{z_l}}\lambda_k\overline{\lambda_l} \geq 0. \tag{1}$$

Ahlfors [1], page 3, gives an elegant proof of this characterization of the unit ball of H^∞.

In this note we shall present a characterization of the unit ball of H^2. Our main tool will be an explicit solution of the "minimal interpolation problem" for H^2 (see [2], page 141). As a byproduct we obtain a new proof of Pick's theorem.

DESCRIPTION OF THE MAIN RESULT

Let $z_1, z_2, \ldots, z_n$ be a sequence in Δ, and let b be the Blaschke product generated by the sequence

$$b(z) = \prod_{j=1}^{n}\frac{z-z_j}{1-\overline{z}_j z}. \tag{2}$$

We shall prove that the following conditions are equivalent for continuous functions f on Δ:

1) f lies in the unit ball of H^2.
2) For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in Δ we have

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{f(z_k)\overline{f(z_l)}}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. \tag{3}$$

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Preliminaries

For mutually distinct points $z_1, z_2, \ldots, z_n$ in Δ and for $w_1, w_2, \ldots, w_n$ in $\mathbb{C}$ we define

$$\Lambda = \{f \in H^2 : f(z_j) = w_j,\ j = 1, 2, \ldots, n\}.$$

Λ is not empty; it contains the Lagrange interpolation polynomial

$$\lambda(z) = l(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) l'(z_k)},$$

where $l(z) = \prod_{j=1}^{n} (z - z_j)$.

In the context of H^p spaces it is more natural to work with the Blaschke interpolation function

$$\beta(z) = b(z) \sum_{k=1}^{n} \frac{1 - \overline{z}_k z}{z - z_k} \cdot \frac{w_k}{b'(z_k)(1 - |z_k|^2)},$$

with $b(z)$ defined as in (2). Of course $\beta \in \Lambda$. However, for our purposes we are better off with

$$\varphi(z) = b(z) \sum_{k=1}^{n} \frac{w_k}{(z - z_k) b'(z_k)}. \tag{4}$$

$\varphi \in \Lambda$, and φ is analytic on some neighbourhood of $\overline{\Delta}$. Λ is a hyperplane in H^2. With φ and b defined as in (4) and (2) we have

$$\Lambda = \{\varphi + bg;\ g \in H^2\}.$$

Theorem 2. *φ is the unique solution of the "minimal interpolation problem", i.e., for every $f \in \Lambda \backslash \{\varphi\}$ we have $\|f\|_2 > \|\varphi\|_2$.*

Proof. It suffices to show that $\varphi \perp (f - \varphi)$ for every $f \in \Lambda$ (since under those circumstances $\|f\|^2 = \|\varphi\|^2 + \|f - \varphi\|^2$).

From the decomposition $f = \varphi + bg$ we have

$$\begin{aligned} \langle f - \varphi, \varphi \rangle &= \langle bg, \varphi \rangle = \frac{1}{2\pi} \int_0^{2\pi} b(e^{it}) g(e^{it}) \overline{\varphi(e^{it})} dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} b(e^{it}) g(e^{it}) \overline{b(e^{it})} \sum_{k=1}^{n} \frac{\overline{w}_k}{(e^{-it} - \overline{z}_k) \overline{b'(z_k)}} dt. \end{aligned}$$

Note that $|b(e^{it})|^2 = 1$. Thus,

$$\begin{aligned} \langle f - \varphi, \varphi \rangle &= \sum_{k=1}^{n} \frac{\overline{w}_k}{2\pi \overline{b'(z_k)}} \int_0^{2\pi} g(e^{it}) \frac{e^{it}}{1 - e^{it} \overline{z}_k} dt \\ &= \sum_{k=1}^{n} \frac{\overline{w}_k}{\overline{b'(z_k)}} \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{g(z)}{1 - \overline{z}_k z} dz = 0, \end{aligned}$$

because the integrand is analytic on Δ.

