NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

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ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.
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INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < \|x\| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $| \cdot |$, and $E$ is a normed $K$-vector space, where we assume $\| \cdot \|$ to satisfy the strong triangle inequality $\|x + y\| \leq \max(\|x\|, \|y\|)$. We write $|K^*| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0, r) := \{x \in E : \|x\| \leq r\}$, $B_E := B_E(0, 1)$.

$E'$ is the space of all linear continuous functions $E \to K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $\|x\| = \sup\{|f(x)| : f \in E', |f| \leq \|f\|\}$ (for all $x \in E$), in other words, if $\gamma : E \to E''$ is an isometry. $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology
on $E'$ making all evaluation maps $f \mapsto f(a)$ \((a \in E)\) continuous. For $X \subseteq E'$ we denote its $w'$-closure by $\overline{X}^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w'$-PRECOMPACT SETS

**LEMMA 1.1.** Let $X$ be a bounded subset of $E'$. Then \(\{x \in E : \inf_{f \in X} |f(x)| > 0\}\) is open in $E$.

**Proof.** $X$ is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{x \in E : |f(x)| > \frac{1}{n} \} \subseteq X$ is open. Then so is \(\bigcup_{n \in \mathbb{N}} U_n = \{x \in E : \inf_{f \in X} |f(x)| > 0\}\).

**LEMMA 1.2.** Let $K$ be not locally compact. Let $X \subseteq E'$ and $a \in E$ be such that $X(a) := \{f(a) : f \in X\}$ is precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed and where $g \in E' \setminus U$. Then for any $e > 0$ there exists a $b \in E$ for which $||a - b|| \leq e$ and $\inf_{f \in X} |f(b)| > 0$.

**Proof.** There exists an $r \in |K|$ such that $B_{E'}(0,r) \subseteq U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K'$ given by $a \sim b$ if $|a - \beta| < |\beta|'$ yields an open partition of $C := \{\lambda \in K : |\delta|r \leq |\lambda| \leq re\}$ that is infinite because $K$ is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\lambda \in C$ such that

\[ |f(a) - \lambda| \geq |\lambda| \quad (f \in X).\]

$U$ is $w'$-closed and edged, $g \not\subseteq U$, so by [6], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_{E'}(0,r)$ so $||a - b|| = ||c|| = ||f(c)|| \leq |\gamma||r|^{-1} \leq e$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(c) - \gamma| = |f(c) - g(c)| = |u(c)| < |\gamma|$. This, combined with \((*)\), yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

**COROLLARY 1.3.** Let $K$ be not locally compact, let $E$ be a Banach space. Let $X \subseteq E'$ be $w'$-precompact. Suppose $X \subseteq g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed, $g \in E' \setminus U$. Then \(\{x \in E : \inf_{f \in X} |f(x)| > 0\}\) is open and dense in $E$.

**Proof.** Just combine Lemmas 1.1 ($w'$-precompactness implies $w'$-boundedness hence norm boundedness by completeness) and 1.2.

**DEFINITION 1.4.** Let us call $X \subseteq E'$ $\sigma$-decomposable in $E'$ if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E'$ such that each $U_n$ is $w'$-closed and $X \subseteq \bigcup_{n \in \mathbb{N}} (f_n + U_n)$, $g \not\subseteq \bigcup_{n \in \mathbb{N}} (f_n + U_n)$.

**THEOREM 1.5.** (SEPARATION THEOREM) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subseteq E'$ be $w'$-precompact and $\sigma$-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.

**Proof.** Without loss, assume $g = 0$. Let \(\{f_n + U_n : n \in \mathbb{N}\}\) be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set \(\{x \in E : \inf_{f \in X} |f(x)| > 0\}\) is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem \(\{x \in E : f(x) \neq 0 \text{ for all } f \in X\} \supset \bigcap_{n \in \mathbb{N}} \{x \in E : \inf_{f \in X_n} |f(x)| > 0\} \neq \emptyset$.

