Analysis of the expected shortfall of aggregate dependent risks

Stan Alink, Matthias Löwe, Mario V. Wüthrich

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Stan Alink* Matthias Löwe† Mario V. Wüthrich‡

Abstract

We consider $d$ identically and continuously distributed dependent risks $X_1, \ldots, X_d$. Our main result is a theorem on the asymptotic behaviour of expected shortfall for the aggregate risks: there is a constant $c_d$ such that for large $u$ we have $\mathbb{E} \left[ \sum_{i=1}^d |X_i| \mid \sum_{i=1}^d X_i \leq -u \right] \sim -uc_d$. Moreover we study diversification effects in two dimensions, similar to our Value-at-Risk studies in [2].

Subject Classification: 62E20, 62H20, 62P05

Keywords: Archimedean copula, dependent random variables, diversification effect, extreme value theory, expected shortfall, Value-at-Risk.

1 Introduction

One of the central topics in modern insurance mathematics and finance is the search for new methods to calculate risk-adjusted solvency requirements for companies. Such methods should in particular be able to cope with all different sorts of risks. Now treating a particular kind of risk is still feasible using analytical tools. The main issue is to model and compute the aggregation effects of different, usually dependent risks.

In [2] and [12] a first step in this direction was undertaken. There $d$ identically distributed dependent risks $X_1, \ldots, X_d$ were considered and results of the following

*Katholieke Universiteit Nijmegen, Subfaculty Wiskunde, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands.
†Westfälischen Wilhelms Universität Münster, Fachbereich Mathematik, Institut für Mathematische Statistik, Einsteinstrasse 62, 48149, Münster, Germany.
‡Winterthur Insurance, Römerstrasse 17, P.O. Box 357, CH-8401 Winterthur, Switzerland.
type were obtained.

\[ P \left[ \sum_{i=1}^{d} X_i \leq -u \right] \sim q_d \cdot P [ X_1 \leq -u ], \quad \text{as } u \to \infty, \tag{1.1} \]

where the constant \( q_d \) quantifies the diversification effect between the dependent risks.

From such analysis of the asymptotic behaviour of quantiles of the aggregate risks we were able to deduce as a main result an asymptotic Value-at-Risk estimate.

However, even though being very popular, Value-at-Risk has some disadvantageous properties, e.g. it is not a coherent risk measure (Value-at-Risk generally misses the subadditivity property, cf. Artzner-Delbaen-Eber-Heath [3] or Alink-Löwe-Wüthrich [2], Theorem 3.5 for \( \beta < 1 \)). Therefore various efforts are undertaken to look for more suitable, coherent risk measures. In many countries the regulators tend to use expected shortfall or worst conditional expectation, which in the case of continuous random variables are equivalent (see Acerbi-Tasche [1]). We do not want to enter the discussion here, about "good" and "bad" risk measures, we simply choose expected shortfall as our risk measure, which is coherent under the assumption that our random variables have continuous marginals (cf. Acerbi-Tasche [1]). I.e. we consider (for small \( p \)'s) \( E[X|X \leq u_p] \), where \( u_p \) is the \( p \)-quantile of \( X \). (To facilitate the analysis, we always assume losses to be negative, i.e. we study lower tails.)

This paper is organized as follows. In Section 2, we briefly describe our model. Section 3 contains the formulation of our main results, while Section 4 is devoted to examples. Finally in Section 5 we give the proofs, which are inspired by our previous results in [2]. We conclude this introduction with a quick reminder on the concept of copulas.

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1.1 Copulas

With expected shortfall as our risk measure, we concentrate on the case of aggregating dependent risks. The dependency of the risks is modelled by copulas. Copulas are simply a convenient description for families of dependent random variables. The concept of copulas was introduced by Sklar [11]. The idea is that the dependence structure of a finite family of random variables is completely determined by their joint distribution function. For any \( d \geq 2 \), a \( d \)-dimensional copula is thus defined as a \( d \)-dimensional distribution function on \([0,1]^d\), with marginals that are uniformly distributed on \([0,1]\).
With the concept of copulas we separate a multivariate distribution function into two parts, one describing the dependence structure and the other one describing the behaviour of the marginals. Moreover, all distribution functions with continuous marginals have a copula associated with them and vice versa. This is the content of Sklar’s theorem [11] (see Joe [7], Nelsen [10] or Section 2 in [2]).

In this article we focus on a special family of copulas, the Archimedean ones:

**Definition 1.1** Choose $d \geq 2$. Let $\phi : [0, 1] \to [0, \infty]$ be strictly decreasing, convex and such that $\phi(0) = \infty$ and $\phi(1) = 0$. Define for $x_i \in [0, 1], i = 1, \ldots, d$:

$$C^\phi(x_1, \ldots, x_d) \overset{\text{def.}}{=} \phi^{-1}\left(\sum_{i=1}^{d} \phi(x_i)\right).$$

(1.2)

The function $\phi$ is called generator of $C^\phi$.

