DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN The Netherlands

THE REACTIVE BARGAINING SET IN EXCHANGE ECONOMIES WITH INDIVISIBILITIES AND MONEY

Marc Meertens, Jos Potters, Hans Reijnierse

Report No. 0403 (February 2004)

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NIJMEGEN
Toernooiveld
6525 ED Nijmegen
The Netherlands

THE REACTIVE BARGAINING SET IN EXCHANGE ECONOMIES WITH INDIVISIBILITIES AND MONEY

M.A. MEERTENS [1], J.A.M. POTTERS [2], J.H. REIJNIERSE [3]

ABSTRACT. The paper investigates the reactive bargaining set, a solution concept for TU-games introduced by Granot (1994), in economies in which agents exchange indivisible goods and one perfectly divisible good (money). Under the assumptions that the preferences of the agents are *quasi-linear* and the initial endowments satisfy the *Total Abundance* condition, which is an abundance condition on the money supply, it is shown that the reactive bargaining set is non-empty. Furthermore, we prove that, in an exchange economy, this bargaining set and the (strong) core coincide if and only if the reactive bargaining set and the core of the underlying TU-game coincide.
KEYWORDS: Reactive bargaining set, Indivisible goods.
JEL-CLASSIFICATION NUMBERS: C71, C78.

INTRODUCTION

In the theory of transferable utility (TU) games several solution concepts can be found in the literature, which are based on the idea of certain bargaining possibilities of the players. The origin of this idea can be found in a paper by Aumann and Maschler (1964) in which they introduced several type of bargaining sets. One variant, which was explicitly studied in Davis and Maschler (1967) turned out to be the most fundamental. More recently, in 1994, Granot introduced the *reactive bargaining set*, which is a refinement of this 'classic' bargaining set (see also Granot and Maschler (1997)). These two bargaining sets have in common that player $\hat{\jmath}$ has to be able to counter an objection (if there is one) of player $\hat{\imath}$. However, the difference between them is in the amount of information player $\hat{\jmath}$ has about the objection of player $\hat{\imath}$. In the bargaining set, player $\hat{\jmath}$ has to give the counter-objection *after* player $\hat{\imath}$ gives the complete objection. But in the reactive bargaining set, player $\hat{\jmath}$ has already to give the coalition he will use to counter *before* player $\hat{\imath}$ has to give the complete objection. So, *only* knowing that player $\hat{\imath}$ raises an objection against him, player $\hat{\jmath}$ has to give the coalition he will use to counter. Granot (1994) proved that the reactive bargaining set of a TU-game is non-empty, provided that the imputation set is non-empty.

[1] Department of Mathematics University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands, e-mail: meertens@math.kun.nl.
[2] Department of Mathematics University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands, e-mail: potters@math.kun.nl.
[3] Department of Economics Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands, e-mail: J.H.Reijnierse@uvt.nl.

In this paper we extend the reactive bargaining set to exchange economies in which agents can buy and sell indivisible goods and in which all payments are made in units of one infinitely divisible good that, following standard use, is referred to as *money*. In these economies, which find their origins in Debreu (1959), we assume that agents have complete, transitive and continuous preferences on the set of consumption bundles under the four additional assumptions that 'more money is better than less money', 'large amounts of money can change preferences', 'the objects are desired' and 'the marginal utility for money is constant'. These preference relations can be represented by utility functions, which are *quasi-linear in money*, i.e. the preferences of the agents can be represented by *reservation values* for subsets of the available indivisible goods.

As in the theory of TU-games the reactive bargaining set of an exchange economy contains the (strong) core whenever the latter is non-empty. Given an abundance condition on the money supply, which is less demanding than the abundance conditions that can be found in Beviá *et al.* (1999) or in Bikhchandani and Mamer (1997), we prove in section 2 that the *reactive bargaining set* is non-empty, even if the (strong) core is empty. However, in case of an exchange economy with more general utilities, i.e which do *not* satisfy the condition of constant marginal utility for money, the reactive bargaining set may be empty, as we demonstrate by an example. Furthermore, it turns out that for a reallocation in the reactive bargaining set, each agent receives a bundle which he appreciates at least as much as his initial endowment. This result does *not* longer hold for the 'classic' bargaining set of Aumann and Maschler (1964). By assigning to every exchange economy a non-negative superadditive TU-game, we prove in section 3 that the reactive bargaining set and the (strong) core of an economy coincide if and only if the reactive bargaining set and the core of this TU-game coincide. In the literature several classes of non-negative superadditive TU-games can be found for which the reactive bargaining set and the core coincide (see Solymosi (1999) for an overview). In section 3 we prove that *every* non-negative superadditive TU-game gives rise to an exchange economy, and therefore each of these classes of TU-games generates exchange economies for which the reactive bargaining set and the core coincide.

Let us start, by giving the exchange economies investigated in this paper along with same basic definitions and results.

1. Preliminaries

The *exchange economies with indivisible goods and money* $\mathcal{E}$ considered in this paper, have the following features:

- There is a finite set of *agents* N, with $n := |N| \geq 2$,
- There is a finite set of indivisible goods Q, with $q := |Q| \geq 1$,

- Each agent $i \in N$ has an *initial endowment* (A_i, m_i), in which $A_i \subseteq Q$ denotes the set of indivisible goods initialy held by agent i and $m_i > 0$ denotes the strictly positive amount of money agent i has in the beginning. We assume that $\{A_i\}_{i \in N}$ is a N-distribution of Q, i.e. $\bigcup_{i \in N} A_i = Q$ and $A_i \cap A_j = \emptyset$ whenever $i \neq j$. We allow, however, $A_i = \emptyset$ for some agents $i \in N$,
- Each agent $i \in N$ has a preference relation $\preceq_i$ on the set $2^Q \times \mathbb{R}_+$ which is assumed to satisfy the following conditions:
  - $\mathbf{C_1}$: $\preceq_i$ is a *complete, transitive* binary relation on $2^Q \times \mathbb{R}_+$,
  - $\mathbf{C_2}$: for all consumption bundles (B, x) and (C, y) with $(B, x) \succ_i (C, y)$, there exists a positive number δ such that $(B, x - \delta) \succ_i (C, y)$ and $(B, x) \succ_i (C, y + \delta)$ (*continuity in money*),
  - $\mathbf{C_3}$: $(B, x) \succ_i (B, y)$ if $x > y$, for all $B \subseteq Q$ (*strict monotonicity in money*),
  - $\mathbf{C_4}$: $(B, 0) \succeq_i (\emptyset, 0)$ for all $B \subseteq Q$ (*non-negative marginal value in goods*),
  - $\mathbf{C_5}$: for all consumption bundles (B, x) and $y \in \mathbb{R}_+$ with $(B, x) \succ_i (\emptyset, y)$, there exists a positive number Δ such that $(B, x) \preceq_i (\emptyset, y + \Delta)$ (*archimedean property*),
  - $\mathbf{C_6}$: for all $B \subseteq Q$ and $x \in \mathbb{R}_+$ with $(B, 0) \sim_i (\emptyset, x)$ it holds that $(B, d) \sim_i (\emptyset, x + d)$ for all $d \in \mathbb{R}_+$ (*marginal utility of money is constant*).