**REMARK.** It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subseteq E$ be weakly precompact and $\sigma$-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \neq f(X)$. Here, $X$ is called $\sigma$-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subseteq \bigcup_{n \in \mathbb{N}} (x_n + U_n)$, a $\not\subseteq \bigcup_{n \in \mathbb{N}} (x_n + U_n)$.
COROLLARY 1.6. Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be $\sigma$-decomposable in $E'$. Suppose $X(a) := \{f(a) : f \in X\}$ is compact for all $a \in E$. Then $X$ is $w'$-compact.

Proof. The map $f \mapsto \{f(a)\}_{a \in E}$ is a homeomorphism of $(E', w')$ onto a subspace of $K^E$. The image of $X$ lies in the compact subset $\prod_{a \in E} X(a)$ so $X$ is $w'$-precompact. Since $E'$ is $w'$-quasicomplete by the $p$-adic Alaoglu Theorem [8], 3.1, it suffices to show that $X$ is $w'$-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \not\in X(a)$. Now $X(a) \subset \overline{X}^{w'}(a) \subset \overline{X(a)} = X(a)$, so $g(a) \not\in \overline{X}^{w'}(a) \iff g \not\in \overline{X}^{w'}$. 

To find examples of $\sigma$-decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. Let $n \in \mathbb{N}$, let $D$ be an $n$-dimensional subspace of $E'$. Then for each $t \in (0, 1)$ there exist $a_1, a_2, \ldots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| > t||f||$ ($f \in D$).

Proof. First assume that the valuation of $K$ is dense. The space $H := \{x \in E : f(x) = 0 \text{ for all } f \in D\}$ has codimension $n$ in $E$. Choose $s \in (t, 1)$ and let $g_1, g_2, \ldots, g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} \leq ||g_i|| \leq t^{-1}$ for $i \in \{1, \ldots, n\}$. There exist $b_1, b_2, \ldots, b_n \in E/H$ such that $g_i(b_j) = \delta_{ij}$ ($i, j \in \{1, \ldots, n\}$). Let $i \in \{1, \ldots, n\}$, let $g = \sum \lambda_i g_i \in (E/H)'$. Then $||g|| \geq \sqrt{s} \max_j |\lambda_j| ||g_i||$ and $|g(b_i)| = |\lambda_i|$ so $|g(b_i)| \leq \max_j |\lambda_j| \leq s \max_j |\lambda_j| ||g|| \leq \sqrt{s}||g||$. So $||g|| < 1$. Thus, with $\pi : E \to E/H$ denoting the canonical quotient map, there exist $a_1, a_2, \ldots, a_n \in B_E$ with $||a_i|| = ||b_i||$ for each $i$. The adjoint $\pi'$ of $\pi$ maps $(E/H)'$ isometrically onto $D$. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $||g|| = ||f||$. We have, writing $g = \sum \lambda_i g_i$, $\max_{1 \leq i \leq n} |f(a_i)| = \max_j |g(b_i)| = \max_j |\lambda_i| \geq t \max_j |\lambda_i| ||g||$ $= t ||\Sigma \lambda_i g_i|| = t ||g|| = t ||f||$. 

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the $b_i$ have norm $\leq 1$ (rather than $< 1$), but one can use that $E/H$ is a strict quotient i.e. there exist $a_1, a_2, \ldots, a_n \in E$ with $||a_i|| = ||b_i||$ and $\pi(a_i) = b_i$ for each $i$.

LEMMA 1.8. Let $D$ be a subspace of $E'$, $D$ of countable type. Then there is a sequence $a_1, a_2, \ldots, a_\mathbb{N} \in B_E$ such that $\max_{a \in D} |f(a)| > t||f||$ for all $f \in D$.

Proof. Let $D_1 \subset D_2 \subset \ldots$ be finite-dimensional subspaces of $D$, $\bigcup_{n} D_n$ is dense in $D$. Let $t \in (0, 1)$. By Lemma 1.7 there exists a finite set $F_n \subset B_E$ such that $\max_{a \in F_n} |f(a)| > t||f||$ for all $f \in D_n$. So, for $F' := \bigcup_{n} F_n$ we obtain 

\[(*) \quad \max_{a \in F'} |f(a)| > t||f|| \quad (f \in \bigcap_{n} D_n).\]

