Cardinality and Mackey Topologies of Non-Archimedean Banach and Fréchet Spaces(*)

by

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Summary. Let $K$ be a non-Archimedean complete non-trivially valued field. One obtains the cardinality of non-Archimedean Fréchet spaces of countable type over $K$. This enables us to get a new characterization of the spherical completeness of $K$ in terms of the Hahn-Banach theorem and the Mackey-Arens theorem.

1. Introduction. Throughout the paper $K$ is a non-Archimedean valued field which is complete with respect to the metric induced by the non-trivial valuation $|.|$. If $E = (E, \tau)$ is a locally convex space (lcs) over $K$ with topology $\tau$ we denote by $P(E)$ or $P$ the family of (non-Archimedean) $\tau$-continuous seminorms. By $E'$ and $E^*$ we denote the topological and algebraic dual of $E$, respectively. For the basic notions and properties about lcs we refer to [21] when $K$ is spherically complete and to [17] when $K$ is not spherically complete. We recall only the following. A non-empty subset $B$ of a vector space $E$ (over $K$) is called absolutely convex if $ax + by \in B$ whenever $x, y \in B$ and $\alpha, \beta \in K$, $|\alpha| \leq 1$, $|\beta| \leq 1$. If $B$ is absolutely convex then the edged hull $B^e$ of $B$ is defined as follows. If $K$ has discrete valuation,


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(*) To the memory of Professor Andrzej Alexiewicz.

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then \( B^n = B \); if the valuation is dense then \( B^n = \bigcap_{|\lambda| > 1} \lambda B \) (cf. [17]).

If \( E \) is an lcs and \( p \in \mathcal{P}(E) \) by \( E_p \) we denote the quotient space \( E / \text{Ker} p \) endowed with the natural norm. The space \( E \) is said to be of countable type if, for every continuous seminorm \( p \) on \( E \) the normed space \( E_p \) is of countable type. Recall that a normed space \( E \) is of countable type if there exists a countable subset \( X \) of \( E \) such that the closure of the linear hull \( [X] \) of \( X \) in \( E \) equals \( E \); this means that either \( \dim E < \infty \) or the completion of \( E \) is isomorphic to the space \( c_0 \) of null \( K \)-sequences (cf. [16]). The weak topology \( \sigma(E, E') \) on \( E \) is of countable type (cf. [17]). A seminorm \( p \) on a \( K \)-vector space \( E \) is polar if \( p = \sup \{|f| : f \in E^*, |f| \leq p\} \). An lcs \( E \) is polar if there is a basis of continuous seminorms consisting of polar seminorms (cf. [17] also for examples). An lcs \( E \) is polar iff the polar neighbourhoods of zero form a neighbourhood basis of zero for the topology of \( E \) (cf. [17], Proposition 5.2). A metrizable and complete lcs will be called a Fréchet space. Two locally convex topologies on \( E \) will be called compatible if they define the same continuous linear functionals on \( E \).

It is natural to ask if \( E \) admits the Mackey topology \( \mu(E, E') \), i.e. the finest locally convex topology on \( E \) whose topological dual is still \( E' \).

In ([21], Theorem 4.17), van Tiel proved that the Mackey topology exists if \( K \) is spherically complete; in this case \( \mu(E, E') \) is determined by the seminorms \( x \mapsto \sup \{|f(x)| : f \in A\} \), where \( A \) runs through the collection of all subsets of \( E' \) which are bounded and \( c \)-compact in the weak topology \( \sigma(E', E) \). In [6] we proved that whenever \( K \) is not spherically complete then the space \( \ell^\infty \) of bounded \( K \)-sequences does not admit the Mackey topology; nevertheless, the weak topology \( \sigma(\ell^\infty, c_0) \) is of countable type and \( \ell^\infty \) admits the finest locally convex topology of countable type to be compatible with \( \sigma(\ell^\infty, c_0) \), this follows from Theorem 2.1 of [18] (cf. also [5], Theorem 2) and our Corollary 3.2.

