The Jacobian Conjecture: linear triangularization for homogeneous polynomial maps in dimension three

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Abstract

Let k be a field of characteristic zero and $F : k^3 \to k^3$ a polynomial map of the form $F = x + H$, where H is homogeneous of degree $d \geq 2$. We show that the Jacobian Conjecture is true for such mappings. More precisely, we show that if JH is nilpotent there exists an invertible linear map T such that $T^{-1}HT = (0, h_2(x_1), h_3(x_1, x_2))$, where the h_i are homogeneous of degree d.
As a consequence of this result, we show that all generalized Drużkowski mappings $F = x + H = (x_1 + L_1^d, \ldots, x_n + L_n^d)$, where L_i are linear forms for all i and $d \geq 2$, are linearly triangularizable if JH is nilpotent and $\operatorname{rk} JH \leq 3$.

Introduction

The Jacobian Conjecture asserts that every polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ satisfying the Jacobian hypothesis, i.e. $\det JF \in \mathbb{C}^*$ is invertible. It was shown in [1] and [14] that it suffices to prove the Jacobian Conjecture for all polynomial maps of the form $F = x + H$, where $H = (H_1, \ldots, H_n)$ and each H_i is a homogeneous polynomial of some fixed degree d (which we may assume to be 3). For such F the Jacobian hypothesis $\det JF \in \mathbb{C}^*$ is well-known to be equivalent to the nilpotency of the matrix JH ([1] or [6]). Therefore one is naturally led to the study of nilpotent Jacobians. A fundamental open problem in this respect is the following, which was formulated as a conjecture problem by various authors ([5], [6], [8], [9], [10]).

Homogeneous Dependence Problem

$HDP(n)$. Let $H = (H_1, \ldots, H_n) : k^n \to k^n$ be homogeneous of degree $d \geq 2$ such that JH is nilpotent. Are the rows of JH linearly dependent over k or equivalently are the H_i linearly dependent over k (k is a field of characteristic zero).

Affirmative answers are known in the following cases:
$\operatorname{rk} JH \leq 1$ (also if H is not homogeneous), [1], [6]. In particular, this holds for the case $n = 2$. The case $n = 3$ and $d = 3$ (Wright [13], 1993) and $n = 4$, $d = 3$ (Hubbers, [8], 1994, see also [6]). One of the main results of this paper (Theorem 1.2) gives an affirmative answer for $n = 3$ (d arbitrary). As a consequence we will show that in dimension 3 the Jacobian

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Conjecture is true for all polynomial maps of the form $F = x + H$ with H homogeneous (of degree d). More precisely we show that those maps are linearly triangularizable, i.e. there exists $T \in Gl_3(k)$ such that $T^{-1}FT = (x_1, x_2 + h_2(x_1), x_3 + h_3(x_1, x_2))$, where h_2 and h_3 are homogeneous of degree d. This generalizes the case $d = 3$ obtained by Wright in [13].

1 The main results and some preliminaries

Throughout this paper k is a field of characteristic zero. The main result is

Theorem 1.1 *Let $H = (H_1, H_2, H_3) : k^3 \to k^3$ be homogeneous of degree $d \geq 2$. If JH is nilpotent then there exists $T \in Gl_3(k)$ such that $T^{-1}HT = (0, h_2(x_1), h_3(x_1, x_2))$, where the h_i are homogeneous of degree d. In particular the polynomial map $F = x + H$ is invertible if* $\det JF \in k^*$

The proof of this result consists of two cases: $(JH)^2x = 0$ and $(JH)^2x \neq 0$. To see the first case we give some easy generalities on homogeneous polynomial maps. So let $H := (H_1, \ldots, H_n) : k^n \to k^n$ be a homogeneous polynomial map of degree $d \geq 2$. Let I_H denote the (prime) ideal of relations between the H_i, i.e. the set of all $R \in k[y] := k[y_1, \ldots, y_n]$ such that $R(H) = 0$. Then I_H is a homogeneous ideal. Consequently, writing $H_i = g\widetilde{H}_i$, where $g := \gcd(H_i)$, we get that $I_H = I_{\widetilde{H}}$. So obviously $\dim k[y]/I_H = \dim k[y]/I_{\widetilde{H}}$. Hence $\operatorname{trdeg}_k k(H) = \operatorname{trdeg}_k k(\widetilde{H})$ which by [6, 1.2.9] implies that $\operatorname{rk} JH = \operatorname{rk} J\widetilde{H}$.
Next we associate to H the k-derivation D_H by the formula

$$D := D_H = \sum H_j \partial_j.$$

Observe that $Dx_i = H_i$ and that $D^2x_i = \sum H_j \partial_j(H_i)$ is the i-th component of $JH \cdot H$. Since by Euler's formula $H = \frac{1}{d} JH \cdot x$, it follows that

$$D^2 x_i = \text{the } i\text{-th component of } \frac{1}{d}(JH)^2 \cdot x. \tag{1}$$

Proposition 1.2 *If H is homogeneous, then $(JH)^2x = 0$, if and only if $x + H$ is a quasi-translation, i.e. $x + H$ is invertible with inverse $x - H$. Furthermore, if H is homogeneous and $x + H$ is a quasi-translation, then $H \circ H = 0$ and* $\operatorname{rk} JH \leq n - 2$.

