BARRELLED-LIKE SPACES IN p-ADIC ANALYSIS

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Abstract

Several notions of barrelledness for locally convex spaces over non-archimedean valued fields are discussed in this paper and the relation between them is studied. We give examples showing that they are different notions in general. In particular, we solve (Theorem 2.6) the question raised by the second author in [13]: Do there exist polar spaces that are polarly barrelled but not barrelled?

On the other hand, the concepts of orthogonality as well as elementary and edged set are used to prove that for a wide class of spaces of countable type the different versions of barrelled spaces considered in the paper coincide. We obtain in this way (and with different proofs) the non-archimedean counterparts of well-known results in the theory of barrelled spaces over the real or complex field.


INTRODUCTION

Like in the classical case (i.e., locally convex spaces over the real or complex field see e.g. [1]) we consider in this paper the notion of barrelled (resp. $N_0$-barrelled) space over a non-archimedean valued field by requiring that every pointwise bounded family (resp. sequence) of non-archimedean continuous seminorms is equicontinuous. Also, the concept of $\ell^\infty$-barrelled space is obtained if one replaces "sequence of continuous seminorms" by "sequence of continuous linear functionals".

But in addition the concept of polar seminorm leads us to consider in the non-archimedean case the polar versions of the above notions by taking a family (resp. sequence) of polar seminorms.

The main purpose of this paper is to study the relationship between these different forms of "barrelledness". We show (mainly in Section 3) that they don't coincide in

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general. It is interesting to point out that, in contrast to the classical situation, the polar versions of barrelledness are in general different from the corresponding ones considering arbitrary seminorms (Theorem 2.6).

On the other hand, we use the concept of orthogonality to prove that for spaces of countable type, $\ell^\infty$-barrelled $\Leftrightarrow$ $\mathcal{N}_0$-barrelled (Proposition 3.2). Also, the concepts of edged and elementary set are used to prove that for spaces strictly of countable type (of which perfect sequence spaces are a particularly interesting example, see Section 4), $\ell^\infty$-barrelled $\Leftrightarrow$ barrelled (Theorem 3.5). We obtain in this way (and with different proofs) the non-archimedean counterparts of two well-known results in the theory of barrelled spaces over the real or complex field (see e.g. [1]). Further, we show (Theorem 5.4) that for spaces of finite type (which are not in general strictly of countable type) we again have that $\ell^\infty$-barrelled $\Leftrightarrow$ barrelled.

1 PRELIMINARIES

Throughout this paper $K$ is a non-archimedean valued field that is complete with respect to the metric induced by the non-trivial valuation $|\cdot|$. We set $|K| := \{|\lambda| : \lambda \in K\}$ and $\overline{|K|} :=$ the closure of $|K|$ in $\mathbb{R}$.

For the basic notions and properties concerning normed and locally convex spaces over $K$ we refer to [18] and [13] respectively. However we recall the following.

1. Let $E$ be a $K$-vector space and let $A$ be an absolutely convex subset of $E$ (i.e., $A$ is a $B_K$-module where $B_K = \{\lambda \in K : |\lambda| \leq 1\}$). $A$ is called edged if for each $x \in E$, the set $\{|\lambda| : \lambda \in K, \lambda x \in A\}$ is closed in $|K|$. We define $A^e$ to be the smallest edged subset of $E$ that contains $A$. If the valuation on $K$ is dense we have $A^e = \bigcap_{|\lambda| > 1} \lambda A$ (and $A^e = A$ if the valuation is discrete). $A$ is called elementary if there exist $m \in \mathbb{N}$, $x_1, \ldots, x_m \in E$ and absolutely convex sets $C_1, \ldots, C_m$ of $K$ such that $A = C_1x_1 + \ldots + C_mx_m$.

For $X, Y \subset E$ we set $X \setminus Y := \{x \in X : x \notin Y\}$. The absolutely convex hull of $X$ is denoted by $\text{co}(X)$ and its linear hull by $[X]$.

A (non-archimedean) seminorm on $E$ is a map $p : E \rightarrow \mathbb{R}$ satisfying:

i) $p(x) \in |K|$

ii) $p(\lambda x) = |\lambda| p(x)$

iii) $p(x + y) \leq \max(p(x), p(y))$

for all $x, y \in E$, $\lambda \in K$ (see [13]). Observe that if $K$ carries a dense valuation $i)$ is equivalent to $p(x) \geq 0$ whereas for a discretely valued field $K$ condition $i)$ is equivalent to $p(x) \in |K|$. It is easily seen that if $p : E \rightarrow [0, +\infty)$ satisfies $i)$ and $ii)$ and $K$ has a discrete valuation the formula $q(x) = \inf\{s \in |K| : p(x) \leq s\}$ defines a seminorm $q$ for which $pq \leq p \leq q$ where $\rho = \max\{|\lambda| : \lambda \in K, |\lambda| < 1\}$. A seminorm $p$ for which $p(x) = 0 \Rightarrow x = 0$ is called a norm and is usually denoted by $\|\cdot\|$ rather than $p$. 

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For a seminorm \( p \) on \( E \) we denote by \( E_p \) the vector space \( E/K \exp \) endowed with the canonical norm. Following [13] we say that \( p \) is polar if \( p = \sup \{|f| : f \in E^*, |f| \leq p\} \), where \( E^* \) is the algebraic dual of \( E \).

2. Let \((E, \|\|)\) be a normed space. If \( A, B \subset E \), \( \text{dist}(A, B) = \inf\{\|a - b\| : a \in A, b \in B\} \) will be the distance between \( A \) and \( B \). \( E \) is called of countable type if \( E \) has a countable subset whose linear hull is dense in \( E \). Recall that in this case for every \( t \in (0, 1), E \) contains a \( t \)-orthogonal basis \((x_n)_{n} \) (i.e., every \( x \in E \) can be written uniquely as \( x = \sum_n \lambda_n x_n, \lambda_n \in K \) for all \( n \), and \( \|x\| \geq t \max_n |\lambda_n| \|x_n\| \). If \( K \) is spherically complete this property also holds for \( t = 1 \) ([18], Lemma 5.5). Instead of \( 1 \)-orthogonal we will write orthogonal.

3. Now assume that \( E \) is a Hausdorff locally convex space over \( K \) with topological dual \( E' \). By \( \sigma(E, E') \) (resp. \( \sigma(E', E) \)) we will mean the weak topology on \( E \) (resp. \( E' \)) with respect to the dual pair \(< E, E' >\). \( E \) is called reflexive if the canonical map \( j_E : E \rightarrow E'' \) is a linear homeomorphism from \( E \) onto its bidual \( E'' \) (where \( E' \) and \( E'' \) have the strong topologies).

A set \( A \subset E \) is called polar if \( A \) coincides with its bipolar with respect to the above dual pair. Every polar set is edged. If \( p \) is a continuous seminorm on \( E \), then \( p \) is polar if and only if \( \{x \in E : p(x) \leq 1\} \) is a polar set ([13], Proposition 3.4.ii)). In particular, if \( A \) is an absolutely convex subset of \( E \), then the formula \( p_A : [A] \rightarrow \mathbb{R}, p_A(x) = \inf\{|\lambda| : x \in \lambda A\} \) (\( x \in [A] \)) defines its associated seminorm \( p_A \) on \( [A] \). If \( A \) is a zero neighbourhood in \( E \), then \( p_A \) is polar if and only if \( A^c \) is a polar subset of \( E \).

Following [13] we say that \( E \) is polar if its topology is defined by a family of polar seminorms. Also, \( E \) is called of countable type (resp. of finite type) if for every continuous seminorm \( p \) on \( E \) the normed space \( E_p \) is of countable type (resp. of finite dimension). Recall that if \( E \) is of countable type then every weakly convergent sequence in \( E \) is convergent ([13], Theorem 4.4 and Proposition 4.11) and that \( E \) is of finite type if and only if its topology is a weak topology ([16], Theorem 2). Spaces of countable type form the non-archimedean counterpart of the transseparable spaces considered in the case of locally convex spaces over the real or complex field (see e.g. [1], p. 53).

