Muon Colliders, Monte Carlo and Gauge Invariance

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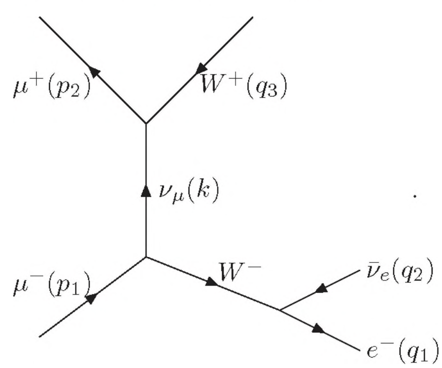
Abstract

If calculated in the standard way, the cross section for the collision of two unstable particles turns out to diverge. This is because part of such a cross section is proportional to the size of the colliding beams. The effect is called the "linear beam size effect". We present a way of including this linear beam size effect in the usual Monte Carlo integration procedure. Furthermore we discuss the gauge breaking that this may cause.

1 Introduction

The cross section for the collision of two unstable particle generally diverges. This happens for instance in the Feynman graph



. (1)

The lower half of this graph looks like the decay of a muon. Consequently the kinematics of the process allows the momentum k to be on its mass shell. After all that is what one gets from the decay of a muon: a muon neutrino on its mass shell. The factor $1/(k^2 + i\epsilon)$ that occurs in the matrix element causes a divergence of the total cross section.

In [1] and [4] this problem has been studied in detail, and it has been shown that this divergence is softened into a finite peak if the incoming particles are described carefully enough. In this paper we give a prescription for including this peak in Monte Carlo simulations. Typically such modifications may result in a violation of gauge invariance in the amplitude. We study this effect in detail.

2 Asymptotic States

In the context of scalar particles, Veltman [2] has shown that the S-matrix satisfies unitarity and causality only if one restricts the in/out states to the stable particles. Because of this, when considering collisions of unstable particles, we should use Feynman graphs that take the production process of the unstable particle into account. We are going to show that we actually do not need to worry about this as long as the wave packets of the unstable particles are much smaller in size than the decay length of the unstable particle. A complete amplitude for the production and collision of two muons looks like

$$\begin{aligned}\mathcal{A} = (2\pi)^4 i \int \frac{d^4p_1'}{(2\pi)^4}\frac{d^4p_2'}{(2\pi)^4} e^{-i\tau_1 p_1'\cdot p_1/m_\mu} e^{-i\tau_2 p_2'\cdot p_2/m_\mu}\delta^4(p_1'+p_2'-q_1-q_2) \\ \psi_{p_2}(p_2')\frac{i(-\not{p}_2'+m_\mu)}{(p_2')^2-m_\mu^2+im_\mu\Gamma_\mu}\mathcal{M}_{\rm coll}\frac{i(\not{p}_1'+m_\mu)}{(p_1')^2-m_\mu^2+im_\mu\Gamma_\mu}\psi_{p_1}(p_1')\end{aligned} \tag{2}$$

where

$$\psi_{p_1}(p_1') = \int \tilde{d}p_a'\,\tilde{d}p_b'\,\phi_{p_a}(p_a')\phi_{p_b}(p_b')(2\pi)^4\delta^4(p_a'+p_b'-p_1')\mathcal{M}_{\rm prod} \tag{3}$$

may be viewed as the wave function of the unstable particle. Notations like $\phi_{p_a}(p_a')$ stand for the wave function of a particle that is peaked in momentum space around the value p_a evaluated at p_a' The above expression for the wave function of an unstable particles assumes that the unstable particle is produced in a two-to-one process. We assume this only for the sake of simplicity of notations. If there are other outgoing or incoming particles their wave functions can easily be added. Also note the factors $e^{-i\tau_i p_i'\cdot p_i/m_\mu}$. These factors are translations of the wave function. The point of these translations is that they make sure that the unstable particles are produced away from the spot where they collide. The invariant distance that the unstable particle travels before colliding is τ_i.

