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Diversification of aggregate dependent risks

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Diversification of aggregate dependent risks

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Abstract

We give a new proof of the central result in [19]: For $d$ identically and continuously distributed risks $X_1, \ldots, X_d$, and large $u$ there is a constant $q_d$ such that $P \left[ \sum_{i=1}^{d} X_i \leq -u \right] \sim q_d \cdot P[ X_1 \leq -u ]$. $q_d$ describes the diversification effect. For $d = 2$ we give explicit formulas for $q_d$ and describe their behaviour with respect to the dependence strength.

Subject Classification: 62E20, 62H20, 62P05

Keywords: Archimedean copula, dependent random variables, diversification effect, extreme value theory, tail dependence.

1 Introduction

Worldwide, regulators look for new methods to calculate solvency requirements for insurance companies (Europe, Switzerland, Australia, Canada, revision of the US RBC, etc.). It is generally understood that the new methods should consider all risks and that risk-adjusted solvency capitals should be calculated. Usually the risks are classified into different categories. In each category one is then able to analyze the risks (e.g. using an analytical approach). The main difficulty comes in when one tries to aggregate the different (dependent) categories and when one tries to quantify the diversification between the different categories. In the current work we give a partial answer to such questions: Consider $d$ identically distributed dependent risks $X_1, \ldots, X_d$, then we obtain results of the following type

$$P \left[ \sum_{i=1}^{d} X_i \leq -u \right] \sim q_d \cdot P[ X_1 \leq -u ], \quad \text{as } u \to \infty, \quad (1.1)$$

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i.e. the constant $q_d$ quantifies the diversification effect between the dependent risks. Moreover it tells how the $d$-dimensional case is related to the one-dimensional case. For $d = 2$ we give explicit formulas for $q_d$, which give the connection between the diversification effect and the dependence strength.

The modelling of stochastic dependencies has shown to be particularly important in extreme value theory, where a profound knowledge of the complete dependence structure of the underlying random variables is needed to come to the right conclusions. In particular, it was understood in recent research (see e.g. Embrechts-McNeil-Straumann [9], Frees-Valdez [11], Juri-Wüthrich [14]) that simple measures of dependence such as the correlation coefficient are insufficient to cover the full range of possible consequences of dependent events.

A way to describe the full dependence structure of dependent random variables is the so-called copula approach. Copulas are simply a convenient way to describe joint distributions of two or more random variables. They were introduced in the seminal paper by Sklar [18], who showed that all finite dimensional probability laws have a copula function associated with it and describing the dependency of its marginal distributions. His ideas can be traced back to Fréchet, see e.g. [10]. We give the mathematical definition of a copula as well as examples in Section 2 below (standard copula literature is e.g. Joe [12] and Nelsen [16]). For an extensive discussion of copula methods the reader is referred to Dall’Aglio-Kotz-Salinetti’s book [5], in particular Schweizer [17] therein.

Many applications of copulas to actuarial sciences can be found in literature, as e.g. Carrière-Frees-Valdez [3]. Many authors have tried to find upper and lower bounds for expressions like formula (1.1) (see e.g. Dhaene-Denuit [7], Denuit-Genest-Marceau [6], Bäuerle-Müller [1], Cossette-Denuit-Marceau [4], of course this list is not complete). We choose a different approach: instead of finding bounds, we rather analyze the asymptotic properties. We find some universal behaviour (weak convergence theorems) that enables us to analyze different classes of models. The dependence structure is described using the copula framework. Successful steps in this direction have been undertaken e.g. by Wüthrich [19] or Juri-Wüthrich [13].

The first of these two papers is also the starting point for our investigations. There one sees that the extreme value behaviour of a sum of correlated, identically distributed random variables – where the correlation comes from a copula – scales like the extreme value behaviour of one variable with the same distribution. The aim of the present paper is twofold: On the one hand we give a different proof for Wüthrich’s result, on the other hand we also derive properties of the proportionality factor.
The paper is organized as follows. In the next section we define the notion of a copula. We also restate Sklar’s famous theorem and give examples of copulas. In Section 3 we give our main result, the asymptotic behaviour (1.1), moreover we provide the properties of the limiting constant \( q_d \) for \( d = 2 \). In Section 4 we give a practical example. Finally, in Section 5 we prove our results.

2 Copulas

The concept of copulas as a description of dependent random variables was introduced by Sklar [18]. The idea is that the dependence structure of a finite family of random variables is completely determined by their joint distribution function.

Let us thus define a copula as follows.

**Definition 2.1** Let \( d \geq 2 \). A \( d \)-dimensional copula is a \( d \)-dimensional distribution function on \([0,1]^d\), with marginals that are uniformly distributed on \([0,1]\).

The idea behind the concept of copulas is to separate a multivariate distribution function into two parts, one describing the dependence structure and the other one describing marginal behaviour. Moreover, all distribution functions with continuous marginals have a copula associated with them and vice versa:

**Theorem 2.2 (Sklar [18])** For a given joint distribution function \( F \) with continuous marginals \( F_1, \ldots, F_d \) there exists a unique copula \( C \) satisfying

\[
F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).
\]  

Conversely, for a given copula \( C \) and marginals \( F_1, \ldots, F_d \) we have that (2.1) defines a distribution with marginals \( F_i \).

Note that the copula of a random vector \((X_1, \ldots, X_d)\) is invariant under strictly increasing transformations (cf. Nelson [16]). Let us first give examples of copulas.

**Example 2.3** There are several special copulas, e.g.

- The **independent** copula \( C_I(x_1, \ldots, x_d) \defeq x_1 \cdot \ldots \cdot x_d \) corresponds to a finite set of independent random variables.

- The so-called **comonotonic** copula \( C_U(x_1, \ldots, x_d) \defeq \min\{x_1, \ldots, x_d\} \). The comonotonic copula corresponds to total positive dependence of the corresponding random variables.

In this article we focus on a special family of copulas, the Archimedean ones:
**Definition 2.4** Choose $d \geq 2$. Let $\phi : [0, 1] \rightarrow [0, \infty]$ be strictly decreasing, convex and such that $\phi(0) = \infty$ and $\phi(1) = 0$. Define for $x_i \in [0, 1], i = 1, \ldots, d$:

\[
C^\phi(x_1, \ldots, x_d) \overset{df}{=} \phi^{-1} \left( \sum_{i=1}^{d} \phi(x_i) \right).
\] (2.2)

The function $\phi$ is called generator of $C^\phi$.

We remark that for $d \geq 3$, $C^\phi$ in general is not a copula, i.e. $C^\phi$ in general is not a distribution function. In order to give necessary and sufficient conditions for which $C^\phi$ is a copula, the following definition is important: We say a function $\phi^{-1}$ is completely monotonic on $[0, \infty)$ if for all $k \in \mathbb{N}, x > 0$, the following expressions exist and

\[
(-1)^k \frac{d^k}{dx^k}\phi^{-1}(x) \geq 0.
\] (2.3)

The following theorem gives necessary and sufficient conditions for having a copula.

**Theorem 2.5 (Kimberling [15])** $C^\phi$ is a copula for all $d \geq 2$ if and only if the generator $\phi$ has an inverse $\phi^{-1}$ which is completely monotonic on $[0, \infty)$.

If $\phi$ allows for the definition of a copula, this copula gets a name:

**Definition 2.6** If $\phi^{-1}$ is completely monotonic on $[0, \infty)$ we call the generated copula $C^\phi$ a (strict) Archimedean copula.

