NON-ARCHIMEDEAN EBERLEIN-ŠMULIAN THEORY

T. KIYOSAWA
Faculty of Education
Shizuoka University
Ohya, Shizuoka, 422 Japan

W.H. SCHIKHOF
Department of Mathematics
University of Nijmegen, Toernooiveld
6525 ED Nijmegen, The Netherlands

(Received December 15, 1994 and in revised form June 8, 1995)

ABSTRACT. It is shown that, for a large class of non-archimedean normed spaces $E$, a subset $X$ is weakly compact as soon as $f(X)$ is compact for all $f \in E'$ (Theorem 2.1), a fact that has no analogue in Functional Analysis over the real or complex numbers. As a Corollary we derive a non-archimedean version of the Eberlein-Šmulian Theorem (2.2 and 2.3, for the 'classical' theorem, see [1], VIII, §2 Theorem 1 and Corollary, page 219).

KEY WORDS AND PHRASES. Non-archimedean Banach space, weak compactness.
1991 AMS SUBJECT CLASSIFICATION. 46S10

INTRODUCTION

Let $E$ be a two-dimensional normed space over $\mathbb{R}$ or $\mathbb{C}$ and let $X := \{x \in E : 0 < ||x|| \leq 1\}$. Each $f \in E'$ has zeros on $X$, so $f(X) = f(\{0\} \cup X)$ is compact, while obviously $X$ is not. The same story can be told when we replace $\mathbb{R}$ or $\mathbb{C}$ by a complete non-trivially valued non-archimedean field $K$ that is locally compact. However, if $K$ is not locally compact then, under reasonable conditions, for a subset $X$ of a normed space $E$ over $K$ compactness of $f(X)$ for all $f \in E'$ implies weak compactness of $X$ (we point out that if such an $X$ has more than one point it cannot be convex). To prove this curious fact (in §2) we shall develop some machinery in §1.

PRELIMINARIES

Throughout $K$ is a non-trivially non-archimedean valued field which is complete with respect to the metric induced by the valuation $| |$, and $E$ is a normed $K$-vector space, where we assume $|| |$ to satisfy the strong triangle inequality $||x + y|| \leq \max(||x||, ||y||)$. We write $|K^x| := \{|\lambda| : \lambda \in K, \lambda \neq 0\}$, $B_E(0,r) := \{x \in E : ||x|| \leq r\}$, $B_E := B_E(0,1)$.

$E'$ is the space of all linear continuous functions $E \rightarrow K$. Equipped with the norm $f \mapsto \sup\{|f(x)| : x \in B_E\}$ it is a Banach space (i.e. a complete normed space). $E$ is called normpolar if the norm is polar i.e. if $||x|| = \sup\{|f(x)| : f \in E', |f| \leq ||f||\}$ (for $x \in E$), in other words, if $g : E \rightarrow E''$ is an isometry. $E'$ is always normpolar. We assume throughout this note that $E$ is normpolar.

A subset $A$ of a (normed) space $E$ is absolutely convex if it is a module over $B_K$. A set $X \subset E$ is convex if it is either empty or an additive coset of an absolutely convex set. A subset $A$ of $E$ is called edged if it is absolutely convex and, in case the valuation of $K$ is dense, $A = \bigcap\{\lambda A : \lambda \in K, |\lambda| > 1\}$. The weak topology $w = \sigma(E, E')$ is the weakest topology on $E$ making all $f \in E'$ continuous. The weak-star topology $w' = \sigma(E', E)$ is the weakest topology
on $E'$ making all evaluation maps $f \mapsto f(a)$ $(a \in E)$ continuous. For $X \subset E'$ we denote its $w'$-closure by $X^{w'}$.

For other notions used in this paper we refer to [4].

1. SEPARATION OF $w'$-PRECOMPACT SETS

**Lemma 1.1.** Let $X$ be a bounded subset of $E'$. Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open in $E$.

**Proof.** $X$ is equicontinuous, so for each $n \in \mathbb{N}$ the set $U_n := \{x \in E : |f(x)| > \frac{1}{n} \text{ for all } f \in X\}$ is open. Then so is $\bigcup_{n} U_n = \{x \in E : \inf_{f \in X} |f(x)| > 0\}$.