Now we are going to use the assumption that the wave packets are much smaller than the decay length. This has as a consequence that in momentum space the wave packets are much broader than the decay length. Because of this we may assume that they are constant functions of $(p_1')^2$ resp $(p_2')^2$ over a range of several times $m_\mu\Gamma_\mu$. Therefore it is possible to integrate the expression for $\mathcal{A}$ given above over the values of $(p_1')^2$ and $(p_2')^2$. We only have to integrate the factors contained in the quantity F that is defined to be given by

$$F = e^{-i\tau_1 p_1'\cdot p_1/m_\mu} e^{-i\tau_2 p_2'\cdot p_2/m_\mu}\frac{1}{[(p_2')^2-m_\mu^2+im_\mu\Gamma_\mu]}\frac{1}{[(p_1')^2-m_\mu^2+im_\mu\Gamma_\mu]}. \tag{4}$$

We integrate along a path parameterized as $p'_{1,2}(t)$. This parameterization is done according to

$$\begin{aligned} p'_1(t) &= p'_1(0) + tc; \\ k'(t) &= k'(0) + tc; \\ p'_2(t) &= p'_2(0) - tc. \end{aligned} \tag{5}$$

If we want to integrate over the value of $(p'_1)^2$ we choose c to be a four vector that is a linear combination of p_1, p_2 and k such that it is orthogonal to the latter two vectors but not to p_1. This parameterization is chosen such in order to satisfy momentum conservation and furthermore to be on a constant k^2-plane in order not to get difficulties with the singularity at $k^2 = 0$. After doing this integral and an analogous one over the value of $(p'_2)^2$, we find the result

$$\begin{aligned} \mathcal{A} = (2\pi)^4 i \int \tilde{d}p'_1\, \tilde{d}p_2\, e^{-i\tau_1 p'_1\cdot p_1/m_\mu} e^{-\Gamma_\mu \tau_1/2} e^{-i\tau_2 p'_2\cdot p_2/m_\mu} e^{-\Gamma_\mu\tau_2/2} \\ \delta^4(p'_1 + p'_2 - q_1 - q_2)\psi_{p_2}(p'_2) u_i(p'_2)\bar{u}_i(p'_2)\mathcal{M}_{\rm coll} u_j(p'_1)\bar{u}_j(p'_1)\psi_{p_1}(p'_1). \end{aligned} \tag{6}$$

This is (except for the decay factors $e^{-\Gamma_\mu\tau_i/2}$) exactly the same as if we had started with incoming muons on their mass shell. The conclusion is that if we have wave packets that are much smaller than the decay length of the unstable particles we may treat them as if they were asymptotic states.

This has no bearing on the question of gauge invariance. Matrix elements depend on the masses of the particles. If masses are chosen such that the muon is no longer unstable (by assuming the electron to be heavier than the muon, so that the decay is forbidden), the matrix element is gauge invariant, so it must also be if masses are chosen accordingly to their measured values.

3 The Linear Beam Size Effect

We observed that the divergence in the cross section is caused by a peak in the matrix element in momentum space. A sharp peak in momentum space means a long range effect in position space. Indeed, the decay product of a decaying muon can travel over arbitrary distances. The consequence is that the cross section becomes proportional to the size of the beam. In colliders the longitudinal beam size is much larger than the transverse one. Consequently, the cross section is actually proportional to the transverse beam size, to be denoted by a. The precise definition of this quantity can be found in [1]. In the same paper a more rigorous version of this qualitative argument was given. In [4] it was shown that the quantities used in the rigorous argument can replaced by covariant ones.

The part of the cross section proportional to the beam size is given by

$$\sigma_{\text{semi-sing}} = a\pi \int d\sigma_{\rm red} \frac{1}{|k_\perp|} \delta(k^2 - m^2), \tag{7}$$

where $\sigma_{\rm red}$ is the cross section with the offending propagators $(k^2 - m^2 \pm i\epsilon)^{-1}$ removed. $k_\perp$ is by definition given by $k + \alpha p_1 + \beta p_2$ with α and β chosen such that $k_\perp \cdot p_{1,2} = 0$.

