Homogeneous quasi-translations and an article of P. Gordan and M. Nöther

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Introduction

In this paper, I present some results about a special type of polynomial maps, the so-called quasi-translations. A quasi-translation is a map $x + H$ such that $x - H$ is its inverse. In 1876 in [6], P. Gordan and M. Nöther studied quasi-translations to understand singular Hessians better, because quasi-translations can be made from Hessians with determinant zero in the following manner:

Let f be a polynomial such that the Hessian $\mathcal{H}f := \mathcal{J}\nabla f$ has determinant zero. Then the components of ∇f are algebraically dependent over $\mathbb{C}$, say that $R(\nabla f) = 0$ with $R \neq 0$. Now

$$x + (\nabla R \circ \nabla f)$$

happens to be a quasi translation.

In [1], it is shown that

$$D := \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$$

with $H_i := R_{x_i}(\nabla f)$ satisfies $D^2 x_i = 0$ for all i. In [5], it is shown that $D^2 x_i = 0$ is equivalent to $x + H$ being a quasi-translation.

We will reprove most of the results of [6] from P. Gordan and M. Nöther (as far as they can not be found in [1]), since they are written in an old-fashioned, not very readable style.

1 Homogeneous quasi-translations $x + H$ with $\operatorname{rk} \mathcal{J}H \le 2$

Let H be homogeneous of degree d such that $D := \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$ satisfies

$$D^2 x_i = DH_i = 0 \quad (1 \le i \le n)$$

i.e. $\mathcal{J}H \cdot H = 0$.

In [6], P. Gordan and M. Nöther characterized such H for which $\operatorname{rk}\mathcal{J}H \le 2$. They proved the following result with geometric methods:

Theorem 1.1. *Assume that $D = \sum_{i=1}^{n} H_i \partial_i$ is a derivation such that $D^2 x_i = 0$ for all i. Assume further that H is homogeneous of degree d and the rank of $\mathcal{J}H$ is at most 2 and define $g = \gcd\{H_1, H_2, \ldots, H_n\}$.*

Then there is a linear transformation T and an $s \ge \min\{2, n-1\}$ such that $\tilde{H} = T^{-1} \circ g^{-1} H \circ T$ satisfies $\tilde{H}_1 = \tilde{H}_2 = \cdots = \tilde{H}_s = 0$ and $\tilde{H}_{s+1}, \ldots, \tilde{H}_n \in \mathbb{C}[x_1, x_2, \ldots, x_s]$.

If $\mathcal{J}H \cdot H = 0$, then H is an eigenvector of $\mathcal{J}H$ and $\mathcal{J}H$ is nilpotent [5, Proposition 1.1 iii)], i.e. all eigenvalues of $\mathcal{J}H$ are zero. But the reverse holds as well, so $x + H$ is a quasi-translation, if and only if H is an eigenvector of $\mathcal{J}H$ and $\mathcal{J}H$ is nilpotent. In particular, if $x + H$ is a quasi-translation, then

$$\mathcal{J}H \cdot H = \operatorname{tr}\mathcal{J}H \cdot H$$

The following theorem characterizes all maps H with $rk\mathcal{J}H \le 2$ that satisfy this property and therefore generalizes theorem 1.1:

Theorem 1.2. *Assume that H is homogeneous. Then the following statements are equivalent:*

i) $\operatorname{rk}\mathcal{J}H \le 2$ *and* $\mathcal{J}H \cdot H = \operatorname{tr}\mathcal{J}H \cdot H$,

ii) There exists a linear transformation T and an $s \ge \min\{2, n-1\}$, such that $\tilde{H} = T^{-1} \circ H \circ T$ is of the form $\tilde{H} = g \cdot h(p, q)$, with $\tilde{H}_1 = \tilde{H}_2 = \cdots = \tilde{H}_s = 0$ and $p, q \in \mathbb{C}[x_1, x_2, \ldots, x_s]$.

We first show that ii) implies i) in the above theorem. So assume ii). Since $H_i \in \mathbb{C}[g^{1/\deg h} p, g^{1/\deg h} q]$ for all i, $\operatorname{rk}\mathcal{J}H = \operatorname{rk}\mathcal{J}\tilde{H} = \operatorname{trdeg}_{\mathbb{C}} \tilde{H} \le 2$ follows. Furthermore, if we put $\tilde{D} = \sum_{i=1}^{n} \tilde{H}_i \frac{\partial}{\partial x_i}$, then it follows from $\tilde{H}_1 = \tilde{H}_2 = \cdots = \tilde{H}_s = 0$ and $p, q \in \mathbb{C}[x_1, x_2, \ldots, x_s]$ that $p, q \in \ker D$, whence

$$\begin{aligned}
\mathcal{J}\tilde{H}_i \cdot \tilde{H} &= \tilde{D}\tilde{H}_i \;=\; h_i(p,q)\tilde{D}g \\
&= \tilde{H}_i \sum_{j=1}^{n} h_j(p,q) \frac{\partial}{\partial x_j} g \\
&= \tilde{H}_i \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \tilde{H}_j \\
&= \tilde{H}_i \cdot \operatorname{tr}\mathcal{J}\tilde{H}
\end{aligned}$$

So $\mathcal{J}\tilde{H} \cdot \tilde{H} = \operatorname{tr} \mathcal{J}\tilde{H} \cdot \tilde{H}$, whence

$$\begin{aligned}
\mathcal{J}H \cdot H &= T\mathcal{J}\tilde{H}|_{T^{-1}x}T^{-1} \cdot T\tilde{H}|_{T^{-1}x} \\
&= T\mathcal{J}\tilde{H}|_{T^{-1}x}\tilde{H}|_{T^{-1}x} \\
&= T \operatorname{tr} \mathcal{J}\tilde{H}|_{T^{-1}x}\tilde{H}|_{T^{-1}x} \\
&= T \operatorname{tr} \mathcal{J}\tilde{H}|_{T^{-1}x}T^{-1} \cdot T\tilde{H}|_{T^{-1}x} \\
&= \operatorname{tr} \mathcal{J}H \cdot H
\end{aligned}$$

and i) follows.

We will prove the implication i) $\Rightarrow$ ii) in the remainder of this section.

Corollary 1.3. *Assume H satisfies i) of theorem 1.2 and define $g = \gcd\{H_1, H_2, \ldots, H_n\} = 1$. Then $x + g^{-1}H$ is a quasi-translation.*

We shall show later that for quasi-translation $x + H$,

$$\operatorname{rk} \mathcal{J}H \leq \max\{n-2, 1\} \tag{1}$$

So if $n \leq 4$, then $\operatorname{rk} \mathcal{J}H \leq 2$.

Corollary 1.4. *Assume $x+H$ is a homogeneous quasi-translation in dimension $n \leq 4$. Then H satisfies i) and ii) of theorem 1.2*

The map

$$H = (0, 0, x_2 g, x_1 g)$$

with $g = (x_1x_3 - x_2x_4)$ is a typical example of a quasi-translation in dimension 4.