The following question is still open (cf. [17]). Let \( E \) be a polar lcs over a non-spherically complete field. Does the polar Mackey topology exist on \( E \) (i.e. the strongest polar topology on \( E \) compatible with the original one)? We know only that the answer is positive for polarly barrelled and polarly bornological spaces (cf. [17], Corollary 7.9).

In this note (applying ideas of [7] and [8]) we extend results of [6] and [7] by showing the following.

**Theorem 1.1.** Let \( E \) be a vector space over \( K \) with \( \dim E \geq K^\# \), where \( K^\# \) denotes the cardinal number of \( K \) and \( \dim E \) denotes the algebraic dimension of \( E \). Let \( \Delta = \{(x, x) : x \in E\} \subset E \times E \). The following assertions are equivalent.

1. \( K \) is spherically complete.
2. For every locally convex topology \( \tau \) on \( E \) the space \((E, \tau) \) admits the
Mackey topology.

(3) There exists a locally convex topology \( \tau \) on \( E \) with \( (E, \tau)' \neq E^* \) such that \( (E, \tau) \) admits the Mackey topology.

(4) \( (E, \tau)' \neq \{0\} \) for every (complete) Hausdorff locally convex topology \( \tau \) on \( E \).

(5) For every \( f \in E^* \) and seminorms \( p, q \) on \( E \) such that \( |f| \leq \max\{p, q\} \), there exist \( f_1, f_2 \in E^* \) such that \( f = f_1 + f_2 \) and \( f_1 \) is \( p \)-continuous and \( f_2 \) is \( q \)-continuous.

(6) The diagonal \( \Delta \) has the following Hahn-Banach property. For every seminorm \( p \) on \( E \times E \) and every \( f \in \Delta^* \) such that \( |f| \leq p \), there exists a linear extension \( h \) of \( f \) to \( E \times E \) such that \( |h| \leq p \).

We will see, however (see Remark 3.4 (a)) that (6) implies (1) for any vector space \( E \) with \( \dim E \geq 1 \).

Since every linear functional on a vector space \( E \) is continuous with respect to the finest locally convex topology, Theorem 1.1 yields also the following.

**Corollary 1.2.** Let \( E \) be a lcs with \( \dim E \geq K^\# \). Then \( E \) admits the Mackey topology iff \( K \) is spherically complete or every linear functional on \( E \) is continuous.

We will see (Proposition 2.2) that \( \dim E \geq K^\# \) for any infinite-dimensional Fréchet space. This together with Theorem 1 yields the following.

**Corollary 1.3.** Let \( K \) be not spherically complete. Then no infinite-dimensional Fréchet space admits the Mackey topology.

We recall that by ([17], Corollary 7.9), every polar Fréchet space admits the polar Mackey topology.

The problem seems to be much more complicated when we are looking for the polar Mackey topology for non-metrizable spaces. We prove the following.

**Theorem 1.4.** The following conditions on \( K \) are equivalent.

(1) For every (polar) lcs over \( K \) the polar Mackey topology exists.

(2) For every lcs \( E \) over \( K \) of finite type (i.e. each continuous seminorm on \( E \) has finite-codimensional kernel) and bounded complete absolutely convex subsets \( A \) and \( B \) the closure \( A + B \) of \( A + B \) is complete.

(3) For every index set \( I \), every pair of closed, bounded, absolutely convex subsets \( A \) and \( B \) of \( K^I \) we have \( A + B \subset [A + B] \).

(4) If \( p, q \) are polar seminorms on some \( K \)-vector space \( E \) and if \( f \in E^* \) such that \( |f| \leq \max\{p, q\} \), then \( f = f_1 + f_2 \), where \( f_1, f_2 \in E^* \), \( f_1 \) is \( p \)-continuous, \( f_2 \) is \( q \)-continuous.
In Theorem 1.1 we used essentially the assumption \( \dim E \geq K^\# \). It is natural to ask whether there is a relation between the cardinality of \( K \) and the linear dimension of any Fréchet space \( E \) over \( K \). We will see (Propositions 2.2 and 2.5) that \( \dim E \geq K^\# \) for any infinite-dimensional Fréchet space and that \( E \) contains a closed subspace \( F \) such that \( \dim(E/F) = K^\# \).