The importance of Archimedean copulas in practice lies in the fact that they are easy to construct, but still we obtain a rich family of dependence structures. Usually, Archimedean copulas depend on one parameter, only. This makes it easier to estimate copulas from data. One of the best studied Archimedean copulas is the Clayton copula with parameter $\alpha > 0$. It is generated by $\phi(t) = t^{-\alpha} - 1$ and takes the form

\[
C^{Cl, \alpha}(x_1, \ldots, x_d) \overset{df}{=} (x_1^{-\alpha} + \ldots + x_d^{-\alpha} - d + 1)^{-1/\alpha}.
\] (2.4)

The limit $\alpha \rightarrow 0$ leads to independence, while $\alpha \rightarrow \infty$ leads to comonotonicity. For more examples we refer to Joe [12] and Nelsen [16].

### 3 An extreme value theorem and corollaries

In Wüthrich [19] an extreme-value theorem is proven, that basically states that the extreme value behaviour of a sum of dependent random variables with identical marginals scales like the extreme value behaviour of one such variable. The formula
for the limiting proportionality constant is rather complicated though. Below we
give an alternative proof that leads to a more transparent description of the limiting
constants and allows to analyze properties of these constants.

In order to be able to state the theorem formally recall the following definition (a
standard reference on regular variation is Bingham-Goldie-Teugels [2]):

**Definition 3.1** A function \( f \) is called regularly varying at some point \( x^- \) (or \( x^+ \),
respectively) with index \( \alpha \in \mathbb{R} \) if for all \( t > 0 \)

\[
\lim_{s \rightarrow x^-} \frac{f(st)}{f(s)} = t^\alpha,
\]

(or \( \lim_{s \rightarrow x^+} \frac{f(st)}{f(s)} = t^\alpha \), respectively).

### 3.1 Main theorem

The main theorem of extreme value theory states that extreme value behaviour of
a sequence of i.i.d. random variables is either degenerate or in exactly one of the
following three classes (see [8], Theorem 3.2.3.): Fréchet, Weibull or Gumbel, i.e. there
are essentially three different types of marginal behaviours. Their characterizations are
given in the following theorem.

Let \( c \) denote the left end-point of the one-dimensional distribution \( F \), where ap­
propriate (i.e., in the Weibull and Gumbel case).

**Theorem 3.2** Let \( d \geq 2 \), \( \alpha, \beta > 0 \), there are constants \( q_d^F(\alpha, \beta), q_d^W(\alpha, \beta), q_d^G(\alpha) \)
such that following holds true: Assume \( X = (X_1, \ldots, X_d) \) has real-valued identically
distributed random components, with continuous marginal \( F(x) = P(X_i < x) \) and \( X \) has Archimedean copula \( C^\phi \), where \( \phi \) is regularly varying at \( 0^+ \) with index \( -\alpha \). Then

a) (The Fréchet case) If \( F \) is regularly varying at \( -\infty \) with index \( -\beta \), then

\[
\lim_{u \rightarrow -\infty} \frac{1}{F(-u)} P \left( \sum_{i=1}^d X_i \leq -u \right) = q_d^F(\alpha, \beta), \quad \text{with}
\]

\[
q_d^F(\alpha, \beta) = \lim_{\epsilon \rightarrow 0} \int_{\sum_{i=1}^d x_i \geq \epsilon, x_i \leq 1/\epsilon} \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d x_i^{-\alpha} \right)^{-1/\alpha} dx_1 \ldots dx_d. \quad (3.3)
\]

b) (The Weibull case) If there is a \( c > -\infty \) such that \( s \mapsto F(c - 1/s) \) is regularly
varying at \( -\infty \) with index \( -\beta \), then

\[
\lim_{u \rightarrow -\infty} \frac{1}{F(c+1/u)} P \left( \sum_{i=1}^d X_i \leq dc + 1/u \right) = q_d^W(\alpha, \beta), \quad \text{with}
\]

\[
q_d^W(\alpha, \beta) = \lim_{\epsilon \rightarrow 0} \int_{\sum_{i=1}^d x_i \geq \epsilon, x_i \leq 1/\epsilon} \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d x_i^{-\alpha} \right)^{-1/\alpha} dx_1 \ldots dx_d. \quad (3.4)
\]
\[
q_d^W(\alpha, \beta) = \lim_{\varepsilon \to 0} \int_{\sum_i x_i \leq 1, x_i \leq 1/\varepsilon} \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d x_i^{-\alpha \beta} \right)^{-1/\alpha} \, dx_1 \ldots dx_d. \tag{3.5}
\]

c) (The Gumbel case) If there is a \( c \geq -\infty \) and a positive function \( s \mapsto \alpha(s) \) such that for \( t \in \mathbb{R} \) one has \( \lim_{u \to c} F(u + ta(u))/F(u) = e^t \), then

\[
\lim_{u \to c} \frac{d}{du} \left( \prod_{i=1}^d X_i \right) = q_d^G(\alpha) \cdot e^{\frac{\alpha - 1}{\alpha}}.
\]

\[
q_d^G(\alpha) = \int_{\sum_i x_i \leq 1} \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d e^{-x_i^{\alpha}} \right)^{-1/\alpha} \, dx_1 \ldots dx_d. \tag{3.7}
\]

Remarks 3.3

- The parameter \( \alpha \) plays the role of the dependence strength. It is essentially a measure for the dependence in the tails (compare to the tail dependence results in Juri-Wüthrich [13], Theorem 3.9).

- For analyzing the asymptotic behaviour of \( \sum_{i=1}^d X_i \) we only need to know the marginals \( X_i \) and the “dependence strength” \( \alpha \). I.e. with Theorem 3.2 we can avoid explicitly choosing the dependence structure (copula), which is notoriously difficult object (see also Embrechts-McNeil-Straumann [9]), but still obtain appropriate asymptotic results. This is like usually in extreme value theory, the asymptotic results divide into different classes/distributions where one only needs to estimate certain parameters.

- The limiting distributions found in (3.3), (3.5) and (3.7) have Clayton copula, this is not surprising in view of the results presented in Juri-Wüthrich [13].

3.2 Properties of the limiting constants for \( d=2 \)

The new characterizations of the limiting constants \( q_d^F(\alpha, \beta), q_d^W(\alpha, \beta) \) and \( q_d^G(\alpha) \) still look complex. Nevertheless, they allow explicit calculations in \( d=2 \) and they have nice monotonicity properties (presented below).

For \( \alpha \neq 0 \) and \( y \geq 0 \) define: \( f_\alpha(y) \overset{def.}{=} (1 + y^\alpha)^{-1/\alpha - 1} \). Then we can prove

Lemma 3.4 For \( \alpha > 0 \), \( f_\alpha(y) \) is a probability density on \([0, \infty)\).

Theorem 3.5 (Fréchet case) For \( \alpha > 0 \) and \( Y_\alpha \sim f_\alpha \) we have

\[
q_d^F(\alpha, \beta) = 1 + E \left( f_{1/\beta}(Y_\alpha) \right) = 1 + E \left( \left( 1 + Y_\alpha^{-1/\beta} \right)^{\beta - 1} \right). \tag{3.8}
\]

Moreover:
\* \( q_F^F(\alpha, \beta) \) is strictly increasing in \( \beta \).

\* For \( \beta > 1 \), \( q_F^F(\alpha, \beta) \) is strictly increasing in \( \alpha \).

\* \( q_F^F(\alpha, 1) = 2 \)

\* For \( \beta < 1 \), \( q_F^F(\alpha, \beta) \) is strictly decreasing in \( \alpha \).