The above formula gives the part of the cross section proportional to the beam size, but it would be more convenient if the linear beam size effect could be incorporated in the usual Monte Carlo integration procedure. This can indeed be done by doing the substitution

$$\frac{1}{k^2 - m^2 \pm i\epsilon} \rightarrow \frac{1}{k^2 - m^2 \pm i|k_\perp|/a}. \tag{8}$$

If we use the approximation

$$\frac{1}{(k^2 - m^2)^2 + |k_\perp|^2/a^2} \sim \frac{a\pi}{|k_\perp|}\delta(k^2 - m^2), \tag{9}$$

these two expressions become equal. This approximation only needs to be valid around the peak at $k^2 = m^2$, which will generally be the case. The only property that is needed for this to be true is that the reduced cross section $d\sigma_{\rm red}$ does not vary much in k^2 at the value m^2 on momentum-squared scales of the order of $|k_\perp|/a$. The contribution of regions of phase space away from this peak can be as large or larger as the result due to the peak. In [1] the matrix element was split up into a part due to the peak and a part due to the rest of the phase space to account for this. Our $i|k_\perp|/a$-prescription gives a good approximation of the matrix element away from the peak at $k^2 = m^2$, so it makes a more or less arbitrary split-up of the cross section unnecesary.

3.1 Gauge Invariance

The above prescription breaks gauge invariance. We study the process $\mu^- + \mu^+ \rightarrow e^- + \bar{\nu}_e + W^+$. To do this, six Feynman graphs with γ, $W^\pm$ and Z^0 as fundamental bosons are needed. The propagators of the massive bosons must be given a width. This does affect the gauge invariance of the amplitude. In [3] it was shown that just using the $iM\Gamma$-prescription may lead to grossly inaccurate results. However, in this paper we want to focus on the effect of the gauge breaking caused by our $i|k_\perp|/a$-prescription. For this reason we use the pole scheme for the massive bosons, so that they do not break the gauge. What flavour of this scheme we actually used can be found in appendix A. It turns out that in the R_ξ-gauge, no gauge dependence due to the $i|k_\perp|/a$-prescription is found, although we actually broke gauge invariance. I.e., the results do not depend on the gauge parameter ξ. This can be understood from the Feynman graph displayed in equation 1. The gauge dependence comes in via a term proportional to $(q_1+q_2)_\mu(q_1+q_2)_\nu$ that occurs in the W^--propagator. However, this term disappears because one of these factors q_1+q_2 is to be contracted with the current containing the outgoing fermions. These are to be taken massless, so consequently this does not contribute, regardless of the gauge breaking that may occur at the other side of the W^--propagator. To see that our prescription actually breaks gauge invariance we used the axial gauge. In this gauge the undressed propagator of the W-particle is given by

$$\Delta(k)_{\nu\mu} = \frac{-i\left(g_{\mu\nu} - \frac{n_\nu k_\mu + n_\mu k_\nu}{n\cdot k} + k_\nu k_\mu \frac{n^2}{(n\cdot k)^2}\right)}{k^2 - M_W^2 + i\epsilon}. \tag{10}$$

The expression for the squared matrix element can be rewritten in such a way that all gauge breaking terms are proportional to $|k_\perp|/a$ or the square of this quantity. The axial gauge is not very easy to work with in practice, because one either has propagators that mix longitudinal gauge bosons with would-be Goldstone bosons or, if propagators are diagonalized, rather complicated expressions for the vertices. Details are discussed in [7]. To find the gauge breaking terms in the unitary gauge we calculate the difference

$$|\mathcal{M}|^2_{\text{gauge-break}} = |\mathcal{M}|^2_{\text{unitary gauge}} - |\mathcal{M}|^2_{\text{gauge invariant}}. \tag{11}$$

The gauge invariant quantity is calculated by using the axial gauge and the gauge invariant prescription

$$\mathcal{M}_{\text{gauge invariant}} = \frac{\operatorname{Res}_{k^2=m^2} \mathcal{M}}{k^2 - m^2 + i|k_\perp|/a} + \mathcal{M}_{\text{regular}} \tag{12}$$