It will be convenient to have an explicit expression for $\|\varphi\|_2$:

$$\begin{aligned}\|\varphi\|_2^2 &= \frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{it})|^2 dt = \frac{1}{2\pi}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{w_k\overline{w}_l}{b'(z_k)\overline{b'(z_l)}}\int_0^{2\pi}\frac{dt}{(e^{it}-z_k)(e^{-it}-\overline{z}_l)} \\ &= \frac{1}{2\pi i}\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{w_k\overline{w}_l}{b'(z_k)\overline{b'(z_l)}}\int_\Gamma\frac{dz}{(z-z_k)(1-\overline{z}_l z)} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{w_k\overline{w}_l}{1-z_k\overline{z}_l}\ \frac{1}{b'(z_k)\overline{b'(z_l)}}.\end{aligned}$$

There are, of course, many other expressions for $\|\varphi\|_2$.

Theorem 3.

$$\|\varphi\|_2 = \max\left\{\left|\sum_{k=1}^{n}\frac{w_k f(z_k)}{b'(z_k)}\right| : f\in H^2, \|f\|_2\le 1\right\}.$$

Proof.

$$\sum_{k=1}^{n}\frac{w_k f(z_k)}{b'(z_k)} = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)\varphi(z)}{b(z)}dz;$$

hence, by Schwarz's inequality we have

$$\left|\sum_{k=1}^{n}\frac{w_k f(z_k)}{b'(z_k)}\right| \le \frac{1}{2\pi}\int_0^{2\pi}|f(e^{it})|\cdot|\varphi(e^{it})|dt \le \|f\|_2\cdot\|\varphi\|_2 \le \|\varphi\|_2.$$

Equality holds for the function $f : z \to \frac{1}{\|\varphi\|_2}\sum_{k=1}^{n}\frac{\overline{w}_k}{(1-\overline{z}_k z)\overline{b'(z_k)}}$.

An immediate result from Theorem 2 is

Corollary. *For every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points of Δ we have*

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}} \le 1.$$

Proof. Take $w_1 = w_2 = \ldots = w_n = 1$. Then $1 \in \Lambda$ and since

$$\|1\|_2 = 1,$$

we have

$$1 \ge \|\varphi\|_2^2 = \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

The equality sign certainly occurs if $0 \in \{z_1, z_2, \ldots, z_n\}$:

$$1 = \varphi(0)^2 \le \frac{1}{2\pi}\int_0^{2\pi}|\varphi(e^{it})|^2 dt = \|\varphi\|_2^2 = \sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1}{1-z_k\overline{z}_l}\cdot\frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

If $0 \notin \{z_1, z_2, \ldots, z_n\}$, there is strict inequality.

Because of the uniqueness of φ there can be equality only if

$$b(z)\sum_{k=1}^{n}\frac{1}{(z-z_k)b'(z_k)} = 1.$$

In this identity for rational functions we let $z \to \infty$. Since $z_j \neq 0$, $\lim_{z\to\infty} b(z)$ has a finite value. Therefore, the left-hand side has limit zero.

Remark. The corollary shows that a function satisfying (1) also satisfies (3).

The fact that $\varphi \in \Lambda$ has an interesting reformulation. We start with a lemma.

Lemma 1. *The partial fraction decomposition of* φ *is*

$$\varphi(z) = \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1-\overline{z}_l z)(1-\overline{z}_l z_k) b'(z_k)\overline{b'(z_l)}}. \tag{5}$$

Proof. An elegant way to prove this is to compute both sides of the following identity.