In the case $d = 2$ this definition automatically implies that $C^\phi$ is a copula. In the case $d \geq 3$, a further assumption is required for $C^\phi$ to be a copula: If for all $k$ and $x > 0$ the $k$-th derivative of the inverse of $\phi, x^{-\alpha} \phi^{-1}(x)$, exists and satisfies

$$(-1)^k \frac{d^k}{dx^k} \phi^{-1}(x) \geq 0,$$

then $C^\phi$ is a distribution function, and hence a copula (cf. [9] and [2]).

Copulas of this type will be called (strict) Archimedean copulas.

The importance of Archimedean copulas in practice lies in the fact that they are easy to construct, but still we obtain a rich family of dependence structures. Usually, Archimedean copulas depend on one parameter, only. This makes it easier – though still very difficult – to estimate copulas from data. One of the best studied Archimedean copulas is the **Clayton copula** with parameter $\alpha > 0$. It is generated by $\phi(t) = t^{-\alpha} - 1$ and takes the form

$$C^{\text{Cl}, \alpha}(x_1, \ldots, x_d) \overset{\text{def.}}{=} (x_1^{-\alpha} + \ldots + x_d^{-\alpha} - d + 1)^{-1/\alpha}.$$

(1.4)

The limit $\alpha \to 0$ leads to independence, while $\alpha \to \infty$ leads to comonotonicity, i.e. complete positive dependence. For more examples we refer to Joe [7] and Nelsen [10].

With the notion of a copula in our hands our main results in this article can be described as follows. Assume the risks $X_1 \ldots X_d$ have the same continuous marginal distribution function $F$ and $(X_1, \ldots, X_d)$ has an Archimedean copula. Then we are
able to compute the asymptotic behaviour of expected shortfall, i.e. we are able to compute the decay of

\[ E \left[ \sum_{i=1}^{d} X_i \mid \sum_{i=1}^{d} X_i \leq -u \right] \]

as \( u \) tends to infinity (we always model losses as negative numbers).

As in the case of extreme value theorems which were proved in [2] it is possible to distinguish three different cases: the Fréchet case, the Gumbel case, and the Weibull case, of which only the two (most) interesting one, the Fréchet and the Gumbel case will be considered here.

### 2 The model

As already mentioned in the introduction we study a multivariate model describing the diversification effect when aggregating \( d \) dependent risks. The dependence structure will be given by an Archimedean copula, and losses are assumed to be negative. More precisely our assumptions read as follows:

**Assumption 2.1** We assume that the random vector \((X_1, \ldots, X_d)\) satisfies:

1) All coordinates \( X_i \) are negative and have the same continuous marginal

\[ F(x) = P[X_1 \leq x]. \]

2) \((X_1, \ldots, X_d)\) has an Archimedean copula with generator \( \phi \).

3) This generator \( \phi \) is regularly varying at \( 0^+ \) with index \(-\alpha\), where \( \alpha > 0 \).

For the last assumption let us recall the following definition (a standard reference on regular variation is Bingham-Goldie-Teugels [4]):

**Definition 2.2** A function \( f \) is called regularly varying at some point \( x^+ \) (or \( x^- \), respectively) with index \( \alpha \in \mathbb{R} \) if for all \( t > 0 \)

\[ \lim_{s \to x^+} \frac{f(st)}{f(s)} = t^\alpha, \]  
(2.1)

(or \( \lim_{s \to x^-} \frac{f(st)}{f(s)} = t^\alpha \), respectively).

### 3 Results

In this section we formulate our central results. Depending on the extreme value behaviour of the underlying risks, we distinguish two cases: the Fréchet case and the
Gumbel case. (In fact there are three cases, namely Fréchet, Gumbel and Weibull case, cf. Embrechts-Klüppelberg-Mikosch [5], Theorems 3.2.3 and 3.4.13. But from a practical point of view the Weibull case is less interesting since it deals with bounded random variables.)

3.1 Fréchet case

In the Fréchet case we look at (dependent) random variables that have a Fréchet-type distribution: their marginal distributions are regularly varying at $-\infty$ with parameter $-\beta$, for some $\beta > 0$. In our case we additionally assume that $\beta > 1$. The latter assumption is needed in order for the random variables to have a (finite) mean, which is hopefully the case in an insurance portfolio, because otherwise there is no finite pure risk premium.