Proof.

i) Assume that H is homogeneous and $x - H$ is the inverse of $x + H$. Then $H(x + H) = H$. Using this equation we get by induction on n that $H(x + nH) = H$ for all $n \in \mathbb{N}$ (just make the substitution $x \to x + H$). Consequently, $H(x + tH) = H$, where t is a polynomial indeterminate. Differentiating to t and substituting $t = 0$ gives $JH \cdot H = 0$. Now apply Eulers formula to get $(JH)^2 \cdot x = 0$.
ii) Assume H is homogeneous and $(JH)^2x = 0$. By (1), D is locally nilpotent and $\exp D = x + H$ with inverse $\exp(-D) = x - H$. Looking at the component of highest degree in the equation $(x + H) \circ (x - H) = x$ we get $H \circ H = 0$.

iii) Observe that $D_H = gD_{\tilde{H}}$. Since D_H is locally nilpotent, it follows from [6, 1.3.34 and 1.3.35] that $D^2_{\tilde{H}}(x_i) = 0$ for all i. So by i), $\tilde{H} \circ \tilde{H} = 0$. If $\operatorname{rk} JH = n-1$ then, as observed above, $\operatorname{rk} J\tilde{H} = n-1$, whence $\dim k[y]/I_{\tilde{H}} = n-1$. So $I_{\tilde{H}}$ is a prime ideal generated by one irreducible polynomial R. Since $\tilde{H} \circ \tilde{H} = 0$ we get $\tilde{H}_i \in I_{\tilde{H}}$ for all i, so R divides all $\tilde{H}_i$, contradicting the fact that $\gcd \tilde{H}_i = 1$ □

Corollary 1.3 *Theorem 1.1 holds if* $(JH)^2x = 0$.

Proof. By 1.2 we get $\operatorname{rk} JH \leq 1$, so $\operatorname{trdeg}_k k(H) \leq 1$. We may assume that $H_3 \neq 0$. Then in particular H_1 and H_3 are algebraically dependent over k and hence linearly dependent over k (by the homogeneity of the H_i). Say $H_1 = c_1H_3$ and similarly $H_2 = c_2H_3$ for some $c_i \in k$. Put

$$T := \begin{pmatrix} 1 & 0 & -c_1 \\ 0 & 1 & -c_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $THT^{-1} = (0, 0, h_3)$. Since the Jacobian of this matrix is nilpotent, the trace of this Jacobian equals zero, i.e. $\partial_3(h_3) = 0$, which implies that $h_3 \in k[x_1, x_2]$ □

The proof of 1.1 in case $(JH)^2x \neq 0$ is based on

Theorem 1.4 *The homogeneous dependence problem has an affirmative answer for* $n = 3$.

In case $(JH)^2x = 0$ we just proved 1.4. However if $(JH)^2x \neq 0$ the proof is much more involved and will be postponed to the next section. Using 1.4 we are now able to give

Proof of 1.1

By 1.3 we may assume that $(JH)^2x \neq 0$. Following Wright in [13] we may furthermore assume that the formulas of (4) below hold, where the terms are ordered lexicographically according $x_1 > x_2 > x_3$ (for more details see the beginning of the next section). It is proved there that for such a H the nilpotency of JH implies that $H_1 = 0$. Consequently $J_{x_2,x_3}(H_2, H_3)$ is nilpotent as well. Viewing H_2, H_3 in $k(x_1)[x_2, x_3]$ it then follows from the fact that the two-dimensional Dependence Problem has an affirmative answer (see [6], 7.1.7i)), that there exist $c_1, c_2 \in k[x_1]$, not both zero such that

$$c_1 \cdot (H_2 - H_2(0,0)) + c_2 \cdot (H_3 - H_3(0,0)) = 0 \tag{2}$$

We may assume that $\gcd(c_1, c_2) = 1$. So the elements $c_i(0) \in k$ are not both zero. Writing c_1 and c_2 as a sum of homogeneous components and using that the H_i are homogeneous of the same degree d, it follows from (2) that $c_1(0)H_2 + c_2(0)H_3 = cx_1^d$, for some $c \in k$. Looking at the term $x_1^{d-1}x_2$ in this equation gives $c_2(0) = 0$ and $c_1(0)H_2 = cx_1^d$, whence $H_2 = \frac{1}{d}x_1^d$. Since $H_1 = 0$ it then follows from $\operatorname{tr} JH = 0$ that $\partial_3(H_3) = 0$ i.e. $H_3 \in k[x_1, x_2]$, which shows that H is on triangular form □