A set \( A \) in \( E \) is called compactoid if for each neighbourhood \( U \) of \( 0 \) in \( E \) there exists a finite set \( H \) in \( E \) such that \( A \subset U + \text{co}(H) \).

In the sequel \( E, F \) will be Hausdorff locally convex spaces over \( K \). By \( L(E, F) \) we will denote the vector space of all continuous linear maps from \( E \) into \( F \). The vector subspace of \( L(E, F) \) consisting of all \( T \in L(E, F) \) whose rank is finite-dimensional will be denoted by \( F^t(E, F) \).
2 BARRELLED-LIKE SPACES

Like in the classical case (i.e. locally convex spaces over the real or complex field, see e.g [1]) we can consider several concepts of barrelledness (compare Proposition 2.3 for the equivalent formulations in terms of barrels).

Definition 2.1 .

a) E is called barrelled if every pointwise bounded family of continuous seminorms on E is equicontinuous.

b) E is called $\mathbb{N}_0$-barrelled if every pointwise bounded sequence of continuous seminorms on E is equicontinuous.

c) E is called $\ell^\infty$-barrelled if every pointwise bounded sequence in $E'$ is equicon­ tinuous.

But, in addition, the concept of polar seminorm leads us to consider in the non­archimedean case the following polar versions of barrelled and $\mathbb{N}_0$-barrelled space.

Definition 2.2 E is called polarly barrelled (resp. polarly $\mathbb{N}_0$-barrelled) if every point­wise bounded family (resp. sequence) of polar continuous seminorms on E is equicon­ tinuous.

Obviously every Fréchet space and every inductive limit of a sequence of Fréchet spaces is a barrelled space. Also, by Theorem 9.6 of [13], every reflexive (and hence every Montel) space is polarly barrelled (for several interesting examples of this kind of spaces see [13] and our Section 4).

One can easily see that if $\{U_i\}_{i \in I}$ is a family of absolutely convex zero neighbour­ hoods in E and $U = \bigcap_{i \in I} U_i$, then U is a barrel (resp. a zero neighbourhood) in E if and only if $\{p_{U_i}\}_{i \in I}$ is a pointwise bounded (resp. equicontinuous) family of con­tinuous seminorms on E. Applying this fact, it is very easy to obtain the following descriptions of the above concepts in terms of barrels.

Proposition 2.3 .

i) E is barrelled if and only if every barrel in E (which is always the intersection of a family of closed absolutely convex zero neighbourhoods in E) is a zero neighbourhood in E.

ii) E is $\mathbb{N}_0$-barrelled if and only if every barrel in E which is the intersection of a sequence of absolutely convex zero neighbourhoods in E is a zero neighbourhood in E. (Observe that every absolutely convex zero neighbourhood in E is closed).

iii) E is $\ell^\infty$-barrelled if and only if every barrel in E which is the intersection of a sequence of (polar) absolutely convex weak zero neighbourhoods in E is a zero neighbourhood in E.
iv) $E$ is polarly barrelled if and only if every polar barrel in $E$ (which is always the intersection of a family of polar absolutely convex weak zero neighbourhoods in $E$) is a zero neighbourhood in $E$.

v) $E$ is polarly $\aleph_0$-barrelled if and only if every (polar) barrel in $E$ which is the intersection of a sequence of polar absolutely convex zero neighbourhoods in $E$ is a zero neighbourhood in $E$.

Remarks 2.4 Also, it is straightforward to verify that:

1. $E$ is polarly barrelled if and only if every pointwise bounded family in $E'$ is equicontinuous ([13], Proposition 6.3). If in addition $E$ is polar then $E$ is polarly barrelled if and only if the topology on $E$ coincides with the strong topology $\beta(E, E')$ (which is the topology of uniform convergence on the pointwise bounded subsets of $E'$).

2. $E$ is polarly $\aleph_0$-barrelled if and only if every pointwise bounded subset of $E'$ which is the union of a sequence of equicontinuous subsets of $E'$, is equicontinuous.

We clearly have the following diagram.

\[
\begin{array}{c c}
\text{barrelled} & \Rightarrow & \text{polarly barrelled} \\
\downarrow & & \downarrow \\
\aleph_0 - \text{barrelled} & \Rightarrow & \text{polarly } \aleph_0 - \text{barrelled} \\
\downarrow & & \downarrow \\
\ell^\infty - \text{barrelled} & & \ell^\infty - \text{barrelled}
\end{array}
\]

Also observe that if $E$ is $\ell^\infty$-barrelled, then $E'$ is $\sigma(E', E)$-sequentially complete (but not conversely, see Remark 5.5.1).

If every continuous seminorm on $E$ is polar (e.g., when $K$ is spherically complete, or when $E$ is of countable type, see [13]), then

$$E \text{ polarly } (\aleph_0-)\text{barrelled} \Rightarrow E \text{ } (\aleph_0-)\text{barrelled.} \quad (1)$$

Further, in the next section we will see that for a wide class of spaces of countable type the five versions of barrelledness considered in Definitions 2.1 and 2.2 coincide (Theorem 3.5).

But we will show (mainly in the next section) that the converses of the arrows appearing in the above diagram are not true in general.

First of all we prove that, in contrast to the classical situation, there are spaces $E$ for which (1) does not hold. This gives an affirmative answer to the problem raised by the second author in [13], p. 208: Do there exist polar spaces that are polarly barrelled but not barrelled? In fact, we are going to construct (Theorem 2.6) a polarly barrelled space which is not even $\aleph_0$-barrelled. To do that, the following lemma will be a useful tool.
Lemma 2.5 Suppose that $K$ is not spherically complete. For each $n = 1, 2, \ldots$
let $S_n := \{X \subset \mathbb{N} : \#X \cap \{1, \ldots, k\} = O(k^{1-1/n}) \text{ when } k \to \infty\}$ (# indicating cardinality) and let $E_n$ be the closed subspace of $\ell^\infty$ consisting of all $x = (x_1, x_2, \ldots) \in \ell^\infty$ such that for every $\epsilon > 0$ the set $\{m \in \mathbb{N} : |x_m| \geq \epsilon\}$ is in $S_{n+1}$. We endow each $E_n$ with the norm topology $\tau_n$ inherited from the norm $\| \cdot \|_\infty$ of $\ell^\infty$. Then, $(E_n)_n$ is a strictly increasing sequence of (polar) Banach spaces containing $c_0$ and satisfying, for each $n \in \mathbb{N}$, the following properties:

i) Every $f \in E_n'$ that vanishes on $c_0$ vanishes identically on $E_n$.

ii) $f \in E_n'$ if and only if there exists a (unique) $y = (y_m)_m \in c_0$ such that $f(x) = \sum_m x_m y_m$ for all $x = (x_m)_m \in E_n$, and in this case $\|f\| = \|y\|_\infty$.

iii) Every $f \in E_n'$ can be uniquely extended to an element $g \in E_{n+1}'$ and $\|g\| = \|f\|$.

Proof. i): Let $x = (x_m)_m \in E_n$ and let $\epsilon > 0$ be given; we prove that $|f(x)| \leq \epsilon \|f\|$.
We have that the set $X = \{m \in \mathbb{N} : |x_m| \geq \epsilon\}$ is in $S_{n+1}$ and so $D := \{z = (z_m)_m \in \ell^\infty : z_m = 0 \text{ whenever } m \notin X\}$ is a subspace of $E_n$ for which $\text{dist}(x, D) \leq \epsilon$.
So, it suffices to prove that $f = 0$ on $D$. This is clear if $X$ is finite. If $X$ is infinite and has the form $X = \{s_1, s_2, \ldots\}$ with $s_1, s_2, \ldots \in \mathbb{N}$ and $s_1 < s_2 < \ldots$, then the map $\phi : \ell^\infty \to D$, $x = (x_1, x_2, \ldots) \mapsto \phi(x)$, where

\[ \phi(x)_m = 0 \text{ if } m \notin X \text{ and } \phi(x)_m = x_j \text{ if } m = s_j \text{ for some } j \]

is a linear surjective isometry for which $\phi(c_0) = D \cap c_0$. It follows that $D/D \cap c_0$ is isometrically isomorphic to $\ell^\infty/c_0$ and so $(D/D \cap c_0)' = \{0\}$ ([18], Corollary 4.3), which implies that $f = 0$ on $D$.

ii): Obviously, for every $y = (y_m)_m \in c_0$, $f : E_n \to K$, $x = (x_m)_m \in E_n \mapsto \sum_m x_m y_m$ is a continuous linear functional on $E_n$ with $\|f\| = \|y\|_\infty$.