The set of all preference relations satisfying the conditions C_1–C_6 is denoted by $\mathcal{R}$. Conditions C_3 states that 'more money is better than less money', condition C_4 states that 'the indivisible goods are desired', condition C_5 states that 'large amounts of money can change preferences' and condition C_6 states that 'money can be used to transfer utility from one agent to another', since the marginal utility of money does not depend on the agent nor its wealth. We start by proving that each preference relation $\preceq \in \mathcal{R}$ can be presented by a quasi-linear utility function.

Proposition 1.1. *For each preference relation $\preceq \in \mathcal{R}$ there exists a unique map $V : 2^Q \to \mathbb{R}_+$ with the properties:*

(i) $V(\emptyset) = 0$ *(ii)* $(B, x) \preceq (C, y)$ *if and only if* $V(B) + x \leq V(C) + y$.

Proof:
We prove that,

> For every $B \subseteq Q$ and every $x \in \mathbb{R}_+$ there exists exactly one real number $u_B(x) \in \mathbb{R}_+$ such that $(B, x) \sim (\emptyset, u_B(x))$.

Let $B \subseteq Q$ and $x \in \mathbb{R}_+$. Define, $\mathcal{K} := \{y \in \mathbb{R}_+ \mid (B, x) \preceq (\emptyset, y)\}$ and define $\mathcal{G} := \{y \in \mathbb{R}_+ \mid (B, x) \succeq (\emptyset, y)\}$. Then $\mathcal{K}$ is non-empty, according to C_5 and $0 \in \mathcal{G}$ according to C_3 and C_4. Moreover, both sets are closed according to C_2 and by completeness of $\preceq$ we have $\mathcal{K} \cup \mathcal{G} = \mathbb{R}_+$. Hence $\mathcal{K} \cap \mathcal{G} \neq \emptyset$.

If $y_1, y_2 \in \mathcal{K} \cap \mathcal{G}$, then $(\emptyset, y_1) \sim (B, x) \sim (\emptyset, y_2)$. This yields, by transitivity of $\preceq$, $(\emptyset, y_1) \sim (\emptyset, y_2)$. And therefore, by C_3, we have that $y_1 = y_2$. Hence, there exists an unique number $y \in \mathbb{R}_+$ such that $(B,x) \sim (\emptyset, y)$.

Define $V(B) := u_B(0)$ for all $B \subseteq Q$, then $V : 2^Q \to \mathbb{R}_+$ is uniquely determined and $V(\emptyset) = 0$. Furthermore, by C_6, we have that $u_B(x) = u_B(0+x) = u_B(0) + x$ for all $B \subseteq Q$ and $x \in \mathbb{R}_+$. Hence, $u_B(x) = V(B) + x$ for all $B \subseteq Q$ and $x \in \mathbb{R}_+$. From this it immediately follows that the map $V : 2^Q \to \mathbb{R}_+$ also satisfies property (ii) mentioned in the proposition. □

Remark. Conversely, if $\preceq$ is a binary relation on $2^Q \times \mathbb{R}_+$ that can be represented by the utility function, $U(B,x) := V(B) + x$, in which $V : 2^Q \to \mathbb{R}_+$ with $V(\emptyset) = 0$, then $\preceq$ belongs to $\mathcal{R}$. ◦

The values $V_i \in \mathbb{R}_+$ are called the *reservation values* of agent $i \in N$. Hence, an exchange economy $\mathcal{E}$ is characterized by the tuple $\langle N, Q, (A_i, m_i, V_i)_{i \in N} \rangle$.

Definition. Let $S \subseteq N$ be a coalition. A *S-redistribution* is a set $\{B_i\}_{i \in S}$ with $\bigcup_{i \in S} B_i = \bigcup_{i \in S} A_i$ and $B_i \cap B_j = \emptyset$ whenever $i \neq j$. So, we allow $B_i = \emptyset$ for some agents $i \in S$. Let $\{B_i\}_{i \in S}$ be a S-redistribution and $x \in \mathbb{R}^S_+$ such that $\sum_{i \in S} x_i = \sum_{i \in S} m_i$ then the set $\{(B_i, x_i)\}_{i \in S}$ is called a *S-reallocation.* ◦

If $\{(B_i, x_i)\}_{i \in N}$ is a N-reallocation then a S-reallocation $\{(C_i, y_i)\}_{i \in S}$ is called a *weak improvement* upon $\{(B_i, x_i)\}_{i \in N}$ if $V_i(C_i) + y_i \geq V_i(B_i) + x_i$ for all $i \in S$ and $V_{\hat{\imath}}(C_{\hat{\imath}}) + y_{\hat{\imath}} > V_{\hat{\imath}}(B_{\hat{\imath}}) + x_{\hat{\imath}}$ for at least one agent $\hat{\imath} \in S$. A S-reallocation $\{(C_i, y_i)\}_{i \in S}$ is called a *strong improvement* upon $\{(B_i, x_i)\}_{i \in N}$ if $V_i(C_i) + y_i > V_i(B_i) + x_i$ for all $i \in S$. As usual, a N-reallocation $\{(B_i, x_i)\}_{i \in N}$ is called *Pareto efficient/weakly Pareto efficient* if coalition N has *no* weak improvement/strong improvement upon $\{(B_i, x_i)\}_{i \in N}$. A N-reallocation $\{(B_i, x_i)\}_{i \in N}$ is a *strong core reallocation/core reallocation* if *no* coalition S has a weak improvement/strong improvement upon $\{(B_i, x_i)\}_{i \in N}$.

Given an exchange economy $\mathcal{E}$ we define a TU-game $(N, v_{\mathcal{E}})$ with the value for coalition $S \subseteq N$ as follows,

$$v_{\mathcal{E}}(S) \;:=\; \max\{\sum_{i \in S} V_i(C_i) \mid \{C_i\}_{i \in S} \text{ a } S\text{-redistribution}\},$$

i.e. the *maximum social welfare* in the sub-economy in which only the actions of agents in coalition S are considered. Observe that the TU-game $(N, v_{\mathcal{E}})$ is *superadditive*, i.e. $v_{\mathcal{E}}(S \cup T) \geq v_{\mathcal{E}}(S) + v_{\mathcal{E}}(T)$ whenever $S \cap T = \emptyset$. Furthermore, since $V_i(C) \geq V_i(\emptyset) = 0$ for all $i \in N$ and $C \subseteq Q$, it is also *non-negative.*

Next, we give an abundance condition on the money supply in the economy $\mathcal{E}$, i.e. a condition on the amounts of money m_i initialy held by the agents $i \in N$.