**Lemma 1.2.** Let $K$ be not locally compact. Let $X \subset E'$ and $a \in E$ be such that $X(a) := \{f(a) : f \in X\}$ is precompact. Suppose $X \subset g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed and where $g \in E' \setminus U$. Then for any $\varepsilon > 0$ there exists a $b \in E$ for which $|a - b| \leq \varepsilon$ and $\inf_{f \in X} |f(b)| : f \in X > 0$.

**Proof.** There exists an $r \in |K^*|$ such that $B_{E'}(0, r) \subset U$. Choose $\delta \in K, 0 < |\delta| < 1$. The equivalence relation $\sim$ on $K^*$ given by $\alpha \sim \beta$ iff $|\alpha - \beta| < |\beta|$ yields an open partition of $C := \{\lambda \in K : |\delta| \leq |\lambda| \leq |\delta|\}$ that is infinite because $K$ is not locally compact. By precompactness $X(a)$ cannot meet each equivalence class and there exists a $\eta \in C$ such that

\[|f(a) - \gamma| \geq |\gamma| \quad (f \in X).\]

$U$ is $w'$-closed and edged, $g \notin U$, so by [6], 4.8 there exists a $c \in E$ such that $g(c) = \gamma, |f(c)| < |\gamma|$ for all $f \in U$. Set $b := a - c$. We have $|f(c)| \leq |\gamma|$ for all $f \in B_{E'}(0, r)$ so $|a - b| = ||c|| = |b(c)| \leq |\gamma| - 1 \leq \varepsilon$. For each $f \in X$, writing $f = g + u$ where $u \in U$, we obtain $|f(c) - \gamma| = |f(c) - g(c)| = |u(c)| < |\gamma|$. This, combined with (*), yields $|f(a) - \gamma| > |f(c) - \gamma|$ for all $f \in X$, so $|f(b)| = |f(a) - f(c)| = \max(|f(a) - \gamma|, |f(c) - \gamma|) = |f(a) - \gamma| \geq |\gamma|$. It follows that $\inf_{f \in X} |f(b)| > 0$.

**Corollary 1.3.** Let $K$ be not locally compact, let $E$ be a Banach space. Let $X \subset E'$ be $w'$-precompact. Suppose $X \subset g + U$ where $U$ is an edged zero neighbourhood in $E'$, $U$ $w'$-closed, $g \in E' \setminus U$. Then $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open and dense in $E$.

**Proof.** Just combine Lemmas 1.1 ($w'$-precompactness implies $w'$-boundedness hence norm boundedness by completeness) and 1.2.

**Definition 1.4.** Let us call $X \subset E'$ $\sigma$-decomposable in $E'$ if for each $g \in E' \setminus X$ there exist $f_1, f_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E'$ such that each $U_n$ is $w'$-closed and $X \subset \bigcup_n f_n + U_n, g \notin \bigcup_n (f_n + U_n)$.

**Theorem 1.5.** (Separation Theorem) Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be $w'$-precompact and $\sigma$-decomposable in $E'$. Then for each $g \in E' \setminus X$ there exists an $a \in E$ such that $g(a) \neq f(a)$ for all $f \in X$.

**Proof.** Without loss, assume $g = 0$. Let \{f_n + U_n : n \in \mathbb{N}\} be a covering of $X$ like in Definition 1.4. By Corollary 1.3 for each $n \in \mathbb{N}$ the set $\{x \in E : \inf_{f \in X} |f(x)| > 0\}$ is open and dense in $E$, where $X_n := X \cap (f_n + U_n)$. By completeness and the Baire Category Theorem $\{x \in E : f(x) \neq 0 \text{ for all } f \in X\} = \bigcap_n \{x \in E : \inf_{f \in X_n} |f(x)| > 0\} \neq \emptyset$.