Conversely, let $f \in E_n'$. Since $c_0 \subset E_n$ we can apply Exercise 3.Q of [18] to derive the existence of $y = (y_m)_m \in \ell^\infty$ such that $f(x) = \sum_m x_m y_m$ for all $x = (x_m)_m \in c_0$. Suppose $y \notin c_0$. There exists $\epsilon > 0$ such that the set $\{m \in \mathbb{N} : |y_m| \geq \epsilon\}$ is infinite and so it contains an infinite element $X = \{s_1, s_2, \ldots\} \in S_{n+1}$ (where $s_1, s_2, \ldots \in \mathbb{N}$ and $s_1 < s_2 < \ldots$). Then $\phi : \ell^\infty \to E_n$ given by the formula (2) defines a linear isometry from $\ell^\infty$ into $E_n$, and so $f \circ \phi \in (\ell^\infty)'$. By Theorem 4.22 of [18], there exists $\alpha = (\alpha_m)_m \in c_0$ such that $(f \circ \phi)(x) = \sum_m x_m \alpha_m$ for all $x = (x_1, x_2, \ldots) \in \ell^\infty$. In this case $|\alpha_j| = |\epsilon| \geq \epsilon$ for all $j = 1, 2, \ldots$, a contradiction. Hence $y = (y_m)_m \in c_0$ and $f(x) = \sum_m x_m y_m$ for all $x = (x_m)_m \in c_0$. Now the conclusion follows from the fact that $c_0$ is weakly dense in $E_n$ (see i)).

Finally, iii) is a direct consequence of i) and ii).

By using this Lemma we now can prove:
Theorem 2.6 Suppose that $K$ is not spherically complete. For each $n = 1, 2, \ldots$, let $E_n$ be as in Lemma 2.5 and let $E = \bigcup_n E_n$ endowed with the norm topology $\tau$ inherited from the norm $\|\|_\infty$ on $\ell^\infty$. Then, $E$ is a polarly barrelled space which is not $\aleph_0$-barrelled.

Proof. First we prove that $\tau$ is the largest element of the set $T$ of all polar topologies on $E$ for which all the canonical inclusions from $E_n$ into $E$ are continuous.

Clearly $\tau \in T$. Also, let $\varrho \in T$; we prove $\varrho \leq \tau$. Let $p$ be a polar $\varrho$-continuous seminorm on $E$ and let $\mathcal{F} = \{f \in (E, \varrho) : |f| \leq p\}$. The restriction of $p$ to $E_1$ is a $\tau$-continuous seminorm in $E_1$ and so there exists a constant $c > 0$ such that $p(x) \leq c\|x\|_\infty$ for all $x \in E_1$. If $f \in \mathcal{F}$, then $|f(x)| \leq c\|x\|_\infty$ for all $x \in E_1$ and by Lemma 2.5 it follows that $|f(x)| \leq c\|x\|_\infty$ for all $x \in E$. We obtain that for every $x \in E$, $p(x) = \sup\{|f(x)| : f \in \mathcal{F}\}$, which implies that $p$ is a $\tau$-continuous seminorm on $E$ and so that $\varrho \leq \tau$.

Now, we are going to show that $E$ is polarly barrelled. To prove that, let $\mathcal{P}$ be a pointwise bounded family of polar $\tau$-continuous seminorms on $E$. Then, $q := \sup\{p : p \in \mathcal{P}\}$ is a polar seminorm on $E$ whose restriction to each $E_n$ is continuous. By the above, $q$ is $\tau$-continuous.

Finally, suppose that $E$ is $\aleph_0$-barrelled. Since, by Baire Category Theorem $E$ is not complete, there exists an $a \in \hat{E} \setminus E$ (where $\hat{E}$ is the completion of $E$). Also, for each $x \in E$, $\text{dist}(x, E_n) = 0$ for large $n$ and so $p : E \to \mathbb{R}$, $x \mapsto \sup_n \text{dist}(x, E_n)/\text{dist}(a, E_n)$ is a well-defined seminorm on $E$ which, by $\aleph_0$-barrelledness, is continuous. Let $\hat{p} : \hat{E} \to \mathbb{R}$ be the continuous extension of $p$ and let $(x_m)_m$ be a sequence in $E$ with $\lim_m x_m = a$. For each $n = 1, 2, \ldots$, we have

$$\hat{p}(a) \geq \lim_n \text{dist}(x_m, E_n)/\text{dist}(a, E_n) = n,$$

a contradiction. Hence, $E$ is not $\aleph_0$-barrelled.

## 3 BARRELLEDNESS AND SPACES OF COUNTABLE TYPE

One of the purposes of this section is to prove that if $E$ is of countable type then $E$ is $\ell^\infty$-barrelled if and only if $E$ is (polarly) $\aleph_0$-barrelled (Proposition 3.2.ii)). The proof given for transseparable spaces in the classical case (see e.g. [1], Proposition 8.2.24) can be adapted to the non-archimedean situation (see Remark 3.3). However, the good behaviour of normed spaces of countable type (see Lemma 3.1, which is an interesting result by itself) allows to give in the non-archimedean case a simpler proof of the above fact.

Lemma 3.1 Let $p$ be a seminorm on a $K$-vector space $E$ such that $E_p$ is of countable type. Then, there exists a sequence $(f_n)_n$ in $E^*$ such that $p = \sup_n |f_n|$.
Proof. We may assume that $p$ is a norm and also that $(E, p)$ is complete.

If the valuation on $K$ is discrete then $(E, p)$ has an orthogonal basis $(e_n)_n$ with $p(e_n) = 1$ for all $n$ and then the result follows.

Now, suppose that the valuation on $K$ is dense. For each $m \in \mathbb{N}$ let $(e_{mn})_n$ be a $(1 - 1/2m)$-orthogonal basis of $(E, p)$ with $1 \leq p(e_{mn}) \leq (1 - 1/2m)^{-1}$ for all $n$ and let $g_{mn} \in (E, p)'$ given by $g_{mn}(x) = \lambda_{mn}c_m$ ($n \in \mathbb{N}$), where $x = \sum_n \lambda_{mn}e_{mn} \in E$ (the convergence of this series in $(E, p)$) and where $c_m \in K$ is such that $(1 - 1/2m)^2 \leq |c_m| \leq (1 - 1/2m)$.

Then for each $m \in \mathbb{N}$ and for each $x = \sum_n \lambda_{mn}e_{mn} \in E$ we have

\[
(1 - 1/2m)^3p(x) \leq (1 - 1/2m)^3 \sup_n |\lambda_{mn}|p(e_{mn}) \leq \sup_n |g_{mn}(x)| \leq \sup_n |c_m| |\lambda_{mn}|p(e_{mn}) \leq |c_m| (1 - 1/2m)^{-1}p(x)
\]

and so $p(x) = \sup_{m,n} |g_{mn}(x)|$.

Proposition 3.2

i) If $E$ is metrizable, then $E$ is barrelled (resp. polarly barrelled) if and only if $E$ is $K_0$-barrelled (resp. polarly $K_0$-barrelled).

ii) If $E$ is of countable type, then $E$ is $K_0$-barrelled if and only if $E$ is $\ell^\infty$-barrelled.

iii) If $E$ is metrizable and of countable type, then

$E$ is barrelled $\iff$ $E$ is $\ell^\infty$-barrelled.