**Theorem 3.1 (Fréchet case)** Assume Assumption 2.1 and that $F$ is regularly varying at $-\infty$ with parameter $-\beta$, $\beta > 1$. We have

$$\lim_{u \to \infty} -\frac{1}{u} E\left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] = c_d^F(\alpha, \beta), \quad (3.1)$$

where

$$c_d^F(\alpha, \beta) = \frac{\beta}{\beta - 1}. \quad (3.2)$$

**Remark 3.2** Note that $c_d^F(\alpha, \beta)$ is constant in $\alpha$ and $d$.

Hence we find the following asymptotic behaviour: As $u \to \infty$ we have

$$E\left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] \approx -\frac{\beta}{\beta - 1} u, \quad (3.3)$$

which is essentially the asymptotic behaviour of the conditional expectation of the Pareto distribution (see Katamara’s Theorem, [5] Theorem A3.6). The dependence strength comes now in via the following observation: For the expected shortfall, conditioned on an event with probability $p$ we obtain the following result: Denote by $-u_p$ the $p$-quantile of $\sum_{i=1}^{d} X_i$. From the above theorem and our results in [2], Theorem 3.2, we get for small $p$

$$E\left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u_p \right. \right] \approx -\frac{\beta}{\beta - 1} u_p \approx \frac{\beta}{\beta - 1} F^{-1} \left( \frac{p}{F_d^d(\alpha, \beta)} \right), \quad (3.4)$$
where
\[ q_d^F(\alpha, \beta) = \int_{x_i > 0, \sum x_i x_i > 1} \left[ \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d x_i^{-\alpha}\right)^{-1/\alpha} \right] dx_1 \ldots dx_d. \] (3.5)

For \( d = 2 \), \( q_d^F(\alpha, \beta) \) can explicitly be calculated (see Theorem 3.5 in [2]):
Choose \( Y_\alpha \sim f_\alpha = (1 + x^\alpha)^{-1}/\alpha \), \( \alpha > 0 \) and \( x > 0 \), then
\[ q_d^F(\alpha, \beta) = 1 + E \left[ \left( 1 + Y_\alpha^{-1/\beta} \right)^{\beta-1} \right]. \] (3.6)

For \( \beta > 1 \), \( q_d^F(\alpha, \beta) \) is increasing in \( \alpha \) (see Theorem 3.5 in [2]). Hence we have found:

**Corollary 3.3** Choose \( d = 2 \) and assume that \((X_1, X_2)\) satisfies the assumptions of Theorem 3.1. For \( p \to 0 \) we have
\[ E \left[ X_1 + X_2 | X_1 + X_2 \leq -u_p \right] \approx \frac{\beta}{\beta - 1} F^{-1} \left( \frac{p}{q_d^F(\alpha, \beta)} \right), \] (3.7)
where the right-hand side of (3.7) is strictly decreasing in \( \alpha \).

This shows that the right-hand side of (3.7) is decreasing in \( \alpha \), i.e. the bigger \( \alpha \), the smaller the diversification effect. This is not surprising since \( \alpha \) measures the dependence strength in the tails (see Juri-Wüthrich [8]). In the bivariate situation a coefficient for the dependence strength in the tails is so-called tail dependence coefficient \( \lambda \) (see Embrechts-McNeil-Straumann [6]). For Archimedean copulas we have \( \lambda = 2^{-1/\alpha} \) (see [8], Theorem 3.9), which is increasing in \( \alpha \).

### 3.2 Gumbel case

In the Gumbel case we look at (dependent) random variables that have a Gumbel-type distribution: there is a \( c \geq -\infty \) and a positive measurable function \( s \mapsto a(s) \) such that for \( t \in \mathbb{R} \) one has for marginals \( F \) that \( \lim_{u \to c} F(u + ta(u))/F(u) = e^t \).

**Theorem 3.4 (Gumbel case)** Under Assumption 2.1 and \( F \) of Gumbel type we have that
\[ \lim_{u \to c} \frac{1}{a(u)} E \left[ \sum_{i=1}^d X_i \sum_{i=1}^d X_i \leq du + a(u) \right] - \frac{du}{a(u)} = c_d^G(\alpha), \] (3.8)
where
\[ c_d^G(\alpha) = \frac{1}{q_d^G(\alpha)} \int_{\sum x_i \leq 1} \left( \sum_{i=1}^d x_i \right) \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d e^{-x_i \alpha} \right)^{-1/\alpha} dx_1 \ldots dx_d. \] (3.9)
with \( q^G_d \) given by
\[
q^G_d(\alpha) = \int \frac{d^d}{dx_1 \ldots dx_d} \left( \sum_{i=1}^d e^{-x_i/\alpha} \right)^{-1/\alpha} dx_1 \ldots dx_d. \tag{3.10}
\]

In particular we get
\[
c^G_d(\alpha) = 1 + \frac{E \left[ Y_\alpha^{-1/2} \log Y_\alpha \right]}{E \left[ Y_\alpha^{-1/2} \right]} = -1, \tag{3.11}
\]

where \( Y_\alpha \) has probability density \( f_\alpha = (1 + x^\alpha)^{-1/\alpha-1} \) on \( x > 0 \).