Definition. An exchange economy $\mathcal{E}$ satisfies the *Total Abundance* (TA) condition if every coalition $S \subseteq N$ has a S-redistribution $\{C_i\}_{i \in S}$ such that,

$$\sum_{i \in S} V_i(C_i) \;=\; v_{\mathcal{E}}(S) \quad \text{and} \quad V_i(C_i) \;\leq\; V_i(A_i) + m_i \text{ for all } i \in S.$$

Such a S-redistribution $\{C_i\}_{i\in S}$ is said to *satisfy the TA-condition for coalition S.* ◦

Remark. The TA-condition is a weaker form than abundance conditions that can be found in the literature, namely $V_i(C) \leq V_i(A_i) + m_i$ or even $V_i(C) \leq m_i$ for all $i \in N$ and $C \subseteq Q$ (see for instance Bikhchandani and Mamer (1997) and Beviá *et al.* (1999)). ◦

Lemma 1.2. *Let $\mathcal{E}$ be an exchange economy which satisfies the TA-condition. If the N-reallocation $\{(B_i, x_i)\}_{i\in N}$ is a core reallocation, then it is also a strong core reallocation.*

Proof: Let $\{(B_i, x_i)\}_{i\in N}$ be a N-reallocation which is not element of the strong core, i.e. there exists a S-reallocation $\{(C_i, y_i)\}_{i\in S}$ such that,

$$\begin{aligned} V_i(C_i) + y_i &\geq V_i(B_i) + x_i \quad \text{for all } i \in S, \\ V_{\hat{\imath}}(C_{\hat{\imath}}) + y_{\hat{\imath}} &> V_{\hat{\imath}}(B_{\hat{\imath}}) + x_{\hat{\imath}} \quad \text{for at least one } \hat{\imath} \in S. \end{aligned}$$

Let $\{C'_i\}_{i\in S}$ be a S-redistribution which satisfies the TA-condition for coalition S and define, $y'_i := V_i(C_i) + y_i - V_i(C'_i) + \frac{1}{|S|}\sum_{i\in S}[V_i(C'_i) - V_i(C_i)]$ for all $i \in S$.

Observe that $\sum_{i\in S} y'_i = \sum_{i\in S} m_i$ and $V_i(C'_i) + y'_i \geq V_i(C_i) + y_i$ for all $i \in S$. Furthermore, we may assume that $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ for all $i \in S$. Otherwise, the coalition $\{i\}$ has a strong improvement. Therefore,

$$\begin{aligned} y'_i &\geq V_i(B_i) + x_i - V_i(C'_i) \geq V_i(A_i) + m_i - V_i(C'_i) \geq 0 \quad \text{for all } i \in S, \\ y'_{\hat{\imath}} &> V_{\hat{\imath}}(B_{\hat{\imath}}) + x_{\hat{\imath}} - V_{\hat{\imath}}(C'_{\hat{\imath}}) \geq V_{\hat{\imath}}(A_{\hat{\imath}}) + m_{\hat{\imath}} - V_{\hat{\imath}}(C'_{\hat{\imath}}) \geq 0. \end{aligned}$$

Hence, there exists a S-reallocation $\{(C'_i, y'_i)\}_{i\in S}$ which is a weak improvement upon $\{(B_i, x_i)\}_{i\in N}$ such that $y'_{\hat{\imath}} > 0$ for the agent $\hat{\imath} \in S$ with $V_{\hat{\imath}}(C'_{\hat{\imath}}) + y'_{\hat{\imath}} > V_{\hat{\imath}}(B_{\hat{\imath}}) + x_{\hat{\imath}}$. By using the strict monotonicity and continuity in money, it is a straightforward exercise to transform this weak improvement $\{(C'_i, y'_i)\}_{i\in S}$ into a strong improvement upon $\{(B_i, x_i)\}_{i\in N}$. Hence, $\{(B_i, x_i)\}_{i\in N}$ is also not contained in the core. □

So, for exchange economies which satisfy the TA-condition the difference between core and strong core disappears. We write $\mathcal{C}(\mathcal{E})$ for the set of (strong) core reallocations of such an economy $\mathcal{E}$. However, for these economies the difference between Pareto efficiency and weak Pareto efficiency remains, as the following example demonstrates. We write $\mathcal{P}(\mathcal{E})$ for the set of Pareto efficient N-reallocations.

Example. Let $\mathcal{E} := \langle N, Q, (A_i, m_i, V_i)_{i\in N}\rangle$ be an exchange economy with $N := \{1, 2\}$, $Q := \{\alpha, \beta\}$, $\{(A_i, m_i)\}_{i\in N} := \{(\alpha, 1), (\beta, 1)\}$ and the reservation values V_i for $i = 1, 2$ given by,

	α	β	$\alpha\beta$
agent 1	7	8	15
agent 2	7	9	16

The economy $\mathcal{E}$ satisfies the TA-condition, since $v_{\mathcal{E}}(12) = V_1(A_1) + V_2(A_2)$. However, the N-reallocation $\{(\alpha, 2), (\beta, 0)\}$ is a weak improvement upon the N-reallocation

$\{(\beta,1)\,,\,(\alpha,1)\}$, but it cannot be transformed into a strong improvement. Hence, $\{(\beta,1)\,,\,(\alpha,1)\} \notin \mathcal{P}(\mathcal{E})$, but nevertheless it is weakly Pareto efficient. ◦

The following proposition illustrates that every core reallocation of an economy $\mathcal{E}$, which satisfies the TA-condition, gives rise to a core element of the TU-game $(N, v_{\mathcal{E}})$ and vice versa.

Proposition 1.3. *Let $\mathcal{E}$ be an exchange economy which satisfies the TA-condition.*

(i) If $\{(B_i, x_i)\}_{i\in N} \in \mathcal{C}(\mathcal{E})$, then $(V_i(B_i) + x_i - m_i)_{i\in N} \in \mathcal{C}(v_{\mathcal{E}})$,

(ii) If $X \in \mathcal{C}(v_{\mathcal{E}})$, then there exists a N-redistribution $\{B_i\}_{i\in N}$ such that $\{(B_i\,,\, X_i + m_i - V_i(B_i))\}_{i\in N} \in \mathcal{C}(\mathcal{E})$.