\* \( \lim_{\alpha \to 0} q_F^F(\alpha, \beta) = 2^{\beta} \) as well as \( \lim_{\alpha \to 0} q_F^F(\alpha, \beta) = 2 \).

Remarks 3.6

\* The behaviour of \( q_F^F(\alpha, \beta) \) is illustrated in Figure 1.

\* For \( \beta > 1 \) there is a "positive" diversification effect, i.e. \( q_F^F(\alpha, \beta) \) is strictly increasing in the dependence strength \( \alpha \). At the first sight it is confusing, that this does not hold true for \( \beta \leq 1 \). One interpretation for this phenomenon is that for \( \beta \leq 1 \) we have no finite mean of the marginals, i.e. there is no finite risk premium for such risks. Therefore it is better to have only one such risk in our portfolio than two (of course in practice there is no such risk in our portfolio because we can not ask for an infinite premium).

\* For \( \beta \in \mathbb{N} \setminus \{0\} \) we have

\[
q_F^F(\alpha, \beta) = \sum_{k=0}^{\beta} \binom{\beta}{k} \frac{\Gamma \left( \frac{\beta - k}{\alpha \beta} + 1 \right) \Gamma \left( \frac{k}{\alpha \beta} + 1 \right)}{\Gamma (1 + 1/\alpha)}.
\] (3.9)

Theorem 3.7 (Weibull case) For \( \alpha > 0 \) and \( Y_\alpha \sim f_\alpha \) we have

\[
q_W^W(\alpha, \beta) = E \left( \left(1 + Y_\alpha^{1/\beta} \right)^{-\beta-1} \right).
\] (3.10)

The limiting constant \( q_W^W(\alpha, \beta) \) is strictly increasing in \( \alpha \) and strictly decreasing in \( \beta \). Moreover \( q_W^W(\alpha, \beta) \leq 1 \) for all \( \alpha, \beta > 0 \), and it holds

\[
\lim_{\alpha \to \infty} q_W^W(\alpha, \beta) = 2^{-\beta} \quad \text{as well as} \quad \lim_{\alpha \to 0} q_W^W(\alpha, \beta) = 0.
\]

Remark 3.8 The behaviour of \( q_W^W(\alpha, \beta) \) is shown in Figure 2, for different \( \alpha \) and \( \beta \). Again we have decreasing diversification for increasing \( \alpha \). 


Figure 1: $q = q_F(\alpha, \beta)$ as a function of $\alpha$, for different $\beta$'s.

Figure 2: $q = q_W(\alpha, \beta)$ as a function of $\alpha$, for different $\beta$'s.
Figure 3: \( q = q_G^{(\alpha)} \) as a function of \( \alpha \).

**Theorem 3.9 (Gambel case)** For \( \alpha > 0 \) we have

\[
q_G^{(\alpha)} = e^{1/2} \cdot \frac{\Gamma^2 \left( 1 + 1/(2\alpha) \right)}{\Gamma \left( 1 + 1/\alpha \right)} = e^{1/2} \left( \frac{q_F^{(\alpha, 2)}}{2} - 1 \right). \tag{3.11}
\]

Furthermore \( q_G^{(\alpha)} \) is strictly increasing in \( \alpha \) and

\[
\lim_{\alpha \to 0} q_G^{(\alpha)} = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} q_G^{(\alpha)} = e^{1/2}. \tag{3.12}
\]

And the behaviour of \( q_G^{(\alpha)} \) is shown in Figure 3, as a function of \( \alpha \).

### 3.3 Conclusions

We find that for \( d \) identically and continuously distributed risks \( X_1, \ldots, X_d \), the probability to suffer a large loss by their sum scales like the probability to suffer a large loss by just one of them. In formulas

\[
P \left[ \sum_{i=1}^d X_i \leq -u \right] \sim q_d(\alpha) \cdot P \left[ X_1 \leq -u \right], \quad \text{as } u \to \infty \tag{3.13}
\]
Moreover, the constant \( q_d(\alpha) \) describes the diversification effect: the larger the dependence strength \( \alpha \) the smaller the diversification effect (Weibull, Gumbel and Fréchet case for \( \beta > 1 \)).

The limiting constant \( q_d \) only depends on the choice of the marginals and on the choice of the dependence strength \( \alpha \), i.e. we do not need to specify the whole dependence structure (copula) to apply our results. As soon as we can estimate \( \alpha \) and the marginals we can apply our theorems to estimate asymptotic quantiles, of course this is a major simplification of the problem (an example is presented in the next section).

4 An Example

We model two motor liability portfolios \( X_1 \) and \( X_2 \). Our goal is to merge them to one big portfolio, and we want to measure the diversification effect we can expect by merging the two portfolios.

Assume \( X_1 \) and \( X_2 \) have Archimedean copula generated by a regularly varying function with index \( -\alpha \) at \( 0^+ \) (\( \alpha > 0 \)). Moreover assume that \( -X_1 \) and \( -X_2 \) have translated Pareto marginals with translation \( V_1 = 880 \) and \( V_2 = 820 \), i.e. \( Y_i = -(X_i + V_i) \) is Pareto distributed with \( \theta = 80 \) and \( \beta = 3 \) for \( i = 1, 2 \).

Choose \( p = 99.5\% \) and define Value-at-Risk

\[
\text{VaR}_{X_i} \overset{def}{=} -\sup\{x; \ P[X_i > x] > p\} + E[X_i].
\] (4.2)

Hence we have

<table>
<thead>
<tr>
<th></th>
<th>portfolio 1</th>
<th>portfolio 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation ( V_i )</td>
<td>880</td>
<td>820</td>
</tr>
<tr>
<td>mean ( E[-X_i] )</td>
<td>1'000</td>
<td>940</td>
</tr>
<tr>
<td>variational coefficient</td>
<td>6.9%</td>
<td>7.3%</td>
</tr>
<tr>
<td>( \text{VaR}_{X_i} )</td>
<td>347.8</td>
<td>347.8</td>
</tr>
</tbody>
</table>

As shown one can easily calculate quantiles for solvency purposes. The main difficulty is to calculate solvency requirements for two such aggregated portfolios. We use Theorem 3.2 and find for \( u \) large (\( V = V_1 + V_2 \))

\[
P[X_1 + X_2 \leq -u] = P[X_1 + X_2 + V \leq -u + V] \sim q^2_d(\alpha, \beta) \left( \frac{\theta}{u - V} \right)^\beta. \]

(4.3)
Define $\text{VaR}_{X_1+X_2}(\alpha)$ as in (4.2). Hence the Value-at-Risk of $X_1+X_2$ is now a function of the dependence strength $\alpha$ and can be approximated by (4.3). We obtain

$$\text{VaR}_{X_1+X_2}(\alpha) \approx V_{X_1+X_2}(\alpha) \overset{def.}{=} \theta \left( \frac{q_2^F(\alpha, \beta)}{1-p} \right)^{1/\beta} + E[X_1+X_2]. \quad (4.4)$$

Since we have a nice expression for $q_2^F(\alpha, \beta)$ (Theorem 3.5), we can numerically approximate the Value-at-Risk for different $\alpha$ (see Figure 4), and thus the decrease in Value-at-Risk when diversifying a portfolio, i.e. the diversification effect is defined as $1 - V_{X_1+X_2}(\alpha)/(\text{VaR}_{X_1} + \text{VaR}_{X_2})$ where $\text{VaR}_{X_1} + \text{VaR}_{X_2}$ corresponds to total positive dependence (see Figure 5). In this picture one can see that our approximation is not sharp for small $\alpha$, but this is not bad, since one can calculate the VaR directly for independent portfolios ($\alpha = 0$). In the tabular at the end of this section we use this direct method for $\alpha = 0$ only.