**Remark 3.5** Note that \( c^G_d(\alpha) \) is constant in \( \alpha \).

We can now do similar considerations as in the Fréchet case, assume that \( F \) is strictly increasing, then for \( u \) close to \( c \):
\[
E \left[ \sum_{i=1}^d X_i \sum_{i=1}^d X_i \leq du + a(u) \right] \approx du + c^G_d(\alpha)a(u)
= dF^{-1}(F(u + c^G_d(\alpha)a(u)/d)) \approx dF^{-1} \left( e^{c^G_d(\alpha)/d}F(u) \right)
\approx dF^{-1} \left( \frac{e^{c^G_d(\alpha)/d} \sum_{i=1}^d X_i \leq du + a(u)}{q^G_d(\alpha)} \right), \tag{3.12}
\]

where in the last step we have used formula (5.22) of [2].

Denote by \( u_p \) the \( p \)-quantile of \( \sum_{i=1}^d X_i \). Then for small \( p \) we get
\[
E \left[ \sum_{i=1}^d X_i \sum_{i=1}^d X_i \leq u_p \right] \approx dF^{-1} \left( \frac{p \cdot \exp \{c^G_d(\alpha)/d\}}{q^G_d(\alpha)} \right), \tag{3.13}
\]

hence expected shortfall can be approximated asymptotically.

Using Theorem 3.9 of [2] we find:

**Corollary 3.6** Choose \( d = 2 \) and assume that \( (X_1, X_2) \) satisfies the assumptions of Theorem 3.4. For \( p \to 0 \) we have
\[
E \left[ X_1 + X_2 \mid X_1 + X_2 \leq u_p \right] \approx 2F^{-1} \left( \frac{p \cdot \exp \{-1/2\}}{q^G_d(\alpha)} \right), \tag{3.14}
\]

where the right-hand side of (3.14) is strictly decreasing in \( \alpha \).
3.3 Conclusions

In Corollaries 3.3 and 3.6 we are able to study the asymptotic behaviour of expected shortfall, which gives upper and lower bounds for small $p$. The remarkable thing is that the estimate only depends on the marginals $F$ and on the dependence strength $\alpha$. I.e. in the Archimedean situation we can avoid the difficulty of choosing an explicit model (copula) for the dependence structure. All we need to estimate are the marginals and the (tail) dependence strength $\alpha$ (or the tail dependence coefficient $\lambda = 2^{-1/\alpha}$, resp.). As expected, the bounds are decreasing for increasing dependence strength $\alpha$, i.e. the larger the dependence strength, the smaller the diversification effect.

4 Example

We revisit the example given in [2]. In [2] we took two dependent motor liability portfolios $X_1$ and $X_2$. As risk measure we considered Value-at-Risk at a certain probability level. Using Value-at-Risk we studied then the diversification effect when merging these two dependent portfolios to one big portfolio $X_1 + X_2$. Here we examine the same example, but this time we choose expected shortfall as our risk measure (which in our continuous setup is a coherent risk measure).

Assume $X_1$ and $X_2$ have Archimedean copula generated by a regularly varying function with index $-\alpha$ at $0^+$ ($\alpha > 0$). Moreover assume that $-X_1$ and $-X_2$ have translated Pareto marginals with translation $V_1 = 880$ and $V_2 = 820$, i.e. $Y_i = -(X_i + V_i)$ is Pareto distributed with $\theta = 80$ and $\beta = 3$: for $i = 1, 2$,

$$P[X_i \leq x] = P[X_i + V_i \leq x + V_i] = \left(\frac{\theta}{-(x + V_i)}\right)^\beta \quad \text{for } x \leq -(\theta + V_i). \quad (4.1)$$

We define expected shortfall for $p \in (0, 1)$:

$$\text{ES}_{X_i}(p) = -E[X_i | X_i < u_p(X_i)] + E[X_i], \quad (4.2)$$

where $u_p(X_i)$ is the $p$-quantile of $X_i$.