Proof: (i) Let $\{(B_i, x_i)\}_{i\in N} \in \mathcal{C}(\mathcal{E})$ and define $X_i := V_i(B_i) + x_i - m_i$ for all $i \in N$. Suppose $X(S) < v_{\mathcal{E}}(S)$ for some coalition $S \subseteq N$ (observe that also $S = N$ may be one of the possibilities). Let $\{C_i\}_{i\in S}$ be a S-redistribution which satisfies the TA-condition for coalition S. Then,

$$\sum_{i\in S}[V_i(B_i) + x_i] \;=\; X(S) + \sum_{i\in S} m_i \;<\; v_{\mathcal{E}}(S) + \sum_{i\in S} m_i \;=\; \sum_{i\in S}[V_i(C_i) + m_i]$$

So, for all $i \in S$ there is a number $y_i \in \mathbb{R}$ such that $V_i(C_i) + y_i = V_i(B_i) + x_i$. Take $i \in S$, then by the core-condition for coalition $\{i\}$ we have,

$$V_i(B_i) + x_i \;\geq\; V_i(A_i) + m_i.$$

Therefore, $y_i = V_i(B_i) + x_i - V_i(C_i) \;\geq\; V_i(A_i) + m_i - V_i(C_i) \;\geq\; 0$ for all $i \in S$. Furthermore, $\sum_{i\in S} y_i < \sum_{i\in S} m_i$. Define $\varepsilon := \sum_{i\in S}[m_i - y_i] > 0$, then the S-reallocation $\{(C_i, y_i + \frac{\varepsilon}{|S|})\}_{i\in S}$ is a strong improvement upon $\{(B_i, x_i)\}_{i\in N}$. Contradiction.

(ii) Let $X \in \mathcal{C}(v_{\mathcal{E}})$. Take a N-redistribution $\{B_i\}_{i\in N}$ which satisfies the TA-condition for coalition N and define $x_i := X_i + m_i - V_i(B_i)$ for all $i \in N$. Observe that $\sum\limits_{i\in N} x_i = \sum\limits_{i\in N} m_i$ and $x_i \geq V_i(A_i) + m_i - V_i(B_i) \geq 0$ for all $i \in N$.

Suppose the coalition $S \subset N$ has an improvement, i.e. there exists a S-reallocation $\{(C_i, y_i)\}_{i\in N}$ such that, $V_i(C_i) + y_i > V_i(B_i) + x_i$ for all $i \in S$. This means that $V_i(C_i) + y_i > X_i + m_i$ for all $i \in S$. Because $\sum_{i\in S} y_i = \sum_{i\in S} m_i$, this yields, $v_{\mathcal{E}}(S) \geq \sum\limits_{i\in S} V_i(C_i) > X(S)$. But $X \in \mathcal{C}(v_{\mathcal{E}})$. Contradiction. □

So, if an economy $\mathcal{E}$ satisfies the TA-condition and the corresponding TU-game $(N, v_{\mathcal{E}})$ is balanced, then the core $\mathcal{C}(\mathcal{E})$ is non-empty. In the next section, we give another set of N-reallocations which can been seen as a substitute of the core in the case the latter is empty, that is, the *reactive bargaining set.*

2. The reactive bargaining set in economies

The reactive bargaining set is a solution concept for TU-games which was introduced by Granot (1994). It is a refinement of the bargaining set for TU-games introduced by Aumann and Maschler (1964) (see also Davis and Maschler (1967)). This solution concept can be extended to economies in an obvious way. To do so, we first need the definition of an *objection* and of a *counter-objection*. For all $i \neq j \in N$ we write $\Gamma_{ij} := \{S \subset N \mid i \in S \subset N \setminus j\}$.

Definition. An *objection* of agent $\hat{\imath}$ against agent $\hat{\jmath}$ with respect to a N-realloca-tion $\{(B_i, x_i)\}_{i \in N}$ is pair $(S, \{(C_i, y_i)\}_{i \in S})$ with $S \in \Gamma_{\hat{\imath}\hat{\jmath}}$ and $\{(C_i, y_i)\}_{i \in S}$ a S-reallocation such that $U_i(C_i, y_i) > U_i(B_i, x_i)$ for all $i \in S$. ○

Given an objection of agent $\hat{\imath}$ against agent $\hat{\jmath}$, we now can give the definition of a counter-objection.

Definition. Given an objection $(S, \{(C_i, y_i)\}_{i \in S})$ of agent $\hat{\imath}$ against agent $\hat{\jmath}$ with respect to a N-reallocation $\{(B_i, x_i)\}_{i \in N}$, a *counter-objection* of agent $\hat{\jmath}$ against agent $\hat{\imath}$ is a pair $(T, \{(D_i, z_i)\}_{i \in T})$ with $T \in \Gamma_{\hat{\jmath}\hat{\imath}}$ and $\{(D_i, z_i)\}_{i \in T}$ a T-reallocation such that,

$$\begin{aligned} U_i(D_i, z_i) &\geq U_i(B_i, x_i) \quad \text{for all } i \in T \setminus S, \\ U_i(D_i, z_i) &\geq U_i(C_i, y_i) \quad \text{for all } i \in T \cap S. \end{aligned}$$

An objection is called *justified*, if such a counter-objection does *not* exist. ○

Now we can give the formal definition of the *reactive bargaining set* in an exchange economy $\mathcal{E}$.

Definition. Let $\mathcal{E}$ be an exchange economy. A Pareto efficient N-reallocation $\{(B_i, x_i)\}_{i \in N} \in \mathcal{P}(\mathcal{E})$ is an element of the *reactive bargaining set* $\mathcal{M}_r(\mathcal{E})$ if for all agents $\hat{\imath}$ and $\hat{\jmath}$, there exists a coalition $T \in \Gamma_{\hat{\jmath}\hat{\imath}}$ such that for every objection $(S, \{(C_i, y_i)\}_{i \in S})$ of agent $\hat{\imath}$ against $\hat{\jmath}$, there exists a T-reallocation $\{(D_i, z_i)\}_{i \in T}$ such that the pair $(T, \{(D_i, z_i)\}_{i \in T})$ is a counter-objection. ○

So, given that agent $\hat{\imath}$ raises an objection against agent $\hat{\jmath}$, before knowing *how* the objection of agent $\hat{\imath}$ completely looks like, agent $\hat{\jmath}$ has to give the coalition he will use to counter this objection. Observe, that in this definition we included that in order for a N-reallocation to be in the reactive bargaining set, it should be Pareto efficient. If one would choose to leave out this assumption, then the initial endowments set $\{(A_i, m_i)\}_{i \in N}$ is clearly an element of the reactive bargaining set.