that gives a gauge invariant quantity in the spirit of the pole scheme. This calculation was done in the axial gauge to check that we actually get a quantity that does not depend on the gauge vector n. The algebra was done using the C++ computer algebra library GiNaC. C.f., [8]. We find that the quantity $|\mathcal{M}|^2_{\text{gauge-break}}$ is, compared to the rest of the cross section, a factor $|k_\perp|/(as) \sim 1/(a\sqrt{s})$ smaller. Numerically that is a factor $7 \cdot 10^{-14}$ for $\sqrt{s} = 150\,\text{GeV}$ and $a = \sqrt{\pi} \cdot 10\,\mu\text{m}$ (which is a reasonable value). In ref [3] it was shown that gauge breaking effects can get enhanced by a factor as large as s/m_e^2, but even if this would happen, the gauge breaking due to our handling of the linear beam size effect remains negligible (note that in the context of muon colliders one would actually expect a factor s/m_μ^2 for the case discussed in [3]).

4 Conclusions

The linear beam size effect can be incorporated in the usual Monte Carlo integration procedure by doing to substitution

$$\frac{1}{k^2 - m^2 \pm i\epsilon} \to \frac{1}{k^2 - m^2 \pm i|k_\perp|/a} \tag{13}$$

in the propagator that causes the divergence. This can be done in a gauge invariant way, but in the unitary gauge the gauge breaking effect is so small that it is safe not to worry about the gauge dependence. The gauge breaking effect of the $iM\Gamma$-prescription is much larger than that due to the $i|k_\perp|/a$.

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A The Pole Scheme

To describe resonances, as observed from the W and Z particles, one needs a resummed propagator that contains a factor $(p^2 - M^2 + iM\Gamma)^{-1}$. The problem with this propagator is that it breaks gauge invariance, which means that observable quantities depend on the gauge choice. The pole scheme (c.f., [5] and [6]) is one of the ways to solve this. To use it, we first observe that both the position of the pole and its residue are gauge invariant quantities. They must be because they can be determined by experiment. The consequence is that every matrix element that involves such a pole can be written as

$$\mathcal{M} = \frac{F(p^2 = M^2 - iM\Gamma)}{p^2 - M^2 + iM\Gamma} - \frac{F(p^2) - F(p^2 = M^2 - iM\Gamma)}{p^2 - M^2 + iM\Gamma} + \mathcal{M}_{\text{rest}}, \tag{14}$$

where the first term is gauge invariant, as are the second and third together.

In practice things are a bit more involved than sketched in the previous paragraph. A matrix element generally depends on more that just p^2 and thus a prescription is needed to tell us what happens to all the other quantities that occur in the matrix element if we put p^2 equal to $M^2 - iM\Gamma$. We follow the method outlined in [5]. Our matrix element contains strings of gamma matrices with spinors at the beginning and end. These are canonicalized to ensure that all strings of gamma matrices are linearly independent. This means that if we have, say, a $\not{p}$ and a $\not{q}$ in some string of gamma matrices, we can also have the same string of gamma matrices with the $\not{p}$ and $\not{q}$ interchanged. We then have to decide which of these two comes in front. The anti-commutation relations that one has for gamma matrices are then used to do this. Also the relations $\not{p}u(p) = mu(p)$ and $\not{p}v(p) = -mv(p)$ are used whenever applicable.

After this has been done, the strings of gamma matrices that remain are linearly independent. They are said to form a set of independent covariants. If the matrix element is going to be gauge invariant, each coefficient of such a string of gamma matrices must separately be gauge invariant. So equation 14 is not used for the full matrix element but actually for the invariant coefficient that occur in front of the different products of strings of gamma matrices. In order to do this, it is also necessary to eliminate one of the outgoing/incoming momenta by using momentum conservation for the entire matrix element. All inner products between in- or outgoing momenta in the matrix element are expressed in a smallest complete set of lorentz invariant variables. In the case of the outgoing momenta shown in the graph in equation 1 the set consisting of

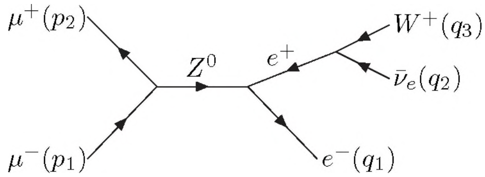
$$\begin{aligned} s &= (p_1 + p_2)^2; \\ t &= (p_1 - q_1 - q_2)^2; \\ x &= (q_1 + q_2)^2; \\ y &= (p_1 + p_2 - q_2)^2; \\ z &= (p_1 - q_1)^2, \end{aligned} \tag{15}$$

can be chosen. If one uses that the squares of incoming and outgoing momenta are given by the appropriate masses squared, all inner products between momenta are determined by specifying the variables (s, t, x, y, z). Now setting the square of some internal momentum in some Feynman graph equal to some value is a well-defined operation, except for some caveats that we discuss next.