Now $F := \bigcup_{f \in F'} F'$ is countable and $(*)$ implies $||f|| = \sup_{a \in F} |f(a)|$ for all $f \in \bigcup_{n} D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. Let $X \subset E'$ be such that $X(a) := \{f(a) : f \in X\}$ is separable for each $a \in E$ and $\{X\}$ is of countable type. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. Then $D := \{g\} \cup X$ is of countable type so by Lemma 1.8 there exist $a_1, a_2, \ldots, a_\mathbb{N} \in B_E$ such that 

\[(*) \quad \max_{a \in E} |h(a)| > t \quad (h \in D).\]

For each $m, n \in \mathbb{N}$ the set $U_{mn} := \{h \in E' : |h(a_m)| \leq \frac{1}{m}\}$ is an edged $w'$-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering $F_m$ no member of which contains $g$. Then $\bigcup_{m} F_m$ still avoids $g$; it remains to be shown that it covers $X$. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all
m, n so \( |f(a_n) - g(a_n)| = 0 \) for all n. Now \( f - g \in D \), so by (\(*\)) we obtain \( \|f - g\| = 0 \) i.e. \( f = g \), a contradiction since \( g \not\in X \).

**Corollary 1.10.** Let \( X \subseteq E' \). If \( X \) is norm precompact, or \( X \) is \( w'\)-precompact and \( |X| \) is of countable type, then \( X \) is \( \sigma\)-decomposable in \( E' \).

**Proposition 1.11.** Let \( X \subseteq E' \) be such that \( X(a) \) is separable for each \( a \in E \). Suppose that for each \( h \in \overline{X} \) the set \( X \cup \{h\} \) is \( w'\)-metrizable. Then \( X \) is \( \sigma\)-decomposable in \( E' \).

**Proof.** Let \( g \in E' \setminus X \). If \( g \not\in \overline{X} \) then there exists a \( w'\)-zero neighbourhood \( U \) such that \((g + U) \cap X = \emptyset \). We may assume that \( U \) is of the form \( \{f \in E' : |f(a_1)| \leq \varepsilon, \ldots, |f(a_n)| \leq \varepsilon\} \) for some \( \varepsilon > 0 \), \( n \in \mathbb{N} \), \( a_1, \ldots, a_n \in E \). Then \( U \) is \( w'\)-closed and edged. By separability of \( X(a_1) \times \ldots \times X(a_n) \) only countably many of the cosets \( f + U : f \in X \) cover \( X \) and none of them contains \( g \). Now let \( g \in \overline{X} \). By \( w'\)-metrizability there exist \( w'\)-neighbourhoods of zero \( U_1 \supset U_2 \supset \ldots \) such that \( X \cap (g + U_n) = \emptyset \). We may suppose that the \( U_n \) are \( w'\)-closed and edged. By separability, like above, for each \( n \) the set \( X \setminus (g + U_n) \) is covered by countably many additive cosets of \( U_n \) none of them containing \( g \). Their union is a countable covering of \( X \) avoiding \( g \).

### 2. EBERLEIN-ŠMULIAN THEORY

We now apply the theory of §1. Recall ([5], p. 57) that \( E \) is said to have property \( (*) \) if for each subspace \( D \) of countable type, every \( f \in D' \) has an extension \( \overline{f} \in E' \). By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete \( K \) has \( (*) \). For general \( K \), spaces with a base, in particular spaces of countable type, have \( (*) \) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that \( E \) is assumed to be normpolar.

**Theorem 2.1.** Let \( K \) be not locally compact, let \( X \) be a subset of \( E \) such that \( f(X) \) is compact for all \( f \in E' \). Then each one of the following properties implies that \( X \) is weakly compact and weakly metrizable.

\begin{itemize}
  \item[(i)] \( E \) has property \( (*) \).
  \item[(ii)] \( E' \) is of countable type.
  \item[(iii)] \( |X| \) is of countable type.
\end{itemize}

Moreover, in case (i) \( X \) is norm compact and the weak and norm topology coincide on \( X \).