Before we prove theorem 1.2, we first formulate a result for homogeneous Jacobians of rank 2:

Theorem 1.5. *Assume $\operatorname{rk} \mathcal{J}H = 2$ and let $g := \gcd(H_i)$. Then there exist $h_i \in k[t_1, t_2]$ homogeneous of the same degree s or zero and p and q in $k[x]$ homogeneous of the same degree r such that $H_i = gh_i(p, q)$ for all i.*

Furthermore, both p and q are irreducible.

Proof. This theorem is formulated as Theorem 2.1 in [2], except the last sentence: the property that both p and q are irreducible. Assume that p and q have minimal degree. We distinguish two cases:

- $p + \lambda q$ is reducible for all $\lambda \in \mathbb{C}$.
  Since $g = \gcd\{H_1, H_2, \ldots, H_n\}$, $p+tq$ does not have divisors which degree with respect to t is zero. So $p + tq$ is irreducible. Now it follows from Theorem 18 in [8, p. 79] (see also Theorem 2.2 in [2]) that

$$p + tq = \sum_{i=1}^{s} a_i(t)(p^*)^i(q^*)^{s-i} \tag{2}$$

for some $s \geq 2$ and $p^*, q^* \in \mathbb{C}$.

Assume $\deg p^* \neq \deg q^*$, say that $\deg p^* > \deg q^*$. Take $\lambda \in \{0, 1\}$ and let i_λ be the largest i such that $a_i(\lambda) \neq 0$. Then

$$p + \lambda q = \overline{p + \lambda q} = a_{i_\lambda}(\lambda)(\overline{p^*})^{i_\lambda}(\overline{q^*})^{s-i_\lambda}$$

where $\overline{f}$ is the largest degree homogeneous part of f. Since $\deg p = \deg(p + q)$, we have $i_0 = i_1$, whence $\gcd\{p, p+q\} \neq 1$, contradicting $\gcd\{p, q\} = 1$.

So $\deg p^* = \deg q^*$. If $\overline{p^*}$ and $\overline{q^*}$ are linearly dependent, then we can replace q^* by a linear combination of p^* and q^* to get $\deg p^* > \deg q^*$, without affecting (2). So $\overline{p^*}$ and $\overline{q^*}$ are linearly independent.

Since

$$\sum_{i=1}^{s} a_i(\lambda) z_1^i z_2^{s-i}$$

is homogeneous and bivariate, it decomposes in linear factors, whence

$$\sum_{i=1}^{s} a_i(\lambda)(\overline{p^*})^i(\overline{q^*})^{s-i} \tag{3}$$

decomposes in linear combinations of $\overline{p^*}$ and $\overline{q^*}$. But each of these combinations is nonzero, so the expression in (3) is nonzero as well for all λ, in particular for $\lambda = 0$ and $\lambda = 1$. It follows that

$$p = \sum_{i=1}^{s} a_i(0)(\overline{p^*})^i(\overline{q^*})^{s-i}$$

and

$$p + q = \sum_{i=1}^{s} a_i(1)(\overline{p^*})^i(\overline{q^*})^{s-i}$$

so H_i/g can be expressed as a homogeneous polynomial in $\overline{p^*}$ and $\overline{q^*}$ for each i . Since $\deg \overline{p^*} = \deg \overline{q^*} < \deg p = \deg q$, and p and q were chosen of minimal degree, we have a contradiction.

- $p + \lambda q$ is irreducible for certain $\lambda \in \mathbb{C}$.
  Let $p^* = q$ and $q^* = p + \lambda q$. Then H is of the form

  $$H = gh^*(p^*, q^*)$$

  Furthermore, q^* is irreducible and according to the above, there exists a $\lambda^* \in \mathbb{C}$ such that $p^* + \lambda q^*$ is irreducible. Now replace p by $p^* + \lambda q^*$ and q by q^* to get the desired result. □

Lemma 1.6. *Let $H = (H_1, H_2, \ldots, H_n)$ be homogeneous and assume* $\operatorname{rk} \mathcal{J}H = 2$ *and* $\mathcal{J}H \cdot H = \operatorname{tr} \mathcal{J}H \cdot H$. *Then there are at least two independent linear relations between the components of H.*

Proof.

i) Assume first that $H_i \in \mathbb{C}[x_1, x_2]$ for all i. Then $\operatorname{tr} \mathcal{J}(H_1, H_2)$ is an eigenvalue of $\mathcal{J}(H_1, H_2)$. Since $\operatorname{tr} \mathcal{J}(H_1, H_2)$ is the sum of all eigenvalues of $\mathcal{J}(H_1, H_2)$ by definition, the other eigenvalue of $\mathcal{J}(H_1, H_2)$ is zero. So H_1 and H_2 are algebraically dependent and thus $n \geq 3$, for $\operatorname{rk} \mathcal{J}H = 2$. Since H_1 and H_2 are homogeneous of the same degree, there exists a homogeneous relation between them. This homogeneous relation decomposes into linear components, whence H_1 and H_2 are linearly dependent. So we may assume that $H_1 = 0$. It follows from $\mathcal{J}H \cdot H = \operatorname{tr} \mathcal{J}H \cdot H = (H_2)_{x_2} \cdot H$ that $\mathcal{J}H_3 \cdot H = (H_2)_{x_2} \cdot H_3$. But also $\mathcal{J}H_3 \cdot H = (H_3)_{x_2} \cdot H_2$. Apparently, either $H_2 = 0$ or
$$\frac{\partial}{\partial x_2} \frac{H_3}{H_2} = 0$$
in which case H_3 is a scalar multiple of H_2. So we have two independent linear relations between the H_i in case $H_i \in \mathbb{C}[x_1, x_2]$ for all i.

ii) Assume next the general case. From theorem 1.5, it follows that H is of the form $g \cdot h(p, q)$, where $g = \gcd\{H_1, H_2, \ldots, H_n\}$ and p and q are homogeneous of the same degree and irreucible. Furthermore, they are relatively prime by definition of g.