2. Dimension of Fréchet spaces. It is known that \( \dim E \geq 2^{\aleph_0} \) for any infinite-dimensional real or complex Fréchet space; \( \dim E = 2^{\aleph_0} \), if \( E \) is additionally separable (cf. [12]). In [13] Popov showed that for every nonatomic measure space \( (\Omega, \Sigma, \mu) \) with \( \mu(\Omega_1) > 2^{\aleph_0} \) there exists a subset \( \Omega_1 \) of \( \Omega \) with \( \mu(\Omega_1) > 0 \) such that the real space \( L^p(\Omega_1) \), \( 0 < p < 1 \), has no Hausdorff \( 2^{\aleph_0} \)-dimensional quotient; recall that \( L^p(\Omega_1) \) has trivial topological dual.

We start with the following observation.

**Lemma 2.1.** \( \dim \ell^\infty = \aleph_0 \).

**Proof.** Since \( \dim K^\# = (K^\#)^{\aleph_0} \) (cf. [10], 5(3), p. 75) and \( \ell^\infty \subset K^\# \), we have \( \dim \ell^\infty \leq (K^\#)^{\aleph_0} \). On the other hand, we can proceed as in the proof of 5(3) of ([10], p. 75) and obtain \( \dim \ell^\infty \geq K^\# \). Now the conclusion follows from \( (K^\#)^{\aleph_0} = K^\# \) (cf. [15], Corollary 3.9). We give a direct proof of this fact: take any \( x = (x_n) \in K^\# \). For every \( n \in \mathbb{N} \) there exists a unique two-sided sequence \( (a_{jn} : j \in \mathbb{Z}) \) of a full set of representatives in \( \{x \in K : |x| \leq 1\} \) modulo \( \{x \in K : |x| \leq |\pi|, 0 < |\pi| < 1, \text{ such that } a_{jn} = 0 \text{ for large } j \in \mathbb{N} \) and \( x_n = \sum_{j=-\infty}^{\infty} a_{jn} \pi^j \) (cf. [19], Theorems 12.1 and 12.3). Let \( \sigma : \mathbb{N} \to \mathbb{Z} \times \mathbb{Z} \) be a bijection. Then the map \( T : ((x_n)) \mapsto \sum_{n \in \mathbb{N}} a_{\sigma(n)} \pi^n \) of \( K^\# \) into \( K^\# \) is (as easily seen) injective.

**Proposition 2.2.** Let \( E \) be a sequentially complete lcs over \( K \) containing an infinite-dimensional bounded set \( B \). Then there exists a continuous injective linear map \( T : \ell^\infty \to E \). In particular, \( \dim E \geq K^\# \) for any infinite-dimensional Fréchet space \( E \).

**Proof.** Since the linear span of the closed absolutely convex hull of \( B \) endowed with the Minkowski functional norm topology is an infinite-dimensional Banach space, we may assume that \( E \) is an infinite-dimensional Banach space. Hence \( E \) contains a subspace isomorphic to \( c_0 \). Let \( (e_n) \) be a Schauder basis of this subspace; we may assume that \( e_n \to 0 \). Then the map \( T : \ell^\infty \to E, T((b_n)) = \sum_{n=1}^{\infty} b_n e_n \) is continuous and injective.

**Examples 2.3.** By Proposition 2.2 infinite-dimensional Fréchet spaces have the dimension at least \( K^\# \). Proposition 2.2 applies also to get the same conclusion for the following spaces. (1) Any \((LF)\)-space (i.e. the inductive limit space of an increasing sequence of infinite-dimensional Fréchet
spaces, see [12], for the definition). (2) Any topological product of infinite-dimensional Fréchet spaces. (3) Any perfect sequence space $A \neq \emptyset$ endowed with its natural topology $\tau(A, \Lambda^*)$, where $\Lambda^*$ is the Köthe dual of $\Lambda$ and $\varphi$ is the space of all sequences in $K$ with only finitely many of non-zero coordinates. By ([1], Proposition 7), where the assumption that $K$ is spherically complete is not necessary, the space $\Lambda$ is complete. Moreover, if $(\alpha_n) \in \Lambda$, $\alpha_n \neq 0$, $n \in \mathbb{N}$, then $\{(\beta_n) : |\beta_n| \leq |\alpha_n|, n \in \mathbb{N}\}$ is an infinite-dimensional bounded set in $\Lambda$.