2 A structure theorem for nilpotent Jacobians of rank ≤ 2

Throughout this section we have the following notation:
k is an algebraically closed field of characteristic zero, $k[x] := k[x_1, \ldots, x_n]$, where $n \geq 3$ and $H := (H_1, \ldots, H_n) : k^n \to k^n$ a homogeneous polynomial map of degree $d \geq 2$. The main result of this section is:

Theorem 2.1 *Assume* $\operatorname{rk} JH \leq 2$ *and let* $g := \gcd(H_i)$*. Then there exist* $h_i \in k[t_1, t_2]$ *homogeneous of the same degree* s *or zero and* p *and* q *in* $k[x]$ *homogeneous of the same degree* r *such that* $H_i = g h_i(p, q)$ *for all* i*.*

The proof of theorem 2.1 is based on the following version of Bertini's theorem, see [11, p. 79]:

Theorem 2.2 *Let* $F(x, y) \in k[x_1, \ldots, x_n, y_1, \ldots, y_m]$*. Assume that* F *is irreducible over* $k(y)$ *and* $\deg_y F = 1$*. If* $F(x, \lambda)$ *is reducible for all* $\lambda \in k^m$*, then there exist an* $s \geq 2$*,* $p, q \in k[x]$ *and* $a_i(y) \in k[y]$ *such that*

$$F(x, y) = \sum_{i=0}^{s} a_i(y) p^i q^{s-i}$$

Proof of theorem 2.1.

i) We may assume that $g = 1$: namely write $H_i = g\widetilde{H}_i$. Then $\gcd(\widetilde{H}_i) = 1$. Furthermore, as observed in §1, $\operatorname{rk} J\widetilde{H} = rkJH$. So we may replace H by $\widetilde{H}$.

ii) Replacing H by $T \circ H$ for some $T \in \operatorname{GL}_n(k)$, we may assume that $H_1, H_2, \ldots, H_m$ are linearly independent over k, and $H_{m+1} = H_{m+2} = \cdots = H_n = 0$. If $m = 1$, then $h_1 = 1$, and we can take $p = x_1$ and $q = x_2$. If $m = 2$, then we can take $p = H_1$ and $q = H_2$.

iii) Assume $m \geq 3$. Consider the triple H_1, H_2, H_3 and let $R(H_1, H_2, H_3) = 0$ be a non-trivial homogeneous relation. Write

$$R = R_0(z_2, z_3) + R_1(z_2, z_3) z_1 + \cdots$$

its development after powers of z_1. From $R(H) = 0$ we get that H_1 divides $R_0(H_2, H_3)$. Write $R_0 = \prod_i (\alpha_i z_2 + \beta_i z_3)$ using that R_0 is homogeneous. If H_1 is irreducible then it divides $\alpha_i H_2 + \beta_i H_3$ for some i, whence $\alpha_i H_2 + \beta_i H_3 = cH_1$ for some $c \in k$ (look at degrees), which contradicts the linear independence of the H_i over k. So H_1 is reducible.

In a similar way we get more generally

$$\lambda_1 H_1 + \cdots + \lambda_m H_m \text{ is reducible for all } \lambda = (\lambda_1, \ldots, \lambda_m) \neq 0 \text{ in } k^m \tag{3}$$

(namely if for example $\lambda_1 \neq 0$, replace the n-tuple $(H_1, \ldots, H_m)$ by $(\lambda_1 H_1 + \cdots + \lambda_m H_m, H_2, \ldots, H_m)$ and apply the previous argument).