**Remark:** It is not hard, by modifying 1.1 - 1.5, to prove the following dual form of this separation theorem. Let $K$ be not locally compact, let $X \subset E$ be weakly precompact and $\sigma$-decomposable in $E$ (see below). Then for each $a \in E \setminus X$ there exists an $f \in E'$ such that $f(a) \notin f(X)$. Here, $X$ is called $\sigma$-decomposable in $E$ if for each $a \in E \setminus X$ there exist $x_1, x_2, \ldots \in X$ and edged zero neighbourhoods $U_1, U_2, \ldots$ in $E$ such that each $U_n$ is weakly closed and $X \subset \bigcup_n (x_n + U_n), a \notin \bigcup_n (x_n + U_n)$. 


COROLLARY 1.6. Let $K$ be not locally compact, let $E$ be a Banach space, let $X \subset E'$ be \sigma-decomposable in $E'$. Suppose $X(a) := \{ f(a) : f \in X \}$ is compact for all $a \in E$. Then $X$ is $w'$-compact.

Proof. The map $f \mapsto (f(a))_{a \in E}$ is a homeomorphism of $(E', w')$ onto a subspace of $K^E$. The image of $X$ lies in the compact subset $\prod_{a \in E} X(a)$ so $X$ is $w'$-precompact. Since $E'$ is $w'$-quasicomplete by the p-adic Alaoglu Theorem [8], 3.1, it suffices to show that $X$ is $w'$-closed. To this end, let $g \in E' \setminus X$. By Theorem 1.5 there exists an $a \in E$ such that $g(a) \not\in X(a)$. Now $X(a) \subset \overline{X'}(a) \subset \overline{X(a)} = X(a)$, so $g(a) \not\in \overline{X'}(a)$ i.e. $g \not\in \overline{X'}$.

To find examples of $\sigma$-decomposable sets (in 1.9-1.11) we need the following Lemmas.

LEMMA 1.7. Let $n \in \mathbb{N}$, let $D$ be an $n$-dimensional subspace of $E'$. Then for each $t \in (0, 1)$ there exist $a_1, a_2, \ldots, a_n \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| \geq t||f|| \quad (f \in D)$.

Proof. First assume that the valuation of $K$ is dense. The space $H := \{ x \in E : f(x) = 0 \text{ for all } f \in D \}$ has codimension $n$ in $E$. Choose $s \in (t, 1)$ and let $g_1, \ldots, g_n$ be a $\sqrt{s}$-orthogonal base of $(E/H)'$ such that $s^{-1} \leq ||g_i|| \leq t^{-1}$ for $i \in \{1, \ldots, n\}$. There exist $b_1, \ldots, b_n \in E/H$ such that $g_i, b_i = \delta_{ij}, i, j \in \{1, \ldots, n\}$. Let $s \in \{1, \ldots, n\}$, let $g = \sum_{i, j} \lambda_i g_i, b_i \in (E/H)'$. Then $||g|| \geq \sqrt{s} \max_{j \in \{1, \ldots, n\}} |g_j| ||b_j||$ and $|g(b_i)| = |\lambda_i|$ so $|g(b_i)| \leq \max_{j \in \{1, \ldots, n\}} |\lambda_j| \leq s \max_{j \in \{1, \ldots, n\}} |\lambda_j| ||g_j|| \leq \sqrt{s} ||g||$. So $||g|| < 1$. Thus, with $\pi : E \to E/H$ denoting the canonical quotient map, there exist $a_1, \ldots, a_n \in B_E$ with $||a_i|| = ||b_i||$ and $\pi(a_i) = b_i$ for each $i$. The adjoint $\pi'$ of $\pi$ maps $(E/H)'$ isometrically onto $D$. Now let $f \in D$. Then $f = \pi'(g)$ where $g \in (E/H)'$, $||g|| = ||f||$. We have, writing $g = \sum_{j=1}^n \lambda_j g_j, \max_{1 \leq i \leq n} |f(a_i)| = \max_{1 \leq i \leq n} |g(b_i)| = \max_{1 \leq i \leq n} |\lambda_i| \geq t \max_{1 \leq i \leq n} |\lambda_i| ||g_i|| \geq t ||\sum_{1 \leq i \leq n} \lambda_i g_i|| = t ||g|| = t ||f||$.