**Corollary 2.4.** If $E$ is an infinite-dimensional Fréchet space over $K$ of countable type, then $\dim E = K^\#$.

**Proof.** Obviously, $\dim E \geq K^\#$. The space $E$ is isomorphic to a subspace of a countable product of Banach spaces of countable type; hence $E$ is isomorphic to a subspace of the product $c_0^\mathbb{N}$. Now $\dim E \leq \dim c_0^\mathbb{N} \leq (K^\#)^\mathbb{N} = K^\#$ which completes the proof.

**Proposition 2.5.** Every infinite-dimensional Fréchet space $E = (E, \tau)$ contains a closed subspace $F$ such that $\dim(E/F) = K^\#$.

**Proof.** If $\tau = \sigma(E, E')$, then $E$ is isomorphic to the space $K^\mathbb{N}$ (cf. [20], Theorem 7). Now suppose that $\tau \neq \sigma(E, E')$. Then there exists a continuous seminorm $p$ on $E$ such that $\dim E_p = \infty$. Let $\|\cdot\|_p$ be the corresponding norm on $E_p$. Let, for some $t \in (0, 1)$, $(e_n)$ be a $t$-orthogonal sequence in $E_p$ with $a < \|e_n\|_p \leq 1$, $n \in \mathbb{N}$, $0 < a < 1$. Let $(f_n)$ be a similar sequence in $\ell^\infty/c_0$. Define a continuous linear map $T : \{e_n : n \in \mathbb{N}\} \rightarrow [f_n : n \in \mathbb{N}]$ by $T(\sum_{n \in \mathbb{N}} \lambda_n e_n) = \sum_{n \in \mathbb{N}} \lambda_n f_n$. Since $\ell^\infty/c_0$ is spherically complete ([16], Theorem 4.1), there exists by Ingleton's theorem (cf. [16], Theorem 4.8), a continuous linear extension $T_0 : E_p \rightarrow \ell^\infty/c_0$ of $T$. Let $F := \ker(T_0 \circ \pi_p)$, where $\pi_p : E \rightarrow E_p$ is the quotient map. Then $F$ is closed, so $E/F$ is a Fréchet space, infinite-dimensional, and $\dim E/F = \dim(\text{Im} T_0) \leq \dim(\ell^\infty/c_0) \leq K^\#$. By Proposition 2.2, $\dim E/F \geq K^\#$.

**Remark 2.6.** (1) In ([4], Proposition 2.6) De Grande-De Kimpe proved that any non-normable Fréchet space $E$ has a subspace isomorphic to $K^\mathbb{N}$. Moreover, she showed that if $K$ is spherically complete, this subspace is complemented in $E$. The latter result is even true for non-spherically complete $K$ if the dual $E'$ separates the points of $E$ (cf. [20], Corollary 9.1 (iv)).

(2) Proposition 2.5 suggests also the following question (still open for the real or complex Banach spaces, cf. [11,14]). Which infinite-dimensional Fréchet spaces over $K$ admit infinite-dimensional quotients of countable type? We shall say that $E$ has a Quotient if $E$ has such a quotient. We have the following partial answer.
(a) A Banach space has a Quotient iff $E$ contains a complemented copy of $c_0$ (cf. [3]).

(b) If the valuation of $K$ is discrete, then any infinite dimensional Banach space has a Quotient. This follows from ([16], Corollary 4.14).

(c) The space $\ell^\infty$ does not have a Quotient if $K$ has a dense valuation. Indeed, by ([16], Corollary 5.19) every continuous linear map of $\ell^\infty$ onto $c_0$ is compact.

(d) If $E$ is spherically complete and $K$ has a dense valuation, then $E$ does not have a Quotient. This follows from ([16], Corollary 5.20).