Observe that if a N-reallocation is contained in the core $\mathcal{C}(\mathcal{E})$ no agent can raise an objection and therefore $\mathcal{C}(\mathcal{E}) \subseteq \mathcal{M}_r(\mathcal{E})$, whenever the core is non-empty. Next, we prove that for exchange economies satisfying the TA-condition the reactive bargaining set $\mathcal{M}_r(\mathcal{E})$ is non-empty, even if the core is empty.

Theorem 2.1. *Let $\mathcal{E} := \langle N, Q, (A_i, m_i, V_i)_{i \in N} \rangle$ be an exchange economy which satisfies the TA-condition, then $\mathcal{M}_r(\mathcal{E}) \neq \emptyset$.*

Proof: Let $\mathcal{E}$ be an exchange economy which satisfies the TA-condition. The reactive bargaining set $\mathcal{M}_r(v_{\mathcal{E}})$ of the TU-game $(N, v_{\mathcal{E}})$ is non-empty (Granot (1994)). Let $X \in \mathcal{M}_r(v_{\mathcal{E}})$ and take a N-redistribution $\{B_i\}_{i \in N}$ which satisfies the TA-condition for coalition N. Define,

$$x_i \;:=\; X_i + m_i - V_i(B_i) \quad \text{for all } i \in N.$$

Observe that $\sum_{i\in N} x_i = \sum_{i\in N} m_i$ and because $X_i \geq v_{\mathcal{E}}(i) = V_i(A_i)$ for all $i \in N$ it follows that $x_i \geq 0$ for all $i \in N$. Hence, $\{(B_i, x_i)\}_{i\in N}$ is a N-reallocation. It is straightforward that this N-reallocation is Pareto efficient. Moreover, we prove that this N-reallocation is contained in the reactive bargaining set $\mathcal{M}_r(\mathcal{E})$.

Take $\hat{\imath} \neq \hat{\jmath} \in N$. Because $X \in \mathcal{M}_r(v_{\mathcal{E}})$ there exist a coalition $T \in \Gamma_{\hat{\jmath}\hat{\imath}}$ such that for every objection (S, Y) from agent $\hat{\imath}$ against agent $\hat{\jmath}$ there exist a vector $Z \in \mathbb{R}^T$ with $Z(T) = v_{\mathcal{E}}(T)$ such that (T, Z) is a counter-objection.

Let $(S, \{(C_i, y_i)\}_{i\in S})$ be an objection from agent $\hat{\imath}$ against agent $\hat{\jmath}$ with respect to the N-reallocation $\{(B_i, x_i)\}_{i\in N}$. Define, $Y_i := V_i(C_i) + y_i - m_i$ for all $i \in S$. Then,

$$Y(S) \;=\; \sum_{i\in S} V_i(C_i) + \sum_{i\in S} y_i - \sum_{i\in S} m_i \;=\; \sum_{i\in S} V_i(C_i) \;\leq\; v_{\mathcal{E}}(S).$$

Furthermore,

$$Y_i \;=\; V_i(C_i) + y_i - m_i \;>\; V_i(B_i) + x_i - m_i \;=\; X_i \quad \text{for all } i \in S.$$

So, (S, Y) is an objection from player $\hat{\imath}$ against $\hat{\jmath}$ with respect to $X \in \mathcal{M}_r(v_{\mathcal{E}})$. Hence, there exists a vector $Z \in \mathbb{R}^T$ such that (T, Z) is a counter-objection. Take a T-redistribution $\{D_i\}_{i\in T}$ which satisfies the TA-condition for coalition T and define $z_i := Z_i + m_i - V_i(D_i)$ for all $i \in T$. Clearly, $\sum_{i\in T} z_i = \sum_{i\in T} m_i$ and because $Z_i \geq X_i \geq v_{\mathcal{E}}(i) = V_i(A_i)$ for all $i \in T$ it follows that $z_i \geq 0$ for all $i \in T$. Furthermore,

$$\begin{aligned} V_i(D_i) + z_i &= Z_i + m_i \;\geq\; Y_i + m_i \;=\; V_i(C_i) + y_i \quad \text{for all } i \in T \cap S,\\ V_i(D_i) + z_i &= Z_i + m_i \;\geq\; X_i + m_i \;=\; V_i(B_i) + x_i \quad \text{for all } i \in T \setminus S. \end{aligned}$$

Hence, the pair $(T, \{(D_i, z_i)\}_{i\in T})$ is a counter-objection. □

In case of exchange economies in which the preference relations $\{\preceq_i\}_{i\in N}$ are represented by utility functions which are *not* quasi-linear, one still can define the reactive bargaining set. However, in this case it may be empty. ○

Example. Let $N := \{1, 2, 3\}$, $Q := \{\alpha_1, \alpha_2, \alpha_3\}$, $A_i := \{\alpha_i\}$ and $m_i := 12$ for $i = 1, 2, 3$. The utility functions $U_i : 2^Q \times \mathbb{R}_+ \to \mathbb{R}$ are,

$$\begin{aligned} U_1(B, x) &:= 7\cdot\lfloor \tfrac{|B|}{2} \rfloor + x,\\ U_2(B, x) &:= 6\cdot\lfloor \tfrac{|B|}{2} \rfloor + (\tfrac{1}{100}\cdot\lfloor \tfrac{|B|}{2} \rfloor + 1)\cdot x,\\ U_3(B, x) &:= 5\cdot\lfloor \tfrac{|B|}{2} \rfloor + x. \end{aligned}$$

We write $\lfloor a \rfloor$ for the integral part of $a \in \mathbb{R}$. Observe that each utility function satisfies the conditions C_1–C_6, except the utility function U_2 of agent 2, which does not satisfy condition C_6.

Suppose $\{(B_i, x_i)\}_{i\in N} \in \mathcal{M}_r(\mathcal{E})$. Since this N-reallocation is in particular Pareto efficient we may assume without loss of generality that $B_k = Q$ for some agent $k \in N$.

Claim: $x_i = x_j > 12$ *for the two remaining agents* $i, j \in N \setminus \{k\}$.

Proof: Suppose $x_i \leq 12$. If $x_j \leq x_k$, then $x_i + x_j \leq 24$ and it is easy to verify that the pair, $\big(\{i,j\}\ ,\ \{(\emptyset, x_i+\varepsilon)\,,\,(\{\alpha_i,\alpha_j\}, 24-x_i-\varepsilon)\}\big)$, with $\varepsilon > 0$ sufficiently small, is a justified objection of agent i against agent k. On the other hand, if $x_j > x_k$, then similarly $x_i + x_k < 24$ and the pair, $\big(\{i,k\}\ ,\ \{(\emptyset, x_i+\varepsilon)\,,\,(\{\alpha_i,\alpha_k\}, 24-x_i-\varepsilon)\}\big)$, with again $\varepsilon > 0$ sufficiently small, is a justified objection of agent i against agent j. Both cases yield a contradiction.