iv) Introduce m new variables $y_1, \ldots, y_m$ and define

$$F(x,y) := y_1 H_1(x) + \cdots + y_m H_m(x).$$

Then for all $0 \neq \lambda \in k^m$ we get $\deg_x F(x,\lambda) = \deg_x F(x,y)$. Since $\deg_y F(x,y) = 1$ and $\gcd(H_i) = 1$, it follows that $F(x,y)$ is irreducible in $k[x,y]$. From (3), we get that $F(x,\lambda)$ is reducible for all λ. It then follows from theorem 2.2 that there exist $p, q \in k[x]$ and an $s \geq 2$ such that

$$F(x,y) = \sum_{j=0}^{s} a_j(y) p(x)^{s-j} q(x)^j.$$

Let e_i denote the i-th standard basis vector of k^m. Then

$$H_i(x) = F(x,e_i) = \sum_{j=0}^{s} a_j(e_i) p(x)^{s-j} q(x)^j = h_i(p,q)$$

where

$$h_i(t_1,t_2) = \sum_{j=0}^{s} a_j(e_j) t_1^{s-j} t_2^j.$$

v) We show that p and q are homogeneous of the same degree. Assume the contrary. Let

$$p = p_e + \cdots + p_f \quad \text{and} \quad q = q_e + \cdots + q_f$$

be the decompositions in homogeneous parts, with p_e or $q_e \neq 0$ and p_f or $q_f \neq 0$. Then $e < f$ and $h_i(p,q) = h_i(p_e,q_e) + \cdots + h_i(p_f,q_f)$. Since all $h_i(p,q)$ are homogeneous of the same degree it follows from $se < sf$ that either $h_i(p_e,q_e) = 0$ for all i or $h_i(p_f,q_f) = 0$ for all i, say $h_i(p_e,q_e) = 0$ for all i. Let $\lambda_i t_1 + \mu_i t_2$ be a factor of $h_i(t_1,t_2)$ such that $\lambda_i p_e + \mu_i q_e = 0$. We may assume $p_e \neq 0$. Consequently $\mu \neq 0$ and $c := -q_e/p_e = \lambda_i/\mu_i \in k$. Hence $\lambda_i t_1 + \mu_i t_2 = \mu_i(ct_1 + t_2)$ i.e. $ct_1 + t_2$ divides $h_i(t_1,t_2)$ for all i, and hence $cp + q$ divides $h_i(p,q)$ for all i which contradicts the fact that $\gcd(h_i(p,q)) = 1$. So apparently p and q are homogeneous of the same degree, say r. Obviously $r \geq 1$ for if $r = 0$ then $p, q \in k$ and hence the H_i are linearly dependent over k □

3 The proof of theorem 1.4

First observe that in order to prove theorem 1.4 we may assume that $k = \mathbb{C}$ (using Lefschetz principle). Furthermore by 1.3 we may assume that JH is nilpotent and $(JH)^2 x \neq 0$. Our aim is to show that after a suitable linear coordinate change the first component of H equals zero, which completes the proof of theorem 1.4. To find such a coordinate change we start with an idea introduced by Wright in [13]: since $(JH)^2 x \neq 0$ we can choose $v \in \mathbb{C}^3$ with $(JH)(v)^2 v \neq 0$. To such a vector associate the matrix

$$T_v := (v \;\; (JH)(v)v \;\; (JH)(v)^2 v).$$

One easily verifies, using $(JH)(v)^3 = 0$, that the columns of T_v are linearly independent over $\mathbb{C}$, so T_v is invertible. Put

$$H_v := T_v^{-1} H T_v.$$

Observe that JH_v is also nilpotent. However H_v is nicer than H in the sense that (as one easily verifies)

$$(JH_v)(e_1) = J_2 := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So, replacing H by H_v, we may assume that $(JH)(e_1) = J_2$.

From now on in this section, we will write $C[x, y, z]$ instead of $\mathbb{C}[x_1, x_2, x_3]$. Since $(JH)(e_1) = J_2$, we get the following if we write each H_i as a sum of monomials ordered lexicographically according to $x > y > z$:

$$\begin{cases} H_1 = 0x^d + 0x^{d-1}y + 0x^{d-1}z + \cdots \\ H_2 = \frac{1}{d}x^d + 0x^{d-1}y + 0x^{d-1}z + \cdots \\ H_3 = 0x^d + 1x^{d-1}y + 0x^{d-1}z + \cdots \end{cases} \tag{4}$$

where " $\cdots$ " stands for terms lower in the lexicographical ordering. The remainder of this section is devoted to showing that $H_1 = 0$. For that purpose, we assume that $H_1 \neq 0$ in order to arrive at a contradiction.