3. Proof of Theorem 1.1. We start with the following Lemma.

Lemma 3.1. Let $(E, \tau)$ be a lcs over $K$. Let $\{\tau_\alpha\}_{\alpha \in A}$ be a family of locally convex topologies on $E$ compatible with $\tau$. Then the topology $\xi = \sup \tau_\alpha$ is compatible with $\tau$ provided $K$ is spherically complete or every topology $\tau_\alpha$ is of countable type.

Proof. If $K$ is spherically complete we proceed as in the proof of Theorem 1 of [7]. Now assume that every $\tau_\alpha$ is of countable type. Observe that $(E, \xi)$ is of countable type. In fact $(E, \xi)$ is isomorphic to the diagonal $\Delta$ of the product $\prod_{\alpha \in A} (E, \tau_\alpha)$. The last product and also $\Delta$ (endowed with the relative topology) are spaces of countable type by ([17], Proposition 4.12). Since spaces of countable type are strongly polar, Theorem 4.2 of [17] combined with our Lemma 1 of [7] (and its proof) yields the following. Let $f \in (E, \xi)'$, $\varepsilon > 0$. There exist seminorms $p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}$ continuous in $\tau_{\alpha_1}, \tau_{\alpha_2}, \ldots, \tau_{\alpha_n}$, respectively, such that $|f| \leq \max\{p_{\alpha_1}, p_{\alpha_2}, \ldots, p_{\alpha_n}\}$. Then there are $f_1, f_2, \ldots, f_n \in E^*$ such that $f = f_1 + f_2 + \ldots + f_n$ and $|f_i| \leq (1 + \varepsilon)p_{\alpha_i}$, $1 \leq i \leq n$. Then (by assumption) $f \in (E, \tau)'$.

Corollary 3.2 [5, 18]. Every lcs $(E, \tau)$ over $K$ admits the finest locally convex topology $\mu$ of countable type compatible with $\tau$.

Proof. Let $(\tau_\alpha)_{\alpha \in A}$ be the family of all locally convex topologies on $E$ of countable type finer than $\sigma(E, E')$ and compatible with $\tau$; recall that $(E, \sigma(E, E'))$ is one of them. Then using Lemma 3.1 one gets that $\mu = \sup_{\alpha \in A} \tau_\alpha$ is compatible with $\tau$ and $(E, \mu)$ is of countable type.

Remark 3.3. It is known (cf. [9]) that when $K$ is not spherically complete, the space $(\ell^\infty, \sigma(\ell^\infty, c_0))$ has a Schauder basis but $\ell^\infty$ (with respect to the norm topology) is not of countable type. Using Lemma 3.1 (and its proof) one deduces also that if $(E, \tau)$ is an lcs such that $(E, \sigma(E, E'))$ has a Schauder basis $(e_n)$, then $E$ admits the finest locally convex topology $\xi$ such that $\sigma(E, E') \leq \xi \leq \tau$ and $(e_n)$ is a Schauder basis for $(E, \xi)$. In fact, $\xi$
is the finest compatible topology of countable type between \( \sigma(E,E') \) and \( \tau \) (observe that \( \xi \) and \( \sigma(E,E') \) have the same convergent sequences by ([17], Proposition 4.11)).

Now we are ready to prove our Theorem 1.1.

