DEPARTMENT OF MATHEMATICS
RADBOUD UNIVERSITY NIJMEGEN The Netherlands

MULTIMAGIC SQUARES, CUBES AND HYPERCUBES

Harm Derksen, Christian Eggermont, Arno van den Essen

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DEPARTMENT OF MATHEMATICS
RADBOUD UNIVERSITY NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

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Abstract

In this paper we give the first method for constructing n-multimagic squares for any n. We give an explicit formula in the last section.

Introduction

Magic squares have been studied over 4000 years. Recently some exciting new results have been found considering these squares (see for example [3], [1], [2], [4] and the nice book of Pickover ([5]). In this paper we will concentrate on so called multimagic squares.

An $m \times m$ matrix M consisting of natural numbers is called an n-multimagic square (where n is a fixed positive natural number) if for each $1 \le d \le n$ the matrix M^{*d}, obtained by raising each element of M to the d-th power, is a magic square i.e. the sum of all elements in each column, row and main diagonal gives the same number; the so-called magic number. Such a matrix M is called normal if its matrix elements consist of the consecutive integers $1, 2, \ldots, m^2$. Throughout this paper we always consider normal magic squares (of course if $m > 1$ and $d > 1$ then the matrix M^{*d} is not normal).

The first 2-multimagic square was constructed by Pfeffermann in 1890: it has order 8 ([9], [11]). In 1905 the first 3-multimagic square was constructed by Tarry: it has order 128. Recently in 2001 both a 4- and a 5-multimagic square were constructed by Boyer and Viricel resp. of order 512 and 1024 ([10], [11] where they also give a nice history on the subject). The record up to now was a 6-multimagic square of order 4096 constructed by Pan Fengchu in October 2003 ([12]).
In this paper we give a constructive procedure to make a large class of n-multimagic squares for each positive integer $n \ge 2$. The problem of finding such squares is reduced to any easy linear algebra problem which is solved in section 3. A more explicit solution is described in section 5. This solution is used to give an explicit formula for n-multimagic squares for all $n \ge 3$. In particular it gives the first 7-multimagic squares, of order 13^7 and 8-multimagic squares of order 17^8 etc.
The method described for constructing n-multimagic squares can easily be extended to n-multimagic cubes and hypercubes. We refer to section 4 for all definitions and more details.

1 Preliminaries

Throughout this paper R denotes a finite ring with q elements. For $c \in R$ we call a bijection $N : R \to \{0, 1, \ldots, q-1\}$ of type c if

$$N(a) = q - 1 - N(-a + c), \text{ for all } a \in R.$$

Lemma 1.1 *i) If 2 is a unit in R, then for every $c \in R$ there exists a bijection N of type c.*
ii) If 2 is not a unit in R, then for every $c \in R^*$ there exists a bijection N of type c.

Proof. i) For $c \in R$ define $\varphi = \varphi_c : R \to R$ by $\varphi(a) = -a + c$ for all $a \in R$. Then $\varphi^2 = 1_R$. So all orbits of φ have length ≤ 2 and a is a fixed point of φ iff $2a = c$.
ii) Let 2 be a unit in R. Then φ has exactly one fixed point (namely $a_0 = 2^{-1}c$) and hence exactly one orbit of length one, denoted $O(a_0)$. Since R is the disjoint union of the orbits of φ, it follows from i) that $R = O(a_0) \cup \bigcup_{i=1}^{s} O(a_i)$, where $\#O(a_i) = 2$ for all $1 \leq i \leq s$. Then define $N(a_0) = s$, $N(a_i) = i - 1$ and $N(\varphi(a_i)) = q - 1 - N(a_i)$ $(= q - i)$, for all $1 \leq i \leq s$. Then N is as desired.
iii) Let 2 not be a unit in R. Then for $c \in R^*$ $\varphi = \varphi_c$ has no fixed points (for if $\varphi(a) = a$, then $2a = c \in R^*$, so $2 \in R^*$ contradiction). So $R = \bigcup_{i=1}^{s} O(a_i)$ with $\#O(a_i) = 2$ for all i. Then define $N(a_i) = i - 1$ and $N(\varphi(a_i)) = q - 1 - N(a_i)$ $(= q - i)$ for all i. Then N is as desired. $\square$