Now, if the valuation is discrete we can modify the above proof by taking $s = t = 1$. Then the $b_i$ have norm $\leq 1$ (rather than $< 1$), but one can use that $E/H$ is a strict quotient i.e. there exist $a_1, \ldots, a_n \in E$ with $||a_i|| = ||b_i||$ and $\pi(a_i) = b_i$ for each $i$.

LEMMA 1.8. Let $D$ be a subspace of $E'$, $D$ of countable type. Then there is a sequence $a_1, a_2, \ldots \in B_E$ such that $\max_{1 \leq i \leq n} |f(a_i)| \geq t ||f||$ for all $f \in D$.

Proof. Let $D_1 \subset D_2 \subset \ldots$ be finite-dimensional subspaces of $D$, $\bigcup D_n$ is dense in $D$. Let $t \in (0, 1)$. By Lemma 1.7 there exists a finite set $F_n \subset B_D$ such that $\max_{1 \leq i \leq n} |f(x)| \geq t ||f||$ for all $f \in D_n$.

So, for $F^i := \bigcup_{n \in \mathbb{N}} F^i_n$ we obtain

$$(*) \quad ||f|| \geq \sup_{a \in F^i} |f(a)| \geq t ||f|| \quad (f \in \bigcup_{n \in \mathbb{N}} D_n).$$

Now $F := \bigcup_{t \in \mathbb{Q}(0, 1)} F^i$ is countable and $(*)$ implies $\sup_{a \in F} |f(a)| \geq t ||f||$ for all $f \in \bigcup_{n \in \mathbb{N}} D_n$, hence, by continuity, for all $f \in D$.

PROPOSITION 1.9. Let $X \subset E'$ be such that $X(a) := \{ f(a) : f \in X \}$ is separable for each $a \in E$ and $|X|$ is of countable type. Then $X$ is $\sigma$-decomposable in $E'$.

Proof. Let $g \in E' \setminus X$. Then $D := \{g\} \cup X$ is of countable type by Lemma 1.8 there exist $a_1, a_2, \ldots \in B_E$ such that

$$(*) \quad ||h|| = \sup_{a \in F^i} \max_{1 \leq i \leq n} |h(a_i)| \quad (h \in D).$$

For each $m, n \in \mathbb{N}$ the set $U_{mn} := \{ h \in E' : ||h(a_n)|| \leq \frac{1}{m} \}$ is an edged $w'$-zero neighbourhood. Its cosets, except for $g + U_{mn}$, cover $X \setminus (g + U_{mn})$ and by separability of $X(a_n)$ there exists a countable subcovering $F_{mn}$ no member of which contains $g$. Then $\bigcup_{m,n} F_{mn}$ still avoids $g$; it remains to be shown that it covers $X$. Suppose $f \in X$ is not covered. Then $f \in g + U_{mn}$ for all
T. KIYOSAWA AND W. H. SCHIKOF

Thus, \(j(X)\) is \(^\kappa\)-compact. Observe that

Moreover, in case (i), \(X\) is norm compact and the weak and norm topology coincide on \(X\).

**PROPOSITION 1.11.** Let \(X \subset E'\) be such that \(X(a)\) is separable for each \(a \in E\). Suppose that for each \(h \in \overline{X}^{\kappa}\) the set \(X \cup \{h\}\) is \(w'\)-metrizable. Then \(X\) is \(\sigma\)-decomposable in \(E'\).

**Proof.** Let \(g \in E' \setminus X\). If \(g \notin \overline{X}^{\kappa}\) then there exists a \(w'\)-zero neighbourhood \(U\) such that \((g + U) \cap X = \emptyset\). We may assume that \(U\) is of the form \(\{f \in E' : |f(a_1)| \leq \epsilon, \ldots, |f(a_n)| \leq \epsilon\}\) for some \(\epsilon > 0\), \(n \in \mathbb{N}\), \(a_1, \ldots, a_n \in E\). Then \(U\) is \(w'\)-closed and edged. By separability of \(X(a_1) \times \ldots \times X(a_n)\) only countably many of the cosets \(f + U : f \in X\) cover \(X\) and none of them contains \(g\). Now let \(g \in \overline{X}^{\kappa}\). By \(w'\)-metrizability there exist \(w'\)-neighbourhoods of zero \(U_1 \supset U_2 \supset \ldots\) such that \(X \cap \bigcap (g + U_n) = \emptyset\). We may suppose that the \(U_n\) are \(w'\)-closed and edged. By separability, like above, for each \(n\) the set \(X \setminus (g + U_n)\) is covered by countably many additive cosets of \(U_n\) none of them containing \(g\). Their union is a countable covering of \(X\) avoiding \(g\).