**Proof of Theorem 1.1.** (1) => (2) follows from Lemma 3.1 and (2) => (3) from the following observation. Let \( (x_\alpha) \) be a Hamel basis for \( E \). Then the topology \( \tau \) defined by the norm \( ||x|| = \max |t_\alpha| \), \( x = \sum t_\alpha x_\alpha \), where \( t_\alpha \in K \), is as in (3). Now we prove (3) => (4). Assume that \( (E,\tau) \) is a lcs as in (3) and \( E \) admits a locally convex topology \( \varphi_1 \) such that \( (E,\varphi_1)' = \{0\} \). Take \( f_0 \in E^* \setminus E' \) and \( x_0 \neq 0 \) in \( E \) such that \( f_0(x_0) = 2 \). Consider the map \( T : E \to E \) defined by \( T(x) = x - f_0(x)x_0 \), then \( T^2 = \text{id} \). Let \( \varphi_2 \) be the image by \( T \) of \( \varphi_1 \). Consider the topologies \( \psi_i = \sup \{\sigma(E,E'),\varphi_i\} \), \( i = 1,2 \). Observe that both topologies \( \psi_i \) are compatible with the original one of \( E \) : fix \( i \in \{1,2\} \). Let \( f \in (E,\psi_i)' \). Then there exist \( f_1, f_2, \ldots, f_n \in E' \) and a \( \varphi_i \)-continuous seminorm \( p \) such that \( |f| \leq \max \{|f_1|, \ldots, |f_n|, p\} \). Let \( H = \bigcap_{k=1}^n \{x \in E : f_k(x) = 0\} \). Then \( |f| \leq p \) on \( H \). Since \( H \) is a finite-dimensional subspace of \( E \), there exists a continuous linear extension of \( f \) to the space \( (E,\varphi_i) \). Since \( (E,\varphi_i)' = \{0\} \), we have \( f = 0 \) on \( H \). Hence \( f \) is a linear combination of \( f_1, \ldots, f_n \), so \( f \in E' \). On the other hand \( f_0 \) is continuous with respect to the topology \( \sup \{\varphi_1, \varphi_2\} \). In fact, if \( (x_\alpha) \) is a null net in \( E \) with respect to the topology \( \varphi = \sup \{\varphi_1, \varphi_2\} \), then \( T(x_\alpha) = x_\alpha - f_0(x_\alpha)x_0 \to 0 \) in \( \varphi_2 \) and \( x_\alpha \to 0 \) in \( \varphi_2 \). Now \( x_0 \neq 0 \), so \( f_0(x_\alpha) \to 0 \). By assumption (the Mackey topology \( \mu(E,E') \) exists) we have \( \sigma(E,E') \leq \sup \{\psi_1, \psi_2\} \leq \mu(E,E') \). This implies that \( f_0 \in E' \), a contradiction.

(4) => (1). Assume that \( K \) is not spherically complete. Since \( \dim E \geq K^\# \) and also \( \dim \ell^\infty/c_0 = K^\# \), there exists an index set \( A \) and a family of vector subspaces \( E_\alpha \) of \( E \), \( \alpha \in A \), such that \( E = \bigoplus_{\alpha \in A} E_\alpha \) (algebraically) with \( \dim E_\alpha = \dim \ell^\infty/c_0 \) for all \( \alpha \in A \). Endow every \( E_\alpha \) with the isomorphic copy of the original topology of \( \ell^\infty/c_0 \). Let \( \phi \) be the Hausdorff locally convex direct sum topology on \( E = \bigoplus_{\alpha \in A} E_\alpha \). Then \( (E,\phi) \) is a Hausdorff and complete lcs, cf. [21] (where the assumption that \( K \) is spherically complete is not necessary). From \( (\ell^\infty/c_0)' = \{0\} \) we obtain \( (E,\phi)' = \{0\} \).

(1) => (5). It is a direct consequence of our Lemma 1 of [7]. (5) => (2). If \( \mu(E,E') \) is the supremum topology of all locally convex topologies on \( E \) compatible with the original one, then \( \mu(E,E') \) is compatible with the original one; hence \( \mu(E,E') \) is the Mackey topology of \( E \). (1) => (6). This follows from Ingleton's Theorem, (cf. [21], Theorem 3.5). (6) => (5). We proceed similarly as in the proof of Lemma 1 of [7].