Proof. i): Let $E$ be a metrizable $K_0$ (resp. polarly $K_0$)-barrelled space and suppose there exists a pointwise bounded family $\{p_i\}_{i \in I}$ of (polar) continuous seminorms on $E$ which is not equicontinuous. Then $p(x) = \sup_i p_i(x)$ ($x \in E$) is a seminorm on $E$ which is not continuous. By metrizability, we can find a sequence $(x_n)_n$ in $E$ with $\lim_n x_n = 0$ and an $\varepsilon > 0$ such that $p(x_n) > \varepsilon$ for all $n$. Hence, for each $n$ there is an $i_n \in I$ for which $p_{i_n}(x_n) > \varepsilon$. Thus, $\{p_{i_n}\}_n$ is a pointwise bounded sequence of (polar) continuous seminorms on $E$ which is not equicontinuous: a contradiction.

ii): It follows directly from Lemma 3.1.

Property iii) is a direct consequence of i) and ii).

Remark 3.3 As we have already announced, it is possible to adapt to the non-archimedean case the proof given of Proposition 3.2.ii) for transseparable spaces over the real or complex field (see e.g. [1], Proposition 8.2.24).

Indeed, since $E$ is of countable type, every equicontinuous subset $B$ of $E'$ is $\sigma(E', E')$-metrizable ([5], Lemma 2.4) and by [13], Proposition 8.2 it is contained in the $\sigma(E', E)$-closed absolutely convex hull of a sequence $(f_n)_n$ in $\pi\sigma(B)$ (where $\pi \in K$ with $|\pi| > 1$ is fixed). Then, the result follows from (1) and Remark 2.4.2.

On the other hand, one can easily see that, applying the Hahn-Banach Theorem, the conclusion of Lemma 3.1 also holds when the ground field is the real or complex
one and $E_p$ is separable. Hence, this Lemma provides an alternative (in fact shorter) proof of Proposition 3.2.ii) for real or complex transseparable spaces.

In the classical case statement (3) in Proposition 3.2 is satisfied for every separable locally convex space $E$ (see e.g. [1], Corollary 8.2.20). One might expect this property also to hold in the non-archimedean case for spaces of countable type, but this is not true in general (see Example 3.8.2). We therefore consider the following subclass, which includes the metrizable locally convex spaces of countable type (Remark 3.9.1).

**Definition 3.4** $E$ is called **strictly of countable type** if there exists a countable set in $E$ whose linear hull is dense in $E$ (our terminology differs from that of [3]).

The main result of this section assures that for spaces strictly of countable type, statement (3) of Proposition 3.2 remains true (observe that there are non-metrizable spaces which are strictly of countable type: take $c_0$ endowed with the weak topology).

**Theorem 3.5** If $E$ is strictly of countable type, then $E$ is barrelled $\iff E$ is $\ell^\infty$-barrelled.

The proof of Theorem 3.5 differs substantially from the classical one for separable spaces. We need some preliminary machinery for this proof.

**Lemma 3.6** Suppose that $E$ is of countable type. Let $T$ be a closed absolutely convex subset of $E$. Let $D$ be a finite-dimensional subspace of $E$ and let $V$ be an elementary edged subset of $D$ such that $T \cap D \subset V$. Then, there exists a countable set $S$ in $E'$ such that

$$\sup_{f \in S} |f(x)| \leq 1 \text{ for all } x \in T \text{ and } \sup_{f \in S} |f(x)| > 1 \text{ for all } x \in D \setminus V.$$ 

**Proof.** $V$ has the form $C_1 e_1 + \ldots + C_n e_n$ (for some $n \in \mathbb{N}$), where $\{e_1, \ldots, e_n\}$ is an algebraic basis of $D$ and where $C_1, \ldots, C_n$ are edged sets (possibly $\{0\}$) in $K$ (see [12], Lemma 2.2 and Proposition 2.10).

1) First, we are going to see that for every $i \in \{1, \ldots, n\}$ for which $C_i \neq K$ (or equivalently $\text{diam } C_i < \infty$, where $\text{diam } C_i$ denotes the diameter of the set $C_i$), there exists a sequence $(g_{im})_m$ in $E'$ such that, for each $m$,

$$|g_{im}| \leq 1 \text{ on } (T + \sum_{j \neq i} K e_j)^e \text{ and } |g_{im}(\mu_{im} e_i)| > 1,$$

where $(\mu_{im})_m$ is a sequence in $K$ chosen such that:

a) $|\mu_{i1}| > |\mu_{i2}| > \ldots \text{ and } \lim_m |\mu_{im}| = \text{diam } C_i$, if the valuation on $K$ is dense or if $C_i = \{0\}$;

b) $\mu_{i1} = \mu_{i2} = \ldots$ and $|\mu_{i1}| = \min\{|\lambda| : \lambda \in K, |\lambda| > \text{diam } C_i\}$, otherwise.
To prove the existence of this sequence \((g_{im})_m\) in \(E'\), we first claim that with our choice of \((\mu_{im})_m\) we have that, for each \(m\),
\[
\mu_{im}e_i \notin (T + \sum_{j \neq i} K e_j)^e
\]  
Indeed, suppose that in case a) \((5)\) is not true for some \(m\) and choose \(\nu \in K\) with \(\text{diam} C_i < |\nu| < \mu_{im}\). Then, \(\mu_{im}e_i \in (\mu_{im}/\nu) (T + \sum_{j \neq i} K e_j)\), and so there exist \(\xi_1, \ldots, \xi_n \in K\) with \(\xi_i = \nu\), such that \(\sum_{j=1}^n \xi_j e_j \in T \cap D \subset V\). In particular, \(\nu \in C_i\), which is a contradiction.

Analogously, assume that in case b) \((5)\) is not true. Observe that in this case \((T + \sum_{j \neq i} K e_j)^e = (T + \sum_{j \neq i} K e_j)\), so if \((5)\) fails there exist \(\xi_1, \ldots, \xi_n \in K\) with \(|\xi_i| > \text{diam} C_i\) such that \(\sum_{j=1}^n \xi_j e_j \in T \cap D \subset V\), again a contradiction.

Now, since \((T + \sum_{j \neq i} K e_j)^e\) is closed ([15], Theorem 1.4.ii)) and edged and \(E\) is of countable type we deduce that this set is polar ([13], Theorems 4.4 and 4.7). Applying \((5)\) we derive the existence, for each \(m \in \mathbb{N}\), of a \(g_{im} \in E'\) satisfying \((4)\).

2) Now, we claim that the countable set \(S = \{g_{im} : i \in \{1, \ldots, n\}, C_i \neq K, m \in \mathbb{N}\} \cup \{0\}\) satisfies the required conditions.

By \((4)\), it is clear that \(\sup_{f \in S} |f(x)| \leq 1\) for all \(x \in T\) and \(g_{im}(e_j) = 0\) for all \(j \neq i\) and for all \(m\).

Now, let \(x = \xi_1 e_1 + \ldots + \xi_n e_n \in D \setminus V\). Then, there is at least one \(i\) for which \(\xi_i \notin C_i\) (and hence \(C_i \neq K\)). Also, from the construction of the \(\mu_{im}\) it is clear that there exists an \(m \in N\) such that \(|\mu_{im}| \leq |\xi_i|\). Then, \(|g_{im}(x)| = |g_{im}(\xi_i e_i)| = |\xi_i^{-1} g_{im}(\mu_{im} e_i)| > |\xi_i| \frac{1}{|\mu_{im}|} \geq 1\) and we are done.

The following result is a generalization of the previous lemma.

**Proposition 3.7** Suppose that \(E\) is of countable type. Let \(T\) be a closed and edged subset of \(E\). Let \(D\) be a finite-dimensional subspace of \(E\). Then, there exists a countable set \(S\) in \(E'\) such that \(\sup_{f \in S} |f(x)| \leq 1\) for all \(x \in T\) and \(\sup_{f \in S} |f(x)| > 1\) for all \(x \in D \setminus T\).

**Proof.** By Theorem 4.8 of [12], there exist countable many elementary edged sets \(V_1 \supset V_2 \supset \ldots \) in \(D\) such that \(\cap_n V_n = T \cap D\). By Lemma 3.6, for each \(n\), there exists a countable set \(S_n\) in \(E'\) such that
\[
\sup_{f \in S_n} |f(x)| \leq 1 \text{ for all } x \in T \text{ and } \sup_{f \in S_n} |f(x)| > 1 \text{ for all } x \in D \setminus V_n.
\]
Then the countable set \(S = \bigcup_n S_n\) satisfies the conditions.