**Proof.** The natural isometry \( j : E \to E'' \) is easily seen to be a homeomorphism of \( E \) with the weak topology onto \( j(E) \) with the restriction of the \( w'\)-topology \( \sigma(E'', E') \). We show that \( j(X) \) is \( \sigma\)-decomposable in \( E'' \). First note that the predual \( E' \) is normpolar. In case (i), from weak precompactness of \( X \) it follows that \( X \) is norm precompact by [7], Th. 3 (the assumption made throughout [7] that \( E \) is complete is easily seen to be superfluous here). So \( j(X) \) is norm precompact in \( E'' \) and therefore \( \sigma\)-decomposable by Corollary 1.10. For case (ii) observe that every \( (w'\)-) bounded subset of \( E'' \) is \( w'\)-metrizable ([8], 6.1) which applies to \( j(X) \cup \{\theta\} \) for any \( \theta \in E'' \). For each \( f \in E' \) the set \( j(X)(f) = f(X) \) is compact hence separable so \( j(X) \) is \( \sigma\)-decomposable in \( E'' \) by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, \( j(X) \) is \( \sigma\)-decomposable, and from Corollary 1.6 we conclude that \( j(X) \) is \( w'\)-compact, so \( X = j^{-1}(j(X)) \) is \( w\)-compact. Observe that \( X \) is \( w\)-bounded hence bounded by normpolarity ((6), 7.7).

We have seen in passing that \( j(X) \) is \( w'\)-metrizable in case (ii), so \( X \) is weakly metrizable. Now let \( X \) satisfy (iii). Then \( j(X) \) is of countable type so by Lemma 1.8 there exist \( f_1, f_2, \ldots \in B_E' \) such that \( \|j(x)\| = \sup_{n \in \mathbb{N}} |f_n(x)| \) for all \( x \in X \). The formula \( d(x, y) = \sup_{n \in \mathbb{N}} |f_n(x) - f_n(y)|2^{-n} \) defines an ultrametric \( d \) on \( X \) (if \( d(x, y) = 0 \) then \( |f_n(x) - f_n(y)| = 0 \) for all \( n \) so \( \|x - y\| = 0 \)). By boundedness of \( X \) the induced topology is weaker than the weak topology on \( X \), but by
weak compactness these topologies coincide and so $X$ is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on $X$ the weak and norm topology coincide, and that therefore $X$ is norm compact and $w$-metrizable.

**REMARKS.**

1. If $K$ is not spherically complete the space $\ell^\infty$ does not have property $(\ast)$ ([4], 4.15 $(\delta) \Rightarrow (\gamma)$) but since $(\ell^\infty)' \simeq c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $\ell^\infty \widehat{\otimes} \ell^\infty$ ([3], 2.3) and the space $D$ of [4], 4.1.

2. Let $K$ be not spherically complete, let $E := \ell^\infty$, let $X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty$, when $e_1, e_2, \ldots$ are the unit vectors. Then (ii) and (iii) above hold. $X$ is weakly compact (since $\lim_{n \to \infty} e_n = 0$ weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := \mathbb{N}$ with the product topology. Then $Ef = \mathbb{N}$. Let $X := \{e_1, e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the unit vectors of $\mathbb{N}$. Then $E$ is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large $n$, so $f(X)$ is finite (hence compact) and contains 0. Yet, $X$ is not (weakly) compact as $0 = w - \lim_{n \to \infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (p-adic Eberlein-Smulian Theorem I) Let $K$ be not locally compact and let $X$, $E$ satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

1. $X$ is weakly compact.
2. $X$ is weakly sequentially compact.
3. $X$ is weakly countably compact.

**Proof.** Each one of the properties (a), (b), (c) implies compactness of $f(X)$ for all $f \in E'$. By Theorem 2.1 $X$ is weakly metrizable and from that the equivalence of (a), (b), (c) follows easily.