**Remark 3.4.** (1) Observe that the implication (6) => (1) is true for any
vector space $E$ over $K$ with $\dim E \geq 1$. In fact, assume that $K$ is not spherically complete. First, suppose that $\dim E = 1$, i.e. the space $E$ is algebraically isomorphic to $K$. According to ([16], p. 68) there exists a norm $\| \cdot \|$ on $K^2$, a one-dimensional space $D$ and $g \in D'$ with $|g(x)| \leq \|x\|$, $x \in D$, that does not admit an extension $g_0 \in (K^2, \| \cdot \|)'$ with $|g_0(x)| \leq \|x\|$, $x \in K^2$. Now let $T : K \times K \rightarrow E \times E$ be a linear bijection such that $T(D) = \Delta$. Set $f = g \circ T^{-1}|\Delta$ and $p(x) = \|T^{-1}(x)\|$, $x \in E \times E$. Then $f \in \Delta^*$, $|f(z)| \leq p(z)$ for all $z \in \Delta$. Suppose that there exists on $E \times E$ a linear extension $f_0$ of $f$ with $|f_0| \leq p$. Then $f_0 \circ T$ is a linear extension of $g$ satisfying $|f_0(Ty)| \leq p(Ty) = \|y\|$ for all $y \in K^2$, which is a contradiction. Now suppose that $\dim E > 1$. Take a non-zero element $a \in E$ and an algebraic complement $S$ of $[a]$. Take on $E \times E$ the seminorm $(s, \lambda a) \times (s', \gamma a) \mapsto p(\lambda a, \gamma a)$, where $p : [a] \times [a] \rightarrow \mathbb{R}$ is as above, and a linear functional $h : \Delta \rightarrow K$ defined by $(s, \lambda a) \times (s, \lambda a) \mapsto f(\lambda a, \lambda a)$, where $f$ is as above. Clearly $|h| \leq p$ on $\Delta$ but $h$ cannot be extended to a linear functional $h_0$ on $E \times E$ with $|h_0| \leq p$ by the first part.

(2) From the proof of Theorem 1.1, (3) $\Rightarrow$ (4), one deduces also the following. Let $(E, \tau)$ be an lcs with trivial topological dual. Then there exists a family $(\tau_\alpha)_{\alpha \in A}$ of locally convex topologies on $E$ such that every $(E, \tau_\alpha)$ is isomorphic to $(E, \tau)$ and $(E, \sup_{\alpha \in A} \tau_\alpha)' = E^*$. Therefore, if $E \neq \{0\}$, then $(E, \tau)$ does not admit the Mackey topology.

(3) We do not know any example of an lcs $E$ over a non-spherically complete $K$ with $\aleph_0 < \dim E < K^\#$ which admits the Mackey topology. Note that any lcs $(E, \tau)$ with $\dim E = \aleph_0$ admits the Mackey topology. Indeed, if $(\tau_\alpha)_{\alpha \in A}$ is the family of all locally convex topologies on $E$ compatible with $\tau$, then $(E, \sup_{\alpha \in A} \tau_\alpha)$ is of countable type ([17], Examples 4.5). Now Lemma 3.1 completes the proof.

(4) Using Corollary 1 we deduce also that the spaces considered in Examples 2.3 admit the Mackey topology iff $K$ is spherically complete.

4. Proof of Theorem 1.4. Let $E$ be a polar lcs. By a special covering of $E'$ (cf. [17], Definition 7.3) we mean a covering $G$ of $E'$ such that

(a) each member of $G$ is edged, $\sigma(E', E)$-bounded, $\sigma(E', E)$-complete;
(b) for each $A, B \in G$ there is a $C \in G$ such that $A \cup B \subset C$;
(c) for each $A \in G$ and $\lambda \in K$ there is a $B \in G$ with $\lambda A \subset B$.

For a special covering $G$ of $E'$ the $G$-topology on $E$ is the topology induced by the seminorms $x \mapsto \sup\{|f(x)| : f \in A\}$, where $A$ runs through $G$.

In order to prove Theorem 2 we shall need the following.

Lemma 4.1. Let $E$ be a polar lcs and suppose that the polar Mackey topology $\mu(E, E')$ exists. Then the family $G$ of all edged and absolutely convex
complete compactoids in \((E', \sigma(E', E))\) is a special covering and \(\mu(E, E')\) equals the \(G\)-topology.