Hence, $x_i, x_j > 12$ and thus $x_k < 12$. Suppose $x_j > x_i$, then again the pair, $\big(\{i,k\}$, $\{(\emptyset, x_i+\varepsilon)\,,\,(\{\alpha_i,\alpha_k\}, 24-x_i-\varepsilon)\}\big)$, with $\varepsilon > 0$ sufficiently small, is a justified objection of agent i against agent j. Contradiction.

Because $x_i = x_j > 12$ and $\{(B_i, x_i)\}_{i\in N}$ is in particular Pareto efficient, it follows that $k = 1$, i.e. $B_1 = Q$ and thus $x_2 = x_3 > 12$.

For every $\varepsilon > 0$ sufficiently small, agent 1 can raise an objection against agent 2 in which he offers agent 3 the bundle $(\emptyset, 24-x_1-\varepsilon)$ (and keeps the bundle $(\{\alpha_1,\alpha_2\}, x_1+\varepsilon)$ for himself). Agent 2 can raise an objection against agent 1 in which he offers agent 3 the bundle $(\emptyset, 24-\frac{100}{101}\cdot(x_2-6)-\varepsilon)$ (and keeps the bundle $(\{\alpha_2,\alpha_3\}, \frac{100}{101}\cdot(x_2-6)+\varepsilon)$ for himself). There is a justified objection, unless $U_3(\emptyset, 24-x_1) = U_3(\emptyset, 24-\frac{100}{101}\cdot(x_2-6))$. This yields,

$$2\cdot x_2 - 12 \;=\; x_2 + x_3 - 12 \;=\; 24 - x_1 \;=\; 24 - \tfrac{100}{101}\cdot(x_2-6).$$

Hence, $x_2 = x_3 = 14\frac{4}{151}$ and thus $x_1 = 7\frac{143}{151}$. But in this case agent 3 can raise an objection against agent 1 in which he offers agent 2 the bundle $(\{\alpha_2,\alpha_3\}, 9\frac{147}{151}-\varepsilon)$ with $\varepsilon > 0$ sufficiently small. The best agent 1 can offer agent 2, in order to counter this objection, is the bundle $(\emptyset, 16\frac{8}{151})$. However,

$$U_2(\{\alpha_2,\alpha_3\}, 9\tfrac{147}{151}-\varepsilon) \;=\; 6 + \tfrac{101}{100}\cdot[9\tfrac{147}{151}-\varepsilon] \;>\; 16\tfrac{8}{151} \;=\; U_2(\emptyset, 16\tfrac{8}{151}).$$

So, the objection is justified and thus $\mathcal{M}_r(\mathcal{E}) = \emptyset$. ∘

3. The reactive bargaining set and the core

In theorem 2.1 we proved that an imputation in the reactive bargaining set $\mathcal{M}_r(v_\mathcal{E})$ of the TU-game $(N, v_\mathcal{E})$ gives rise to a N-reallocation in the reactive bargaining set of the exchange economy $\mathcal{E}$. The converse of this statement is also true. To prove this, we first need the following lemma, which states that every N-reallocation in the reactive bargaining set is *individual rational.*

Lemma 3.1. *Let* $\mathcal{E}$ *be an exchange economy which satisfies the TA-condition. If* $\{(B_i, x_i)\}_{i\in N} \in \mathcal{M}_r(\mathcal{E})$, *then* $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ *for all* $i \in N$.

Proof: Let $\{(B_i,x_i)\}_{i\in N}\in\mathcal{M}_r(\mathcal{E})$ and define the sets,

$$\begin{aligned} I &:= \{i\in N \mid V_i(B_i)+x_i < V_i(A_i)+m_i\},\\ J &:= \{j\in N \mid V_j(B_j)+x_j > V_j(A_j)+m_j\}. \end{aligned}$$

Suppose $I\neq\emptyset$. Because in particular $\{(B_i,x_i)\}_{i\in N}\in\mathcal{P}(\mathcal{E})$ it follows that $J\neq\emptyset$. Take $\hat{\imath}\in I$, then agent $\hat{\imath}$ is able to raise an objection against every agent in J.

A coalition $T\subseteq N\setminus\{\hat{\imath}\}$ can *improve* if there exists a T-reallocation $\{(D_i,z_i)\}_{i\in T}$ such that $V_i(D_i)+z_i\geq V_i(B_i)+x_i$ for all $i\in T$.

Claim: If a coalition T can improve, then $J\not\subseteq T$.

Proof: Suppose $J\subseteq T$ for some coalition $T\subseteq N\setminus\{\hat{\imath}\}$. There exists a T-reallocation $\{(D_i,z_i)\}_{i\in T}$ such that $V_i(D_i)+z_i\geq V_i(B_i)+x_i$ for all $i\in T$. Furthermore, $V_i(A_i)+m_i\geq V_i(B_i)+x_i$ for all $i\in N\setminus T$ and $V_i(A_i)+m_i>V_i(B_i)+x_i$ for all $i\in I$. Hence, the N-reallocation $\{\,(A_i,m_i)_{i\in N\setminus T}\,,\,(D_i,z_i)_{i\in T}\,\}$ is a weak improvement upon $\{(B_i,x_i)\}_{i\in N}$. Contradiction.

Because $\{(B_i,x_i)\}_{i\in N}\in\mathcal{M}_r(\mathcal{E})$ it follows that for every $j\in J$ there exists a coalition $T_j\in\Gamma_{j\hat{\imath}}$ which can improve. Observe that if two disjoint coalitions T and T' can improve, then also the coalition $T\cup T'$ can improve and thus, according to the claim, there exists an agent $j\in J\setminus(T\cup T')$. Hence, there exists a coalition $T_j\in\Gamma_{j\hat{\imath}}$ and an agent $j'\in J\setminus T_j$ such that $T_j\cap T_{j'}\neq\emptyset$. Without loss of generality we may assume that $j\notin T_{j'}$. We write $T:=T_j$ and $T':=T_{j'}$.