Hence we have for $p = 0.5$

<table>
<thead>
<tr>
<th>portfolio 1</th>
<th>portfolio 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>translation $V_i$</td>
<td>880</td>
</tr>
<tr>
<td>mean $E[-X_i]$</td>
<td>1'000</td>
</tr>
<tr>
<td>variational coefficient</td>
<td>6.9%</td>
</tr>
<tr>
<td>$u_p(X_i)$</td>
<td>$-1'347.8$</td>
</tr>
<tr>
<td>$\text{ES}_{X_i}(p)$</td>
<td>581.8</td>
</tr>
</tbody>
</table>
Now we merge these two dependent portfolios to one big portfolio and we study expected shortfall as a function of the dependence strength $\alpha$:

$$ES_{X_1+X_2}(p; \alpha) = -E \left[ X_1 + X_2 \mid X_1 + X_2 < u_{p}^a(X_1 + X_2) \right] + E[X_1 + X_2], \quad (4.3)$$

where $u_{p}^a(X_1 + X_2)$ is the $p$-quantile of $X_1 + X_2$. Using Corollary 3.3 we see that we have the following approximation for small $p$

$$ES_{X_1+X_2}(p; \alpha) \approx \frac{\beta}{\beta-1} \theta \left( \frac{q^p_{\alpha, \beta}(\alpha, \beta)}{p} \right)^{1/\beta} - 2 \cdot \theta \frac{\beta}{\beta-1} \overset{\text{def}}{=} E_{X_1+X_2}(\alpha). \quad (4.4)$$

If we evaluate $E_{X_1+X_2}(\alpha)$ for different $\alpha$'s ($p = 0.5\%$) we obtain the following table (note that in the independent case we calculated the exact values, rather than the approximated values):

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>indep.</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>3.0</th>
<th>4.0</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-E[X_1 + X_2]$</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
<td>1'940</td>
</tr>
<tr>
<td>$E_{X_1+X_2}(\alpha)$</td>
<td>771</td>
<td>978</td>
<td>1'092</td>
<td>1'126</td>
<td>1'140</td>
<td>1'152</td>
<td>1'157</td>
<td>1'164</td>
</tr>
<tr>
<td>Div.eff.ES($\alpha$)</td>
<td>33.7%</td>
<td>16.0%</td>
<td>6.2%</td>
<td>3.2%</td>
<td>2.0%</td>
<td>1.0%</td>
<td>0.6%</td>
<td>0%</td>
</tr>
<tr>
<td>Div.eff.VaR($\alpha$)</td>
<td>31.6%</td>
<td>17.8%</td>
<td>6.9%</td>
<td>3.6%</td>
<td>2.2%</td>
<td>1.1%</td>
<td>0.6%</td>
<td>0%</td>
</tr>
</tbody>
</table>

$\alpha = \infty$ belongs to the comonotonic case (total positive dependence), Div.eff.ES($\alpha$) measures the diversification effect of the expected shortfall for $\alpha$-dependent random variables $X_1 + X_2$ relative to the comonotonic case, and Div.eff.VaR($\alpha$) gives the comparison to the results obtained in [2] for Value-at-Risk.

Not surprisingly, we see that the diversification effect decreases for increasing dependence strength $\alpha$. One also observes that the decrease is rather fast, i.e. already introducing slight dependencies in the tails reduces the diversification savings substantially.

For small $\alpha$, $p$ should be even smaller than 0.5% in order for the approximation to be sharp. This is not a serious problem, however, since we can calculate the expected shortfall and the diversification effect directly in the independent case.
Figure 1: The expected shortfall as a function of $\alpha$.

Figure 2: The diversification effect as a function of $\alpha$. 
5 Proofs

Proof of Theorem 3.1.

We use the following representation to calculate expected values:

\[
-\frac{1}{u} E \left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] = \int_{0}^{\infty} \frac{-1}{u} P \left( \sum_{i=1}^{d} X_i > z \left| \sum_{i=1}^{d} X_i \leq -u \right. \right) dz \\
= 1 + \int_{1}^{\infty} \frac{F(-u)}{P \left( \sum_{i=1}^{d} X_i \leq -u \right)} \int_{1}^{\infty} P \left[ \sum_{i=1}^{d} X_i < -zu \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] dz.
\]

Remark that \( zu \geq u \) due to \( z \geq 1 \). Hence for all \( \delta > 0 \) and all \( u \) sufficiently large we have (using twice Theorem 3.2 of [2])

\[
-\frac{1}{u} E \left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] \geq 1 + (1 - \delta) \int_{1}^{\infty} \frac{F(-zu)}{F(-u)} dz,
\]

and

\[
-\frac{1}{u} E \left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] \leq 1 + (1 + \delta) \int_{1}^{\infty} \frac{F(-zu)}{F(-u)} dz.
\]

Next we use that \( F \) is regularly varying with index \( -\beta \), hence \( \lim_{u \to \infty} F(-zu)/F(-u) = z^{-\beta} \). Moreover for \( u \) large, \( F(-zu)/F(-u) \) is uniformly bounded by an integrable function (see Katamara’s Theorem, [5] Theorem A3.6), hence using the dominated convergence theorem

\[
\lim_{u \to \infty} -\frac{1}{u} E \left[ \sum_{i=1}^{d} X_i \left| \sum_{i=1}^{d} X_i \leq -u \right. \right] = \frac{\beta}{\beta - 1}.
\]

Here in the last step we need the assumption \( \beta > 1 \). This finishes the proof of Theorem 3.1.