Figure 4: Asymptotic Value-at-Risk $V_{X_1+X_2}(\alpha)$ for different $\alpha$, compared to independent portfolios and comonotonic portfolios.
Figure 5: Diversification effect as a function of $\alpha$, compared to independent portfolios and comonotonic portfolios.
5 The Proofs

In this section we provide the proofs to the statements in the previous sections.

5.1 Proof of the extreme value theorem

As announced above we give a new proof of Theorem 3.2. We work out the details for the Fréchet case and indicate where the proofs in the Weibull and Gumbel case differ.

5.1.1 Fréchet case

Lemma 5.1 (Fréchet) Let \( d \geq 2, \alpha > 0 \) and \( \beta > 0 \). Let \( X = (X_1, \ldots, X_d) \) have Archimedean copula \( C^\phi \), where \( \phi \) is a regularly varying function at \( 0^+ \) with index \(-\alpha\). Moreover assume that all \( X_i \) have the same, continuous marginal \( F(x) \) that is regularly varying at \(-\infty\) with index \(-\beta\). Furthermore, let \( \varepsilon \in (0, 1), \alpha_1 \in (0, 1/\varepsilon) \) and \( x_2, \ldots, x_d \geq 0 \). Then:

\[
\lim_{u \to \infty} P(X_i < -u/x_i, i = 1, \ldots, d \mid X_1 < -\varepsilon u) = \left( \sum_{i=1}^d x_i^{-\alpha_1} \right)^{-1/\alpha} \varepsilon^\beta.
\]

Proof. Since \( \phi \) and \( F \) are regularly varying, the following holds: For every \( \varepsilon > 0 \) there is a \( u_0 \) such that for all \( i \) and \( u > u_0 \):

\[
F(-u/x_i) \leq (x_i + \delta)^\beta F(-u), \quad \text{and} \quad (\varepsilon + \delta)^{-\beta} F(-u) \leq F(-\varepsilon u),
\]

and \( F(-u) \) is so close to 0 that:

\[
\phi((x_i + \delta)^\beta F(-u)) \geq ((x_i + \delta)^\beta + \delta)^{-\alpha} \phi(F(-u)), \quad \text{and}
\]

\[
\sum_{i=1}^d ((x_i + \delta)^\beta + \delta)^{-\alpha} \phi F(-u) \leq \phi \left( \left( \sum_{i=1}^d (x_i + \delta)^\beta + \delta \right)^{-\alpha} - \delta \right)^{-1/\alpha} F(-u).
\]

Now we show the upper bound:
\[
\limsup_{u \to -\infty} \mathbb{P}(X_i \leq -u/x_i, i = 1, \ldots, d \mid X_1 \leq -\varepsilon u)
\]

\[
= \limsup_{u \to -\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{d} \phi \circ F(-u/x_i)\right)}{F(-\varepsilon u)}
\]

\[
\leq \limsup_{u \to -\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{d} \phi \left((x_i + \delta)^\beta F(-u)\right)\right)}{(\varepsilon + \delta)^{-\beta} F(-u)}
\]

\[
\leq \limsup_{u \to -\infty} \frac{\phi^{-1}\left(\sum_{i=1}^{d} ((x_i + \delta)^\beta + \delta)^{-\alpha} \phi F(-u)\right)}{(\varepsilon + \delta)^{-\beta} F(-u)}
\]

\[
\leq \frac{\left(\sum_{i=1}^{d} ((x_i + \delta)^\beta + \delta)^{-\alpha} - \delta\right)^{-1/\alpha}}{(\varepsilon + \delta)^{-\beta}},
\]

where for the first inequality we applied (5.2), for the second inequality we applied (5.3), and for the third inequality we applied (5.4). Since this holds for all $\delta > 0$, we get the upper bound. The lower bound is proven similarly. ■

Note that

\[
G_{\varepsilon, \beta}(x_1, \ldots, x_d) \overset{def}{=} \left(\sum_{i=1}^{d} x_i^{-\alpha}\right)^{-1/\alpha} e^\beta
\]

is a distribution function on $(0, 1/\varepsilon) \times (0, \infty)^{d-1}$. Let $g_{\varepsilon, \beta}^x$ be its density function and define:

\[
G(\varepsilon) \overset{def}{=} \varepsilon^{-\beta} \int_{\sum_{i=1}^{d} x_i \geq 1, x_i \leq 1/\varepsilon} g_{\varepsilon, \beta}^x(x_1, \ldots, x_d) dx_1 \ldots dx_d
\]

\[
= \int_{\sum_{i=1}^{d} x_i \geq 1, x_i \leq 1/\varepsilon} \frac{\partial^d}{dx_1 \ldots dx_d} \left(\sum_{i=1}^{d} x_i^{-\alpha}\right)^{-1/\alpha} dx_1 \ldots dx_d.
\]

Since $G(\varepsilon)$ is increasing for $\varepsilon \downarrow 0$, one can define $G(0) = \lim_{\varepsilon \downarrow 0} G(\varepsilon) \leq \infty$.

**Proof of Theorem 3.2. (The Fréchet case).** The key idea is to connect

\[
\mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u \mid X_1 \leq -\varepsilon u\right) \quad \text{with} \quad \mathbb{P}(X_i \leq -u/x_i, i = 1, \ldots, d \mid X_1 \leq -\varepsilon u)
\]

in the following way: $\lim_{u \to -\infty} \mathbb{P}(\sum_{i=1}^{d} X_i \leq -u \mid X_1 \leq -\varepsilon u) = \varepsilon^\beta G(\varepsilon)$. This is done by taking random variables $Y_{1(u)}^{(u)}, \ldots, Y_{d(u)}^{(u)}$ with distribution function

\[
H(x_1, \ldots, x_d) \overset{def}{=} \mathbb{P}(X_i \leq -u/x_i, i = 1, \ldots, d \mid X_1 \leq -\varepsilon u)
\]
and random variables $Y_1, \ldots, Y_d$ with distribution function $G^{\alpha, \beta}(x_1, \ldots, x_d)$. From Lemma 5.1 it follows that $(Y_1^{(u)}, \ldots, Y_d^{(u)})$ converges in distribution to $(Y_1, \ldots, Y_d)$, as $u \to \infty$, and thus

$$\mathbb{P}(\sum_{i=1}^{d} 1/Y_i \geq 1) = \mathbb{P}(\sum_{i=1}^{d} X_i \leq -u | X_1 \leq -\varepsilon u)$$

converges (again as $u \to \infty$) to

$$\mathbb{P}(\sum_{i=1}^{d} 1/Y_i \geq 1) = \int_{\sum_{i=1}^{d} 1/x_i \geq 1, x_i \leq 1/\varepsilon} g^{\alpha, \beta}(x_1, \ldots, x_d) dx_1 \cdots dx_d = \varepsilon^\beta G(\varepsilon).$$

For the lower bound we see that

$$\liminf_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u\right) \geq \liminf_{u \to \infty} \frac{F(-\varepsilon u)}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u \mid X_1 \leq -\varepsilon u\right) \geq \liminf_{u \to \infty} \frac{F(-\varepsilon u)}{F(-u)} \varepsilon^\beta G(\varepsilon) = G(\varepsilon),$$

where we used again that $F$ is regularly varying. Since $\varepsilon > 0$ was arbitrary

$$\liminf_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u\right) \geq G(0).$$