The utility levels $(u_i)_{i\in T\cap T'}$ are called *feasible* if there exists a T-reallocation $\{(D_i,z_i)\}_{i\in T}$ such that, $V_i(D_i)+z_i\geq u_i$ for all $i\in T\cap T'$ and $V_i(D_i)+z_i\geq V_i(B_i)+x_i$ for all $i\in T$ *or* there exists a T'-reallocation $\{(D_i',z_i')\}_{i\in T'}$ such that $V_i(D_i')+z_i'\geq u_i$ for all $i\in T\cap T'$ and $V_i(D_i')+z_i'\geq V_i(B_i)+x_i$ for $i\in T'$.

Since the coalition T and T' are able to improve such feasible utility levels exist. Let $(u_i^{\max})_{i\in T\cap T'}$ be the feasible utility levels such that,

$$\sum_{i\in T\cap T'} u_i^{\max} \geq \sum_{i\in T\cap T'} u_i \quad \text{for all feasible utility levels } (u_i)_{i\in T\cap T'}.$$

Say, these feasible utility levels $(u_i^{\max})_{i\in T\cap T'}$ are realized by the T-reallocation $\{(D_i^{\max},z_i^{\max})\}_{i\in T}$. Take $0<\varepsilon<\min\{m_{\hat{\imath}},V_{\hat{\imath}}(A_{\hat{\imath}})+m_{\hat{\imath}}-V_{\hat{\imath}}(B_{\hat{\imath}})-x_{\hat{\imath}}\}$, then the pair,

$$\left(\{\hat{\imath}\}\cup T\ ,\ \{(A_{\hat{\imath}},m_{\hat{\imath}}-\varepsilon)\,,\,(D_i^{\max},z_i^{\max}+\tfrac{\varepsilon}{|T|})_{i\in T}\}\right)$$

is an objection from agent $\hat{\imath}$ against agent j'. Hence, there exists a T'-reallocation $\{(D_i',z_i')\}_{i\in T'}$ such that,

$$\begin{aligned} V_i(D_i')+z_i' &\geq V_i(D_i^{\max})+z_i^{\max}+\tfrac{\varepsilon}{|T|} > u_i^{\max} && \text{for all } i\in T\cap T',\\ V_i(D_i')+z_i' &\geq V_i(B_i)+x_i && \text{for all } i\in T'\setminus T. \end{aligned}$$

But this means that $\sum_{i\in T\cap T'} V_i(D'_i) + z'_i > \sum_{i\in T\cap T'} u_i^{\max}$. Contradiction. □

In the proof of lemma 3.1 we explicitly used the fact that in the reactive bargaining set the agent $\hat{\jmath}$ has to give the coalition T, which he will use to counter, in advance. It might be of interest to point out that the result of lemma 3.1 *no* longer holds in case of the 'classic' bargaining set $\mathcal{M}(\mathcal{E})$, i.e. in case the objector has to give the complete objection *before* agent $\hat{\jmath}$ has to give this coalition T.

Example. Let $N := \{1,\ldots,4\}$, $Q := \{\alpha_1,\ldots,\alpha_4\}$, $A_i := \{\alpha_i\}$, $m_i := 7$ for all $i \in N$ and the reservation values $V_i : 2^Q \to \mathbb{R}$ for all $i \in N$ given by,

$$V_i(C) \;:=\; V(C) \;=\; \begin{cases} 9 & \text{if } |C| = 4, \\ 7 & \text{if } |C| = 3 \text{ or } (|C| = 2 \text{ and } \alpha_1 \notin C), \\ 2 & \text{if } |C| = 2 \text{ and } \alpha_1 \in C, \\ 0 & \text{else.} \end{cases}$$

Claim: $\{(\emptyset, 6\frac{1}{2}), (\emptyset, 10\frac{1}{6}), (\{\alpha_2,\alpha_3\}, 3\frac{1}{6}), (\{\alpha_1,\alpha_4\}, 8\frac{1}{6})\} \in \mathcal{M}(\mathcal{E})$.

Proof: The agents 2,3,4 are interchangeable and can be treated equally. Moreover, none of them has a justified objection against agent 1, since $V_1(A_1)+m_1 > V_1(B_1)+x_1$. It is left to the reader to check that the agents 2,3,4 do not have justified objections against one and other.

Agent 1 can only raise an objection against an agent $\hat{\jmath} \in \{2,3,4\}$ via $S = N \setminus \hat{\jmath}$. If $(S, \{(C_i, y_i)\}_{i\in S})$ is an objection of agent 1 against agent $\hat{\jmath}$, then there exists an agent $i \in S \setminus 1$ such that $V_i(C_i) + y_i < \frac{1}{2}\cdot[7 + 21 - 6\frac{1}{2}] = 10\frac{3}{4}$. Hence, the pair $(\{i\hat{\jmath}\}, \{(\{\alpha_i, \alpha_{\hat{\jmath}}\}, 3\frac{3}{4}), (\emptyset, 10\frac{1}{4})\}$ is a counter-objection of agent $\hat{\jmath}$. ○

With lemma 3.1 we now can prove that every N-reallocation in the reactive bargaining set gives rise to an element in the reactive bargaining set in the corresponding TU-game.

Proposition 3.2. *Let $\mathcal{E}$ be an exchange economy which satisfies the TA-condition. If $\{(B_i, x_i)\}_{i\in N} \in \mathcal{M}_r(\mathcal{E})$, then $(V_i(B_i) + x_i - m_i)_{i\in N} \in \mathcal{M}_r(v_{\mathcal{E}})$.*

Proof: Let $\{(B_i, x_i)\}_{i\in N} \in \mathcal{M}_r(\mathcal{E})$ and define $X_i := V_i(B_i) + x_i - m_i$ for all $i \in N$. Then, by lemma 3.1, $X_i \geq V_i(A_i) = v_{\mathcal{E}}(i)$ for all $i \in N$.

Claim: $X(N) = v_{\mathcal{E}}(N)$.

Proof: Suppose $\sum_{i\in N} V_i(B_i) < v_{\mathcal{E}}(N)$. Let $\{C_i\}_{i\in N}$ be a N-redistribution which satisfies the TA-condition for coalition N. Define for all $i \in N$,

$$y_i \;:=\; V_i(B_i) + x_i - V_i(C_i) + \tfrac{1}{n}\sum_{i\in N}[V_i(C_i) - V_i(B_i)].$$

Because $V_i(B_i) + x_i \geq V_i(A_i) + m_i$ for all $i \in N$ it follows that $y_i \geq 0$ for all $i \in N$. Furthermore, $\frac{1}{n}\sum_{i\in N}[V_i(C_i) - V_i(B_i)] > 0$. Hence, the N-reallocation $\{(C_i, y_i)\}_{i\in N}$

is a strong improvement upon $\{(B_i, x_i)\}_{i \in N}$. Contradiction.