This is enough material to prove Theorem 3.5.
PROOF (OF THEOREM 3.5)

Let $E$ be an $\ell^\infty$-barrelled space which is strictly of countable type and let $\{x_1, x_2, \ldots\}$ be a countable set in $E$ whose linear hull in dense in $E$.

Let $T$ be a barrel in $E$. We want to prove that $T$ is a zero neighbourhood in $E$ (Proposition 2.3.i)). For that we can assume that $T$ is edged. By Proposition 3.7 we have, for each $n$, a countable set $S_n$ in $E'$ such that

$$\sup_{f \in S_n} |f(x)| \leq 1 \text{ for all } x \in T \quad (6)$$

and

$$\sup_{f \in S_n} |f(x)| > 1 \text{ for all } x \in D_n \setminus T \quad (7)$$

where $D_n$ is the linear hull of $\{x_1, \ldots, x_n\}$.

Then, $S := \bigcup_n S_n$ is a countable pointwise bounded subset of $E'$. By $\ell^\infty$-barrelledness, $U = \{x \in E : |f(x)| \leq 1 \text{ for all } f \in S\}$ is a zero neighbourhood in $E$. By (6), we clearly have that $T \subseteq U$. Also, suppose that there exists $x \in U \setminus T$. Since $U \setminus T$ is open, there exists a $y \in \bigcup_n D_n$ such that $y \in U \setminus T$, which contradicts (7). Hence, $T = U$, which implies that $T$ is a zero neighbourhood in $E$.

Examples 3.8 In Theorem 2.6 we showed that the converses of the horizontal arrows appearing in our diagram (see Section 2) are not true in general. Now we are going to show that the same happens for the vertical arrows of that diagram.

1. Example of an $\ell^\infty$-barrelled space which is not polarly $\aleph_0$-barrelled (compare Example 8.2.38 of [1]).

Choose a set $I$ such that $\#I > \#\ell^\infty$. Let $E = \ell^\infty(N \times I)$ as a $K$-vector space. Let $C$ be the collection of all countable (pointwise) bounded subsets of $(E, \|\|_\infty)'$, where $\|\|_\infty$ is the canonical norm on $E$. For each $S \in C$ we define a seminorm $p_S$ on $E$ by

$$p_S(x) = \sup\{\|f(x)\| : f \in S\} \quad (x \in E)$$

and also, for each $m \in \mathbb{N}$ we define a seminorm $p_m$ on $E$ by

$$p_m(x) = \sup\{|x_{(n,i)}| : n \in \{1, 2, \ldots, m\}, i \in I\} \quad (x = (x_{(n,i)})_{n,i} \in E).$$

Then, the family of polar seminorms $\{p_S : S \in C\} \cup \{p_m : m \in \mathbb{N}\}$ defines a Hausdorff locally convex topology $\tau$ on $E$ for which it is easily seen that $(E, \tau)$ is $\ell^\infty$-barrelled (observe that $(E, \tau)' = (E, \|\|_\infty)'$). However, $(E, \tau)$ is not polarly $\aleph_0$-barrelled.

Indeed, we shall see that $\|\|_\infty = \sup_m p_m$ is not $\tau$-continuous. Suppose it was. Then, there would exist an $M > 0$, an $S = \{f_1, f_2, \ldots\} \in C$ and an $m \in \mathbb{N}$ such that for all $x = (x_{(n,i)})_{n,i} \in E$,
\[\|x\|_\infty \leq M \max\{\sup_j f_j(x), \sup_i \{x_{(n,i)} : n \leq m, i \in I\}\}.\] (8)

Let \(D\) be the subspace of \(E\) consisting of all \(x \in \ell_\infty(N \times I)\) that vanish on \(\{(1) \times I\} \cup \{(2) \times I\} \cup \ldots \cup \{(m) \times I\}\). Then, on one hand \((D, \|\cdot\|_\infty)\) is isometrically isomorphic to \((E, \|\cdot\|_\infty)\). But on the other hand we have by (8) that \(\|x\|_\infty \leq M \sup_j |f_j(x)|\) for all \(x \in D\), which implies that the map \((D, \|\cdot\|_\infty) \to \ell_\infty, x \in D \mapsto (f_1(x), f_2(x), \ldots)\) is a linear homeomorphism from \((D, \|\cdot\|_\infty)\) onto its image. Hence, \(\#(N \times I) = \#I \leq \#\ell_\infty < \#I\), a contradiction.

2. Example of a \(N_0\)-barrelled (hence \(\ell_\infty\)-barrelled) space of countable type which is not (polarly) barrelled.

Take an uncountable set \(I\) and let \(E = c_0(I)\) as a \(K\)-vector space. Let \(\tau\) be the locally convex topology on \(E\) generated by the polar seminorms \(p_C\), \((C \subset I, C\) countable\), given by \(p_C(x) = \max_{i \in C} |x_i|\) \((x = (x_i)_{i \in I} \in E)\).

It is easy to see that \((E, \tau)\) is of countable type. Also, since \((E, \|\cdot\|_\infty)\) is not of countable type (where \(\|\cdot\|_\infty\) is the canonical norm on \(E\)), we deduce that \(\|\cdot\|_\infty = \sup\{p_C, C \subset I, C\) countable\} is not \(\tau\)-continuous. Then, \((E, \tau)\) is not polarly barrelled.

To prove \(N_0\)-barrelledness of \((E, \tau)\), it suffices, by Proposition 3.2.ii), to show that this space is \(\ell_\infty\)-barrelled. So, let \((f_1, f_2, \ldots)\) be a pointwise bounded sequence in \((E, \tau)'\). Then \(M := \sup_n \|f_n\| < \infty\), where \(\|f_n\|\) is the norm of \(f_n\) as an element of \(\ell_\infty(I)\). Hence, for each \(n = 1, 2, \ldots\) there exists a countable subset \(C_n\) of \(I\) for which

\[|f_n(x)| \leq M \max_{m \in C_n} |x_m|\] for all \(x = (x_i)_{i \in I} \in E\).

Then, for the countable set \(C = \bigcup_n C_n \subset I\) we have that \(\sup_n |f_n(x)| \leq M p_C(x)\) for each \(x = (x_i)_{i \in I} \in E\). Therefore, the sequence \((f_1, f_2, \ldots)\) is \(\tau\)-equicontinuous and we are done.

Remarks 3.9 1. We clearly have that

\(E\) strictly of countable type \(\Rightarrow\) \(E\) of countable type

and that if \(E\) is a normed space, then the converse is also true.

More in general, if \(E\) is a metrizable space, then

\(E\) strictly of countable type \(\iff\) \(E\) of countable type

([1], p. 53).

2. But it follows from Theorem 3.5 that the non-metrizable space of countable type constructed in Example 3.8.2 is not strictly of countable type (for a direct proof of this last fact see [1], Example 2.5.2).
4 PERFECT SEQUENCE SPACES

A particularly interesting class of locally convex spaces being strictly of countable type, which we are going to consider in this section, are the perfect sequence spaces endowed with the associated normal topology (see below for the definition of this topology). Recall that a sequence space (i.e., a vector subspace of $K^N$) $\Lambda$ is called perfect if $\Lambda^{\times \times} = \Lambda$, where $\Lambda^{\times} := \{(b_n)_n \in K^N : \lim_n a_n b_n = 0$ for all $(a_n)_n \in \Lambda\}$ is the Köthe-dual of $\Lambda$.

This kind of spaces plays an important role for the development of a $p$-adic Quantum Mechanics (see e.g., [10] and [6]). For instance, if $B$ is an infinite matrix consisting of strictly positive real numbers $b^k_n (n, k \in N)$ and satisfying the conditions $b^k_n \leq b^{k+1}_n$ for all $n, k$, the non-archimedean Köthe space $K(B)$ associated with the matrix $B$ and defined by

$$K(B) = \{(\alpha_n)_n \in K^N : \lim_n |\alpha_n| b^k_n = 0, \text{ for all } k = 1, 2, \ldots\}$$

is a perfect sequence space (see [4]). For $b^k_n = k^n$, $K(B)$ is the space of entire functions on $K$, which is needed for the definition of a non-archimedean Laplace transform in [10] and [6].