For the upper bound choose $\varepsilon < 1/d$. Then

$$\limsup_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u\right) = \limsup_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u, X_1 \leq -\varepsilon u\right) + \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u, X_1 > -\varepsilon u\right).$$

For the first term we have:

$$\limsup_{u \to \infty} \frac{1}{F(-u)} \mathbb{P}\left(\sum_{i=1}^{d} X_i \leq -u, X_1 \leq -\varepsilon u\right) = G(\varepsilon).$$

(5.11)
For the second term:

\[
\limsup_{u \to -\infty} \frac{1}{F(-u)} \mathbb{P} \left( \sum_{i=1}^{d} X_i \leq -u, X_1 > -\varepsilon u \right) \\
\leq \limsup_{u \to -\infty} \frac{1}{F(-u)} \sum_{i=2}^{d} \mathbb{P} \left( X_i \leq -u/d, X_1 > -\varepsilon u \right) \\
= \limsup_{u \to -\infty} \frac{d-1}{F(-u)} \mathbb{P} \left( X_2 \leq -u/d, X_1 > -\varepsilon u \right) \\
= \limsup_{u \to -\infty} \frac{d-1}{F(-u)} \left( \mathbb{P}(X_2 \leq -u/d) - \mathbb{P}(X_2 \leq -u/d, X_1 \leq -\varepsilon u) \right) \\
= (d-1) \left( d^\beta - (d^{-\alpha \beta} + \varepsilon^\alpha)^{-1/\alpha} \right), \tag{5.12}
\]

where in the last equation we repeatedly use the fact that \( \phi \) and \( F \) are regularly varying. Since \( \varepsilon \in (0, 1/d) \) we let \( \varepsilon \downarrow 0 \) and arrive at:

\[
\limsup_{u \to -\infty} \frac{1}{F(-u)} \mathbb{P} \left( \sum_{i=1}^{d} X_i \leq -u \right) \leq G(0), \tag{5.13}
\]

which is the upper bound we claimed. This finishes the proof of Theorem 3.2. \( \blacksquare \)

### 5.1.2 Weibull case

The Weibull case is very similar. Lemma 5.1 is replaced by the following lemma:

**Lemma 5.2 (Weibull)** Let \( d \geq 2, \alpha > 0 \) and \( \beta > 0 \). Let \( X = (X_1, \ldots, X_d) \) have Archimedean copula \( C^x \), where \( x \) is a regularly varying function at \( 0^+ \) with index \( -\alpha \). Let all \( X_i \) have the same, continuous marginal \( F(x) \) such that there is a constant \( c \) such that \( s \mapsto F(c - 1/s) \) is regularly varying at \( -\infty \) with index \( \beta \). Furthermore, let \( \varepsilon \in (0, 1), x_1 \in (0, 1/\varepsilon) \) and \( x_2, \ldots, x_d \geq 0 \). Then:

\[
\lim_{u \to -\infty} \mathbb{P}(X_i \leq c + x_i/u, i = 1, \ldots, d \mid X_1 \leq c + 1/\varepsilon u) = \left( \frac{\sum_{i=1}^{d} x_i^{-\alpha \beta}}{\varepsilon^{-\beta}} \right)^{-1/\alpha}. \tag{5.14}
\]

**Proof of Lemma 5.2 and Theorem 3.2 (the Weibull case).**

The proof of Lemma 5.2 follows, mutatis mutandis, the lines of the proof of Lemma 5.1 in the Fréchet case. The only change for the proof of Theorem 3.2 is that now we take \( Y_i^{(u)} \), \( i = 1, \ldots, d \) with distribution function

\[
H^*(x_1, \ldots, x_d) \overset{def}{=} \mathbb{P}(X_i \leq c + x_i/u, i = 1, \ldots, d \mid X_1 \leq c + 1/\varepsilon u),
\]

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and this time
\[ \Pr \left( \sum_{i=1}^{d} X_i \leq dc + 1/u \mid X_1 \leq c + 1/\varepsilon u \right) = \Pr \left( \sum_{i=1}^{d} Y_i^{(u)} \leq 1 \right), \]

such that
\[ \lim_{u \downarrow c} \Pr \left( \sum_{i=1}^{d} X_i \leq dc + 1/u \mid X_1 \leq c + 1/\varepsilon u \right) = \int_{\sum_{i=1}^{d} x_i \leq c + 1/\varepsilon} g^\alpha_\varepsilon(x_1, \ldots, x_d) dx_1 \ldots dx_d \]

where \( g^\alpha_\varepsilon(x_1, \ldots, x_d) \) again is the density function associated with \( G^{\alpha, \beta}(x_1, \ldots, x_d) \).

Thus in this case \( q_W^{\alpha, \beta}(\alpha, \beta) = \lim_{\varepsilon \to 0} G^*(\varepsilon) \), where
\[ G^*(\varepsilon) \overset{\text{def}}{=} \varepsilon^{-\beta} \int_{\sum_{i=1}^{d} x_i \leq c + 1/\varepsilon} g^\alpha_\varepsilon(x_1, \ldots, x_d) dx_1 \ldots dx_d \quad (5.15) \]

This finishes the proofs in the Weibull case. ■

### 5.1.3 Gumbel case

Eventually, the Gumbel case is slightly different.

**Lemma 5.3 (Gumbel)** Let \( d \geq 2, \alpha > 0 \). Let \( X = (X_1, \ldots, X_d) \) have Archimedean copula \( C^\phi \), where \( \phi \) is a regularly varying function at \( 0^+ \) with index \(-\alpha\). Let all \( X_i \) have the same, continuous marginal \( F(x) \) such that there is a constant \( c \) and a positive function \( s \mapsto a(s) \) such that \( \lim_{u \downarrow c} F(u + ta(u))/F(u) = e^f \), for all \( t \in \mathbb{R} \).

Furthermore, let \( \varepsilon \in (0, 1), x_1 \in (-\infty, 1/e) \) and \( x_2, \ldots, x_d \in \mathbb{R} \). Then:

\[ \lim_{u \downarrow c} \Pr \left( X_i \leq u + x_i a(u), i = 1, \ldots, d \mid X_1 \leq u + a(u)/\varepsilon \right) = e^{-1/\varepsilon} \left( \sum_{i=1}^{d} e^{-\alpha x_i} \right)^{-1/\alpha} . \quad (5.16) \]

**Proof.** Again the proof follows the proof of Lemma 5.1. But this time we have to change more: With the help of Gumbel-case variants for inequalities (5.2), (5.3) and (5.4) and again for \( \delta > 0 \) the inequalities become (details are left to the reader):

\[ \limsup_{u \to \infty} \Pr \left( X_i \leq u + x_i a(u), i = 1, \ldots, d \mid X_1 \leq u + a(u)/\varepsilon \right) \]

\[ \leq \limsup_{u \to \infty} \frac{\delta^{-1} \left( \sum_{i=1}^{d} \phi \left( e^{x_i + \delta} F(-u) \right) \right)}{e^{1/\varepsilon - \delta} F(-u)} \]

\[ = \left( \left( \sum_{i=1}^{d} e^{x_i + \delta} \right)^{-1/\alpha} - \delta \right)^{1/\alpha} e^{\delta - 1/\varepsilon} . \quad (5.17) \]
With $\delta \downarrow 0$ and a similar lower bound this proves the lemma.