Proposition 3.1 *With the notations of 2.1 and H as in (4) there exist p, q and g of the form*

$$q = x^r + 0x^{r-1}y + \cdots, \; p = 0x^r + 1x^{r-1}y + \cdots \quad \text{and} \quad g = x^t + \cdots$$

($rs + t = d$). Furthermore then

$$h_1(p, q) \equiv 0 \pmod{p^2}, \; h_2(p, q) \equiv \frac{1}{d}q^s \pmod{p} \quad \text{and} \quad h_3(p, q) \equiv q^{s-1}p \pmod{p^2}.$$

Proof. Since $gh_2(p, q) = H_2 = \frac{1}{d}x^d + \cdots$ it follows that $g = x^t + \cdots$ and that we may assume that $q = x^r + \beta x^{r-1}y + \cdots$. Since $x^{d-1}y + \cdots = H_3 = gh_3(p, q)$ it follows that we may assume that $p = x^{r-1}y + \cdots$. Replacing q by $q - \beta p$ we may assume that $\beta = 0$. Looking again at the equations $gh_2(p, q) = \frac{1}{d}x^d + \cdots$ ($x^{d-1}y + \cdots = gh_3(p, q)$) and using that the $h_i(t_1, t_2)$ are homogeneous we obtain that $h_2(p, q) \equiv \frac{1}{d}q^s \pmod{p}$ and $h_3(p, q) \equiv q^{s-1}p \pmod{p^2}$. Finally, looking at the coefficient of x^d $x^{d-1}y$ in the equation

$$0x^d + 0x^{d-1}y + \cdots = H_1 = gh_1(p, q)$$

we get that $h_1(p, q) \equiv 0 \pmod{p^2}$ □

Corollary 3.2 *Notations as in 3.1. Let p_1 be an irreducible factor of $p = x^{r-1}y + \cdots$ of the form $p_1 = x^m y + \cdots$, with $m \geq 0$. Then p_1 divides $H_2 - \frac{1}{d}x^d$.*

Proof. Since $H_1 \equiv 0 \pmod{p^2}$ the elements of the first row of JH are $\equiv 0 \pmod{p}$. It follows that the sum of all 2×2 principal minors is zero. Consequently also the 2×2 principal minor

$$[H_2, H_3] := \det J_{y,z}(H_2, H_3) \equiv 0 \pmod{p}. \tag{5}$$

Using that $H_2 \equiv \frac{1}{d} g q^s \pmod{p}$ and $H_3 \equiv g q^{s-1} p \pmod{p^2}$ we get

$$g q^{s-1} [\frac{1}{d} g q^s, p] \equiv 0 \pmod{p}. \tag{6}$$

Looking at the lexicographical highest order term in p we obtain that $p = p_1 a$ with $\gcd(a, p_1) = 1$. Similarly $\gcd(p_1, g) = \gcd(p_1, q) = 1$. It then follows from (6) that

$$\left[\frac{1}{d} g q^s, p_1\right] \equiv 0 \pmod{p_1}.$$

Observe that $p_1(0) := p_1(y = 0, z = 0) = 0$. So by lemma 3.3 below

$$\frac{1}{d} g q^s - \frac{1}{d} g(0) q(0)^s \equiv 0 \pmod{p_1} \tag{7}$$

i.e. $\frac{1}{d} g q^s \equiv \frac{1}{d} g(0) q(0)^s = \frac{1}{d} x^d \pmod{p_1}$, since $g(0) = x^t$, $q(0) = x^r$ and $t + rs = d$. Since by 3.1 $H_2 \equiv \frac{1}{d} g q^s \pmod{p_1}$ the desired result follows □

Lemma 3.3 *Let A be U.F.D. and $p, g \in A[y, z]$ such that $p(0) = 0$, p is irreducible in $A[x, y]$ and $[p, g] (= \det J_{y,z}(p, g)) \equiv 0 \pmod{p}$. Then p divides $g - g(0)$.*

Proof.

i) Put $D := p_z \partial_y - p_y \partial_z$. So D is an A-derivation on $A[y, z]$. Extend D to a K-derivation on $K[y, z]$, where K is the quotient field of A. By Gauss' lemma, p is irreducible in $K[y, z]$. So by ii) below it follows that $g - g(0) = h \cdot p$ for some $h \in K[y, z]$. Let $c, d \in A \backslash \{0\}$ be such that $ch = d\tilde{h} \in A[y, z]$, $\gcd(c, d) = 1$ and the gcd of all coefficients of $\tilde{h}$ is equal to 1. Then the equation $c(g - g(0)) = d\tilde{h}p$ shows that c is a unit in A (if p_1 is a prime factor of c it divides p, contradicting that p is irreducible in $A[y, z]$). Consequently p divides $g - g(0)$ as desired.