**Proof.** For each \(\mu(E, E')\)-continuous polar seminorm \(p\) on \(E\) set \(A_p = \{ f \in E^* : |f| \leq p \}\). Then by (17), Proposition 7.4 and its proof, the topology \(\mu(E, E')\) is the \(G\)-topology, where \(G = \{ A_p : p \) is a \(\mu(E, E')\)-continuous seminorm on \(E\} \) is a special covering of \(E'\). Let \(A\) be an edged, absolutely convex complete compactoid in \((E \setminus \sigma(E, E))\) is a special covering and \(\mu(E, E)\) equals the \(Q\)-topology.

By (17), Proposition 7.4, the \(Q\)-topology is compatible with the original one of \(E\). Hence \(A\) is in \(G\). (This will show that \(Q\) is the largest possible special covering of \(E\)). Let \(Q = \{(\lambda A + F)^e : \lambda \in K, F \subseteq E, F\) be absolutely convex and \(\sigma(E', E)\)-bounded, \(\dim[F] < \infty\}\). It is easy to check that \(Q\) is a special covering of \(E\); the completeness of \((\lambda A + F)^e\) follows from (18, Theorem 4.1). By (17), Proposition 7.4, the \(Q\)-topology is compatible with the original one of \(E\). Hence \(A\) is in \(G\) and \(A \subseteq A_p\). Since by (17), Proposition 4.10, the set \(A\) is a polar set, we get equality.

Now we are ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** (1) \(\Rightarrow\) (2): Let \((F, \tau)\) be a lcs of finite type; then \(\sigma(F, F') = \tau\). The space \(E := (F', \sigma(F', F))\) admits the polar Mackey topology (by (1)) and by Lemma 4.1 it is the \(G\)-topology, where \(G\) equals the set of all edged, absolutely convex, complete compactoids in \((E', \sigma(E', E))\). By (b) above, for \(A, B \subseteq G\) the set \(A + B\) is complete; (2) now follows because the natural map \((F, \sigma(F, F')) \rightarrow (E', \sigma(E', E))\) is an isomorphism.

(2) \(\Rightarrow\) (3): Note that \([A + B]\) is of finite type, being a subspace of \(K^I\). By (2) one gets that the closure of \(A + B\) in \([A + B]\) is complete; hence the closure of \(A + B\) in \(K^I\) lies in \([A + B]\).

(3) \(\Rightarrow\) (4): Let \((e_i : i \in I)\) be an algebraic base for \(E\) and endow \(E\) with the finest locally convex topology. Then \(E^* = E'\) and \((E', \sigma(E', E))\) is isomorphic to the product \(K^I\). Let \(p, q\) be two polar seminorms on \(E\), then \(A = \{ f \in E' : |f| \leq p \}\) and \(B = \{ f \in E' : |f| \leq q \}\) are edged, absolutely convex, and complete. Hence we may apply (3) to conclude that \(A + B \subseteq [A + B]\). By polarity of \(p\) and \(q\) we have \((A + B)^e = (A + B)^00 = (U \cap V)^0\), where \(U, V\) are the unit balls of \(p, q\), respectively, and so \(U \cap V\) is the unit ball in the normed topology defined by \(\max\{p, q\}\). Hence by (3) one gets \(\{ f \in E' : |f| \leq \max\{p, q\} \} \subseteq (A + B)^e \subseteq [A + B] = [A] + [B]\). Thus, we can write every \(f \in E^*\) with \(|f| \leq \max\{p, q\}\) as \(g + h\), where \(g \in [A]\) and \(h \in [B]\); hence \(g\) is \(p\)-continuous and \(h\) is \(q\)-continuous.

(4) \(\Rightarrow\) (1): It suffices to show that if \(\tau_1\) and \(\tau_2\) are polar locally convex topologies on a lcs \((E, \tau)\) compatible with \(\tau\), then so is \(\sup\{\tau_1, \tau_2\}\). Let \(p\) be a \(\tau_1\)-continuous polar seminorm and \(q\) be a \(\tau_2\)-continuous polar seminorm on \(E\). Let \(f \in E^*, |f| \leq \max\{p, q\}\). Then, by (4), \(f = g + h\), where \(g\) is
$\tau_1$-continuous, $h$ is $\tau_2$-continuous, i.e. $g, h \in (E, \tau)'$. Hence $f$ is $\tau$-continuous and the proof is complete.

REFERENCES