**NOTE.** In Corollary 2.2, (a), (b), (c) are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if $K$ is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over $K$ has $(\ast)$. Hence, $X$ is a norm compactoid (then $X$ is weakly metrizable since the norm and weak topology coincide on $X$ ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0,1]$ and a $t$-orthogonal sequence $e_1, e_2, \ldots$ in $X$ such that $\inf_{n \to \infty} ||e_n|| > 0$. By (c) there is a weak accumulation point $a$ of $\{e_1, e_2, \ldots\}$. This $a$ is in the weak closure $D$ of $\{e_1, e_2, \ldots\}$ which equals the norm closure, so $a = \sum_{i=1}^{\infty} \lambda_i e_i$ where $||\lambda_i e_i|| \to 0$. If $\lambda_j \neq 0$ for some $j$, let $U := \{x \in E : |\delta_j(x)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the $j$th coordinate function $\Sigma \xi_j e_i \mapsto \xi_j$ on $D$. Then $a + U$ is a weak neighbourhood of $a$ but for each $n \in \mathbb{N}, n \neq j$ we have $|\delta_j(a - e_n)| = |\lambda_j|$ so $e_n \notin a + U$, a contradiction. Hence, $a = 0$. But then $\{x \in E : |f(x)| < 1\}$ is a weak neighbourhood of $a$ containing no $e_n$ if $f \in E'$ is such that $f(e_n) = 1$ for all $n$. Contradiction, so $X$ is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces $E$ and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let \( X \subseteq E \). Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) \( X \) is weakly relatively compact. (b) \( X \) is weakly relatively sequentially compact. (c) \( X \) is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets \( X \) of \( \ell^\infty(I) \) where \( I \) has at least the cardinality of the continuum, but is non-measurable, and where \( K \) is not spherically complete. The \( I \)-valued characteristic function of a subset \( S \subseteq I \) is denoted \( \chi_S \) and is given by \( \chi_S(x) := 1 \) if \( x \in S \), \( \chi_S(x) := 0 \) if \( x \in I \setminus S \).

1. Let \( X := \{ \chi_S : S \subseteq I \} \). Then \( X \) is a weakly compact but not weakly sequentially compact subset of \( \ell^\infty(I) \).
   Proof. \( X \) is bounded and since \( \ell^\infty(I)' = c_0(I) \) (cf.4.21) the weak topology on \( X \) is the topology of pointwise convergence. Clearly the map \( f \mapsto (f|_S)_{S \subseteq I} \) is a homeomorphism \( X \to \{0,1\}^I \), hence \( X \) is weakly compact. To prove that \( X \) is not weakly sequentially compact, let \( \phi : I \to Y \) be a surjection where \( Y := \{ \chi_A : A \subseteq N \} \subseteq \ell^\infty \). The formula \( \phi(x) = (\chi_{S_1}(x), \chi_{S_2}(x), \ldots) \) \( (x \in I) \) defines subsets \( S_1, S_2, \ldots \) of \( I \). If \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a subsequence of \( \xi_{S_1}, \xi_{S_2}, \ldots \), then, by surjectivity of \( \phi \), there is an \( x \in I \) for which \( (\chi_{S_1}(x), \chi_{S_2}(x), \ldots) = (1,0,1,0,1,\ldots) \), so the subsequence is not weakly convergent.

2. Let \( Z := \{ \chi_S : S \subseteq I, S \text{countable} \} \subseteq \ell^\infty(I) \). Then \( Z \) is weakly sequentially compact but not weakly compact.
   Proof. Clearly the weak closure of \( Z \) equals \( X \) of above, so \( Z \) is not weakly compact. On the other hand, if \( \xi_{S_1}, \xi_{S_2}, \ldots \) is a sequence in \( Z \) then \( S := \cup S_n \) is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of \( S \), hence at all points of \( I \), to an element of \( Z \).

3. Let \( T := \{ \xi_{i_n} : i \in I \} \subseteq \ell^\infty(I) \). Then \( f(T) \) is compact for all \( f \in \ell^\infty(I)' \) but \( T \) is not weakly countably compact.
   Proof. Let \( f \in \ell^\infty(I)' \). As \( \ell^\infty(I)' = c_0(I) \) we have that \( f(\xi_{i_n}) = 0 \) except for \( i \in \{i_1, i_2, \ldots \} \) where we may assume the \( i_n \in I \) to be distinct. Then \( \xi_{i_n} \rightharpoonup 0 \) weakly so \( T_i := \{0\} \cup \{\xi_{i_n} : n \in N\} \) is weakly compact and \( f(T_i) = f(T) \) is compact. However the only weak accumulation point of \( \{\xi_{i_1}, \xi_{i_2}, \ldots \} \) is \( 0 \not\in T \) so that \( T \) is not weakly countably compact.

REFERENCES