ii) It remains to prove the lemma in case A is a field, say $A = k$. First we assume that k is algebraically closed. Put $B := k[y, z]/(p)$. Then B is a domain and we get the induced k-derivation $\overline{D} : B \to B$ which by the hypothesis satisfied $\overline{D}(\overline{g}) = 0$. If $\overline{g} \notin k$, then $\operatorname{trdeg}_k k(\overline{g}) = 1$ (since k is algebraically closed!) Since also $\operatorname{trdeg}_k Q(B) = 1$ the extension $k(\overline{g}) \subset Q(B)$ is algebraic. Since $\overline{D}$ is zero on $k(\overline{g})$ it is also zero on $Q(B)$ ([6], 1.2.8). In particular $\overline{D}(\overline{y}) = 0$ i.e. $p_z \equiv 0 \pmod{p}$ and $\overline{D}(\overline{z}) = 0$ i.e. $p_y \equiv 0 \pmod{p}$, which gives a contradiction looking at degrees. So $\overline{g} \in k$ i.e. $g - \lambda \in (p)$ for some $\lambda \in k$. Since $p(0) = 0$ we get $\lambda = g(0)$, so $g - g(0) \in (p)$ as desired.

iii) Finally we show that we may assume that k is algebraically closed. Consider $p \in \overline{k}[y,z]$. Then p may become reducible, but, as one easily verifies, all its prime factors only have multiplicity one, say $p = p_1 \dots p_s$. From $[p,q] \equiv 0 \pmod{p}$ it follows that $[p_i, g] \equiv 0 \pmod{p_i}$ for all i. So by ii) $g - g(0) \equiv 0 \pmod{p_i}$ for all i, whence $g - g(0) \equiv 0 \pmod{p}$ □

Corollary 3.4 *Notations as in 3.2. If $(a,b,c) \in \mathbb{C}^3$ is a common zero of p_1 and q, then $a = 0$.*

Proof. By 3.1 $H_2 \in (p,q) \subset (p_1,q)$ (= the ideal generated by p_1 and q). Also by 3.2 $\frac{1}{d}x^d \in (H_2, p_1)$. So $x^d \in (p_1, q)$ □

Proof of theorem 1.4 (finished)

i) Since $(JH)(e_1) = J_2$ we have

$$(JH)(e_1)e_1 = e_2, \ (JH)(e_1)e_2 = e_3 \ \text{ and } \ (JH)(e_1)e_3 = 0. \tag{8}$$

Now let $\varepsilon \geq 0$. Put $v = (1, \varepsilon, 0)$ $T = T_v$ and $H_v = T_v^{-1} H T_v$. From (8) we get $T_{e_1} = I_3$. Consequently, if ε is close to zero the matrix T_v is invertible. By the argument in the beginning of this section $(JH_v)(e_1) = J_2$ and there exist p_v and q_v, homogeneous of degree r as in 3.1. Now we are going to construct such p_v and q_v explicitly (see formula (9) below). Therefore, observe that since $H_i = gh_i(p,q)$ for all i, it follows that

$$\begin{pmatrix} H_{v_1} \\ H_{v_2} \\ H_{v_3} \end{pmatrix} = T^{-1} \left((g \circ T) \cdot \begin{pmatrix} h_1(p \circ T, q \circ T) \\ h_2(p \circ T, q \circ T) \\ h_3(p \circ T, q \circ T) \end{pmatrix} \right).$$

Furthermore $p \circ T$ and $q \circ T$ are homogeneous of degree r. So we can write

$$\begin{aligned} q \circ T &= q_r(\varepsilon)x^r + q_{r-1}(\varepsilon)x^{r-1}y + \cdots, \\ p \circ T &= p_r(\varepsilon)x^r + p_{r-1}(\varepsilon)x^{r-1}y + \cdots \end{aligned}$$

where $q_i(\varepsilon)$ and $p_i(\varepsilon)$ are polynomials in ε. Since, as observed above, $T_{e_1} = I_3$, we get $T_v = I_3$ if $\varepsilon = 0$. So in that case ($\varepsilon = 0$), $q \circ T = q$ and $p \circ T = p$, whence

$$\begin{pmatrix} q_r(0) & q_{r-1}(0) \\ p_r(0) & p_{r-1}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This implies that for ε close to 0 the matrix

$$A_\varepsilon := \begin{pmatrix} q_r(\varepsilon) & q_{r-1}(\varepsilon) \\ p_r(\varepsilon) & p_{r-1}(\varepsilon) \end{pmatrix}$$

is invertible. Consequently we get

$$A_\varepsilon^{-1} \begin{pmatrix} q \circ T \\ p \circ T \end{pmatrix} = \begin{pmatrix} 1 \cdot x^t + 0 \cdot x^{r-1}y + \cdots \\ 0 \cdot x^r + 1 \cdot x^{r-1}y + \cdots \end{pmatrix}.$$

So if we put

$$\begin{pmatrix} q_v \\ p_v \end{pmatrix} = A_\varepsilon^{-1} \begin{pmatrix} q \circ T \\ p \circ T \end{pmatrix}. \tag{9}$$

then we get that p_v, q_v and $g_v := g \circ T$ satisfy the properties of proposition 3.1. Furthermore, both q_v and p_v are $\mathbb{C}$-linear combinations of $q \circ T$ and $p \circ T$.

ii) **Claim:** for all $\varepsilon > 0$ sufficiently close to zero the vector $v = (1, \varepsilon, 0)$ has the property that T_v is invertible and that the first component of $T_v^{-1}\theta$ is non-zero for every non-trivial common zero θ of p and q in $\mathbb{C}^3$.