The caveats are

1. If we have outgoing or incoming vector bosons, we should also treat inner products of the form $p \cdot \epsilon$ with ϵ the polarization vector as linearly independent covariant quantities. Some elements in the set of independent covariants contain a factor $p \cdot \epsilon$. In the case of the axial gauge this set furthermore includes factors that are inner products with the gauge vector n.
2. In the unitary gauge, the inner product of a polarization vector with the momentum of the particle to which the polarization vector belongs is zero. For this reason, these inner products should not appear in the set of independent covariants, nor in the coefficients that multiply them. The same applies to the inner product of polarization vectors with the gauge vector n in the axial gauge.
3. In the axial gauge the property holds that if we have outgoing or incoming vector bosons the matrix element becomes zero if the polarization vector of a vector boson is replaced by its momentum. It is a feature of the axial gauge that this not only happens for massless gauge bosons but also for massive ones. This shows that the set of covariants that we had is not really linearly independent. To see how this can be solved consider a matrix element of the form
$$\mathcal{M} = \epsilon \cdot p_1 F_1 + \epsilon \cdot p_2 F_2 + \epsilon \cdot p_3 F_3. \tag{16}$$
Here the inner products $\epsilon \cdot p_i$ are the covariants and the F_i are the invariant functions. If we know that the relation
$$q \cdot p_1 F_1 + q \cdot p_2 F_2 + q \cdot p_3 F_3 = 0 \tag{17}$$
holds, we can eliminate F_1 from the matrix element. We get
$$\mathcal{M} = \left(\epsilon \cdot p_2 - \epsilon \cdot p_1 \frac{q \cdot p_2}{q \cdot p_1}\right) F_2 + \left(\epsilon \cdot p_3 - \epsilon \cdot p_1 \frac{q \cdot p_3}{q \cdot p_1}\right) F_3. \tag{18}$$
Thus we have actually reduced the set of covariants from three to two in this example. This boils down to doing the substitution
$$\epsilon \to \epsilon - \epsilon \cdot p_1 \frac{q}{q \cdot p_1}. \tag{19}$$
In this substitution the vector p_1 can be chosen to be any linear combination of incoming and outgoing momenta. It is advisable to choose one that does not yield any singularities in the physical phase space due to dividing by $q \cdot p_1$. In the unitary gauge a similar reduction can be carried out. In our calculation we chose to get $(q_1 + q_2) \cdot q_3$ in the denominator. This is equal to $s - x - M_W^2$. This quantity has no poles in the physical phase space.

4. One has to be careful about the set of invariant variables. Actually the set (s, t, x, y, z) has a problem. To see this, consider the Feynman graph



(20)

The internal electron propagator is given by

$$S = \frac{-i(\not{q}_2 + \not{q}_3)}{s - x - y + M_W^2}. \quad (21)$$

In the pole scheme we should determine the residue for the Z-pole. This means that we put $s = M_Z^2$ to lowest order. The maximum value of $x + y$ is s and occurs in the limit that the outgoing electron is produced at rest. We see that the quantity $1/(s - x - y + M_W^2)$ does not have a pole in the physical phase space but if we put $s = M_Z^2$ it does develop a pole. For this reason we did not use the set of parameters (s, t, x, y, z) but instead the set (s, t, ξ, η, z) where $\xi = x/s$ and $\eta = y/s$. This set causes no trouble with spurious singularities.

The problem with spurious singularities, that we encountered in item 3 and 4 can be looked upon as follows. Formula 14 tells us to split the invariant functions in the matrix element. However, we have some freedom in making this split-up. This makes it possible that the pole term has a singularity that is then canceled by the regular term. A sensible choice of such a split-up takes care not to introduce new singularities in the physical phase space.

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