Now let $m \geq 1$. For each $1 \leq j \leq m$ we choose a bijection $N_{(j)} : R \to \{0, 1, \ldots, q-1\}$ of type c_j, for some $c_j \in R$ (this is possible by 1.1). Put $c = (c_1, \ldots, c_m) \in R^m$ and define $N_m : R^m \to \{1, 2, \ldots, q^m\}$ by

$$N_m((a_1, \ldots, a_m)) = 1 + \sum_{j=1}^{m} q^{j-1} N_{(j)}(a_j).$$

Since the coefficients of the q-adic expansion of any natural number are unique and each $N_{(j)}$ is a bijection, it follows that N_m is a bijection. Moreover we have

Lemma 1.2 *$N_m(-a) = q^m + 1 - N_m(a + c)$, for all $a = (a_1, \ldots, a_m) \in R^m$.*

Proof. $N_m(-a) = 1 + \sum_{j=1}^{m} q^{j-1} N_{(j)}(-a_j) = 1 + \sum_{j=1}^{m} q^{j-1}(q - 1 - N_{(j)}(a_j + c_j)) = 1 + \sum_{j=1}^{m} q^j - \sum_{j=1}^{m} q^{j-1} - \sum_{j=1}^{m} q^{j-1} N_{(j)}(a_j + c_j) = (q^m + 1) - (1 + \sum_{j=1}^{m} q^{j-1} N_{(j)}(a_j + c_j)) = q^m + 1 - N_m(a + c)$. $\square$

To conclude this section we will give a result (proposition 1.4) which plays a crucial role in the next section. First some notations. Let $n \geq 1$, $1 \leq s \leq n$ and $L : R^n \to R^s$ an affine map i.e. there exist an R-linear map $L_0 : R^n \to R^s$ and an element $v \in R^s$ such that $L(a) = L_0(a) + v$ for all $a \in R^n$.

Lemma 1.3 *If $L : R^n \to R^s$ is surjective then $\#L^{-1}(y) = q^{n-s}$ for all $y \in R^s$.*

Proof. Let $y \in R^s$. Since L is surjective there exists $a_0 \in R^n$ such that $L(a_0) = y$. So $L^{-1}(y) = a_0 + \ker L_0$. Since L is surjective, so is L_0. It follows that $R^n/\ker L_0 \simeq R^s$, whence $\#\ker L_0 = q^{n-s}$ and consequently $\#L^{-1}(y) = q^{n-s}$. $\square$

Proposition 1.4 *Let $1 \leq s \leq n$ and for each $1 \leq j \leq s$ let $N_{(j)} : R \to \{0, 1, \ldots, q-1\}$ be a bijection of type c_j, for some $c_j \in R$.*
Then for every surjective affine map $L : R^n \to R^s$ and every s-tuple $e_1, \ldots, e_s \geq 1$ we have

$$\sum_{a \in R^n} N_{(1)}(L(a)_1)^{e_1} \ldots N_{(s)}(L(a)_s)^{e_s} = q^{n-s} \left(\sum_{i=0}^{q-1} i^{e_1} \right) \ldots \left(\sum_{i=0}^{q-1} i^{e_s} \right).$$

Proof. Let $y = (y_1, \ldots, y_s) \in R^s$. Then for each $a \in L^{-1}(y)$ we get

$$N_{(1)}(L(a)_1)^{e_1} \ldots N_{(s)}(L(a)_s)^{e_s} = N_{(1)}(y_1)^{e_1} \ldots N_{(s)}(y_s)^{e_s}.$$

So by Lemma 1.3 we obtain

$$\sum_{a \in L^{-1}(y)} N_{(1)}(L(a)_1)^{e_1} \ldots N_{(s)}(L(a)_s)^{e_s} = q^{n-s} N_{(1)}(y_1)^{e_1} \ldots N_{(s)}(y_s)^{e_s}, \ \forall y \in R^s. \tag{1}$$