**Proof of Theorem 3.2 (Gumbel case).** For the proof of the last part of Theorem 3.2 we take $Y_1^{(u)}, \ldots, Y_d^{(u)}$ with distribution function

$$H^\circ (x_1, \ldots, x_d) \overset{\text{def}}{=} \mathbb{P}(X_i \leq u + x_i a(u), i = 1, \ldots, d \mid X_1 \leq u + a(u)/\varepsilon)$$

and $Y_1, \ldots, Y_d$ with distribution function

$$G^\circ_\varepsilon (x_1, \ldots, x_d) \overset{\text{def}}{=} e^{-1/e} \left( \sum_{i=1}^{d} e^{-x_i a} \right)^{-1/\alpha}. \quad (5.18)$$

Then, if $g^\circ_\varepsilon \alpha$ denotes the density of $G^\circ_\varepsilon \alpha$

$$\lim_{\varepsilon \downarrow 0} \mathbb{P} \left( \frac{1}{\varepsilon} \sum_{i=1}^{d} X_i \leq u + a(u) \mid X_1 \leq u + a(u)/\varepsilon \right) = \lim_{\varepsilon \downarrow 0} \mathbb{P} \left( \frac{1}{\varepsilon} \sum_{i=1}^{d} Y_i^{(u)} \leq 1 \right) = \int_{\sum_{i=1}^{d} X_i \varepsilon \leq 1, x_i \leq 1/\varepsilon} g^\circ_\varepsilon \alpha (x_1, \ldots, x_d) dx_1 \ldots dx_d, \quad (5.19)$$

and thus $q^G_d (\alpha) = \lim_{\varepsilon \downarrow 0} G^\circ_\varepsilon (\varepsilon)$, where

$$G^\circ_\varepsilon (\varepsilon) \overset{\text{def}}{=} e^{1/e} \int_{\sum_{i=1}^{d} x_i \leq 1, x_i \leq 1/\varepsilon} g^\circ_\varepsilon \alpha (x_1, \ldots, x_d) dx_1 \ldots dx_d. \quad (5.20)$$

Just as in the Weibull-case the limit is already reached as soon as $\varepsilon \leq 1$, thus:

$$q^G_d (\alpha) = \int_{\sum_{i=1}^{d} x_i \leq 1} \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^{d} e^{-x_i a} \right)^{-1/\alpha} dx_1 \ldots dx_d. \quad (5.21)$$

We have now proved that

$$\lim_{\varepsilon \downarrow 0} 1/F(u) \mathbb{P} \left( \frac{1}{\varepsilon} \sum_{i=1}^{d} X_i \leq u + a(u) \right) = q^G_d (\alpha), \quad (5.22)$$

which finishes the proof of Theorem 3.2 (Gumbel case) \textcircled{\textbullet}
5.2 Limiting constants in the case \( d = 2 \)

5.2.1 Fréchet marginals

Let us choose \((Z_1, Z_2) \sim G_{\alpha, \beta}^\alpha (\text{see formula (5.7)})\). Choose \( \varepsilon < 1 \).

\[
G(\varepsilon) = \varepsilon^{-\beta} \mathbb{P}\left( \frac{1}{Z_1} + \frac{1}{Z_2} \geq 1 \right)
\]
\[
= \varepsilon^{-\beta} \left( \mathbb{P}(Z_1 \leq 1) + \mathbb{P}\left( \frac{1}{Z_1} + \frac{1}{Z_2} \geq 1, 1 < Z_1 \leq 1/\varepsilon \right) \right)
\]
\[
= 1 + \varepsilon^{-\beta} \mathbb{P}\left( Z_2 \leq \frac{Z_1}{Z_1 - 1}, 1 < Z_1 \leq 1/\varepsilon \right).
\]

Inserting the densities we obtain

\[
G(\varepsilon) = 1 + \beta \int_{1/\varepsilon}^{1/\varepsilon} \left( x_1^{-\alpha\beta} + \left( \frac{x_1}{x_1 - 1} \right)^{-\alpha\beta} \right)^{-1/\alpha - 1} x_1^{-\alpha\beta - 1} dx_1
\]
\[
= 1 + \beta \int_{1/\varepsilon}^{1/\varepsilon} x_1^{\beta - 1} \left( 1 + (x_1 - 1)^{\alpha\beta} \right)^{-1/\alpha - 1} dx_1.
\]

(5.23)

Since the function under the integral is of order \( x_1^{-(1+\alpha\beta)} \) as \( x_1 \to \infty \), which is in \( L^1 \), we can let \( \varepsilon \to 0 \) and we find

\[
q_{\alpha\beta}^F(\alpha, \beta) = G(0) = 1 + \beta \int_{1}^{\infty} x_1^{\beta - 1} \left( 1 + (x_1 - 1)^{\alpha\beta} \right)^{-1/\alpha - 1} dx_1.
\]

(5.24)

(5.25)

To analyze the integral, we first substitute \( x_1 - 1 \mapsto z \), and then \( z^\beta \mapsto y \):

\[
q_{\alpha\beta}^F(\alpha, \beta) = 1 + \beta \int_{0}^{\infty} (z + 1)^{\beta - 1} \left( 1 + z^{\alpha\beta} \right)^{-1/\alpha - 1} dz
\]
\[
= 1 + \int_{0}^{\infty} \left( 1 + y^{-1/\beta} \right)^{\beta - 1} (1 + y^{\alpha})^{-1/\alpha - 1} dy.
\]

(5.26)

Hence we have separated the term into a product of two terms, one only depending on \( \alpha \), the other one only depending on \( \beta \). Moreover these terms have the same structure.

Hence, if we define \( f_\alpha(y) \) as above we arrive at

\[
q_{\alpha\beta}^F(\alpha, \beta) = 1 + \int_{0}^{\infty} f_{-1/\beta}(y) \cdot f_\alpha(y) dy.
\]

(5.27)

Proof of Lemma 3.4. Choose \( 0 \leq c_1 < c_2 \leq \infty \). Then

\[
\int_{c_1}^{c_2} f_\alpha(y) dy = \frac{-1}{\alpha} \int_{c_1^{-\alpha}}^{c_2^{-\alpha}} (1 + z)^{-1/\alpha - 1} dz
\]
\[
= \left( 1 + c_2^{-\alpha} \right)^{-1/\alpha} - \left( 1 + c_1^{-\alpha} \right)^{-1/\alpha},
\]

(5.28)
where in the first step we applied the substitution \( y^\alpha \rightarrow z^{-1} \). Letting \( c_1 \rightarrow 0 \) and \( c_2 \rightarrow \infty \) we find that \( f_\alpha \) is indeed a probability density function on \([0, \infty)\). ■ As a direct result we now see that

\[
q_d^F(\alpha, \beta) = 1 + E \left( f_{-1/\beta}(Y_\alpha) \right), 
\]

which is the first statement of Theorem 3.5.