Let us first assume the claim. Then choose ε close to zero as in this claim. Then by (9) the common zeros of p_v and q_v are the common zeros of $p \circ T_v$ and $q \circ T_v$ and hence are all the elements of the form $T_v^{-1}\theta$ where θ runs through all common zeros of p and q. By the claim we may therefore assume (replacing p and q by p_v and q_v) that all common zeros of p and q have their first component non-zero. However by 3.4, choosing a non-trivial common zero of p_1 and q (which is obviously a common zero of p and q) we get a contradiction!

iii) So it remains to prove the claim. The invertibility of T_v follows for small $\epsilon > 0$ since then $v = (1, \epsilon, 0)$ is close to e_1 and $T_{e_1} = I_3$. Next, we show that for small $\epsilon > 0$ and θ as in the claim, the first component of $T_v^{-1}\theta$ is nonzero. For that purpose, we first observe that since $\gcd(p, q) = 1$ p and q have only a finite number of common zeros in $\mathbb{P}^2(\mathbb{C})$. So it suffices to prove that for each $0 \neq \theta \in \mathbb{C}^3$ the first component of $T_v^{-1}\theta$ is non-zero if ε is sufficiently close to zero. So let $\theta = (a, b, c) \neq 0$ in $\mathbb{C}^3$.
In iv), we will show that the first row of T_v^{-1} is of the form

$$(1 + O(\varepsilon) \quad d\lambda(k-1)\varepsilon^k + O(\varepsilon^{k+1}) \quad -\lambda k \varepsilon^{k-1} + O(\varepsilon^k))$$

whence its components all have different order in ϵ, namely 0, k, and $k-1$ respectively (here we use that $k \geq 2$). It follows that for small $\epsilon > 0$, no $\mathbb{C}$-linear combination of these components can be zero (except the trivial combination). In particular, the first component of $T_v^{-1}\theta$ is non-zero for small $\epsilon > 0$.

iv) To compute $T_v^{-1}\theta$ we first compute $T_v = (v \;\; (JH)(v)v \;\; (JH)^2(v)v)$. Observe that by Euler's formula $JH(v)v = dH(v)$, so $T_v = (v \;\; dH(v) \;\; dJH(v)H(v))$.
Since $p = yx^{r-1} + \cdots$ and p^2 divides H_1 (by 3.1) there exists a $k \geq 2$ such that p^k divides H_1 but p^{k+1} does not divide H_1. Consequently $H_1 = \lambda y^k x^{d-k} + \cdots$ (use that $H_1 = gh_1(p, q)$, $g = x^t + \cdots$ and $q = x^r + \cdots$). Since $v = (1, \varepsilon, 0)$ we get $H_1(v) = \lambda\varepsilon^k + O(\varepsilon^{k+1})$. Furthermore $(H_1)_x(v) = O(\varepsilon^k)$, $(H_1)_y(v) = \lambda k \varepsilon^{k-1} + O(\varepsilon^k)$ and $(H_1)_z(v) = O(\varepsilon^{k-1})$. Using the formulas $H_2 = \frac{1}{d} \cdot x^d + 0 \cdot x^{d-1}y + \cdots$ and $H_3 = 0 \cdot x^d + 1 \cdot x^{d-1}y + \cdots$ we get

$$d \begin{pmatrix} H_1(v) \\ H_2(v) \\ H_3(v) \end{pmatrix} = \begin{pmatrix} d\lambda\varepsilon^k + O(\varepsilon^{k+1}) \\ 1 + O(\varepsilon) \\ d\varepsilon + O(\varepsilon^2) \end{pmatrix} \tag{10}$$

and

$$(JH)(v) = \begin{pmatrix} O(\varepsilon^k) & \lambda k \varepsilon^{k-1} + O(\varepsilon^k) & O(\varepsilon^{k-1}) \\ 1 + O(\varepsilon) & O(\varepsilon) & O(1) \\ O(\varepsilon) & 1 + O(\varepsilon) & O(1) \end{pmatrix}.$$

Consequently

$$dJH(v)H(v) = \begin{pmatrix} \lambda k\varepsilon^{k-1} + O(\varepsilon^k) \\ O(\varepsilon) \\ 1 + O(\varepsilon) \end{pmatrix}. \tag{11}$$