Proof of Theorem 3.4. For the lower bound note that

\[
1 - E \left[ \sum_{i=1}^{d} \frac{X_i - u}{a(u)} \left| \sum_{i=1}^{d} X_i \leq du + a(u) \right. \right] = E \left[ 1 - \sum_{i=1}^{d} \frac{X_i - u}{a(u)} \left| \sum_{i=1}^{d} \frac{X_i - u}{a(u)} \leq 1 \right. \right]
\]
has a positive argument in the integral. We define \( Y_i(u) = \frac{X_i - u}{a(u)} \). Hence for all \( \varepsilon > 0 \)

$$
1 - E \left[ \sum_{i=1}^{d} \frac{X_i - u}{a(u)} \right. \sum_{i=1}^{d} X_i \leq du + a(u) \]
$$

$$
= \int_0^\infty P \left[ 1 - \sum_{i=1}^{d} Y_i(u) > z \left| \sum_{i=1}^{d} Y_i(u) \leq 1 \right. \right] dz \quad (5.6)
$$

$$
= \int_0^\infty P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \left| \sum_{i=1}^{d} Y_i(u) \leq 1 \right. \right] dz
$$

$$
= \int_0^\infty \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{P \left[ \sum_{i=1}^{d} Y_i(u) \leq 1 \right]} dz
$$

$$
= \frac{F(u + a(u)/\varepsilon)}{P \left[ \sum_{i=1}^{d} Y_i(u) \leq 1 \right]} \int_0^\infty \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{F(u + a(u)/\varepsilon)} dz. \quad (5.7)
$$

From the Gumbel assumption on \( F \) and formula (5.22) in [2], we find that the first term on the right-hand side in (5.6) satisfies

$$
\lim_{u \to -\infty} \frac{F(u + a(u)/\varepsilon)}{P \sum_{i=1}^{d} Y_i(u) \leq 1} = e^{1/\varepsilon} q_d^2(\alpha). \quad (5.7)
$$

It remains to study the integral. Choose \( M > 1 \) and \( \varepsilon < d \) and divide the integral into two parts:

$$
\int_0^\infty \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{F(u + a(u)/\varepsilon)} dz \quad (5.8)
$$

$$
= \int_0^M \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{F(u + a(u)/\varepsilon)} dz + \int_M^\infty \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{F(u + a(u)/\varepsilon)} dz.
$$

To the first term we apply the dominated convergence theorem, the second term becomes arbitrarily small for large \( M \).

**Term 1.** For \( z > 0 \)

$$
P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right] \leq d \cdot P \left[ Y_1(u) < (1 - z)/d \right] \leq d \cdot F(u + a(u)/d). \quad (5.9)
$$

Hence for all large \( u \) we have that

$$
\frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right]}{F(u + a(u)/\varepsilon)} \leq d \cdot \frac{F(u + a(u)/d)}{F(u + a(u)/\varepsilon)} \leq (d + 1) \exp \{1/d - 1/\varepsilon\}. \quad (5.10)
$$
Henceforth we have found an uniform upper bound, which implies that our function is $L^1$ on $[0, M]$. There remains to prove pointwise convergence in $z$ so that we can apply the dominated convergence theorem to the first term on the right-hand side of (5.8).

We introduce the events $\{Y_i(u) < 1/\varepsilon\}$.

$$
P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \right] = P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \bigg| Y_1(u) < 1/\varepsilon \right] + P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z, Y_1(u) \geq 1/\varepsilon \right] / F(u + a(u)/\varepsilon).
$$  

(5.11)

Lemma 5.3 of [2] states:

$$
\lim_{u \to \infty} P(X_i \leq u + x_i a(u), i = 1, \ldots, d | X_1 \leq u + a(u)/\varepsilon) = e^{-1/\varepsilon} \left( \sum_{i=1}^{d} e^{-\alpha x_i} \right)^{-1/\alpha}.
$$  

(5.12)

When we apply this to the first term on the right-hand side of (5.11), we find

$$
e^{-1/\varepsilon} f_{1,\varepsilon}(z) \overset{\text{def}}{=} \lim_{u \to \infty} P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z \bigg| Y_1(u) < 1/\varepsilon \right]
$$  

(5.13)

To the second term on the right-hand side of (5.11) we give an estimate which is similar to (5.12) in [2].