For an absolutely continuous random variable \( Y_\alpha \) with density function \( f_\alpha \) we can compute

\[
H(c; \alpha) = \mathbb{P}(Y_\alpha \geq c) = 1 - (1 + c^{-\alpha})^{-1/\alpha}. 
\]

Now

\[
\frac{dH(c; \alpha)}{d\alpha} = -\frac{1}{\alpha^2} (1 + c^{-\alpha})^{-1/\alpha - 1} \left((1 + c^{-\alpha}) \log(1 + c^{-\alpha}) - c^{-\alpha} \log c^{-\alpha}\right). 
\]

For the last term in the above expression we know that

\[(1 + x) \log(1 + x) - x \log x = \log(1 + x) + x \log(1 + 1/x) > 0,\]

for all \( x > 0 \). This implies that

\[
\frac{dH(c; \alpha)}{d\alpha} < 0 \quad \text{for all } c \in (0, \infty). 
\]

Moreover \( \lim_{c \to 0} \frac{dH(c; \alpha)}{d\alpha} = 0 \). Hence \( H(c; \alpha) = \mathbb{P}(Y_\alpha \geq c) = 1 - (1 + c^{-\alpha})^{-1/\alpha} \) is strictly decreasing in \( \alpha \) for all \( c > 0 \). We are now ready to prove Theorem 3.5:

**Proof of Theorem 3.5.** Fix \( \beta > 1 \) and \( 0 < \alpha_1 < \alpha_2 \). Now (with (5.29))

\[
q_d^F(\alpha_1, \beta) - q_d^F(\alpha_2, \beta) 
= \left( E \left( \left(1 + Y_{\alpha_1}^{-1/\beta}\right)^{\beta - 1} \right) - E \left( \left(1 + Y_{\alpha_2}^{-1/\beta}\right)^{\beta - 1} \right) \right)
\]

\[
= \int_0^\infty \mathbb{P} \left( \left(1 + Y_{\alpha_1}^{-1/\beta}\right)^{\beta - 1} > x \right) - \mathbb{P} \left( \left(1 + Y_{\alpha_2}^{-1/\beta}\right)^{\beta - 1} > x \right) \, dx
\]

\[
= \int_0^\infty \mathbb{P} \left( Y_{\alpha_1} < \left(x^{1/(\beta - 1)} - 1\right)^{-\beta}\right) - \mathbb{P} \left( Y_{\alpha_2} < \left(x^{1/(\beta - 1)} - 1\right)^{-\beta}\right) \, dx.
\]

Using (5.30)-(5.32) we see that this last term is always negative, implying that \( q_d^F(\alpha_1, \beta) - q_d^F(\alpha_2, \beta) < 0 \) for \( \alpha_1 < \alpha_2 \), hence that \( \alpha \mapsto q_d^F(\alpha, \beta) \) is a strictly increasing function for \( \beta > 1 \). Analogously for \( \beta < 1 \)

\[
q_d^F(\alpha_1, \beta) - q_d^F(\alpha_2, \beta) 
= \int_{\alpha_1}^{\alpha_2} P \left( Y_{\alpha_1} > \left(x^{1/(\beta - 1)} - 1\right)^{-\beta}\right) - P \left( Y_{\alpha_2} > \left(x^{1/(\beta - 1)} - 1\right)^{-\beta}\right) \, dx > 0.
\]
Hence $\alpha \mapsto q_F^\beta(\alpha, \beta)$ is a strictly decreasing function for $\beta < 1$. The case $\beta = 1$ is clear.

Next we prove that $q_F^\beta(\alpha, \beta)$ is strictly increasing in $\beta$. Write $y = (z^{1/\beta} - 1)^\beta$, then

\[ q_F^\beta(\alpha, \beta) = 1 + \int_0^\infty \left(1 + y^{-1/\beta}\right)^{\beta-1} (1 + y^\alpha)^{-1/\alpha-1} dy \quad (5.34) \]

\[ = 1 + \int_1^{\infty} \left(1 + (z^{1/\beta} - 1)^\alpha\right)^{-1/\alpha-1} dz \]

\[ = 1 + \int_1^{\infty} \left(1 + \exp \left\{ \alpha \beta \log(z^{1/\beta} - 1) \right\} \right)^{-1/\alpha-1} dz. \]

Define $h(\beta; z) = \beta \log(z^{1/\beta} - 1)$.

\[
\frac{h(\beta; z)}{d\beta} = \frac{1}{z^{1/\beta} - 1} \left[ (z^{1/\beta} - 1) \log(z^{1/\beta} - 1) - z^{1/\beta} \log z^{1/\beta} \right] < 0 \quad \text{for } z > 1. \quad (5.35)
\]

Hence $h(\cdot; z)$ is strictly decreasing for all $z > 1$, which implies that $q_F^\beta(\alpha, \beta)$ is strictly increasing in $\beta$. This finishes the proof of the first part of Theorem 3.5.

5.2.2 Weibull marginals

Proof of Theorem 3.7. For the Weibull case recall that using (5.15) and (5.7) we can compute for $d = 2$ and $\varepsilon < 1$:

\[ G^\varepsilon(\varepsilon) = \varepsilon^{-\beta} \int_{x_1 + x_2 \leq 1, x_1 \leq 1/\varepsilon} q_{\alpha, \beta}^\varepsilon(x_1, x_2) dx_1 dx_2 \]

\[ = \varepsilon^{-\beta} P[Z_1 + Z_2 \leq 1] \]

\[ = \varepsilon^{-\beta} \int_0^1 \left( \frac{dy}{dx_2} G_{\alpha, \beta}^\varepsilon(x_1, x_2) \right)^{1-x_2} dx_2 \]

\[ = \beta \int_0^1 x_2^{-1-\frac{\alpha}{\beta}} \left( \frac{1 - x_2}{x_2} \right)^{-\frac{\alpha}{\beta} + 1} - 1/\alpha-1 dx_2. \quad (5.36) \]

Substituting $y = (1 - x_2)/x_2$ and $z = y^{-\beta}$ we obtain

\[ \lim_{\varepsilon \to 0} G^\varepsilon(\varepsilon) = \beta \int_0^\infty (y + 1)^{-1-\beta} (y^{-\alpha} + 1)^{-1/\alpha-1} dy \]

\[ = \int_0^{\infty} (1 + z^{1/\beta})^{-1-\beta} (z^\alpha + 1)^{-1/\alpha-1} dz, \quad (5.37) \]
which proves \( q^W_2(\alpha, \beta) = E \left( (1 + \frac{1}{\alpha} Y_1)^{-1} \right) \). Moreover \( q^W_2(\alpha, \beta) \) is strictly increasing in \( \alpha \) (similar proof to Theorem 3.5) and strictly decreasing in \( \beta \) (substitute the role of \( \alpha \) and \( \frac{1}{\beta} \)). This finishes the proof of the first part of Theorem 3.7. ■

5.2.3 Gumbel marginals

**Proof of Theorem 3.9.** For the Gumbel case, we can perform similar calculations using (5.18) and (5.19) (details are again left to the reader):

\[
G^\gamma(\varepsilon) = \int_{-\infty}^{1/\varepsilon} e^{\gamma} \left( 1 + e^{-\alpha(1-2x_1)} \right)^{-1/\alpha - 1} dx_1. \tag{5.38}
\]

Letting \( \varepsilon \to 0 \) and substituting \( e^{-\alpha(1-2x_1)} = y \) we obtain

\[
G^\gamma(0) = \frac{1}{2} \int_0^\infty y^{-1/2} (1 + y^\alpha)^{-1/\alpha - 1} dy. \tag{5.39}
\]

Now we see the first part of Theorem 3.9. The second part follows from Theorem 3.5 since \( q^E_2(\alpha, 2) = 2(1 + e^{-1/2} q^E(\alpha)) \). This finishes the proof of Theorem 3.9. ■