Since by the surjectivity of L, R^n is the disjoint union of the fibres $L^{-1}(y)$, where y runs through R^s, we deduce from (1) that

$$\begin{aligned} \sum_{a \in R^n} N_{(1)}(L(a)_1)^{e_1} \ldots N_{(s)}(L(a)_s)^{e_s} &= \sum_{y \in R^s} \sum_{a \in L^{-1}(y)} N_{(1)}(L(a)_1)^{e_1} \ldots N_{(s)}(L(a)_s)^{e_s} \\ &= \sum_{y \in R^s} q^{n-s} N_{(1)}(y_1)^{e_1} \ldots N_{(s)}(y_s)^{e_s} \\ &= q^{n-s} \left(\sum_{y_1 \in R} N_{(1)}(y_1)^{e_1} \right) \ldots \left(\sum_{y_s \in R} N_{(s)}(y_s)^{e_s} \right). \end{aligned}$$

Since each $N_{(j)} : R \to \{0, 1, \ldots, q-1\}$ is a bijection we get that $\sum\limits_{y_j \in R} N_{(j)}(y_j)^{e_j} = \sum\limits_{i=0}^{q-1} i^{e_j}$, which concludes the proof. $\square$

2 A construction of n-multimagic squares

Let $n \in \mathbb{N}$. The following theorem gives the main tool for constructing n-multimagic squares. We will use the notations introduced in the previous section. More precisely let R be a finite ring with q elements. First we choose $c_1, \ldots, c_n$ in R and n bijections $N_{(1)}, \ldots, N_{(n)} : R \to \{0, 1, \ldots, q-1\}$ of types $c_1, \ldots, c_n$ respectively. With these bijections we define $N_n : R^n \to \{1, 2, \ldots, q^n\}$ as described in section two. This choice will be fixed throughout this section. In a similar way we fix a bijection

$N_{2n} : R^{2n} \to \{1, 2, \ldots, q^{2n}\}$. Then we will define a $q^n \times q^n$ matrix M by describing its entries on the places $(N_n(a), N_n(b))$, with $(a, b) \in R^n \times R^n$. Observe that since N_n is a bijection each pair (i, j) with $1 \leq i, j \leq q^n$ can be written uniquely in the form $(N_n(a), N_n(b))$ for some $(a, b) \in R^n \times R^n$. So indeed M, as in 2.1, defines a $q^n \times q^n$ matrix.

Theorem 2.1 *Let $A, B \in M_{2n,n}(R)$ and $t \in R^{2n}$ be such that all $n \times n$ minors of the matrices $A, B, A{+}B$ and $A{-}B$ are units in R and such that $(A, B) \in Gl_{2n}(R)$. Then the $q^n \times q^n$ matrix M defined by*

$$M_{N_n(a),N_n(b)} = N_{2n}\left((A, B)\binom{a}{b} + t\right)$$

is n-multimagic.

Proof. i) First observe that all matrix elements M_{ij} are different since $(A, B) \in Gl_{2n}(R)$ and N_{2n} is a bijection. Consequently the matrix M consists exactly of all elements of the set $\{1, 2, \ldots, q^{2n}\}$.
ii) Now let $1 \leq d \leq n$. First we want to show that the sum of all elements in any column of M^{*d} is the same i.e. equal to the same constant only depending on q and n. Therefor fix $b \in R^n$. Then the $N_n(b)$-th column of M^{*d} consists of the elements $M^d_{N_n(a),N_n(b)}$, where a runs through R^n (remember that $N_n : R^n \to \{1, 2, \ldots, q^n\}$ is a bijection). So $S_b(d)$, the sum of the elements of the $N_n(b)$-th column of M^{*d} is equal to

$$\sum_{a \in R^n} M^d_{N_n(a),N_n(b)}.$$

To compute $S_b(d)$ first observe that the j-th component of the vector $(A, B)\binom{a}{b}$ is equal to $A_{(j)}.a + B_{(j)}.b$, where $A_{(j)}$ (resp. $B_{(j)}$) denotes the j-th row of A (resp. B). Using the definitions of $M_{N_n(a),N_n(b)}$ and N_{2n} we get

$$S_b(d) = \sum_{a \in R^n} \left(1 + \sum_{j=1}^{2n} C_j(a, b)\right)^d \tag{2}$$

where

$$C_j(a, b) = q^{j-1} N_{(j)}(A_{(j)}.a + B_{(j)}.b + t_j), \text{ for all } 1 \leq j \leq 2n. \tag{3}$$