Replacing H by $T^{-1} \circ H \circ T$ and p resp. q by $p \circ T$ resp. $q \circ T$ for a suitable $T \in \mathrm{GL}_n(\mathbb{C})$, we may assume that
$$\begin{aligned} h_1(p,q) &\equiv p^r \pmod{pq} \\ h_2(p,q) &\equiv 0 \pmod{pq} \\ &\vdots \\ h_{n-1}(p,q) &\equiv 0 \pmod{pq} \\ h_n(p,q) &\equiv q^r \pmod{pq} \end{aligned}$$
Since H is an eigenvector of $\mathcal{J}H$, $h(p, q) = g^{-1}H$ is such an eigenvector as well, say $\mathcal{J}H \cdot h(p, q) = b(x) \cdot h(p, q)$. Put
$$D := \sum_{i=1}^{n} h_i(p, q) \frac{\partial}{\partial x_i}$$
Assume $H_1 \neq 0$. Then
$$D \frac{H_i}{H_1} = \frac{H_1 D H_i - H_i D H_1}{H_1^2} = \frac{H_1 b(x) H_i - H_i b(x) H_1}{H_1^2} = 0$$
but also
$$D \frac{H_i}{H_1} = D \frac{h_i(p/q, 1)}{h_1(p/q, 1)} = \left(\frac{\partial}{\partial y_1} \frac{h_i}{h_1} \right) (p/q, 1) \cdot D \frac{p}{q}$$
and
$$D \frac{H_i}{H_1} = D \frac{h_i(1, q/p)}{h_1(1, q/p)} = \left(\frac{\partial}{\partial y_2} \frac{h_i}{h_1} \right) (1, q/p) \cdot D \frac{q}{p}$$

Since $D(q/p) = -q^2/p^2 D(p/q)$, it follows that either both h_2/h_1 and h_3/h_1 are constant or $D(p/q) = 0$.

In the first case, we have with $H_2 = h_2/h_1 \cdot H_1$ and $H_3 = h_3/h_1 \cdot H_1$ two independent linear relations between the components of H, so assume $D(p/q) = 0$. Then $Dp = p/q \cdot Dq$, so $q \mid Dq$, for $\gcd\{p, q\} = 1$. Since also $Dq \equiv p^r q_{x_1} \pmod{q}$, it follows that $q \mid p^r q_{x_1}$. This is only possible if $q_{x_1} = 0$, i.e. $q \in \mathbb{C}[x_2, \ldots, x_{n-1}, x_n]$, for $\gcd\{p, q\} = 1$. Similarly, $p \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$.

iii) Assume first that either $p \in \mathbb{C}[x_2, \ldots, x_{n-1}]$ or $q \in \mathbb{C}[x_2, \ldots, x_{n-1}]$, say that $q \in \mathbb{C}[x_2, \ldots, x_{n-1}]$. Then $p, q \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$. Let $\hat{g}$ be the leading coefficient of g with respect to x_n and put $\hat{H} = \hat{g}(h_1(p, q), h_2(p, q), \ldots, h_{n-1}(p, q))$. Then it is easy to show that $\hat{H}$ inherits all properties of H, but $\hat{H}$ has one variable less. So if $n \geq 4$, then it follows by induction that there are at least two independent linear relations between the components of H. The case $n = 3$ reduces to i), so there are at least two independent linear relations between the components of H.

iv) Assume next that $p_{x_1} \neq 0$ and $q_{x_n} \neq 0$. Since p and q are irreducible, $\gcd\{p, p_{x_1}\} = \gcd\{q, q_{x_n}\} = 1$. So there is a non-constant $c \in \mathbb{C}[x_2, \ldots, x_{n-1}]$, and there are $a_i \in \mathbb{C}[x_1, x_2, \ldots, x_{n-1}]$ and $b_i \in \mathbb{C}[x_2, \ldots, x_{n-1}, x_n]$ such that
$$a_1 p + a_2 p_{x_1} = b_1 q + b_2 q_{x_n} = c \tag{4}$$
Write $\omega = (\omega_1, \omega_2, \ldots, \omega_{n-1}, \omega_n)$. Now take $\omega_2, \ldots, \omega_{n-1} \in \mathbb{C}$ such that $c(\omega_2, \ldots, \omega_{n-1}) = 1$. Take ω_1 such that $p(\omega) = 0$ and ω_n such that $q(\omega) = 0$. From (4), it follows that $p_{x_1}(\omega) \neq 0$ and $q_{x_n}(\omega) \neq 0$. Since $p_{x_n}(\omega) = q_{x_1}(\omega) = 0$, $(\nabla p)(\omega)$ and $(\nabla q)(\omega)$ are independent.

The following lemma completes the proof of lemma 1.6, since it reduces the general case to the case $H_i \in \mathbb{C}[x_1, x_2]$ of i): □

Lemma 1.7. *Let $H = (H_1, H_2, \ldots, H_n)$ be homogeneous. Assume* $\operatorname{rk} \mathcal{J}H = 2$ *and write $H = gh(p, q)$, where $g = \gcd\{H_1, H_2, \ldots, H_n\}$. Assume that there is an $\omega \in \mathbb{C}^n$ such that $p(\omega) = q(\omega) = 0$ and such that $(\nabla p)(\omega)$ and $(\nabla q)(\omega)$ are independent. Then there is an $\hat{H} = \hat{g}h(\hat{p}, \hat{q})$ such that $\hat{p}$ and $\hat{q}$ are linear, $\hat{g} = \gcd\{\hat{H}_1, \hat{H}_1, \ldots, \hat{H}_n\} \in \mathbb{C}[\hat{p}, \hat{q}]$, and such that the algebraic relations between the H_i correspond to those between the $\hat{H}_i$.*

Furthermore, if $\mathcal{J}H \cdot H = \operatorname{tr} \mathcal{J}H \cdot H$, then $\mathcal{J}\hat{H} \cdot H = \operatorname{tr} \mathcal{J}\hat{H} \cdot H$.

Proof. By way of a suitable linear transformation, we may assume that $\omega = e_1$. Let r be the degree of p and q. Since $p(e_1) = q(e_1) = 0$, the coefficients of x_1^r in p and q are zero. It follows that the coefficient of $x_i x_1^{r-1}$ of p resp. q equals $p_{x_i}(e_1)$ resp. $q_{x_i}(e_1)$, for all i.

Let $\hat{p}$ resp. $\hat{q}$ be the leading coefficient of p resp. q with respect to x_1. Then $\hat{p} = \langle (\nabla p)(e_1), x \rangle$ and $\hat{q} = \langle (\nabla q)(e_1), x \rangle$, for $(\nabla p)(e_1)$ and $(\nabla q)(e_1)$ are independent vectors by assumption. Furthermore, $\hat{p}$ and $\hat{q}$ are algebraically independent linear forms. Since $\operatorname{rk} \mathcal{J}H = 2$, p and q are algebraically independent as well, and each relation between the components of both H and $h(\hat{p}, \hat{q})$ are relations between the components of h already.