For a perfect sequence space with Köthe-dual $\Lambda^{\times}$ a (separating) bilinear form on the dual pair $(\Lambda, \Lambda^{\times})$ is defined by

$$<a, b> = \sum_n a_n b_n \quad a = (a_n)_n \in \Lambda, \; b = (b_n)_n \in \Lambda^{\times}.$$ 

For each $n$, the sequence with 1 in the $n$-th place and 0's elsewhere will be denoted by $e_n$.

For $b = (b_n)_n \in \Lambda^{\times}$, a seminorm $p_b$ on $\Lambda$ is defined by $p_b(a) = \sup_n |a_n b_n|$, $a = (a_n)_n \in \Lambda$. The family of seminorms $\{p_b : b \in \Lambda^{\times}\}$ determines a Hausdorff locally convex topology on $\Lambda$. It is denoted by $\pi(\Lambda, \Lambda^{\times})$ and it is called the normal topology on $\Lambda$. The sequence $(e_n)_n$ forms a Schauder basis in $\langle \Lambda, \pi(\Lambda, \Lambda^{\times}) \rangle$ (recall that a sequence $(x_n)_n$ in a locally convex space $E$ is called a Schauder basis for $E$ if every $x \in E$ can be written uniquely as $x = \sum_n \lambda_n x_n$ where the coefficient functionals $f_n : x \in E \mapsto \lambda_n \in K$ are continuous). Therefore, $\langle \Lambda, \pi(\Lambda, \Lambda^{\times}) \rangle$ is strictly of countable type, in particular, it is of countable type. In fact, $\pi(\Lambda, \Lambda^{\times})$ is the finest topology of countable type on $\Lambda$ compatible with the dual pair $(\Lambda, \Lambda^{\times})$ ([5], Proposition 2.6). Also, $\langle \Lambda, \pi(\Lambda, \Lambda^{\times}) \rangle$ is complete ([2], Proposition 7, where $K$ is assumed to be spherically complete, but this result holds in general).

In this section we always assume that the perfect space $\Lambda$ (resp. $\Lambda^{\times}$) is endowed with the corresponding normal topology $\pi(\Lambda, \Lambda^{\times})$ (resp. $\pi(\Lambda^{\times}, \Lambda)$), which is the topology of uniform convergence on the compactoid sets of $\Lambda^{\times}$ (resp $\Lambda$, [11], Corollary 2.3).
For every infinite matrix $B$ as above, the associated Köthe space $K(B)$ endowed with the corresponding normal topology is a Fréchet (and hence barrelled) space. In fact, every non-archimedean countably normed Fréchet space with a Schauder basis can be identified with some $K(B)$ ([4], Proposition 2.4).

By using Theorem 3.5 we are going to give several characterizations of barrelledness (equivalently, $\ell^\infty$-barrelledness) for perfect sequence spaces, showing that this property is "dual" of the property of being semi-Montel (recall that a locally convex space $E$ is called semi-Montel if every bounded subset of $E$ is compactoid in $E$).

**Theorem 4.1** (Compare Proposition 20 of [2]) The following are equivalent.

i) $\Lambda$ is barrelled.

ii) $\beta(\Lambda, \Lambda^\times) = n(\Lambda, \Lambda^\times)$.

iii) $\Lambda^\times$ is semi-Montel.

iv) Every compactoid subset of $\Lambda$ is $\beta(\Lambda, \Lambda^\times)$-compactoid.

v) Every weakly convergent sequence in $\Lambda$ is $\beta(\Lambda, \Lambda^\times)$-convergent.

vi) The unit vectors $e_1, e_2, \ldots$ form a Schauder basis for $(\Lambda, \beta(\Lambda, \Lambda^\times))$.

vii) $(\Lambda, \beta(\Lambda, \Lambda^\times))$ is of countable type.

**Proof.** We only prove $i) \Rightarrow iii), iv) \Rightarrow v)$ and $vii) \Rightarrow i)$.

$i) \Rightarrow iii)$: Let $B$ a bounded (and hence equicontinuous) subset of $\Lambda^\times = (\Lambda, n(\Lambda, \Lambda^\times))'$. Then, on $B$, the weak*-topology $\sigma(\Lambda^\times, \Lambda)$ coincides with the topology of uniform convergence on the compactoids of $\Lambda$ ([13], Lemma 10.6). Hence, $B$ is compactoid in $\Lambda^\times$ ([13], Lemma 10.5).

$iv) \Rightarrow v)$: Let $(y_n)_n$ be a sequence converging to $y$ weakly on $\Lambda$. Since $\Lambda$ is of countable type, we have that this sequence is also convergent (to the same limit) in the normal topology of $\Lambda$. Then, $A = \{y_1, y_2, \ldots\} \cup \{y\}$ is a compactoid subset of $\Lambda$. By iv) and Theorem 1.4 of [14], on $A$ the weak topology $\sigma(\Lambda, \Lambda^\times)$ coincides with the strong topology $\beta(\Lambda, \Lambda^\times)$. Hence $(y_n)_n$ is $\beta(\Lambda, \Lambda^\times)$-convergent to $y$.

$vii) \Rightarrow i)$: Let $(f_1, f_2, \ldots)$ be a pointwise bounded (and hence bounded) sequence in $\Lambda^\times = (\Lambda, n(\Lambda, \Lambda^\times))'$. Since $(\Lambda, \beta(\Lambda, \Lambda^\times))$ is of countable type, it follows from Theorem 8.5 of [13] that $\{f_1, f_2, \ldots\}$ is a compactoid subset of $\Lambda^\times$. Then, $\{a \in \Lambda : |f_n(a)| \leq 1 \text{ for all } n\}$ is a zero neighbourhood in $\Lambda$. So, $(f_1, f_2, \ldots)$ is equicontinuous. Thus, $\Lambda$ is $\ell^\infty$-barrelled and hence barrelled by Theorem 3.5.

**Remarks 4.2** 1. It follows from Theorem 9.8 of [13] that if $K$ is not spherically complete, property i) of Theorem 4.1 is equivalent to

i') $\Lambda$ is reflexive.

But this is not true in general.

**Example.** Take $\Lambda = c_0$ (so, $\Lambda^\times = \ell^\infty$). One can easily see that the normal topology on $\Lambda$ coincides with the topology defined by the canonical norm on $c_0$. Hence, $\Lambda$ is a barrelled space, which is not reflexive if $K$ is spherically complete ([18], Theorem 4.16).
2. If $K$ is spherically complete, property ii) of Theorem 4.1 is equivalent to

\[ ii' \beta(\Lambda, \Lambda^\times) \text{ is compatible with the duality } (\Lambda, \Lambda^\times). \]

(see the proof of Proposition 20 of [2]).

But this is not true in general.

Example. Suppose that $K$ is not spherically complete and take $\Lambda = \ell^\infty$ (so, $\Lambda^\times = c_0$). Clearly $\beta(\Lambda, \Lambda^\times)$ coincides with the topology defined by the canonical norm on $\ell^\infty$ and hence this topology is compatible with the duality $(\ell^\infty, (\ell^\infty)^\times)$ ([18], Theorem 4.22). However, since $\beta(\Lambda, \Lambda^\times)$ is not of countable type, we have that $\beta(\Lambda, \Lambda^\times) \neq \pi(\Lambda, \Lambda^\times)$.

Recall that a Hausdorff locally convex space $E$ is called Montel if $E$ is polar, polarly barrelled and every closed bounded subset of $E$ is a complete compactoid.

Observe that $\Lambda$ is Montel if and only if $\Lambda$ is barrelled and semi-Montel (or equivalently, $\Lambda$ is reflexive and semi-Montel, [13], Theorems 9.6 and 10.3). Applying these results of [13], Theorem 2.1 of [8] and our Theorem 4.1, we obtain the following descriptions of the perfect sequence spaces that are Montel.