**2. EBERLEIN-ŠMULIAN THEORY**

We now apply the theory of §1. Recall ([5], p. 57) that \(E\) is said to have property \(\ast\) if for each subspace \(D \subset\) of countable type, every \(f \in D'\) has an extension \(\overline{f} \in E'\). By the non-archimedean Hahn-Banach Theorem [4], 4.8 every normed space over a spherically complete \(K\) has \((\ast)\). For general \(K\), spaces with a base, in particular spaces of countable type, have \((\ast)\) ([5], p. 58), and so have strongly polar spaces ([6], 4.2). Recall that \(E\) is assumed to be normpolar.

**THEOREM 2.1.** Let \(K\) be not locally compact, let \(X\) be a subset of \(E\) such that \(j(X)\) is compact for all \(f \in E'\). Then each one of the following properties implies that \(X\) is weakly compact and \(\sigma\)-metrizable.

(i) \(E\) has property \(\ast\).

(ii) \(E'\) is of countable type.

(iii) \([X]\) is of countable type.

Moreover, in case (i) \(X\) is norm compact and the weak and norm topology coincide on \(X\).

**Proof.** The natural isometry \(j : E \to E''\) is easily seen to be a homeomorphism of \(E\) with the weak topology onto \(j(E)\) with the restriction of the \(w'\)-topology \(\sigma(E'', E')\). We show that \(j(X)\) is \(\sigma\)-decomposable in \(E''\). First note that the predual \(E'\) is normpolar. In case (i), from weak precompactness of \(X\) it follows that \(X\) is norm precompact by [7], Th. 3 (the assumption made throughout [7] that \(E\) is complete is easily seen to be superfluous here). So \(j(X)\) is norm precompact in \(E''\) and therefore \(\sigma\)-decomposable by Corollary 1.10. For case (ii) observe that every \((w'\) - bounded subset of \(E''\) is \(w'\)-metrizable ([8], 6.1) which applies to \(j(X) \cup \{\theta\}\) for any \(\theta \in E''\). For each \(f \in E'\) the set \(j(X)(f) = f(X)\) is compact hence separable so \(j(X)\) is \(\sigma\)-decomposable in \(E''\) by Proposition 1.11. For case (iii) we can directly apply Corollary 1.10. Thus, \(j(X)\) is \(\sigma\)-decomposable, and from Corollary 1.6 we conclude that \(j(X)\) is \(w'\)-compact, so \(X = j^{-1}(j(X))\) is \(w'\)-compact. Observe that \(X\) is \(w'\)-bounded hence bounded by normpolarity ([6], 7.7).

We have seen in passing that \(j(X)\) is \(w'\)-metrizable in case (ii), so \(X\) is weakly metrizable. Now let \(X\) satisfy (iii). Then \(j(X)\) is of countable type so by Lemma 1.8 there exist \(f_1, f_2, \ldots \in B_E\) such that \(\|j(x)\| = \sup\|f_n(x)\|\) for all \(x \in X\). The formula \(d(x, y) = \sup\|f_n(x) - f_n(y)\|\) defines an ultrametric \(d\) on \(X\) (if \(d(x, y) = 0\) then \(\|f_n(x) - f_n(y)\| = 0\) for all \(n\) so \(\|x - y\| = 0\)). By boundedness of \(X\) the induced topology is weaker than the weak topology on \(X\), but by
weak compactness these topologies coincide and so $X$ is weakly metrizable. Finally, in case (i) apply [6], 5.12 to conclude that on $X$ the weak and norm topology coincide, and that therefore $X$ is norm compact and $w$-metrizable.