There is a linear coordinate system of $\mathbb{C}[x]$ of the form $x_1, y_2, \dots, y_{n-2}, \hat{p}, \hat{q}$. Let g_1 be the leading coefficient of g with respect to x_1 and define g_i inductively as the leading coefficient of g_{i-1} with respect to y_i, for $i = 2, \dots, n-2$. Then $\hat{g} := g_{n-2} \in \mathbb{C}[p, q]$. Furthermore, $\hat{H}$ can be obtained from H in the same way as $\hat{g}$ was obtained from g, and each step in this construction preserves the properties of H. So $\mathcal{J}\hat{H} \cdot \hat{H} = \operatorname{tr} \mathcal{J}\hat{H} \cdot \hat{H}$. □

Proof of theorem 1.2: We must show the implication i) ⇒ ii). So assume i). Take s maximal such that there is a $T \in \mathrm{GL}_n(\mathbb{C})$ for which $\tilde{H} = T^{-1} \circ H \circ T$ satisfies $\tilde{H}_1 = \tilde{H}_2 = \cdots = \tilde{H}_s = 0$. Put $K = \mathbb{C}(\frac{x_1}{x_s}, \frac{x_2}{x_s}, \dots, \frac{x_{s-1}}{x_s})$. We distinguish three cases:

- $(\tilde{H}_{s+1}, \tilde{H}_{s+2}, \dots, \tilde{H}_n)$ are linearly dependent over K.
  Say that
  $$c_{s+1}\tilde{H}_{s+1} + c_{s+2}\tilde{H}_{s+2} + \cdots + c_n\tilde{H}_n = 0$$
  where $c_i \in K$ for all i. Write $\tilde{H} = gh(p, q)$ and put $\hat{h} = \sum_{i=s+1}^n c_i h_i \in K[y_1, y_2]$. Then $\hat{h}(p, q) = 0$, so either $\hat{h} = 0$ or p and q are algebraically dependent over K.

  If $\hat{h} = 0$, then $h_{s+1}, h_{s+2}, \dots, h_n$ are linearly dependent over K, and hence over $\mathbb{C}$, since $h_i \in \mathbb{C}[y_1, y_2]$ for all i. So $H_{s+1}, H_{s+2}, \dots, H_n$ are linearly dependent over $\mathbb{C}$. This contradicts the maximality of s, however. So $\hat{h}$ is a relation over K between p and q. Since $\hat{h}$ is homogeneous, $\hat{h}$ decomposes in linear factors over $\bar{K}[y_1, y_2]$, where $\bar{K}$ is the algebraic closure of K, and one of these factors is already a relation between p and q. So $p/q \in \bar{K} \cap K(x_s, x_{s+1}, \dots, x_n)$. Since p and q are relatively prime, it follows that $p, q \in K$, and ii) follows.

- $s \geq 2$.
  $\tilde{H}$ is a homogeneous polynomial map over $K = \mathbb{C}(\frac{x_1}{x_s}, \frac{x_2}{x_s}, \dots, \frac{x_{s-1}}{x_s})$, with $n-(s-1) < n$ indeterminates $x_s, x_{s+1}, \dots, x_n$. From Lefschetz' principle, it follows again from lemma 1.6 that there are at least two independent linear relations over K between the components of $(\tilde{H}_s, \tilde{H}_{s+1}, \dots, \tilde{H}_n)$. One such relation is $\tilde{H}_s = 0$, and the other is given by, say
  $$c_{s+1}\tilde{H}_{s+1} + c_{s+2}\tilde{H}_{s+2} + \cdots + c_n\tilde{H}_n = 0$$
  where $c_i \in K$ for all i. Write $\tilde{H} = gh(p, q)$ and put $\hat{h} = \sum_{i=s+1}^n c_i h_i \in K[y_1, y_2]$. Then $\hat{h}(p, q) = 0$, so either $\hat{h} = 0$ or p and q are algebraically dependent over K.

  If $\hat{h} = 0$, then $h_{s+1}, h_{s+2}, \dots, h_n$ are linearly dependent over K, and hence over $\mathbb{C}$, since $h_i \in \mathbb{C}[y_1, y_2]$ for all i. So $H_{s+1}, H_{s+2}, \dots, H_n$ are linearly dependent over $\mathbb{C}$. This contradicts the maximality of s, however. So $\hat{h}$ is a relation over K between p and q. Since $\hat{h}$ is homogeneous, $\hat{h}$ decomposes in linear factors over $\bar{K}[y_1, y_2]$, where $\bar{K}$ is the algebraic closure of K, and one of these factors is already a relation between p and q. So $p/q \in \bar{K} \cap K(x_s, x_{s+1}, \ldots, x_n)$. Since p and q are relatively prime, it follows that $p, q \in K$, and ii) follows.

- $s \leq 1$.
  If $\operatorname{rk} \mathcal{J}H = 1$, then $s = n - 1$, for each pair of components of H is linearly dependent. If $\operatorname{rk} \mathcal{J}H = 2$, then it follows from lemma 1.6 that $s \geq 2$. So if $H \neq 0$, then $s = 1$, $n = 2$ and $\operatorname{rk} \mathcal{J}H = 1$. Now define $\tilde{H}_3 = x_1^d$, where d is the degree of H. The map $(\tilde{H}, \tilde{H}_3) = (0, \tilde{H}_2, \tilde{H}_3)$ satisfies the properties of H, whence $\tilde{H}_2$ and $\tilde{H}_3$ are linearly dependent. So $\tilde{H}_2 \in \mathbb{C}[x_1]$. □

Corollary 1.8. *Assume that $H \in \mathbb{C}[x_1, x_2, x_3]^3$ is homogeneous, such that two of the three eigenvalues of $\mathcal{J}H$ are zero. Then the components of H are linearly dependent.*

Proof. Corollary 1.8 extends Theorem 1.4 of [2], where all eigenvalues of $\mathcal{J}H$ are assumed to be zero, i.e. $\mathcal{J}H$ is nilpotent. The proof of Theorem 1.4 distinguishes two cases: $\mathcal{J}H \cdot H = 0$ and $\mathcal{J}H \cdot H \neq 0$. Since $\mathcal{J}H \cdot H = 0$ implies that $\mathcal{J}H$ is nilpotent [5, Prop. 1.1], the case $\mathcal{J}H \cdot H = 0$ follows from Theorem 1.4 of [2]. So assume $\mathcal{J}H \cdot H \neq 0$.

In the proof of the case $\mathcal{J}H \cdot H \neq 0$ in the proof of Theorem 1.4, of [2], the condition that three eigenvalues are zero instead of two is only required on page 299 of [2], to ensure that the matrix T_v is invertible. It follows from the definiton of T_v that the condition that three eigenvalues are zero can be weakened to the condition that two eigenvalues are zero, provided we add the condition that x, $\mathcal{J}H \cdot x$ and $(\mathcal{J}H)^2 \cdot x$ are independent. So we may assume that x, $\mathcal{J}H \cdot x$ and $(\mathcal{J}H)^2 \cdot x$ are dependent.

i) Assume first that the vectors x and $\mathcal{J}H \cdot x$ are already dependent. Since $\mathcal{J}H \cdot x = dH$, it follows that H is of the form $H = g \cdot x$. This contradicts $\det \mathcal{J}H = 0$.

ii) So assume that x and $\mathcal{J}H \cdot x$ are independent. Since x, $\mathcal{J}H \cdot x$ and $(\mathcal{J}H)^2 \cdot x$ are dependent by assumption, it follows that there are $a(x), b(x) \in \mathbb{C}(x)$ such that
$$(\mathcal{J}H)^2 \cdot x = a(x) \cdot x + b(x) \cdot \mathcal{J}H \cdot x$$
Assume first that $a(x) = 0$. Dividing the above equality by d
$$\mathcal{J}H \cdot H = b(x) \cdot H$$
follows, i.e. H is an eigenvector of $\mathcal{J}H$.