**Corollary 4.3** The following are equivalent.

i) $\Lambda$ is Montel.

ii) $\Lambda$ and $\Lambda^\times$ are reflexive (resp. barrelled, resp. semi-Montel).

iii) $\Lambda^\times$ is Montel.

iv) $\Lambda$ and $(\Lambda, \beta(\Lambda, \Lambda^\times))$ are semi-Montel.

v) Every $(\sigma(\Lambda, \Lambda^\times))$-bounded subset of $\Lambda$ is $\beta(\Lambda, \Lambda^\times)$-compactoid.

**Remarks 4.4** 1. If follows from Theorems 10.3 and 10.4 of [13] that if $K$ is spherically complete, properties i) $\implies$ v) of Corollary 4.3 are equivalent to each one of the following:

vi) $\Lambda$ is reflexive.

vii) $\Lambda^\times$ is reflexive.

But this is not true in general.

Example. Suppose that $K$ is not spherically complete. Take $\Lambda = c_0$ (see example in Remark 4.2.1). We have that $\Lambda$ is a reflexive space ([18], Theorem 4.17) which is not Montel.

2. It is known (see [11], Section 3) that the class of locally convex spaces $E$ having an orthogonal basis and such that $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ are sequentially complete, coincides with the class of spaces $E$ that are linearly homeomorphic to some perfect sequence space (recall that in [3] a Schauder basis $(x_n)_n$ of $E$ is said to be an orthogonal basis in $E$ if its topology is defined by a family $\mathcal{P}$ of non-archimedean seminorms satisfying the condition: if $x \in E$, $x = \sum \lambda_n x_n$ then $p(x) = \max p(\lambda_n x_n)$ for all $p \in \mathcal{P}$).

Hence, Theorem 4.1 (resp. Corollary 4.3) provides us characterizations of barrelledness (resp. Montelness) for those locally convex spaces $E$ with an orthogonal basis for which $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ are sequentially complete.
In this section we study barrelledness for spaces of finite type as related to finite dimensionality of bounded sets. Although these spaces are not in general metrizable or strictly of countable type (Remark 5.5.2 below), we show (Theorem 5.4) that for spaces of finite type barrelledness is equivalent to $\ell^\infty$-barrelledness (Compare Proposition 3.2 and Theorem 3.5).

In the sequel we assume that $E$ is a polar space.

The following two lemmas give characterizations of the properties "$E$ is of finite type" and "every bounded subset of $E$ is finite-dimensional" respectively, in terms of operators.

**Lemma 5.1** The following are equivalent.

i) $E$ is of finite type.

ii) $L(E, \ell^\infty) = F_i(E, \ell^\infty)$ (resp. $L(c_0, E) = F_i(c_0, E)$).

**Proof.** Clearly i) $\Rightarrow$ ii).

ii) $\Rightarrow$ i): Suppose that $L(E, c_0) = F_i(E, c_0)$. Let $(f_1, f_2, \ldots)$ be an equicontinuous sequence in $E'$ with $\lim_n f_n = 0$ in $\sigma(E', E)$. Then, $T \in L(E, c_0)$ given by $T(x) = (f_1(x), f_2(x), \ldots)$ ($x \in E$) has, by ii), finite-dimensional rank.

Then, the weak topology on $E$ coincides with the topology on $E$ of the uniform convergence on the equicontinuous sequences $(f_1, f_2, \ldots)$ in $E'$ with $\lim_n f_n = 0$ in $\sigma(E', E)$ and hence with the original topology on $E$ ([5], Theorem 3.8). This implies that $E$ is of finite type.

**Lemma 5.2** The following are equivalent.

i) Every bounded subset of $E$ is finite-dimensional.

ii) $E$ is sequentially complete and $L(\ell^\infty, E) = F_i(\ell^\infty, E)$ (resp. $L(c_0, E) = F_i(c_0, E)$).

**Proof.** Clearly i) $\Rightarrow$ ii).

ii) $\Rightarrow$ i): It is enough to see that for every bounded sequence $(x_1, x_2, \ldots)$ in $E$, $[x_1, x_2, \ldots]$ is finite-dimensional.

Let $(x_1, x_2, \ldots)$ be a bounded sequence in $E$ and let $(\lambda_1, \lambda_2, \ldots)$ be a sequence in $K \setminus \{0\}$ with $\lim_n |\lambda_n| = 0$. Then, $T : G \longrightarrow E$, $G = \ell^\infty$ (resp. $G = c_0$) given by $T(\alpha) = \sum_n \alpha_n \lambda_n x_n$ ($\alpha = (\alpha_n) \in G$) is a well-defined (by sequential completeness of $E$) continuous linear map. It follows by ii) that the rank of $T$ is finite-dimensional, and hence so is $[x_1, x_2, \ldots]$.

**Remark 5.3** The condition "$E$ is sequentially complete" in property ii) of Lemma 5.2 cannot be dropped in general.
Example. Take $E = c_{00}$ endowed with the canonical norm. By Example 2.7 of 
[7], $L(F, E) = F_i(F, E)$ for every Banach space $F$. But obviously $E$ does not satisfy 
condition i) of the lemma.

Now, with the aid of Lemmas 5.1 and 5.2 we can describe, among all the polar 
spaces spaces $E$, those that are barrelled of finite type.

**Theorem 5.4** The following are equivalent.

i) $E$ is a barrelled space of finite type.

ii) $E$ is an $\ell^\infty$-barrelled space of finite type.

iii) Every pointwise bounded subset of $E'$ is finite-dimensional.

iv) $\sigma(E, E') = \beta(E, E')$.

v) $E$ is $\ell^\infty$-barrelled and $L(E, \ell^\infty) = F_i(E, \ell^\infty)$ (resp. $L(E, c_0) = F_i(E, c_0)$).

vi) $(E', \sigma(E', E))$ is sequentially complete and $L(\ell^\infty, (E', \sigma(E', E))) = 
F_i(\ell^\infty, (E', \sigma(E', E)))$ (resp. $L(c_0, (E', \sigma(E', E))) = F_i(c_0, (E', \sigma(E', E)))$).

**Proof.** The equivalences ii) $\iff$ v) and iii) $\iff$ vi) are direct consequences of 
Lemmas 5.1 and 5.2 respectively.

ii) $\Rightarrow$ iii): Let $(f_1, f_2, \ldots)$ be a pointwise bounded sequence in $E'$, which is 
equicontinuous by $\ell^\infty$-barrelledness. Since $E$ is of finite type, there exist $g_1, \ldots, g_m \in 
E'$ $(m \in \mathbb{N})$ such that

$$
\sup_n |f_n(x)| \leq \sup_{i=1}^m |g_i(x)| \quad \text{for all } x \in E.
$$

Hence $\bigcap_{n=1}^m \ker g_i \subset \bigcap_n \ker f_n$, and so $[f_1, f_2, \ldots] \subset [g_1, \ldots, g_m]$ which implies that 
$[f_1, f_2, \ldots]$ is finite-dimensional and we are done.

iii) $\Rightarrow$ iv): Observe that by iii) every pointwise bounded subset of $E'$ is contained 
in the absolutely convex hull of a finite set of $E'$. Now, iv) follows easily.

iv) $\Rightarrow$ i): Since $E$ is polar, the original topology $\tau$ of $E$ is the topology of the 
uniform convergence on the equicontinuous subsets of $E'$. By iv) it follows that 
$\sigma(E, E') = \tau = \beta(E, E')$. Then, $E$ is an space of finite type which is polarly barrelled 
(Remark 2.4.1) and by (1) it is barrelled.

**Remarks 5.5** 1. There are non $\ell^\infty$-barrelled spaces of finite type $E$ such that $E'$ is 
$\sigma(E', E)$-sequentially complete.

**Example.** Take $E = (\ell^\infty, \sigma(\ell^\infty, c_0))$. Since every weakly convergent sequence in 
c_0 is convergent we have that $E'$ is $\sigma(E', E)$-sequentially complete. But applying 
Theorem 5.4.ii) $\Rightarrow$ iii) we deduce that $E$ is not $\ell^\infty$-barrelled.

2. There are barrelled spaces of finite type which are not strictly of countable 
type and so they are not metrizable (Remark 3.9.1).