$$
\lim_{u \to \infty} \sup \frac{P \left[ \sum_{i=1}^{d} Y_i(u) < 1 - z, Y_1(u) \geq 1/\varepsilon \right]}{F(u + a(u)/\varepsilon)} \
\leq \lim_{u \to \infty} \sup \frac{(d-1) \cdot P[Y_2(u) < (1-z)/d, Y_1(u) \geq 1/\varepsilon]}{F(u + a(u)/\varepsilon)} \
\leq \lim_{u \to \infty} \frac{(d-1) \cdot F(u + a(u)(1-z)/d)}{F(u + a(u)/\varepsilon)} \
\times \left( 1 - \frac{\phi^{-1} \left( \phi(F(u + a(u)(1-z)/d)) + \phi(F(u + a(u)/\varepsilon)) \right)}{F(u + a(u)(1-z)/d)} \right) \
= (d-1)e^{-1/\varepsilon} \left[ e^{(1-z)/d} - \left( e^{-\alpha(1-z)/d} + e^{-\alpha/\varepsilon} \right)^{-1/\alpha} \right] \
\leq (d-1)e^{-1/\varepsilon} e^{(1-z)/d} \left[ 1 - \left( 1 + e^{-\alpha/\varepsilon + \alpha/d} \right)^{-1/\alpha} \right] \overset{\text{def}}{=} e^{-1/\varepsilon} f_{2,\varepsilon}(z).
$$  

(5.14)
Now we come to the last term on the right-hand side of (5.8). For $M > 1$,

$$\int_M^\infty P \left[ \sum_{i=1}^d Y_i(u) < 1 - z \right] dF(u + a(u)/\varepsilon)$$

$$\leq d \int_M^\infty F(u + (1 - z)a(u)/d) dF(u + a(u)/\varepsilon)$$

$$= d F \left( u - \frac{M-1}{d} a(u) \right) \int_{(M-1)/d}^\infty F \left( u - \frac{M-1}{d} a(u) \right) dx$$

$$= d F \left( u - \frac{M-1}{d} a(u) \right) \int_{(M-1)/d}^\infty P \left[ Y_1(u) < -x \right] Y_1(u) < -(M-1)/d dx$$

$$= d F \left( u - \frac{M-1}{d} a(u) \right) \int_{(M-1)/d}^\infty E \left[ -Y_1(u) \right] = E \left[ -Y_1(u) \right] = -Y_1(u) \leq \frac{M-1}{d}.$$  

(5.15)

Next we consider the expectation in the expression above:

$$E \left[ -Y_1(u) \right] = \frac{M-1}{d}$$

$$= \limsup_{u \to c} \frac{1}{a(u)} E \left[ -X_1 - v_M(u) \right] = \limsup_{u \to c} \frac{a(-v_M(u))}{a(u)} = 1,$$

(5.16)

where $v_M(u) = (M-1)a(u)/d - u$. Now we may use the that we are working with marginals which have Gumbel type, henceforth (see [5], formula (3.3.34))

$$\limsup_{u \to c} \frac{1}{a(u)} E \left[ -X_1 - v_M(u) \right] = \limsup_{u \to c} \frac{a(-v_M(u))}{a(u)} = \limsup_{u \to c} \frac{a(-v_M(u) + u)}{a(u)} = 1,$$

(5.17)

where in the last step we have used that $\lim_{u \to c} a'(u) = 0$ (see [5], Theorem 3.3.26 and formula (3.3.31)).

Hence we find for all $\varepsilon < d$ and all $M > 1$ (see (5.13), (5.14), (5.15), (5.17))

$$\limsup_{u \to c} \int_0^\infty P \left[ \sum_{i=1}^d Y_i(u) < 1 - z \right] dF(u + a(u)/\varepsilon)$$

$$\leq e^{-1/\varepsilon} \left( \int_0^M f_{1,\varepsilon}(x) + f_{2,\varepsilon}(x) dx + d e^{-(M-1)/d} (M/d + 1) \right).$$

(5.18)

The function $f_{1,\varepsilon}$ is increasing in $\varepsilon$. Moreover

$$\int_0^M f_{2,\varepsilon}(x) dx = (d-1) e^{1/d} \left[ 1 - \left( 1 + e^{-\alpha/\varepsilon + \alpha/d} \right)^{-1/\alpha} \right].$$