5.2.4 \( \alpha \)-Limiting behaviour

It remains to analyze the limiting behaviour as stated at the end of Theorems 3.5 and 3.7. This is fairly straightforward and most of the work goes into showing that limit and integral can be interchanged. For the \( \alpha \to \infty \) limit in Theorem 3.5 we use monotone convergence. To justify this we first look at \( d/d\alpha f_\alpha(y) \). Differentiating shows that

\[
\frac{d}{d\alpha} f_\alpha(y) > 0 \iff (1 + y^\alpha) \log (1 + y^\alpha) > (1 + \alpha)y^\alpha \log y^\alpha. \tag{5.40}
\]

We see that for \( y \in [0,1) \) the right-hand side is negative and the left-hand side is positive, so \( f_\alpha(y) \) is increasing in \( \alpha \) for \( y \in [0,1] \). For the interval \( [1, \infty) \) we have to do some extra work: Let \( \varepsilon > 0 \). Now for all \( \alpha \) such that \( \alpha^2 - \alpha > \frac{2 \log 2}{\log(1+\varepsilon)} \) and for all \( y \geq 1 + \varepsilon \) it holds that \( \alpha^2 - \alpha > \frac{2 \log 2}{\log(1+\varepsilon)} \). This leads to

\[
\alpha \log y^\alpha > 2 \log 2 + \alpha \log y = \log 4y^\alpha > \log (y^\alpha + 2 + y^{-\alpha})
= \log (y^\alpha + 1) + \log(1 + y^{-\alpha}) > y^{-\alpha} \log (y^\alpha + 1) + \log (y^{-\alpha} + 1)
= (1 + y^{-\alpha}) \log (y^\alpha + 1) - \log y^\alpha, \tag{5.41}
\]

which in turn can be rewritten as

\[
(1 + \alpha)y^\alpha \log y^\alpha > (1 + y^\alpha) \log (1 + y^\alpha). \tag{5.42}
\]
This, together with (5.40) implies that \( f_\alpha(y) \) is decreasing in \( \alpha \) on \([1 + \varepsilon, \infty)\), for sufficiently large \( \alpha \). So we now can use monotone convergence on \([1 + \varepsilon, \infty)\). Here we remark that \( f_\alpha(y) \) is bounded from above by 1 and that \( f^{1/3}_\alpha(y) \) is bounded on \([1, 1 + \varepsilon]\) by a constant \( c \).

\[
\limsup_{\alpha \to \infty} q_2^F(\alpha, \beta) = \limsup_{\alpha \to \infty} \left( 1 + \int_0^\infty f_{-1/\beta}(y) \cdot f_\alpha(y) dy \right)
\]

\[
= 1 + \int_0^1 f_{-1/\beta}(y) \cdot \lim_{\alpha \to \infty} f_\alpha(y) dy + \int_{1+\varepsilon}^\infty f_{-1/\beta}(y) \cdot \lim_{\alpha \to \infty} f_\alpha(y) dy
\]

\[
+ \limsup_{\alpha \to \infty} \int_1^{1+\varepsilon} f_{-1/\beta}(y) \cdot f_\alpha(y) dy
\]

\[
\leq 1 + \int_0^1 f_{-1/\beta}(y) \cdot 1 dy + \int_{1+\varepsilon}^\infty f_{-1/\beta}(y) \cdot 0 dy + \lim_{\alpha \to \infty} \int_1^{1+\varepsilon} c \cdot 1 dy
\]

\[
= 1 + \left[ (1 + y^{1/\beta})^3 \right]_0^{1+\varepsilon} + c\varepsilon = 2^\beta + c\varepsilon . \tag{5.43}
\]

From these calculations one can also see that \( \lim_{\alpha \to \infty} q_2^F(\alpha, \beta) \geq 2^\beta \).

For the \( \alpha \to 0 \) limit let \( \varepsilon, \beta > 0 \). Since

\[
\lim_{y \to \infty} f_{-1/\beta}(y) = \lim_{y \to \infty} (1 + y^{-1/\beta})^{\beta-1} = 1 , \tag{5.44}
\]

there is an \( y_{\varepsilon, \beta} \) such that for all \( y > y_{\varepsilon, \beta} \), \( |f_{-1/\beta}(y) - 1| < \varepsilon \). Then:

\[
q_2^F(\alpha, \beta) \leq 1 + \int_0^{y_{\varepsilon, \beta}} f_{-1/\beta}(y) \cdot f_\alpha(y) dy + \int_{y_{\varepsilon, \beta}}^\infty (1 + \varepsilon) f_\alpha(y) dy \tag{5.45}
\]

\[
= 1 + \int_0^{y_{\varepsilon, \beta}} f_{-1/\beta}(y) \cdot f_\alpha(y) dy + \int_{y_{\varepsilon, \beta}}^\infty \left( 1 - \left(1 + y_{\varepsilon, \beta}^{-\alpha} \right)^{-1/\alpha} \right) (1 + \varepsilon) \cdot f_\alpha(y) dy . \tag{5.28}
\]

As both \( \lim_{\alpha \to 0} (1 + x^{1/\alpha})^{-1/\alpha} = 0 \) and \( \lim_{\alpha \to 0} (1 + x^{-\alpha})^{-1/\alpha} = 0 \) for all \( x > 0 \) by dominated convergence we arrive at.

\[
\lim_{\alpha \to 0} \int_0^{y_{\varepsilon, \beta}} f_{-1/\beta}(y) \cdot f_\alpha(y) dy = \int_0^{y_{\varepsilon, \beta}} f_{-1/\beta}(y) \cdot \left( \lim_{\alpha \to 0} f_\alpha(y) \right) dy = 0 .
\]

The last three equations yield \( \limsup_{\alpha \to 0} q_2^F(\alpha, \beta) \leq 2 + \varepsilon \). Likewise \( \liminf_{\alpha \to 0} q_2^F(\alpha, \beta) \geq 2 - \varepsilon \), which what is claimed in Theorem 3.5.

The considerations leading to the \( \alpha \)-limits in Theorem 3.7 are almost the same:

For the \( \alpha \to \infty \)-limit we can take 1 for \( c \) and the final integral becomes:

\[
\lim_{\alpha \to \infty} q_2^W(\alpha, \beta) = \int_0^1 (1 + y^{1/\beta})^{\beta-1} dy = \lim_{\varepsilon \to 0} \left[ (1 + y^{-1/\beta})^{-\beta} \right]_0^{1+\varepsilon} = 2^{-\beta} .
\]
For the $\alpha \to 0$-limit we remark that (comp. (5.44)) $\lim_{y \to \infty} f_{1/\beta}(y) = \lim_{y \to \infty}(1 + y^{1/\beta})^{-\beta} = 0$ and if we now take $y_{e, \beta}$ such that for all $y > y_{e, \beta}$: $f_{1/\beta}(y) < \varepsilon$ we see that

$$\lim_{\alpha \downarrow 0} \frac{1}{\alpha^2} \leq \lim_{\alpha \downarrow 0} \left( \int_{0}^{y_{e, \beta}} f_{1/\beta}(y) \cdot f_{\alpha}(y) dy + \int_{y_{e, \beta}}^{\infty} \varepsilon f_{\alpha}(y) dy \right)$$

$$= \int_{0}^{y_{e, \beta}} f_{1/\beta}(y) \cdot \left( \lim_{\alpha \downarrow 0} f_{\alpha}(y) \right) dy + \limsup_{\alpha \downarrow 0} \left( 1 + \left(1 + y_{e, \beta}^{-3/2}\right)^{-1/\alpha} \right) \varepsilon = \varepsilon.$$

Eventually (3.12) follows immediately from (3.11).

References