**Example.** For each set $I$, let $E = K^I$ endowed with the product topology (which 
is of finite type, see [16],4). Let $\{e_i : i \in I\}$ be the unit vectors of $K^I$. It is well
known that $c_{00}(I) = \{e_i : i \in I\}$ is algebraically isomorphic to $E'$ through the map $c_{00}(I) \to E' : y = (y_i)_{i \in I} \in c_{00}(I) \mapsto g_y \in E'$, $g_y(x) = \sum_i x_i y_i$ ($x = (x_i)_{i \in I} \in E$).

We identify every $y \in c_{00}(I)$ with its image under this map.

First, we are going to see that for every set $I$, $K^I$ is a barrelled space. For that, take an infinite-dimensional sequence $(y_1, y_2, \ldots)$ in $c_{00}(I)$. Every $y_s$ ($s = 1, 2, \ldots$) can be written as $y_s = \sum_{j=1}^{n_s} \lambda_{s,j} e_j$ ($\lambda_{s,j} \in K$, $n_s \in \mathbb{N}$, $\lambda_{s,n_s} \neq 0$), where we can assume that $n_1 < n_2 < \ldots$.

We construct inductively a sequence $(\xi_s)_s$ in $K$ such that, for each $s \in \mathbb{N}$,

$$|\xi_1 \lambda_{s,n_1} + \xi_2 \lambda_{s,n_2} + \ldots + \xi_s \lambda_{s,n_s}| \geq s \quad (9)$$

and consider $x = (x_i)_{i \in I} \in K^I$ such that $x_i = 0$ if $i \notin \{n_1, n_2, \ldots\}$ and $x_i = \xi_m$ if $i = n_m$ for a certain $m$. By (9), the sequence $(y_s(x))_s$ is not bounded. Hence, every linearly independent sequence in $c_{00}(I) = E'$ is not pointwise bounded. Thus, applying Theorem 5.4 we conclude that $K^I$ is barrelled.

Now, we are going to see that if $\#I > \#K$, then $K^I$ is not strictly of countable type.

Assume that $K^I$ is strictly of countable type and that $\{x_1, x_2, \ldots\}$ is a countable subset whose linear hull is dense on $K^I$. Then, the map $(K^I)' \to K^N : f \mapsto (f(x_1), f(x_2), \ldots)$ is injective, which implies that $\#I \leq K^N = (#K)^{n_s} = #K$ ([17], Corollary 3.9).

3. Every complete space of finite type $E$ is barrelled.

Indeed, by Theorem 7 of [16] a such space $E$ is linearly homeomorphic to a power of $K$ and so it is barrelled (Remark 5.5.2).

4. However, there are barrelled spaces of finite type which are not complete.

Example. Let $\mathcal{T} = \{M \subset \mathbb{N} : \lim_n \frac{\#(M \cap [1,n])}{n} = 0\}$ and let $D$ be the subspace of $K^N$ defined by:

$$D = \{ (\alpha_1, \alpha_2, \ldots) \in K^N : \{n \in \mathbb{N} : \alpha_n \neq 0\} \in \mathcal{T} \}.$$ 

If we endow $D$ with the topology induced by the product topology of $K^N$, then $D$ is an space of finite type (see [16],1). Also, since $D$ is dense in $K^N$ we deduce that $D$ is not complete.

Like in Remark 5.5.2, we identify every element of $c_{00} (= c_{00}(\mathbb{N}))$with its image under the canonical map $c_{00} \to (C^0)^{\mathbb{N}} = D'$.

Now, we prove that $D$ is barrelled. By Theorem 5.4 it suffices to see that for every infinite-dimensional sequence $(f_1, f_2, \ldots)$ in $c_{00}$ there exists an $\alpha \in D$ such that $(f_1(\alpha), f_2(\alpha), \ldots)$ is an unbounded subset of $K$. For that, observe that every $f_s$ ($s = 1, 2, \ldots$) can be written as $f_s = \sum_{j=1}^{n_s} \lambda_{s,j} e_j$ ($\lambda_{s,j} \in K$, $n_s \in \mathbb{N}$, $\lambda_{s,n_s} \neq 0$), where we can assume that $n_1 < n_2 < \ldots$ and $\{n_1, n_2, \ldots\} \in \mathcal{T}$ (as usual, $e_1, e_2, \ldots$ are the unit vectors of $c_{00}$).

We construct inductively a sequence $(\xi_s)_s$ in $K$ such that (9) is satisfied for each $s \in \mathbb{N}$. Let $\alpha \in D$ be given by $\alpha_j = 0$ if $j \notin \{n_1, n_2, \ldots\}$ and $\alpha_j = \xi_m$ if $j = n_m$
for a certain \( m (j \in \mathbb{N}) \). Then, \(|f_s(\alpha)| \geq s\) for all \( s \in \mathbb{N} \), which implies that the set \( \{f_1(\alpha), f_2(\alpha), \ldots\} \) is not bounded.

Since every weakly bounded set of a polar space \( E \) is bounded, we can apply Lemma 5.2 and Theorem 5.4 to derive the following.

**Corollary 5.6** The following are equivalent.

1. \((E', \sigma(E', E))\) is barrelled.
2. \((E', \sigma(E', E))\) is \( \ell^\infty \)-barrelled.
3. Every bounded subset of \( E \) is finite-dimensional.
4. \( \beta(E', E) = \sigma(E', E) \).
5. \( E \) is (weakly) sequentially complete and \( L(\ell^\infty, E) = Fi(\ell^\infty, E) \) (resp. \( L(c_0, E) = Fi(c_0, E) \)).

Property iii) of Corollary 5.6 is very useful to give examples of spaces \( E \) for which \((E', \sigma(E', E))\) (which is always of finite type) is (and is not) barrelled.

**Examples 5.7**

1. No infinite-dimensional metrizable space \( E \) satisfies iii).

2. If \( \Lambda \) is a perfect sequence space containing a sequence \((\alpha_n)_n\) with an infinite number of non-trivial components, then \( \{(\beta_n)_n \in K : |\beta_n| \leq |\alpha_n| \text{ for all } n\} \) is an infinite-dimensional bounded subset of \( \Lambda \) and so property iii) is not true for this \( \Lambda \).

3. If \( E \) is sequentially complete and \( \dim E < \#K \) (where \( \dim E \) denotes the algebraic dimension of \( E \)), then \( E \) satisfies iii) (see [9], Proposition 2.2).

4. Let \((E', \tau_0)\) be a \( K \)-vector space endowed with the strongest locally convex topology \( \tau_0 \) and let \( \tau \) be a polar locally convex topology on \( E \) such that \( \sigma(E, E') \leq \tau \leq \tau_0 \). Then, \((E, \tau)\) satisfies iii) and so \((E', \sigma(E', E))\) is barrelled. Hence, for every polar space \( E \) for which \( E' = E^* \) (e.g. any space with the strongest topology of finite type) we have that \((E', \sigma(E', E))\) is barrelled.

5. However, there are polar spaces \( E \) such that \((E', \sigma(E', E))\) is barrelled and \( E' \neq E^* \).

**Example.** Let \( E = c_00 \) as \( K \)-vector space and let \( e_1, e_2, \ldots \) be the unit vectors of \( E \). It is well known that \( K^N \) is algebraically isomorphic to \( E^* \) through the map \( K^N \rightarrow E^*: y = (y_n)_n \in K^N \mapsto g_y \in E^*, g_y(x) = \sum_n x_n y_n \ (x = (x_n)_n \in E) \). We identify every \( y \in K^N \) with its image under this map.

Let \( Y \) and \( D \) be as in Remark 5.5.4. Let us introduce the topology \( \tau \) on \( E \) defined by the family of seminorms \( \{p_f : f \in D\} \), where for each \( f \in D \) the seminorm \( p_f \) is given by \( p_f(x) = |f(x)| \ (x \in E) \). Then, \((E, \tau)\) is a polar Hausdorff locally convex space for which \( E' = D \) (and hence \( E' \neq E^* \)). Also, we have proved in Remark 5.5.4 that \((E', \sigma(E', E)) = (D, \sigma(D, E))\) is barrelled.
References


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