(5.19)

which converges to 0 for $\varepsilon \to 0$. Hence we find (see (5.6), (5.7), (5.18))
\[
\limsup_{M \to \infty} \limsup_{\epsilon \to 0} \left( 1 - E \left[ \sum_{i=1}^{d} Y_1(u) \right] \sum_{i=1}^{d} Y_1(u) \leq 1 \right) \tag{5.20}
\]

\[
\leq \frac{1}{q_d^2(\alpha)} \int_0^\infty \int_{\sum_{i=1}^{d} x_i < 1-z} \left[ \frac{dt}{dx_1 \cdots dx_d} \left( \sum_{i=1}^{d} e^{-\alpha x_i} \right)^{-1/\alpha} \right] dx_1 \cdots dx_d dz
\]

\[
= \frac{1}{q_d^2(\alpha)} \int \left( 1 - \sum_{i=1}^{d} x_i \right) \left[ \frac{dt}{dx_1 \cdots dx_d} \left( \sum_{i=1}^{d} e^{-x_i \alpha} \right)^{-1/\alpha} \right] dx_1 \cdots dx_d
\]

\[
= 1 - \frac{1}{q_d^2(\alpha)} \int \left( \sum_{i=1}^{d} x_i \right) \left[ \frac{dt}{dx_1 \cdots dx_d} \left( \sum_{i=1}^{d} e^{-x_i \alpha} \right)^{-1/\alpha} \right] dx_1 \cdots dx_d.
\]

Exchanging the two integration finishes to proof of the upper bound. The same lower bound is found only considering the term coming from \( f_{1,\epsilon} \). This finishes the proof of (3.8).

Now, for the case \( d = 2 \) we find

\[
c_d^2(\alpha) = \frac{2}{q_d^2(\alpha)} \int_{x_1 + x_2 \leq 1} x_1 \left[ \frac{d^2}{dx_1 dx_2} \left( \sum_{i=1}^{2} e^{-x_i \alpha} \right)^{-1/\alpha} \right] dx_1 dx_2
\]

\[
= \frac{2}{q_d^2(\alpha)} \int_{-\infty}^{\infty} x e^{-\alpha x} e^{-\alpha(1-x)} e^{-1/\alpha-1} dx
\]

\[
= \frac{2}{q_d^2(\alpha)} \int_{-\infty}^{\infty} x e^x \left( 1 + e^{-\alpha(1-2\alpha)} \right)^{-1/\alpha-1} dx \tag{5.21}
\]

\[
y = e^{-\alpha(1-2\alpha)}
\]

\[
= \frac{2}{q_d^2(\alpha)} \left( \int_{-\infty}^{\infty} (1 + \log(y) y^{-1/2} (1 + y^\alpha)^{-1/\alpha-1} dx \right)
\]

\[
= \frac{2}{q_d^2(\alpha)} \left( \frac{e^{1/2}}{4} E \left[ Y_\alpha^{-1/2} (1 + \log Y_\alpha) \right] \right).
\]

Recall (5.39) from [2]:

\[
q_d^2(\alpha) = \frac{e^{1/2}}{2} E \left[ Y_\alpha^{-1/2} \right], \tag{5.22}
\]

and find:

\[
c_d^2(\alpha) = 1 + \frac{E \left[ Y_\alpha^{1/2} \log Y_\alpha \right]}{E \left[ Y_\alpha^{-1/2} \right]} . \tag{5.23}
\]

This proves the left equality of (3.11); for a proof of the right equality we introduce
\[ \gamma < 0 \text{ and generalize:} \]

\[
\frac{E(Y_\alpha^{-1/2} \log(Y_\alpha))}{E(Y_\alpha^{-1/2})} = \frac{\int_0^\infty y^n \log(y)(1 + y^n)^{-\frac{1}{n} - 1} dy}{\int_0^\infty y^n (1 + y^n)^{-\frac{1}{n} - 1} dy} = \frac{d}{d\gamma} \log \left[ \int_0^\infty y^n (1 + y^n)^{-\frac{1}{n} - 1} dy \right]
\]

\[
\frac{E(Y_\alpha^{-1/2} \log(Y_\alpha))}{E(Y_\alpha^{-1/2})} = \frac{d}{d\gamma} \log \left[ \frac{1}{\alpha} \int_0^\infty (1 + z)^{-\frac{1}{\alpha} - 1} z^{\frac{2}{\alpha} - 1} dz \right]
\]

\[
\frac{E(Y_\alpha^{-1/2} \log(Y_\alpha))}{E(Y_\alpha^{-1/2})} = \frac{d}{d\gamma} \log \left[ \frac{1}{\alpha} \int_0^1 (1 - s)^{\frac{2}{\alpha} - 1} ds \right]
\]

Now we take \( \gamma = -1/2 \) and find:

\[
\frac{E(Y_\alpha^{-1/2} \log(Y_\alpha))}{E(Y_\alpha^{-1/2})} = -2 ,
\]

which, together with (5.23) finishes proof of Theorem 3.4. \( \blacksquare \)

References