Since $\mathcal{J}H \cdot H = 0$ implies that $\mathcal{J}H$ is nilpotent, it follows that $b(x)$ equals the third eigenvalue of $\mathcal{J}H$, i.e. $0 + 0 + b(x) = \operatorname{tr} \mathcal{J}H$. It follows that 1. of theorem 1.2 is fulfilled. So 2. of theorem 1.2 holds as well. Since $s \geq \min\{2, n-1\}$, it follows that $s = 2$ in 2. of theorem 1.2. So $\tilde{H} = (0, 0, g)$ in 2. of theorem 1.2.

iii) We show that the remaining case $a(x) \neq 0$ does not occur. So assume that $a(x) \neq 0$. Since $\mathcal{J}H \cdot x$ is independent of x, it follows that for generic $v_1 \in \mathbb{C}^3$, both $a(v_1) \neq 0$ and v_1 and

$$v_2 := (\mathcal{J}H)(v_1) \cdot v_1$$

are independent. Take v_3 independent of v_1 and v_2 and put

$$T = (v_1|v_2|v_3)$$

Then it is a straightforward exercise to compute that $\mathcal{J}(T^{-1} \circ H \circ T)$ is of the form

$$\mathcal{J}(T^{-1} \circ H \circ T) = \begin{pmatrix} 0 & a(v_1) & * \\ 1 & b(v_1) & * \\ 0 & 0 & * \end{pmatrix}$$

Since $\det \mathcal{J}H = 0$, the last row of the above Jacobian must be zero. So the sum of the determinants of the principal (2×2)-minors in the above Jacobian is $-a(v_1) \neq 0$. This contradicts the assumption that two of the three eigenvalues of $\mathcal{J}H$ are zero, so $a(x) = 0$. □

2 Homogeneous quasi-translations in dimension five

Let $x + H$ be any quasi-translation (possibly a real translation as well). Then $D := \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$ satisfies

$$D^2 x_i = DH_i = 0 \quad (1 \leq i \leq n)$$

and

$$H(x + tH) = H(x) \tag{5}$$

is a polynomial equality, since $H(x+tH) = H((\exp(tD))x) = (\exp(tD))H = H$. If L is a linear form, then $DL = L(H)$, whence

$$L \in \ker D \Longleftrightarrow L(H) = 0 \tag{6}$$

If $T \in \mathrm{GL}_n(\mathbb{C})$ then the map

$$x + T^{-1}H(Tx) = T^{-1}x \circ (x + H) \circ Tx$$

is again a quasi-translation.

Now assume that H is homogeneous of degree $d \geq 1$. If $A \in \ker D$ is homogeneous of degree ≥ 1, then

$$A = (\exp tD)A = A(x_1 + tH_1, x_2 + tH_2, \ldots, x_n + tH_n)$$

and $A(H)$ is the highest degree on the right hand side (the variable t is needed for the case $d = 1$). But the corresponding part on the left hand side is zero, so $A(H) = 0$.

In general, one can write $A \in \ker D$ as a sum of its homogeneous parts, each of which is contained in $\ker D$, and we get

$$A \in \ker D \Longrightarrow A(H) = A(0) \tag{7}$$

We call a derivation $D = \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i}$ *irreducible* if $\gcd\{H_i \mid 1 \le i \le n\} = 1$.

Proposition 2.1. *D is reducible, if and only if $\dim V(H) = n-1$. Furthermore, if $g = \gcd\{H_i \mid 1 \le i \le n\}$, then $\tilde{D} = g^{-1}D$ satisfies $\tilde{D}^2 x_i = 0$ as well.*

Proof. Assume $\dim V(H) = n - 1$. Then $V(H)$ contains an irreducible $(n-1)$-dimensional subvariety $V(g')$. It follows that $g' \mid H_i$ for all i, whence D is not irreducible. The reverse is similar.

Assume $g = \gcd\{H_i \mid 1 \le i \le n\}$. Since D is locally nilpotent, $\ker D$ is factorially closed, and $g \in \ker D$ follows. So $gDf = D(gf)$ for all f and $\tilde{D}^2 x_i = g^{-2} D^2 x_i = 0$. □

The above proposition says that in case D is reducible, i.e. $g = \gcd\{H_i \mid 1 \le i \le n\} \ne 1$, then D can be made irreducible by dividing out g. So we assume from now that D is irreducible. Notice that dividing out g might decrease the degree d to zero. But in that case, the original H satisfies

$$\operatorname{rk} \mathcal{J} H = 1 \tag{8}$$

This case has been dealt with in section 1. So we maintain our assumption that $d \ge 1$.

Since D is irreducible and $d \ge 1$, it follows from proposition 2.1 that $\dim V(H) \le n-2$. Looking at the coefficient of t^d in (5) with $d = \deg H$, $H \circ H = 0$ follows. So $H(\mathbb{C}^n) \subset V(H)$ and we have

$$\operatorname{rk} \mathcal{J} H = \dim H(\mathbb{C}^n) \le \dim V(H) \le n - 2$$

as well. This and (8) imply (1) in section 1.

Lemma 2.2. *Every strictly increasing chain of prime ideals in $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, \ldots, x_n]$ extends to one of length n.*

Proof. Assume without loss of generality that the first ideal is zero. From Krull's maximal ideal theorem, it follows that we may assume that the last prime ideal of the chain is a maximal ideal $\mathfrak{m}$. Using Hilbert's correspondence between zero sets and radicals, we see that $\mathfrak{m}$ is of the form $(x_1 - p_1, x_2 - p_2, \ldots, x_n - p_n)$. It follows that the height of $\mathfrak{m}$ is n. Since $\mathbb{C}[x]$ is catenary, the chain $0 \subsetneq \cdots \subsetneq \mathfrak{m}$ at hand extends to one of length n, as desired. □

Lemma 2.3. *Assume V_1 and V_2 are irreducible varieties of codimensions h_1 and h_2 in $\mathbb{C}^n$ and $V_1 \cap V_2 \neq \{0\}$. Then the codimension of $V_1 \cap V_2$ is at most $h_1 + h_2$.*

Proof. Let $\mathfrak{p}_1 = I(V_1)$ and $\mathfrak{p}_2 = I(V_2)$. Then $(\mathfrak{p}_1(x), \mathfrak{p}_2(y))$ is a prime ideal in $\mathbb{C}[x,y] = \mathbb{C}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n]$ of height $h_1 + h_2$. Since

$$(V_1 \cap V_2) \times (V_1 \cap V_2) \subset V\big(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x - y\big)$$

there exists a be a minimal prime ideal $\mathfrak{q}$ containing $(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x - y)$.