For $z \in \Delta$ we have

$$\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{1-\overline{\zeta}z}\cdot\frac{d\zeta}{\zeta} = \overline{\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\varphi(\zeta)}}{1-\zeta\overline{z}}\cdot\frac{d\zeta}{\zeta}}.$$

The left-hand side is equal to

$$\frac{1}{2\pi i}\int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} dz = \varphi(z),$$

while the right-hand side is equal to the complex conjugate of

$$\frac{1}{2\pi i}\int_\Gamma \overline{b(\zeta)} \sum_{k=1}^{n} \frac{\overline{w}_k}{\overline{(\zeta - z_k) b'(z_k)}} \cdot \frac{1}{1-\zeta\overline{z}}\cdot\frac{d\zeta}{\zeta},$$

i.e., to the complex conjugate of

$$\frac{1}{2\pi i}\int_\Gamma \frac{1}{b(\zeta)} \sum_{k=1}^{n} \frac{\overline{w}_k}{(1-\overline{z}_k\zeta)\overline{b'(z_k)}}\ \frac{d\zeta}{1-\overline{z}\zeta}.$$

Calculation of the residues at the points $z_1, z_2, \ldots, z_n$ lead to (5).

The condition $\varphi \in \Lambda$ implies that $\varphi(z_j) = w_j$, $j = 1, \ldots, n$, i.e.,

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \frac{w_k}{(1-\overline{z}_l z_j)(1-\overline{z}_l z_k) b'(z_k)\overline{b'(z_l)}} = w_j.$$

This is equivalent to the assertion that the matrices

$$B = (\beta_{lk})$$

and its conjugate $\overline{B} = (\overline{\beta}_{lk})$ where

$$\beta_{lk} = \frac{1}{(1-\overline{z}_l z_k) b'(z_k)}$$

are inverses of each other, i.e., B and $\overline{B}$ are unitary.

Proof of the main result

Theorem 4. *Let f be a continuous function on the unit disc in the complex plane. Then the following conditions are equivalent:*

1. *f is analytic and f lies in the unit ball of H^2.*
2. *For every $n \in \mathbb{N}$ and for every sequence $z_1, z_2, \ldots, z_n$ of mutually distinct points in Δ we have*

$$\sum_{k=1}^{n}\sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1. \tag{3}$$

Proof. We split up the proof into two lemmas.

Lemma 2. *Let f belong to the unit ball of H^2, and let a sequence of mutually distinct points $z_1, z_2, \ldots, z_n$ in Δ be given. Then* (3) *holds.*

Proof. Define $w_j = f(z_j)$. f lies in the hyperplane Λ and the element φ of Λ with minimal norm satisfies

$$\|\varphi\|_2 \leq \|f\|_2 \leq 1.$$

Use of the explicit expression for $\|\varphi\|_2$ leads to (3).

Lemma 3. *Let f be continuous and assume that f satisfies* (3)*. Then f is analytic and f lies in the unit ball of H^2.*

Proof. We apply (3) for the case $n = 1$; an easy computation shows that

$$|f(z)| \leq \frac{1}{\sqrt{1 - |z|^2}} \tag{6}$$

for every choice of $z \in \Delta$.

Let $0 < r < \rho < 1$, and let $z_1, z_2, z_3, \ldots$ be an enumeration of the rational points of $\overline{\Delta}_\rho$. For every n there is a function φ_n with

$$\varphi_n(z_j) = f(z_j), \quad j = 1, 2, \ldots, n,$$

and

$$\|\varphi_n\|_2^2 = \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{f(z_k)\overline{f(z_l)}}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1.$$

Thus, φ_n lies in the unit ball of H^2, and so by Lemma 2, we have for every sequence $\zeta_1, \zeta_2, \ldots, \zeta_n$ in Δ

$$\sum_{k=1}^{m}\sum_{l=1}^{m} \frac{\varphi_n(\zeta_k)\overline{\varphi_n(\zeta_l)}}{1 - \zeta_k\overline{\zeta}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}} \leq 1.$$

It follows from (6) that

$$|\varphi_n(\zeta)| \leq \frac{1}{\sqrt{1 - |\zeta|^2}},$$

hence the sequence $\varphi_1, \varphi_2, \ldots$ is uniformly bounded on $\overline{\Delta}_\rho$. Therefore, it contains a locally uniformly convergent subsequence φ_{n_j}. At the points $z_1, z_2, \ldots$ the subsequence converges to f. By the continuity of f and the fact that $\{z_1, z_2, \ldots\}$ is dense in Δ_ρ we see that

$$\lim_{n_j \to \infty} \varphi_{n_j} = f.$$

This shows that f is analytic on Δ_ρ for all $\rho < 1$. Because of uniform convergence on Γ_r, we have

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2 dt = \lim_{n_j\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |\varphi_{n_j}(re^{it})|^2 dt \le 1.$$

Thus, $f \in H^2$ and $\|f\|_2 \le 1$.