**REMARKS.**

1. If $K$ is not spherically complete the space $\ell^\infty$ does not have property $(\ast)$ ([4], 4.15 ($\delta \Rightarrow \gamma$)) but since $(\ell^\infty)' \cong c_0$ ([4], 4.17) it satisfies (ii) of the above Theorem, and so do the non-reflexive space $\ell^\infty \oplus \ell^\infty$ ([3], 2.3) and the space $D$ of [4], 4.1.

2. Let $K$ be not spherically complete, let $E := \ell^\infty$, let $X := \{0\} \cup \{e_1, e_2, \ldots\} \subset \ell^\infty$, when $e_1, e_2, \ldots$ are the unit vectors. Then (ii) and (iii) above hold. $X$ is weakly compact (since $\lim_{n \to \infty} e_n = 0$ weakly) but is obviously not norm compact.

3. The following example indicates that extending Theorem 2.1 to, say, metrizable locally convex spaces is doubtful. Let $E := K^N$ with the product topology. Then $E' \cong \bigoplus K$. Let $X := \{e_1, e_2, \ldots\}$ where $e_1, e_2, \ldots$ are the unit vectors of $K^N$. Then $E$ is of countable type so (i), (ii), (iii) of Theorem 2.1 are (formally) satisfied. For each $f \in E'$ we have $f(e_n) = 0$ for large $n$, so $f(X)$ is finite (hence compact) and contains 0. Yet, $X$ is not (weakly) compact as $0 = w - \lim_{n \to \infty} e_n \notin X$.

The following is now an almost trivial consequence of Theorem 2.1.

**COROLLARY 2.2.** (p-adic Eberlein-Smulian Theorem I) Let $K$ be not locally compact and let $X, E$ satisfy one of the conditions (i), (ii), (iii) of Theorem 2.1. Then the following are equivalent.

(a) $X$ is weakly compact.

(b) $X$ is weakly sequentially compact.

(c) $X$ is weakly countably compact.

**Proof.** Each one of the properties (a), (b), (c) implies compactness of $f(X)$ for all $f \in E'$.

By Theorem 2.1 $X$ is weakly metrizable and from that the equivalence of (a), (b), (c) follows easily.

**NOTE.** In Corollary 2.2, (a), (b), (c) are obviously equivalent to: 'for all $f \in E'$ the image $f(X)$ is compact.'

We have seen in the Introduction that Theorem 2.1 fails if $K$ is locally compact. We now investigate what happens to Corollary 2.2. Note that every normed space over $K$ has $(\ast)$.

**THEOREM 2.3.** (p-adic Eberlein-Šmulian Theorem II) Let $K$ be locally compact, let $X \subseteq E$. Then each one of the above statements (a), (b), (c) is equivalent to 'X is norm compact'.

**Proof.** We have (a) $\Rightarrow$ (c), (b) $\Rightarrow$ (c). It suffices to prove that (c) implies that $X$ is a norm compactoid (then $X$ is weakly metrizable since the norm and weak topology coincide on $X$ ([6], 5.12)). Suppose not. Then by [7], Th. 2 there is a $t \in (0,1]$ and a $t$-orthogonal sequence $e_1, e_2, \ldots$ in $X$ such that $\inf_n \|e_n\| > 0$. By (c) there is a weak accumulation point $a$ of $\{e_1, e_2, \ldots\}$. This $a$ is in the weak closure $D$ of $\{e_1, e_2, \ldots\}$ which equals the norm closure, so $a = \sum_{i=1}^\infty \lambda_i e_i$ where $\|\lambda_i e_i\| \to 0$. If $\lambda_j \neq 0$ for some $j$, let $U := \{x \in E : |\delta_j(x)| < |\lambda_j|\}$ where $\delta_j \in E'$ is an extension of the jth coordinate function $\Sigma \xi_j e_i \mapsto \xi_j$ on $D$. Then $a + U$ is a weak neighbourhood of $a$ but for each $n \in \mathbb{N}$, $n \neq j$ we have $|\delta_j(a - e_n)| = |\lambda_j|$ so $e_n \notin a + U$, a contradiction. Hence, $a = 0$.