Now observe that $(1 + x_1 + \cdots + x_{2n})^d$ can be written as $1 + g$, where g is a sum of terms of the form $\alpha x_{j_1}^{e_1} \ldots x_{j_s}^{e_s}$, where $1 \leq j_1 < j_2 \ldots < j_s \leq 2n$, $e_1, \ldots, e_s \geq 1$ and $e_1 + \cdots + e_s \leq d$ (so in particular $s \leq d \leq n$) and α is a positive integer. So it follows from (2) that $S_b(d)$ only depends on q and n if we can show that for each set of exponents $e_1, \ldots, e_s$ and indices $j_1, \ldots, j_s$ as above, the sum

$$\sum_{a \in R^n} C_{j_1}(a, b)^{e_1} \ldots C_{j_s}(a, b)^{e_s} \tag{4}$$

only depends on q and n (and of course $e_1, \ldots, e_s$, $j_1, \ldots, j_s$). To see this we are going to use Proposition 1.4. Therefor put $J = (j_1, \ldots, j_s)$ and define the affine map $L : R^n \to R^s$ by the formula

$$L(a) = A_{(J)}.a + B_{(J)}.b + t_J$$

where $A_{(J)}$ (resp. $B_{(J)}$) is the $s \times n$ matrix with rows $A_{(j_1)}, \ldots, A_{(j_s)}$ (resp. $B_{(j_1)}, \ldots, B_{(j_s)}$) and $t_{(J)}$ in the column of length s with components $t_{j_1}, \ldots t_{j_s}$. Since, as observed above, $s \leq n$ and all $n \times n$ minors of A are units in R (by hypothesis) it follows that $L : R^n \to R^s$ is surjective. Since by (3) $C_{j_i}(a, b) = q^{j_i - 1} N_{(j_i)}(L(a)_i)$ for all $1 \leq i \leq s$, it follows from Proposition 1.4 that the expression in (4) is equal to

$$q^{n-s}.q^{e_1(j_1-1)+\cdots+e_s(j_s-1)} \left(\sum_{i=0}^{q-1} i^{e_i} \right) \cdots \left(\sum_{i=0}^{q-1} i^{e_s} \right) \tag{5}$$

which indeed only depends on q and n, as desired.
iii) Interchanging the roles of a and b in the argument given in ii) we get that also the sum of all elements in each row of M^{*d} is the same.
iv) Now lets compute the sum of the (main) diagonal elements of M^{*d}. This sum is equal to

$$S_1 = \sum_{a \in R^n} M^d_{N_n(a), N_n(a)}.$$

To compute it just repeat the arguments given in ii) with $b = a$. It then remains to show that the expression in (4) with $b = a$ equals the expression given in (5). Therefor just observe that $C_{j_i}(a, a) = q^{j_i - 1} N_{(j_i)}(L_1(a)_i)$ for all $1 \leq i \leq s$, where $L_1 : R^n \to R^s$ is the affine map defined by

$$L_1(a) = A_{(J)}.a + B_{(J)}.a + t_{(J)} = (A + B)_{(J)}.a + t_{(J)}$$

(recall that $J = (j_1, \ldots, j_s)$). Since by hypothesis all $n \times n$ minors of $A + B$ are units in R, it follows that L_1 is surjective. Then using Proposition 1.4 again we obtain that the expression in (4) with $b = a$ is indeed equal to the expression given in (5).
iv) Finally we consider the sum of all elements from the "second" diagonal of M^{*d}. This sum is equal to

$$S_2 = \sum_{a \in R^n} M^d_{N_n(a), q^n + 1 - N_n(a)}.$$

Since by lemma 1.2 $q^n + 1 - N_n(a) = N_n(-a + c)$ we get

$$S_2 = \sum_{a \in R^n} M^d_{N_n(a), N_n(-a+c)}.$$

Then repeating the arguments in ii) with $b = -a + c$ leads us to define the affine map $L_2 : R^n \to R^s$ defined by

$$L_2(a) = A_{(J)}.a + B_{(J)}(-a + c) + t_{(J)} = (A - B)_{(J)}.a + B_{(J)}.c + t_{(J)}.$$

Also this map is surjective since all $n \times n$ minors of $A - B$ are units in R. So again we find that the expression in (4) with $b = -a + c$ is equal to the expression in (5), which completes the proof of this theorem. $\square$

3 Finding the good matrices

In order to be able to construct effectively n-multimagic squares by the method described in theorem 2.1, we need to show how to find a ring R and matrices A and B in $M_{2n,n}(R)$ which satisfy the conditions of that theorem.