Lemma 2 and Lemma 3 together constitute a proof of the theorem.

Corollary. *For $f \in H^2$ we define*

$$\nu(f) = \sup\left\{\sum_{k=1}^n \sum_{l=1}^n \frac{f(z_k)\overline{f(z_l)}}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}};\right.$$
$$\left. z_1, z_2, \ldots, z_n \text{ mutually distinct points of } \Delta\right\}.$$

Then $\nu(f) = \|f\|_2^2$.

Proof. Assume that $\nu(f) = 1$. Then by Lemma 3 $\|f\|_2^2 \le 1$. If $\|f\|_2^2 < \lambda^2 < 1$ for some λ, then we have $\left\|\frac{1}{\lambda}f\right\| < 1$ but $\nu\left(\frac{1}{\lambda}f\right) > 1$ which is impossible by Lemma 2.

In a similar way we can show that $\|f\|_2^2 = 1$ implies that $\nu(f) = 1$. By the homogeneity of ν and $\|\quad\|_2^2$ it follows that for all $f \in H^2$: $\nu(f) = \|f\|_2^2$.

Pick's theorem

As an application of our results we shall give a proof of Pick's theorem.

Let g belong to the unit ball of H^∞, and let $z_1, z_2, \ldots, z_n$ be a sequence of mutually distinct points in Δ. Let $w_1, w_2, \ldots, w_n$ be an arbitrary sequence of complex numbers. We consider the hyperplanes Λ and Λ_g where

$$\Lambda_g = \{f \in H^2 : f(z_j) = w_j g(z_j), j = 1, 2, \ldots, n\}.$$

Of course, if $f \in \Delta$, then $g \cdot f \in \Delta_g$, and by Theorem 2 applied to Λ_g we have

$$\|gf\|_2^2 \ge \sum_{k=1}^n \sum_{l=1}^n \frac{w_k g(z_k)\overline{w_l g(z_l)}}{1 - z_k\overline{z}_l} \frac{1}{b'(z_k)\overline{b'(z_l)}}.$$

Let φ be, as before, the element of Λ with smallest norm. From $\|g\|_\infty \le 1$ we obtain

$$\|g\varphi\|_2 \le \|\varphi\|_2.$$

Combining these steps leads to

$$\sum_{k=1}^n \sum_{l=1}^n \frac{w_k\overline{w}_l}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}}$$
$$= \|\varphi\|_2^2 \ge \|g\varphi\|_2^2 \ge \sum_{k=1}^n \sum_{l=1}^n \frac{w_k\overline{w}_l g(z_k)\overline{g(z_l)}}{1 - z_k\overline{z}_l} \cdot \frac{1}{b'(z_k)\overline{b'(z_l)}},$$

i.e., to

$$\sum_{k=1}^n \sum_{l=1}^n \frac{1 - g(z_k)\overline{g(z_l)}}{1 - z_k\overline{z}_l} \cdot \frac{w_k\overline{w}_l}{b'(z_k)\overline{b'(z_l)}} \ge 0,$$

and since the sequence $w_1, w_2, \ldots, w_n$ is arbitrary, we have for all choices of λ_1, $\lambda_2, \ldots, \lambda_n$,

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{1-g(z_k)\overline{g(z_l)}}{1-z_k\overline{z}_l}\cdot\lambda_k\overline{\lambda}_l\geq 0.$$

By the choice $n = 1$, $\lambda_1 = 1$ we see that the converse is trivial.

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