But then $\{x \in E : |f(x)| < 1\}$ is a weak neighbourhood of $a$ containing no $e_n$ if $f \in E'$ is such that $f(e_n) = 1$ for all $n$. Contradiction, so $X$ is a norm compactoid.

**REMARK.** Corollary 2.2 for strongly polar spaces $E$ and Theorem 2.3 were first proved directly by the first author.
REMARK. The following 'relative' version of the Eberlein-Šmulian Theorem holds. (Compare [1], VIII §2, Theorem 1). Let $X \subset E$. Suppose one of the conditions (i), (ii), (iii) of Theorem 2.1 is satisfied. Then the following are equivalent. (a) $X$ is weakly relatively compact. (b) $X$ is weakly relatively sequentially compact. (γ) $X$ is weakly relatively countably compact. We leave the easy proof to the reader.

COUNTEREXAMPLES. We show that the previous theory fails for certain subsets $X$ of $\ell^\infty(I)$ where $I$ has at least the cardinality of the continuum, but is non-measurable, and where $K$ is not spherically complete. The $K$-valued characteristic function of a subset $S \subset E$ is denoted $\xi_S$ and is given by $\xi_S(x) := 1$ if $x \in S$, $\xi_S(x) := 0$ if $x \in I \setminus S$.

1. Let $X := \{\xi_S : S \subset I\}$. Then $X$ is a weakly compact but not weakly sequentially compact subset of $\ell^\infty(I)$.

Proof. $X$ is bounded and since $\ell^\infty(I)' \cong c_0(I)$ ([4],4.21) the weak topology on $X$ is the topology of pointwise convergence. Clearly the map $f \mapsto (f(i))_{i \in N}$ is a homeomorphism $X \to \{0,1\}'$, hence $X$ is weakly compact. To prove that $X$ is not weakly sequentially compact, let $\phi : I \to Y$ be a surjection where $Y := \{\xi_A : A \subset N\} \subset \ell^\infty$. The formula $\phi(x) = (\xi_{S_1}(x), \xi_{S_2}(x), \ldots) (x \in I)$ defines subsets $S_1, S_2, \ldots$ of $I$. If $\xi_{S_1}, \xi_{S_2}, \ldots$ is a subsequence of $\xi_{S_1}, \xi_{S_2}, \ldots$ then, by surjectivity of $\phi$, there is an $x \in I$ for which $(\xi_{S_1}(x), \xi_{S_2}(x), \ldots) = (1,0,1,0,1,\ldots)$, so the subsequence is not weakly convergent.

2. Let $Z := \{\xi_S : S \subset I, S\text{ countable}\} \subset \ell^\infty(I)$. Then $Z$ is weakly sequentially compact but not weakly compact.

Proof. Clearly the weak closure of $Z$ equals $X$ of above, so $Z$ is not weakly compact. On the other hand, if $\xi_{S_1}, \xi_{S_2}, \ldots$ is a sequence in $Z$ then $S := \cup S_n$ is countable and by a standard diagonal procedure one obtains a subsequence converging at all points of $S$, hence at all points of $I$, to an element of $Z$.

3. Let $T := \{\xi_{i_n} : i \in I\} \subset \ell^\infty(I)$. Then $f(T)$ is compact for all $f \in \ell^\infty(I)'$ but $T$ is not weakly countably compact.

Proof. Let $f \in \ell^\infty(I)'$. As $\ell^\infty(I)' \cong c_0(I)$ we have that $f(\xi_{i_n}) = 0$ except for $i \in \{i_1,i_2,\ldots\}$ where we may assume the $i_n \in I$ to be distinct. Then $\xi_{i_n} \to 0$ weakly so $T_i := \{0\} \cup \{\xi_{i_n} : n \in \mathbb{N}\}$ is weakly compact and $f(T) = f(T_i)$ is compact. However the only weak accumulation point of $\{\xi_{i_1}, \xi_{i_2}, \ldots\}$ is 0 $\notin T$ so that $T$ is not weakly countably compact.

REFERENCES