The following (well-known) lemma is the crucial tool to solve this problem.

Lemma 3.1 *Let $m \geq 1$ and $Q(x_1, \ldots, x_m)$ a non-zero polynomial in the variables $x_1, \ldots, x_m$ over $\mathbb{Z}$. Then one can find effectively integers $a_1, \ldots, a_m$ such that $Q(a_1, \ldots, a_m) \neq 0$.*

Proof. By induction on m. The case $m = 1$ is obvious since $Q(x_1)$ has at most $\deg Q$ zeros. Now let $m \geq 2$. Write $Q = q_d(x_1, \ldots, x_{n-1})x_m^d + \cdots + q_1(x_1, \ldots, x_{m-1})x_m + q_0(x_1, \ldots, x_{m-1})$ with $q_d \neq 0$. By the induction hypothesis there exist integers $a_1, \ldots, a_{m-1}$ such that $q_d(a_1, \ldots, a_{m-1}) \neq 0$. So the polynomial $q(x_m) = Q(a_1, \ldots, a_{m-1}, x_m)$ in $\mathbb{Z}[x_m]$ is non-zero and has x_m-degree d. Consequently there exists an integer a_m such that $q(a_m) \neq 0$ i.e. $Q(a_1, \ldots, a_{m-1}, a_m) \neq 0$, as desired. □

Proposition 3.2 *Let $n \geq 1$. Then one can find effectively a positive integer $q > 1$ and matrices $A, B \in M_{2n,n}(\mathbb{Z}/q\mathbb{Z})$ such that all $n \times n$ minors of the matrices $A, B, A + B$ and $A - B$ are units in $\mathbb{Z}/q\mathbb{Z}$ and such that $(A, B) \in Gl_{2n}(\mathbb{Z}/q\mathbb{Z})$.*

Proof. Let $A_u = (A_{ij})$ and $B_u = (B_{ij})$ be two universal $2n \times n$ matrices i.e. the elements A_{ij} and B_{ij} are distinct variables. Then each $n \times n$ minor of A_u, B_u, $A_u + B_u$ and $A_u - B_u$ is a non-zero polynomial in the $4n^2$-variable polynomial ring $\mathbb{Z}[A_{ij}, B_{ij}, 1 \leq i \leq 2n, 1 \leq j \leq n]$. Let P be the product of all these minors and let Q be the product of P and the polynomial $\det(A_u, B_u)$. By lemma 3.1 we can find integers a_{ij} and b_{ij} such that $Q(a_{ij}, b_{ij})$ is a non-zero integer. Finally let q be a positive integer > 1 such that $g.c.d(Q(a_{ij}, b_{ij}), q) = 1$ for all i, j. Then one easily verifies that $A = (a_{ij})$ and $B = (b_{ij})$ represent matrices in $M_{2n,n}(\mathbb{Z}/qZ)$ having the desired properties. □

4 Multimagic cubes and hypercubes

In this section we briefly indicate how the method developed in the previous sections can be used to construct multimagic cubes, perfect multimagic cubes and hypercubes. Note that there is no consensus on the definition of multimagic cubes (hypercubes etc.) in the literature. The choice given below can also be found in [6], [7].

4.1 Multimagic cubes

A cube of numbers (resp. the consecutive numbers $1, 2, \ldots, n^3$) is called magic (resp. normal magic) if the sum of all elements in each row, column and pillar is the same and is equal to the sum of all elements of each of the four space diagonals. Furthermore, if $n \geq 1$ such a cube is called n-multimagic if for each $1 \leq d \leq n$ the cube obtained by raising each of its elements to the d-th power is magic.

Completely analogues to the construction of n-multimagic squares in 2.1 we define a $q^n \times q^n \times q^n$ cube by the formula

$$M_{N_n(a),N_n(b),N_n(c)} = N_{3n}\left((A,B,C)\begin{pmatrix} a \\ b \\ c \end{pmatrix} + t\right)$$

where each of the vectors a, b and c runs through R^n, $t \in R^{3n}$ and A, B and C are matrices in $M_{3n,n}(R)$ which satisfy the following properties (which guarantee the matrix M to be an n-multimagic cube):

1. $(A, B, C) \in Gl_{3n}(R)$ (which guarantees that all the natural number $1, 2, \ldots, q^{3n}(= (q^n)^3)$ appear in M).