Since $(x - y)$ has n generators, the ideal $(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x - y)/(\mathfrak{p}_1(x), \mathfrak{p}_2(y))$ in $C[x,y]/(\mathfrak{p}_1(x), \mathfrak{p}_2(y))$ has n generators as well. It follows from the Krull height theorem that the height of $(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x-y)/(\mathfrak{p}_1(x), \mathfrak{p}_2(y))$ is at most n. Consequently, the height of q is at most $h_1 + h_2 + n$.

From lemma 2.2, we obtain that there exists a chain of prime ideals $\mathfrak{q} \subsetneq \cdots$ of lenght $2n - (h_1 + h_2 + n) = n - h_1 - h_2$. Consequently, the Krull dimension of $\mathbb{C}[x,y]/(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x-y)$ is at least $n-h_1-h_2$. But $\mathbb{C}[x,y]/(\mathfrak{p}_1(x), \mathfrak{p}_2(y), x-y)$ is isomorphic to the coordinate ring of $V_1 \cap V_2$. So $V_1 \cap V_2$ has dimension $n-h_1-h_2$ at least, as desired. □

Lemma 2.4. *Assume $x + H$ is a homogeneous quasi-translation and suppose that $\dim H(\mathbb{C}^n) = \dim V(H) \leq \lceil n/2 \rceil$. Let p and q be generic points in the Zariski closure W of $H(\mathbb{C}^n)$. Then there exists an $r \neq 0$ such that $\mathbb{C}p + \mathbb{C}r \subset W$ and $\mathbb{C}q + \mathbb{C}r \subset W$.*

In other words, there exists projective lines $L_p \ni p$ and $L_q \ni q$ in W that have a nonzero intersection.

Proof. Since $H(H) = 0$, $W \subset V(H)$. By $\dim W = \dim V(H)$ we obtain that the interior of $W \subset V(H)$ is non-empty. Since p and q are generic, we may assume that $p, q \in H(\mathbb{C}^n)$ and p, q are contained in the interior of $W \subset V(H)$.

Since $\dim H(\mathbb{C}^n) \leq \lceil n/2 \rceil$, it follows from the fiber theorem that $H^{-1}(p)$ has dimension $n - \lceil n/2 \rceil$ at least. Since H is homogeneous, the codimension of $\mathbb{C}^* H^{-1}(p)$ is less than that of $H^{-1}(p)$. So $\mathbb{C}^* H^{-1}(p)$ has codimension $\lceil n/2 \rceil - 1 < n/2$ at least. The same holds for $\mathbb{C}^* H^{-1}(q)$. Since $0 \in \mathbb{C}H^{-1}(p) \cap \mathbb{C}^* H^{-1}(q)$, it follows from lemma 2.3 that the codimension of the intersection of the Zariski closures of $\mathbb{C}^* H^{-1}(p)$ and $\mathbb{C}^* H^{-1}(q)$ is less than $n/2 + n/2 = n$. So there exists an $r \neq 0$ that is contained in this intersection.

Assume $r' \in \mathbb{C}^* H^{-1}(p)$. Then $H(r') = \lambda p$ for some $\lambda \in \mathbb{C}^*$. It follows that

$$H(r' + tp) = H\big(r' + t\lambda^{-1} H(r')\big) = H(r') = \lambda p$$

and we can derive that $H(r) \in \mathbb{C}p \cap \mathbb{C}q = \{0\}$. So $H(r + tp) = H(r) = 0$ and thus $\mathbb{C}p + \mathbb{C}r \subset V(H)$.

Since p is contained in the interior of $W \subset V(H)$, $\mathbb{C}p + \mathbb{C}r \subset W$. Similarly, $\mathbb{C}q + \mathbb{C}r \subset W$, as desired. □

Assume L_p is a projective line in a variety $W \subset \mathbb{C}^n$. We call L_p a *superline* if for generic $q \in W$, there exists a projective line $L_q \ni q$ such that $L_q \subset W$ and $L_q \cap L_p \neq \{0\}$.

Lemma 2.5. *Assume L_p is a superline in W. Then for all $q \in W$, there exists a projective line $L_q \ni q$ such that $L_q \subset W$ and $L_q \cap L_p \neq \{0\}$. In particular, $W = \bigcup_{q \in W} L_q$.*

Proof. W is the Zariski closure of $\bigcup_{q \in W} L_q$. □

Lemma 2.6. *Assume $x + H$ is a homogeneous quasi-translation and suppose that $\dim H(\mathbb{C}^n) = \dim V(H) = 3 \leq \lceil n/2 \rceil$. Let W be the Zariski closure of $H(\mathbb{C}^n)$ and $p \in W$ generic. Then there exists a projective line $L_p \ni p$ that is a superline of W.*

Proof. If there are infinitely many lines $L_p \ni p$ that are contained in W, then $\bigcup_{p \in W} L_p$ has dimension 3 at least, whence $\mathbb{C}p + \mathbb{C}q \subset W$ for generic q. So every projective line through p that is contained in W is a superline of W in this case. So assume that there are only finitely many lines $L_p \ni p$ that are contained in W. From lemma 2.4, it follows that one of these lines L_p is a superline of W, as desired. □

Theorem 2.7. *Assume $x + H$ is a homogeneous quasi-translation and*

$$\dim H(\mathbb{C}^n) = \dim V(H) = 3 \leq \lceil n/2 \rceil$$

Let W be the Zariski closure of $H(\mathbb{C}^n)$. Then either W is star-shaped with respect to some $p \in W \backslash \{0\}$ or there exists projective lines L_p and L_q such that $W \subset L_p + L_q$.

Proof. Take $p \in W$ generic. Then there exists a superline $L_p \ni p$ of W. If $L_q \cap L_p = \{0\}$, then for every point r of L_q, there exists a projective line L_r such that $L_r \cap L_p \neq \{0\}$, for L_p is a superline. Moreover, the union of all these lines L_r has dimension 3. It follows that $W \subset L_p + L_q$ in this case.

So assume $L_q \cap L_p \neq \{0\}$ for every projective line $L_q \subset W$. Now take q generic. Then there exists a superline $L_q \ni q$ of W. Next, take r generic. Then there exist a projective line $L_r \ni r$ that is contained in W. Now $L_r \cap L_p \neq \{0\}$, but similarly, we may assume that $L_r \cap L_q \neq \{0\}$, since otherwise $W \subset L_q + L_r$. Now there are two cases:

- $L_r \cap L_p \cap L_q \neq \{0\}$ for generic r. Then W is star-shaped with respect to every point of $L_r \cap L_p \cap L_q$.
- $L_r \cap L_p \cap L_q = \{0\}$ for generic r. Then $W \subset L_p + L_q$. Furthermore, W is a three-dimensional linear subspace of $\mathbb{C}^n$ in this case.

This gives the desired result. □

Some of the ideas of the above theorem and its proof can be found on [6, pp. 565-566].