So from (10) and (11) we get

$$T_v = \begin{pmatrix} 1 & d\lambda\varepsilon^k + O(\varepsilon^{k+1}) & \lambda k\varepsilon^{k-1} + O(\varepsilon^k) \\ \varepsilon & 1 + O(\varepsilon) & O(\varepsilon) \\ 0 & d\varepsilon + O(\varepsilon^2) & 1 + O(\varepsilon) \end{pmatrix}.$$

So the first row of the adjoint matrix of T_v is of the form

$$(1 + O(\varepsilon) \quad d\lambda(k-1)\varepsilon^k + O(\varepsilon^{k+1}) \quad -\lambda k\varepsilon^{k-1} + O(\varepsilon^k))$$

as well the first row of T_v^{-1}, due to the adjoint formula for computing the inverse matrix □

4 An application and some final remarks

Before we make some final remarks concerning theorem 1.1 we first give an application. Recall that a polynomial mapping F is called a Keller map if $\det JF \in k^*$. Furthermore a polynomial mapping $H : k^n \to k^n$ is called a *generalized Drużkowski form* if there exists an integer $d \geq 2$ such that each component H_i of H is a d-th power of a linear form. A polynomial mapping $F = x + H$, where H is a generalized Drużkowski form, is called a generalized Drużkowski mapping.
It was recently shown by Cheng in [4] that if H is a Drużkowski form such that JH is nilpotent and $\mathrm{rk}\, JH \leq 2$, then H is linearly triangularizable. We can extend this result to $\mathrm{rk}\, JH \leq 3$. More precisely,

Corollary 4.1 *Let H be a generalized Drużkowski form with JH nilpotent. If $\mathrm{rk}\, JH \leq 3$, then H is linearly triangularizable. In particular, the Jacobian conjecture holds for all corresponding generalized Drużkowski mappings $F = x + H$.*

Proof. This follows directly from theorem 1.1 and Theorem 2 of [4] □

To conclude this paper we make some remarks on possible extensions of theorem 1.1.

- $HDP(3)$ **without the trace condition.**

  In 1.1 we showed that if $H \in k[x_1, x_2, x_3]^3$ is homogeneous and JH is nilpotent, then the components of H are linearly dependent over k and JH is linearly triangularizable. One can ask whether these results can be proved under a weaker condition than the nilpotency of JH. The nilpotency of JH can be split up into the following three subconditions:

  1. the determinant of JH is zero,

2. the sum of the determinants of the three 2×2 principal minors of JH is zero,
3. the trace of JH is zero.

Let us first consider showing linear dependence. Then subcondition 1. is necessary, since without it there is not even algebraic dependence, let alone linear dependence. But it is not enough for linear dependence, even if we add subcondition 3. to it, as the following example makes clear:

$$H = \begin{pmatrix} x_2^2 \\ x_1^2 \\ x_1 x_2 \end{pmatrix}$$

Furthermore since the sum of the determinants of the three 2×2 principal minors of the JH with H as above equals $-4x_1x_2$, the eigenvalues of JH are $0, 2\sqrt{x_1x_2}$ and $-2\sqrt{x_1x_2}$. Since these are not all polynomials, it follows that JH with H as above is also not linearly triangularizable.

So it remains to investigate what happens to the linear triangularizability and the linear dependence in case the Jacobian of H satisfies the subconditions 1. and 2. described above.

First the linear triangularizability: one easily verifies that the Jacobian of

$$H = \begin{pmatrix} 0 \\ x_1^2 x_2 x_3 \\ x_2^2 x_3^2 \end{pmatrix}$$

satisfies the subconditions 1. and 2. Furthermore the k-vector space V spanned by the entries of JH has dimension 6. If JH was linearly triangularizable, then using that it has one eigenvalue zero, one would have that $\dim V \leq 5$, a contradiction.

It therefore remains to see whether subconditions 1. and 2. are sufficient for the linear dependence of the components of H. It turns out that the answer to this question is positive. The proof of this result is given in the paper [3] of the first author.

- **Possible generalizations of theorem 1.1 in case $n \geq 4$.**

Finally we make some comments on possible generalizations of theorem 1.1 to higher dimensions.

First of all, it was already shown by Wright in [13] that in dim ≥ 4 the conditions H homogeneous and JH nilpotent are not sufficient to imply that H is linear triangularizable.

So the final question is: does $HDP(n)$ has an affirmative answer if $n \geq 4$? In [2] the first author shows that the answer to this question is negative for all $n \geq 5$. Therefore it remains to investigate the question: does $HDP(4)$ have an affirmative answer ?

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