2. all $n \times n$ minors of the matrices A, B and C are units in R (which guarantees that for each $1 \leq d \leq n$ the sum of all elements in each column, row and pillar of M^{*d} is the same, and hence equal to the magic sum).

3. all $n \times n$ minors of the matrices $A + B + C$, $-A + B + C$, $A - B + C$ and $A + B - C$ are units in R (which guarantees that for each $1 \leq d \leq n$ the sum of all elements on each of the four space diagonals of M^{*d} is equal to the magic sum).

To find a ring R and matrices A, B and C satisfying the properties 1, 2 and 3 one can use the method described in section 3.

4.2 Perfect multimagic cubes

Recall that a magic cube is called perfect if additionally the diagonals of each orthogonal slice have the magic sum property. Furthermore, if $n \geq 1$ such a cube is called n-multimagic perfect if for each $1 \leq d \leq n$ the cube obtained raising each of its elements to the d-th power is perfect magic.

To guarantee that the n-multimagic cube M constructed above is also n-multimagic perfect we impose on the matrices A, B, C the following conditions

4. all $n \times n$ minors of the matrices $A + B$, $A - B$, $A + C$, $A - C$, $B + C$ and $B - C$ are units in R.

Again the method of section 3 can be used to find matrices A, B and C having the properties 1 to 4 above.

4.3 Hypercubes

From the above it is now clear how to generalize these constructions to hypercubes.

5 Explicit examples

Before we give some examples of multimagic squares we describe an explicit construction of matrices A and B satisfying the properties of theorem 2.1. Therefor we need the following lemmas.

Lemma 5.1 *Let $n \geq 1$ and R a ring such that 2 and 3 are units in R. Furthermore let $P, Q \in M_n(R)$ are such that all $n \times n$ minors of the $2n$ by n matrix $A = \binom{P}{Q}$ are units in R. Then the matrices A and $B = \binom{2P}{-2Q}$ satisfy the properties of theorem 2.1.*

Proof. Since 2 is a unit in R the hypothesis on P and Q implies that also the $n \times n$ minors of B are units in R. Furthermore the $n \times n$ minors of $A + B = \binom{3P}{-Q}$ are also units in R, since 3 and -1 are. Similarly the $n \times n$ minors of $A - B$ are units in R. Finally, using elementary column operators one can reduce the matrix $(A, B) = \begin{pmatrix} P & 2P \\ Q & -2Q \end{pmatrix}$ to the matrix $\begin{pmatrix} P & 0 \\ Q & -4Q \end{pmatrix}$ which is clearly invertible over R since both $\det P$ and $\det(-4Q)$ are units in R. $\square$

Lemma 5.2 *Let $n \geq 2$ and denote by V the $2n \times n$ matrix defined as follows: for each $1 \leq i \leq 2n-2$ the i-th row of V is equal to $(1, i, i^2, \ldots, i^{n-1})$, the $2n-1$-th row of V is equal to $(1, 0, \ldots, 0)$ and the $2n$-th row of V equals $(0, 0, \ldots, 0, 1)$. Furthermore let P denote the $n \times n$ matrix consisting of the first n rows of V and let Q denote the $n \times n$ matrix consisting of the last n rows of V.*
If R is a ring such that $i \in R^*$ for all $1 \leq i \leq 2n-2$, then all $n \times n$ minors of $\binom{P}{Q}$ are units in R.

Proof. Using Vandermonde determinants one easily verifies that each factor appearing in each $n \times n$ minor of V is either $2n-2$ or of the form $i-j$ where $1 \leq j < i \leq 2n-2$, from which the desired result follows. $\square$

As an immediate consequence of the lemmas above we get

Corollary 5.3 *Let $n \geq 2$ and R a ring such that 2 and 3 are units in R. If $i \in R^*$ for all $1 \leq i \leq 2n-2$ then the matrices $A = \binom{P}{Q}$ and $B = \binom{2P}{-2Q}$ with P and Q as in lemma 5.2 satisfy the hypothesis of theorem 2.1.*

In 5.4 - 5.7 below we choose one bijection $N : R \to \{0, 1, \ldots, q-1\}$ of some type $c \in R$ and define for each $m \geq 1$

$$N_m((a_1, \ldots, a_m)) = 1 + \sum_{j=1}^{m} q^{j-1} N(a_j).$$

So in the definition of N_m as given in section 1 we take all $N_{(j)}$ to be equal to N.