The following quasi-translations are examples of each of the three cases in theorem 2.7:

- Case $L_q \cap L_p = \{0\}$:
  $H = (x_5^2(ax_1 - x_5^2x_2), a(ax_1 - x_5^2x_2), x_5^2(ax_3 - x_5^2x_4), a(ax_3 - x_5^2x_4), 0)$ with $a = x_1x_4 - x_2x_3$,
- Case $L_r \cap L_p \cap L_q \neq \{0\}$:
  $H = (x_4^5, bx_4^3, b^2x_4, 0, -b^2x_1 + 2bx_2x_4^2 - x_3x_4^4)$ with $b = x_1x_3 - x_2^2 + x_4x_5$.
- Case $L_r \cap L_p \cap L_q = \{0\}$:
  $H = (x_4^2, x_4x_5, x_1x_5 - x_2x_4, 0, 0)$,

The components of the second example are linearly dependent. If we take $g = x_4$, $p = x_4^2$ and $q = b$, then this example is of the form

$$H = \begin{pmatrix} gh_1(p,q) \\ gh_2(p,q) \\ gh_3(p,q) \\ gh_4(p,q) \\ r \end{pmatrix}$$

and $g \nmid r$. Furthermore, there is no linear conjugation that makes H of the above form with $g \mid r$. The below theorem now claims that the components of H are linearly dependent, and indeed $H_4 = 0$.

Theorem 2.8. *Let $H \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]^5$ be homogeneous of degree $d \geq 0$, such that $\mathcal{J}H \cdot H = 0$. Put $g := \gcd\{H_1, H_2, H_3, H_4, H_5\}$. If the components of H are not linearly dependent over $\mathbb{C}$, then there exists a $T \in \mathrm{GL}_5(\mathbb{C})$ such that $\tilde{H} = T^{-1} \circ H \circ T$, $\tilde{H}$ is of the form*

$$\tilde{H} = g \begin{pmatrix} h_1(p,q) \\ h_2(p,q) \\ h_3(p,q) \\ h_4(p,q) \\ r \end{pmatrix}$$

Furthermore, $x + g^{-1}H$ is a quasi-translation as well (and the components of $g^{-1}H$ are not linearly dependent over $\mathbb{C}$ either).

Proof. Assume the components of H are not linearly dependent over $\mathbb{C}$. Then $d \geq 2$. From theorem 2.7, we obtain that the Zariski closure of $H(\mathbb{C}^5)$ is star-shaped with respect to some nonzero $p \in \mathbb{C}^5$. Take T and $\tilde{H}$ such that $\tilde{H}(\mathbb{C}^5)$ is star-shaped with respect to e_5.

i) Since $\tilde{H}_5$ is algebraically independent over $\mathbb{C}$ of $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$, it follows from $\operatorname{trdeg}_{\mathbb{C}} \tilde{H} = \dim W = 3$ that

$$\operatorname{rk} \mathcal{J}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4) = \operatorname{trdeg}_{\mathbb{C}}(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4) = 2$$

It follows from [2, Th. 2.1] that $(\tilde{H}_1, \tilde{H}_2, \tilde{H}_3, \tilde{H}_4)$ is of the form $gh(p,q)$, where g, p and q are homogeneous polynomials and p and q are relatively prime.

ii) From proposition 2.1, it follows that we may assume that $\tilde{D} := \sum_{i=1}^{5} \tilde{H}_i \frac{\partial}{\partial x_i}$ is irreducible. In order to prove that $g \in \mathbb{C}^*$, we assume that g has an irreducible divisor g_1. Notice that $g_1 \nmid \tilde{H}_5$. Since $\ker \tilde{D}$ is factorially closed,

$$0 = \tilde{D} g_1 \equiv \tilde{H}_5 \frac{\partial}{\partial x_5} g_1 \pmod{g_1}$$

It follows that $g_1 \mid (g_1)_{x_5}$, but $\deg g_1 > \deg (g_1)_{x_5}$, so $(g_1)_{x_5} = 0$, i.e. $g_1 \in \mathbb{C}[x_1, x_2, x_3, x_4]$.

iii) Assume first that $p, q \in \mathbb{C}[x_1, x_2, x_3, x_4]$. Then the components of $(x_1 - H_1, x_2 - H_2, x_3 - H_3, x_4 - H_4)$ do not have x_5, whence

$$\begin{pmatrix} x_1 - H_1 \\ x_2 - H_2 \\ x_3 - H_3 \\ x_4 - H_4 \end{pmatrix} \circ \begin{pmatrix} x_1 + H_1 \\ x_2 + H_2 \\ x_3 + H_3 \\ x_4 + H_4 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

So $(x_1 + H_1, x_2 + H_2, x_3 + H_3, x_4 + H_4)$ is a quasi-translation. Making this quasi-translation irreducible, we get that

$$(x_1, x_2, x_3, x_4) + h(p,q)$$

is a quasi-translation. So by 1.1 and $\operatorname{rk} \mathcal{J}(h(p,q)) \le 2$, there are two independent linear relations between the components of $\tilde{H}$, whence between the components of H as well. So assume that one of p, q is not contained in $\mathbb{C}[x_1, x_2, x_3, x_4]$. Replacing the other by a linear combination of p and q, we may assume that neither p nor q is contained in $\mathbb{C}[x_1, x_2, x_3, x_4]$.

iv) Just as in section 1, we may assume that

$$\begin{aligned} h_1(p,q) &\equiv p^r \pmod{pq} \\ h_2(p,q) &\equiv 0 \pmod{pq} \\ h_3(p,q) &\equiv 0 \pmod{pq} \\ h_4(p,q) &\equiv q^r \pmod{pq} \end{aligned}$$

Since the linear transformation that is used to get this form only takes place at the first four coordinates, the property that $p, q \notin \mathbb{C}[x_1, x_2, x_3, x_4]$ is preserved.

Since $\tilde{D}g_1 = 0$, it follows from

$$\tilde{D}g_1 \equiv q^r (g_1)_{x_4} \pmod{p}$$

and

$$\tilde{D}g_1 \equiv p^r (g_1)_{x_1} \pmod{q}$$

that $p \mid (g_1)_{x_4}$ and $q \mid (g_1)_{x_1}$. Since $p, q \notin \mathbb{C}[x_1, x_2, x_3, x_4]$ and $(g_1)_{x_4}$, $(g_1)_{x_1} \in \mathbb{C}[x_1, x_2, x_3, x_4]$, it follows that $(g_1)_{x_4} = (g_1)_{x_1} = 0$. So $g_1 \in \mathbb{C}[x_2, x_3]$. Since g_1 is homogeneous, g_1 is linear, for it is irreducible. From (6), it follows that g_1 is a linear relation between the components of $\tilde{H}$, whence the components of H are linearly dependent as well.

This completes the proof of theorem 2.8. □

It is not known yet whether in theorem 2.8, the components of H need to be linearly dependent. I promise a bottle of Joustra Beerenburg (Frisian spirit) to the one who first solves the problem whether for quasi-translations $x + H$ in dimension 5 with H homogeneous, the components of H need to be linearly dependent or not.