Take $\hat{\imath} \neq \hat{\jmath} \in N$. Because $\{(B_i, x_i)\}_{i \in N} \in \mathcal{M}_r(v_{\mathcal{E}})$ there exist a coalition $T \in \Gamma_{\hat{\jmath}\hat{\imath}}$ such that for every objection $(S, \{(C_i, y_i)\}_{i \in S})$ from agent $\hat{\imath}$ against agent $\hat{\jmath}$ there exist a T-reallocation $\{(D_i, z_i)\}_{i \in T}$ such that $(T, \{(D_i, z_i)\}_{i \in T})$ is a counter-objection.

Let (S, Y) be an objection from $\hat{\imath}$ against $\hat{\jmath}$ with respect to X. Take a S-redistribu-tion $\{C_i\}_{i \in S}$ which satisfies the TA-condition for coalition S and define,

$$y_i \;:=\; Y_i - V_i(C_i) + m_i \text{ for all } i \in S.$$

Then $\sum_{i \in S} y_i \;=\; \sum_{i \in S} m_i$ and because $Y_i > X_i \geq v_{\mathcal{E}}(i)$ for all $i \in S$ it follows that $y_i \geq 0$ and $V_i(C_i) + y_i \;>\; V_i(B_i) + x_i$ for all $i \in S$ and thus the pair $(S, \{(C_i, y_i)\}_{i \in S})$ is an objection from player $\hat{\imath}$ against $\hat{\jmath}$ with respect to $\{(B_i, x_i)\}_{i \in N} \in \mathcal{M}_r(\mathcal{E})$. Hence, there exists a T-reallocation $\{(D_i, z_i)\}_{i \in T}$ such that $(T, \{(D_i, z_i)\}_{i \in T})$ is a counter-objection. Define $Z_i \;:=\; V_i(D_i) + z_i - m_i$ for all $i \in T$. Clearly, $Z(T) \;\leq\; v_{\mathcal{E}}(T)$. Furthermore,

$$\begin{aligned} Z_i &= V_i(D_i) + z_i - m_i \;\geq\; V_i(C_i) + y_i - m_i \;=\; Y_i \quad \text{for all } i \in T \cap S, \\ Z_i &= V_i(D_i) + z_i - m_i \;\geq\; V_i(B_i) + x_i - m_i \;=\; X_i \quad \text{for all } i \in T \setminus S. \end{aligned}$$

Hence, the pair (T, Z) is a counter-objection. □

Combining the results of proposition 3.2 and proposition 1.3 with the proof of theorem 2.1 we obtain the following result.

Corollary 3.3. *Let $\mathcal{E} := \langle N, Q, (A_i, m_i, V_i)_{i \in N} \rangle$ be an exchange economy which satisfies the TA-condition, then*

$$\mathcal{M}_r(\mathcal{E}) \;=\; \mathcal{C}(\mathcal{E}) \quad \textit{if and only if} \quad \mathcal{M}_r(v_{\mathcal{E}}) \;=\; \mathcal{C}(v_{\mathcal{E}}).$$ □

In case of an exchange economy with four or less agents, the reactive bargaining set and the core coincide, whenever the latter is non-empty. The follows immediately by corollary 3.3 and the result by Solymosi (2002). Recall that an exchange economy $\mathcal{E}$ generates a (non-negative) superadditive TU-game $(N, v_{\mathcal{E}})$ and since in the literature several classes of superadditive balanced TU-games can be found for which the core and the reactive bargaining set coincide (see Solymosi (1999) for an overview), it may be of interest to point out the fact that the converse is also true.

Proposition 3.4. *If (N, v) is a non-negative superadditive TU-game, then there exists an exchange economy $\mathcal{E}$ such that $v_{\mathcal{E}} \;=\; v$.*

Proof: Let (N, v) be a superadditive TU-game, which is non-negative. Define, $Q := \{\alpha_1, \ldots, \alpha_n\}$ and $A_i := \{\alpha_i\}$ for all $i \in N$. The amounts of money $\{m_i\}_{i \in N}$ are not relevant. Furthermore, we define for all $i \in N$ the reservation values $V_i : 2^Q \to \mathbb{R}$ as follows,

$$V_i(B) \;:=\; v(\{j \in N \mid \alpha_j \in B\}) \text{ for all } B \subseteq Q.$$

We prove that $v(S) \;=\; v_{\mathcal{E}}(S)$ for all $S \subseteq N$. Let $S \subseteq N$, observe that any S-redistribution $\{B_i\}_{i \in S}$ obeys,

$$\sum_{i \in S} V_i(B_i) \quad = \quad \sum_{i \in S} v(\{j \in N \mid \alpha_j \in B_i\}) \ \leq \ v(S).$$

The last inequality follows from the superadditivity of (N, v). Hence, according to this inequality we have, $v_{\mathcal{E}}(S) \leq v(S)$.

By giving all indivisible goods within the coalition $S \subseteq N$ to exactly one agent, we also have that $v_{\mathcal{E}}(S) \geq v(S)$ for all $S \subseteq N$. □

Clearly, two different exchange economies may give the same non-negative superadditive TU-game. Even the number of indivisible goods, nor the initial endowments do *not* need to be the same. Nevertheless, given a non-negative superadditive TU-game for which the reactive bargaining set and the core coincide, the same result holds for every exchange economy generated by this TU-game.

References

Aumann RJ and Maschler M (1964) The bargaining set for cooperative games, In: Dresher M. Shapley L.S. Tucker A.W. (eds). Advances in Game Theory, Princeton University Press. Princeton, NJ, 443–476.

Beviá C, Quinzii M and Silva JA (1999) Buying several indivisible goods, *Mathematical Social Sciences* 37, 1–23

Bikhchandani S and Mamer J (1997) Competitive equilibrium in an exchange economy with indivisibilities, *Journal of Economic Theory* 74, 385–413.

Davis M and Maschler M (1967) Existence of stable payoff configurations for cooperative games, In: Shubik M. (eds). Essays in mathematical economies in honour of Oskar Morgenstern, Princeton University Press. Princeton, NJ, 39–52.

Debrue G (1959) Theory of Value, New York: John Wiley & Son Inc.

Granot D (1994) On a new bargaining set for cooperative games and on assignment games, Working Paper, Faculty of Commerce and Business Administration, University of British Columbia, Vancouver, Canada.

Granot D and Maschler M (1997) The Reactive Bargaining Set: Structure, Dynamics and Extension to NTU Games, *International Journal of Game Theory* 26, 76–95.

Solymosi T (1999) On the bargaining set, kernel and core of superadditive games, *International Journal of Game Theory* 28, 229–240.

Solymosi T (2002) The bargaining set of four-person balanced games, *International Journal of Game Theory* 31, 1–11.