Corollary 5.4 *(An explicit formula for n-multimagic squares.) Let $n \geq 3$, q the smallest prime number $\geq 2n-1$, $R = \mathbb{F}_q$ and $N : R \to \{0, 1, \ldots, q-1\}$ the bijection (of type -1) given by $N(\bar{i}) = i$ for all $0 \leq i \leq q-1$. Let A and B be as in 5.3. Then for each $t \in R^{2n}$ the matrix M defined in 2.1 is n-multimagic.*

So in particular for $n = 7$ we get 7-multimagic squares of order 13^7 and for $n = 8$, 8-multimagic squares of order 17^8 etc.

To conclude this paper we give some interesting multimagic squares, constructed using some rings different from the prime fields used in 5.4.

Example 5.5 (An associative bimagic square of order 16.)
Take $R = \mathbb{F}_2[x]/(x^2 + x + 1)$,

$$A = \begin{pmatrix} x & 0 \\ 0 & 1 \\ 1 & 1 \\ x & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & x \\ 1 & 1 \\ 1 & 0 \\ 0 & x \end{pmatrix}$$

$t \in R^4$ arbitrary and $N : R \to \{0, 1, 2, 3\}$ (a bijection of type $x+1$) given by $N(0) = 0$, $N(1) = 2$, $N(x) = 1$ and $N(x + 1) = 3$.
Then the corresponding matrix M defined in theorem 2.1 is bimagic (= 2-multimagic) and associative (= the sum of any pair of matrix elements which are symmetric with respect to the center of the square is equal to $16^2 + 1$).

Example 5.6 (A family of bimagic squares of odd order.)
Take $R = \mathbb{Z}/q\mathbb{Z}$ with $q \geq 3$, q odd,

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \\ 2 & 2 \end{pmatrix},$$

$t \in R^4$ arbitrary and $N : R \to \{0, 1, \ldots, q - 1\}$ the standard bijection (of type -1) as in 5.4. Then the corresponding $q^2 \times q^2$ matrices M as defined in theorem 2.1 are bimagic. In particular this gives a family of bimagic squares of odd order.

Finally, if we choose $q = 3$ and $t = \begin{pmatrix} 2 & 1 & 2 & 0 \end{pmatrix}^T$ we recover the associative 9×9 bimagic square constructed by R.V. Heath (see p. 212 [8]) from before 1974.

Example 5.7 (An associative, pandiagonal, bimagic, ... magic square of order 25.)
Take $R = \mathbb{F}_5$, $N : R \to \{0, 1, 2, 3, 4\}$ the standard bijection (of type -1) and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 \\ 2 & 4 \\ 3 & 4 \\ 3 & 2 \end{pmatrix}, \quad t = \begin{pmatrix} 0 \\ 4 \\ 0 \\ 2 \end{pmatrix}.$$

Then the corresponding 25×25 matrix is associative, pandiagonal, bimagic and has the following properties:

i) Each of the 25 standard 5×5 submatrices is pandiagonal (even with the same magic sum).

ii) For each pair (i, j) $(1 \leq i, j \leq 25)$ the 5×5 matrix obtained by deleting each row with row number not equivalent to i mod 5 and each column with column number not equivalent to j mod 5 is pandiagonal!

More research into different properties and various examples can be found in the thesis of the second author ([13]). The reader is also referred to the website ([14]).

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Harm Derksen
Dept. of Mathematics, Univ. of Michigan
East Hall, 525 East University
Ann Arbor, MI 48109-1109
hderksen@math.lsa.umich.edu

Christian Eggermont
Dept. of Mathematics, Radboud University Nijmegen
6525 ED Nijmegen, The Netherlands
C.Eggermont@science.ru.nl

Arno van den Essen
Dept. of Mathematics, Radboud University Nijmegen
6525 ED Nijmegen, The Netherlands
A.vandenEssen@science.ru.nl