3 Homogeneous singular Hessians

In [6], the homogeneous polynomials $f \in \mathbb{C}[x_1, x_2, \dots, x_n]$ for which $\det \mathcal{H}f = 0$ are classified for all $n \leq 5$, using theorem 1.1 (or theorem 1.2). If $n \leq 4$, then the f's that satisfy the above are exactly those f that can be expressed as a (homogeneous) function in three linear coordinates, see e.g. [3, Cor. 1.3]. In [1], the following results are used to classify the homogeneous polynomials $f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ for which $\det \mathcal{H}f = 0$:

Theorem 3.1. *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ be homogeneous and assume that R is homogeneous and of minimum degree ≥ 1 such that $R(\nabla f) = 0$. Then there are two independent linear relations between the components of $H = \nabla R \circ \nabla f$.*

Theorem 3.2. *Let $f \in \mathbb{C}[x_1, x_2, x_3]$ such that $R(\nabla f) = 0$ (f does not need to be homogeneous). Then there exists a linear relations between the components of $H = \nabla R \circ \nabla f$. More generally, for every quasi-translation $x + H$ in dimension 3, the components of H are linearly dependent.*

The latter theorem, which is also used to classify the $h \in \mathbb{C}[x_1, x_2, x_3]$ for which $\det \mathcal{H}h = 0$, is proved by Z. Wang in [9]. A. van den Essen found another proof which is given below. But first, we give the proof of theorem 3.1, which is essentially that of P. Gordan and M. Nöther in [6, p. 567]:

Lemma 3.3. *Let $f \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ be homogeneous and $R \in \mathbb{C}[x_1, x_2, x_3, x_4]$ be homogeneous and irreducible, such that $R(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4}) = 0$. Assume further that $A \in \mathbb{C}[x_1, x_2, x_3, x_4]$ is homogeneous such that $R(\nabla A) = 0$ and $A(H) = 0$, where $H = \nabla R \circ \nabla f$. Then H_1, H_2, H_3, H_4 are linearly dependent.*

Proof. Since A is homogeneous and $R(\nabla A) = 0$, it follows from [3, Cor. 1.3] that there exists a linear relation L between the components of ∇A. Assume first that $\operatorname{rk} \mathcal{H}A = 3$. Then the relations between the components of ∇A form a prime ideal of height one, which is a principal ideal. Since L is irreducible, (L) must be that principal ideal, and $L \mid R$. So R is linear, which implies that H is constant. In particular, H_1, H_2, H_3, H_4 are linearly dependent.

So assume that $\operatorname{rk} \mathcal{H}A = 2$. Since there exists a linear relation L between the components of ∇A, A can be expressed a a polynomial in three homogeneous linear coordinates. The rank of the (3×3)-Hessian with respect to these three linear coordinates of A cannot be larger than 2, the rank of the original Hessian. So this (3×3)-Hessian is singular as well. It follows again from [3, Cor. 1.3] that A can be expressed as a polynomial in two linear coordinates. Since A is homogeneous and bivariate, A decomposes into linear factors, and one of these factors is already a relation between H_1, H_2, H_3, H_4. $\square$

Proof of theorem 3.1: From lemma 2.7, it follows that we need to distinguish the following two cases:

- The components of H are linearly dependent.
  Without loss of generality, we assume that $H_5 = 0$. Now write
  $$f = A x_5^m + B x_5^{m+1}$$
  such that $A \in \mathbb{C}[x_1, x_2, x_3, x_4] \backslash \{0\}$. Then $A + x_5 B \mid f$. If e is the degree of R, then
  $$Df = \sum_{i=1}^{5} R_{x_i}(\nabla f) f_{x_i} = eR(f) = 0$$
  with $D = \sum_{i=1}^{5} H_i \frac{\partial}{\partial x_i}$ follows from Eulers formula, whence $A + x_5 B \in \ker D$. From (7) and $H_5 = 0$, $A(H) = A(H) + H_5 B(H) = 0$ follows. Since R is assumed to be of minimum degree, it follows from $H_5 = 0$ and $\deg R > \deg R_{x_5}$ that $R_{x_5} = 0$, i.e. $R \in \mathbb{C}[x_1, x_2, x_3, x_4]$. It follows that the coefficient of x_5^{me} of $R(\nabla f)$ is $R(\nabla A)$, for ∇A is the lowest degree part of $(f_{x_1}, f_{x_2}, f_{x_3}, f_{x_4})$ with respect to x_5. Since $R(\nabla f) = 0$, $R(\nabla A) = 0$ as well. From lemma 3.3, it follows that H_1, H_2, H_3, H_4 are linearly dependent. So there are two independent linear relations between the components of H.

- H_5 is algebraically independent of H_1, H_2, H_3, H_4.
  Now let A be the leading coefficient to x_5 of f, i.e.
  $$f = A x_5^m + O(x_5^{m-1})$$
  Since $f \in \ker D$, $f(H) = 0$ follows. But, since H_5 is algebraically independent of H_1, H_2, H_3, H_4, $A(H) = 0$ follows. Furthermore, $R(\nabla A)$ is the leading coefficient with respect to x_5 of $R(\nabla f)$, and hence equal to zero. From lemma 3.3, it follows that H_1, H_2, H_3, H_4 are linearly dependent.

The existence of another linear dependence has been shown in the case above. $\square$

P. Gordan and M. Nöther settle the second case in the proof of theorem 3.1 in a different matter, see [6, p. 568]. In the first case in the proof of theorem 3.1, [4, proposition 5.3 ii)] is proved.

Proof of theorem 3.2: Let d be the degree of H and put

$$\tilde{H} = x_4^d \left(H\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}, \frac{x_3}{x_4}\right), 0\right)$$

Put $x = (x_1, x_2, x_3)$. Since $(x - x_4^{d-1}H, x_4)$ is the inverse polynomial map of $(x + x_4^{d-1}H, x_4)$ due to (5) and $(x_4^{-1}x, x_4)$ is the inverse polynomial map of $(x_4 x, x_4)$,

$$((x, x_4) - \tilde{H}) = (x_4 x, x_4) \circ (x - x_4^{d-1}H, x_4) \circ (x_4^{-1}x, x_4)$$

is the inverse polynomial map of

$$((x, x_4) + \tilde{H}) = (x_4 x, x_4) \circ (x + x_4^{d-1}H, x_4) \circ (x_4^{-1}x, x_4)$$

So $(x, x_4) + \tilde{H}$ is a homogeneous quasi-translation in dimension 4, whence there are two independent linear relations between the components of $\tilde{H}$. One of these relations is $\tilde{H}_4 = 0$ and the other is a linear relation between $\tilde{H}_1, \tilde{H}_2, \tilde{H}_3$. Since $H = (\tilde{H}_1, \tilde{H}_2, \tilde{H}_3) \circ (x, 1)$, the components of H are linearly dependent. $\square$

In [9], Z. Wang characterizes all quasi-translations in dimension 3, which can also be done by way of the above homogeneization technique.

References

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